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Global threshold analysis on a diffusive host–pathogen model with hyperinfectivity and nonlinear incidence functions

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Abstract

In this paper, we are concerned with the mathematical analysis of a host–pathogen model with diffusion, hyperinfectivity and nonlinear incidence. We define the basic reproduction number $\Re_0$ by the spectral radius of the next generation operator, and study the relation between $\Re_0$ and the principal eigenvalue of the problem linearized at the disease-free steady state (DFSS). Under some assumptions, we show the threshold property of $\Re_0$: if $\Re_0 < 1$, then the DFSS is globally asymptotically stable (GAS), whereas if $\Re_0 > 1$, then the system is uniformly persistent and a positive steady state (PSS) exists. Moreover, for the special case where all parameters are constants, we show that the PSS is GAS for $\Re_0 > 1$. Numerical simulation suggests that the spatial heterogeneity could enhance the intensity of epidemic, whereas the diffusion effect could reduce it.

Keywords: Reaction–diffusion model; Bounded spatial domain; Basic reproduction number; Hyperinfectivity; Nonlinear incidence

1. Introduction

1.1. Two diffusive host–pathogen models with distinct dispersal rates

In recent years, diffusive host–pathogen models in a spatially heterogeneous environment towards understanding the dynamics of infectious diseases have been extensively studied [10,41,48]. Infectious diseases occur in heterogeneous environment on account of the variations in environmental conditions, for example, humidity, rainfall, temperature and availability of medical resources. Therefore, assuming that model parameters depend on a spatial variable is meaningful and important. In this research direction, the following diffusive host–pathogen model was

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proposed by Wu and Zou in [48]:

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S \Delta S + \lambda(x) - \beta(x) S P - a(x) S, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial I}{\partial t} = d_I \Delta I + \beta(x) S P - b(x) I, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial P}{\partial t} = c(x) I - m(x) P, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial S}{\partial \vartheta} = \dfrac{\partial I}{\partial \vartheta} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ (S, I, P)(x,0) = (S^0, I^0, P^0)(x) \geq 0, & x \in \Omega. \end{cases} \tag{1.1}$$

At space x and time t, $S := S(x,t)$, $I := I(x,t)$ and $P := P(x,t)$ represent respectively the densities of susceptible hosts, infected hosts and pathogens; The moving pattern of susceptible (resp. infected) hosts is modeled by the diffusion rate $d_S > 0$ (resp. $d_I > 0$); $\lambda(x)$ and $a(x)$ are the recruitment and mortality rate of susceptible hosts; Susceptible hosts get infection from pathogens with transmission rate $\beta(x)$; $b(x)$ and $m(x)$ are the removing and decay rates of infected hosts and pathogens, respectively; $c(x)$ stands for the reproduction rate of pathogens by infected hosts; We assume that the hosts' habitat is the domain Ω, which is a connected and bounded subset of $\mathbb{R}^n$ having smooth boundary $\partial\Omega$. The spatially dependent parameters $\lambda(x)$, $a(x)$, $\beta(x)$, $b(x)$, $c(x)$ and $m(x)$ are supposed to be continuous, strictly positive and uniformly bounded on $\bar{\Omega}$. $\frac{\partial}{\partial \vartheta}$ means the normal derivative along ϑ to $\partial\Omega$. Throughout the paper, if there are no specific requirements, we still denote $x \in \Omega, t > 0$ and $x \in \partial\Omega, t > 0$ as $(x,t) \in \Omega \times (0,\infty)$ and $(x,t) \in \partial\Omega \times (0,\infty)$, respectively. Inspired by the recent work of [10,41], model (1.1) was formulated by the assumptions that: (i) the pathogens remain immobile on $\bar{\Omega}$, so there is no diffusion term in P-equation; (ii) the consumption mechanism of pathogens due to the hosts getting infection from pathogens is neglected; (iii) the diffusion coefficients of susceptible and infected hosts are different due to the mobility patterns of these subpopulations being changed when they get the disease.

In [48], by overcoming the mathematical challenges and difficulties caused by (i) and (ii), i.e., establishing the boundedness of solution and verifying the asymptotic smoothness of the solution semiflow, the authors verified the existence of unique global solutions and a global attractor. By the classic method of next generation operator (NGO), the basic reproduction number (BRN) of (1.1) is obtained and proved as a threshold parameter that (1.1) is uniformly persistent if BRN exceeds one, while the disease-free steady state (DFSS) is globally asymptotically stable (GAS) if BRN is less than one. Combined with the spatial heterogeneity, the different effects of the mobility patterns of hosts as $d_S \to 0$ and $d_I \to 0$ on the disease dynamics were investigated through establishing the asymptotic profiles of positive steady state (PSS). One interesting concentration phenomenon occurs: the infected hosts will gather on some points of Ω if $d_I \to 0$. We also refer interested readers to [7,8,15–17,25–27,47] for relative studies on the asymptotic behaviors of PSS in other cases, especially, the different roles of diffusion rates of hosts in determining the disease dynamics.

The infection mechanism in system (1.1), called mass action mechanism, is represented by the bilinear incidence function $\beta(x)SP$, which was used in [3,10,41] to describe the epidemic outbreak of certain diseases. Considering the fact that cholera transmission also involves human-to-human pathways (e.g., [6,20,29]), in the recent work [43], Wang and Wang explored the following diffusive cholera model:

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S \Delta S + \lambda(x) - \alpha(x) S I - \beta(x) S P - a(x) S, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial I}{\partial t} = d_I \Delta I + \alpha(x) S I + \beta(x) S P - b(x) I, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial P}{\partial t} = c(x) I - m(x) P, & (x,t) \in \Omega \times (0,\infty), \end{cases} \tag{1.2}$$

with the same boundary and initial conditions as for model (1.1), where $\alpha(x)$ is the transmission rate between susceptible and infected hosts. Let $Q^0 = (U(x), 0, 0)$ be the unique DFSS of (1.2), where $U(x)$ satisfies

$$\begin{cases} 0 = d_S \Delta U + \lambda(x) - a(x)U, & x \in \Omega, \\ \dfrac{\partial U}{\partial \vartheta} = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

and $U(x)$ is GAS in $C(\bar{\Omega}, \mathbb{R})$. By defining the BRN of (1.2) as

$$\Re_0 = \sup_{\phi \in H^1(\Omega),\ \phi \neq 0} \left\{ \frac{\int_\Omega (\alpha U + \frac{c\beta U}{m})\phi^2 dx}{\int_\Omega \left(d_I |\nabla \phi|^2 + b\phi^2\right) dx} \right\}, \tag{1.4}$$

the authors in [43] obtained the threshold-type results of (1.2) in the sense that BRN determines the cholera persistence and extinction. Especially, when the BRN exceeds one, at least one PSS exists while DFSS is the global attractor of the system when BRN is less than one. Once the disease persists for a given reaction–diffusion epidemic system, it becomes important to investigate the profile of its spatial distribution, since it plays the role in helping decision-makers to predict the prevalence of disease transmission and to make some effective control of disease eradication.

1.2. Two diffusive cholera epidemic models with hyperinfectivity

Vibrio cholerae is the causative agent of Cholera. Recent laboratory studies have revealed that the newly shed vibrios remain highly toxic and infectious for several hours (see, e.g., [4,12]). It was reported in [21] that the infectivity of the newly shed vibrios are up to 700-fold as compared to the vibrios previously shed into the environment and grown for several months. Along this direction, Harley, Morris and Smith [12] classified the vibrios according to infectivity: hyperinfectious and lower-infectious of *Vibrio cholerae*, denoted by HI vibrios and LI vibrios, respectively. In a recent work, by setting a theoretical river with $x = 0$ and $x = L$ respectively representing two ends of the river, Wang and Wang [42] analyzed a reaction–advection–diffusion cholera model incorporating HI and LI vibrios and spatial heterogeneity. Let $B_1 = B_1(x,t)$ (resp. $B_2 = B_2(x,t)$) be the concentration of HI (resp. LI) vibrios in the water environment. The HI and LI vibrios studied in [42] are governed by the following equations:

$$\begin{cases} \dfrac{\partial B_1}{\partial t} = D_1 \Delta B_1 - \nu_1 \nabla B_1 + c(x)I + \theta_1(x)B_1\left(1 - \dfrac{B_1}{K_1(x)}\right) - \delta(x)B_1, & x \in (0,L),\ t > 0, \\ \dfrac{\partial B_2}{\partial t} = D_2 \Delta B_2 - \nu_2 \nabla B_2 + \delta(x)B_1 + \theta_2(x)B_2\left(1 - \dfrac{B_2}{K_2(x)}\right) - m(x)B_2, & x \in (0,L),\ t > 0, \\ D_i \dfrac{\partial B_i}{\partial x}(0,t) - \nu_i B_i(0,t) = \dfrac{\partial B_i}{\partial x}(L,t) = 0, & t > 0,\ i = 1,2, \\ B_i(x,0) = B_i^0(x) \geq 0, & 0 \leq x \leq L,\ i = 1,2, \end{cases} \tag{1.5}$$

where ν_1 (resp. ν_2) is the convection coefficient of HI (resp. LI) vibrios; $\theta_i(x)$ and $K_i(x)$ represent the intrinsic growth rate and the bacterial maximal capacity of HI and LI vibrios, respectively; $\delta(x)$ is the transfer rate from HI vibrios to LI vibrios; In particular, it was shown that the constant equilibrium is GAS under some special conditions. The authors also investigated the dependence of BRN on parameters. Crucially, the risk of infection will be underestimated if HI vibrios are not considered. Inspired by the work [42], Yang and Wang [49] further highlighted the importance of hyperinfectivity using a nonlocal and time-delayed diffusive model with constant parameters. Here, the mobility of infected individuals in latenty period results in a nonlocal delay. For simplicity, the convection coefficient and the loss of immunity of the cholera epidemic (that is, a flux from recovered individuals to susceptible individuals) are ignored. The model studied in [49] reads as

$$
\begin{cases}
\dfrac{\partial S}{\partial t} = D_S \Delta S + \lambda - aS - S\left(\alpha I + \beta_1 \dfrac{B_1}{B_1 + K_1} + \beta_2 \dfrac{B_2}{B_2 + K_2}\right), & (x,t) \in \Omega \times (0,\infty), \\
\dfrac{\partial I}{\partial t} = D_I \Delta I - (d + \gamma + m_2) I + e^{-d_I \tau} \mathscr{H}_1(x, S, I, B_1, B_2), & (x,t) \in \Omega \times (0,\infty), \\
\dfrac{\partial B_1}{\partial t} = D_1 \Delta B_1 + \xi I - \delta_1 B_1, & (x,t) \in \Omega \times (0,\infty), \\
\dfrac{\partial B_2}{\partial t} = D_2 \Delta B_2 + \delta_1 B_1 - \delta_2 B_2, & (x,t) \in \Omega \times (0,\infty), \\
\dfrac{\partial S}{\partial \vartheta} = \dfrac{\partial I}{\partial \vartheta} = \dfrac{\partial B_1}{\partial \vartheta} = \dfrac{\partial B_2}{\partial \vartheta} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\
(S, I, B_1, B_2)(x, 0) = (S^0, I^0, B_1^0, B_2^0)(x), & x \in \Omega,
\end{cases}
\tag{1.6}
$$

with

$$
\mathscr{H}_1(x, S, I, B_1, B_2) = \int_\Omega \Gamma(D_I \tau, x, y) S(t-\tau, y) \left(\alpha I(t-\tau, y) + \beta_1 \frac{B_1(t-\tau, y)}{B_1(t-\tau, y) + K_1} + \beta_2 \frac{B_2(t-\tau, y)}{B_2(t-\tau, y) + K_2} \right) \mathrm{d}y.
$$

With the consideration that HI and LI vibrios remain immobile in contaminated water, the recent work [44] further extended the works in [42,49] by studying the following model,

$$
\begin{cases}
\dfrac{\partial S}{\partial t} = D_S \Delta S + \lambda(x) - \alpha(x) S I - \mathscr{H}_2(x, S, B_1, B_2) - a(x) S, & (x,t) \in \Omega \times (0,\infty), \\
\dfrac{\partial I}{\partial t} = D_I \Delta I + \mathscr{H}_2(x, S, B_1, B_2) - (d(x) + \gamma(x) + m(x)) I, & (x,t) \in \Omega \times (0,\infty), \\
\dfrac{\partial B_1}{\partial t} = \xi(x) I - \delta_1(x) B_1, & (x,t) \in \Omega \times (0,\infty), \\
\dfrac{\partial B_2}{\partial t} = \delta_1(x) B_1 - \delta_2(x) B_2, & (x,t) \in \Omega \times (0,\infty), \\
\dfrac{\partial S}{\partial \vartheta} = \dfrac{\partial I}{\partial \vartheta} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\
(S, I, B_1, B_2)(x, 0) = (S^0, I^0, B_1^0, B_2^0)(x), & x \in \Omega,
\end{cases}
\tag{1.7}
$$

with

$$
\mathscr{H}_2(x, S, B_1, B_2) = S \left(\beta_1(x) \frac{B_1}{B_1 + K_1(x)} + \beta_2(x) \frac{B_2}{B_2 + K_2(x)} \right).
$$

By investigating the global dynamics of (1.7), the authors also highlighted the importance of restricting the mobility of susceptible humans in helping elimination of disease and the importance of incorporating HI vibrios in avoiding the underestimation of the risk of infection.

1.3. Our model and basic assumptions

Inspired by the recent works on the diffusive host–pathogen models with general incidence functions involving location-dependent parameters (see e.g., [31,32,45,50]), in this paper, we describe the interactions between susceptible individuals and two states of *Vibrio cholerae* with more general incidence rate. Note that the vibrios hyperinfectivity has not been considered in the models in [31,44,45,50]. As a continuation work, we incorporate general nonlinear incidence functions and the vibrios hyperinfectivity into model (1.7). Traditionally, the incidence rate of disease plays a vital role in investigating the dynamics of disease, as it measures how many individuals get infected per unit time. Many specific forms of incidence functions have been proposed in the literature. However, models that include a general class of incidence functions obey the advantage that dynamical results obtained in particular models can be extended to a broader class of models. Based on this, the researchers can pay attention to other biological factors, such as seasonality, spatial heterogeneity and growing domain that could bring more difficulties in analyzing the models. Here, we denote by $u_1 = u_1(x,t)$ (resp. $u_2 = u_2(x,t)$) the density of susceptible (resp. infected) humans and by $v_1 = v_1(x,t)$ (resp. $v_2 = v_2(x,t)$) the concentration of HI (resp. LI) vibrios in the water environment. For convenience, we briefly introduce some assumptions imposed on our main model:

(A1) We suppose that human hosts' habitat is a connected spatial domain $\Omega \subset \mathbb{R}^n,\ n \geq 1$, with sufficiently smooth boundary $\partial\Omega$;

(A2) We assume that the mobility for susceptible and infected humans is allowed only in Ω, and the mobility patterns for human hosts are modeled by the distinct diffusion rates. The recovered individuals never lose immunity as those in [14];

(A3) We denote by $N(x,u_1)$ the growth term of susceptible humans accounting for both production and natural mortality;

(A4) The nonlinear incidence function $f(x,u_1,u_2)$ represents the direct (i.e., human-to-human) transmission rate, and the nonlinear incidence functions $g(x,u_1,v_1)$ and $h(x,u_1,v_2)$ represent respectively the indirect transmission rates between susceptible humans and HI state of vibrios, and between susceptible humans and LI state of vibrios.

Biologically, we assume that the general nonlinear functions $N(x,u_1)$, $f(x,u_1,u_2)$, $g(x,u_1,v_1)$ and $h(x,u_1,v_2)$ satisfy:

(B1) $N(x,u_1) \in C^{0,1}(\bar{\Omega}\times\mathbb{R}^+)$ is decreasing in u_1. There exists a unique $u_1^P \in C^2(\Omega,\mathbb{R}^+)\cap C^1(\bar{\Omega},\mathbb{R}^+)$ such that

$$D_S \Delta u_1^P + N(x,u_1^P) = 0, \quad x \in \Omega, \qquad \frac{\partial u_1^P}{\partial \vartheta} = 0, \quad x \in \partial\Omega.$$

As in [32, Lemma 2.2], we further assume that if $w(x,t)$ satisfies the diffusive equation

$$\begin{cases} \dfrac{\partial w}{\partial t} = D_S \Delta w + N(x,w), & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial w}{\partial \vartheta} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ w(x,0) = w_0(x) \in C(\bar{\Omega},\mathbb{R}^+), & x \in \bar{\Omega}. \end{cases} \tag{1.8}$$

Then, $\lim_{t\to\infty} w(x,t) = u_1^P(x)$ in $C(\bar{\Omega},\mathbb{R}^+)$. Furthermore, if $N(x,v) = N(v)$, independent of x, then $u_1^P(x) \equiv u_1^P$. Clearly, $N(x,u_1)$ satisfies the following two commonly used functional forms: $N(x,u_1) = \Lambda - \tilde{d}u_1$ (see e.g., [22]) and $N(x,u_1) = \Lambda - \tilde{d}u_1 + \tilde{r}u_1(1-u_1/\mathscr{K})$ with $\tilde{d} \geq \tilde{r}$ (see e.g., [28]). Moreover, the characteristics of $N(x,u_1)$ imply that $N(x,u_1) \leq \Lambda - \tilde{d}u_1$ for some $\Lambda > 0$ and $\tilde{d} > 0$;

(B2) $f(x,u_1,u_2), g(x,u_1,v_1), h(x,u_1,v_2) \in C^1(\bar{\Omega}\times\mathbb{R}^+\times\mathbb{R}^+)$ increase monotonically with respect to the second and third variables and concave down with respect to the third variable, that is, when the number of infected humans or the vibrios is large, then the incidence rate will response slowly as compared to linearly to the increase with respect to u_2, v_1 and v_2 [39,50]. Furthermore, $f(x,u_1,0) = f(x,0,u_2) = g(x,u_1,0) = g(x,0,v_1) = h(x,u_1,0) = h(x,0,v_2) = 0$ for all $x \in \bar{\Omega}$ and $u_1,u_2,v_1,v_2 \geq 0$, which means that no infection occurs if one of u_1, u_2, v_1 and v_2 equals zero; For convenience, we let

$$f_{u_1}(x,u_1,u_2) = \frac{\partial f(x,u_1,u_2)}{\partial u_1},\ f_{u_2}(x,u_1,u_2) = \frac{\partial f(x,u_1,u_2)}{\partial u_2},\ g_{u_1}(x,u_1,v_1) = \frac{\partial g(x,u_1,v_1)}{\partial u_1},$$
$$g_{v_1}(x,u_1,v_1) = \frac{\partial g(x,u_1,v_1)}{\partial v_1},\ h_{u_1}(x,u_1,v_2) = \frac{\partial h(x,u_1,v_2)}{\partial u_1} \text{ and } h_{v_2}(x,u_1,v_2) = \frac{\partial h(x,u_1,v_2)}{\partial v_2};$$

Table 1
Parameters used in model (1.9).

Variable or Parameter	Description
$d(x)$	Infected human death rate
$\gamma(x)$	Infected human recover rate
m	Disease-induced mortality rate
$\xi(x)$	Infected human shedding rate
$\delta_1(x)$	Transformation rate from HI vibrios to LI vibrios
$\delta_2(x)$	Losing rate of LI state of vibrios
D_S	Diffusion coefficient of susceptible humans
D_I	Diffusion coefficient of infected humans

(B3) There exist positive constants $K_1, K_2 > 0$ such that, for $u_i > 0, v_i > 0,\ i = 1, 2,\ 0 < f(x, u_1, u_2) < K_1 u_1 u_2$, $0 < g(x, u_1, v_1) < K_2 u_1$ and $0 < h(x, u_1, v_2) < K_2 u_1$ for all $x \in \Omega$. The assumption for f is based on the fact that the contact number of susceptible humans per unit time by infected humans cannot exceed the contact number in the bilnear form.

In the homogeneous case, some specific forms of nonlinear incidence function $f(x, u_1, u_2)$ commonly used in the literature fulfill (**B**2)-(**B**3). Some concrete examples are: bilinear incidence $\alpha(x)u_1u_2$ (see [45,47]); saturation incidence $\alpha(x)u_1u_2/(1 + ku_2)$ (see [9]); Beddington–DeAngelis incidence $\alpha(x)u_1u_2/(1 + k_1u_1 + k_2u_2)$ (see [5]); Crowley–Martin incidence $\alpha(x)u_1u_2/[(1 + k_1u_1)(1 + k_2u_2)]$ (see [51]). The assumptions for g and h also permit the saturation incidence, Beddington–DeAngelis incidence and Crowley–Martin incidence, but do not permit the bilinear incidence.

With these considerations, we shall investigate the dynamics of the following host–pathogen model:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = D_S \Delta u_1 + N(x, u_1) - f(x, u_1, u_2) - g(x, u_1, v_1) - h(x, u_1, v_2), & (x, t) \in \Omega \times (0, \infty), \\ \dfrac{\partial u_2}{\partial t} = D_I \Delta u_2 + f(x, u_1, u_2) + g(x, u_1, v_1) + h(x, u_1, v_2) - (d(x) + \gamma(x) + m)u_2, & (x, t) \in \Omega \times (0, \infty), \\ \dfrac{\partial v_1}{\partial t} = \xi(x)u_2 - \delta_1(x)v_1, & (x, t) \in \Omega \times (0, \infty), \\ \dfrac{\partial v_2}{\partial t} = \delta_1(x)v_1 - \delta_2(x)v_2, & (x, t) \in \Omega \times (0, \infty), \end{cases} \tag{1.9}$$

with

$$\begin{cases} \dfrac{\partial u_1}{\partial \vartheta} = \dfrac{\partial u_2}{\partial \vartheta} = 0, & (x, t) \in \partial\Omega \times (0, \infty), \\ (u_1, u_2, v_1, v_2)(x, 0) = (u_1^0, u_2^0, v_1^0, v_2^0)(x), & x \in \Omega. \end{cases} \tag{1.10}$$

Table 1 lists parameter meanings for the model (1.9). All location-dependent parameters $d(x)$, $\gamma(x)$, $\xi(x)$, $\delta_1(x)$, $\delta_2(x)$, are supposed to be uniformly bounded and strictly positive on $\bar{\Omega}$. The nonnegative initial conditions, $(u_1^0, u_2^0, v_1^0, v_2^0)(x)$, $x \in \bar{\Omega}$ are continuous functions. Constants m, D_S and D_I are strictly positive.

Remark 1. For simplicity, we consider the losing rate δ_2 only for LI vibrios. Even if we consider another losing rate δ_3 for HI vibrios by changing δ_1 in the third equation of (1.9) to $\delta_1 + \delta_3$, the analysis and main results in this paper would be essentially unchanged.

System (1.9) should be studied in a suitable phase space. Let $\mathbb{X} := C(\bar{\Omega}, \mathbb{R}^4)$ be the set of vector valued continuous functions on $\bar{\Omega}$ and let $\|\phi\|_{\mathbb{X}} := \max(\|\phi_1\|_\infty, \|\phi_2\|_\infty, \|\phi_3\|_\infty, \|\phi_4\|_\infty)$, $\forall \phi = (\phi_1, \phi_2, \phi_3, \phi_4)^{\mathrm{T}} \in \mathbb{X}$ be its norm, where T denotes the transpose operation of a vector and $\|\psi\|_\infty := \sup_{x \in \Omega} |\psi(x)|$, $\forall \psi \in C(\bar{\Omega}, \mathbb{R})$. Let

$\mathbb{X}^+ := C(\bar{\Omega}, \mathbb{R}^4_+)$ be the positive cone of $\mathbb{X}$. Throughout of this paper, we set

$$q^+ := \max_{x\in\bar{\Omega}}\{q(x)\} \text{ and } q^- := \min_{x\in\bar{\Omega}}\{q(x)\},$$

where $q \in \{d, \gamma, \xi, \delta_1, \delta_2\}$.

1.4. Statement of our main results on system (1.9)

We first discuss the existence of the solution for system (1.9). Then, we study the global dynamics of (1.9) in terms of BRN. Our first result is concerned with the well-posedness of (1.9). Let $u := (u_1, u_2, v_1, v_2)^{\mathrm{T}}$.

Theorem 1.1. *For any $\phi \in \mathbb{X}^+$, system (1.9) has a unique global nonnegative classical solution $u(\cdot, t; \phi)$ on $t \in [0, \infty)$ with $u(\cdot, 0; \phi) = \phi$. The semiflow $\Upsilon(t) : \mathbb{X}^+ \to \mathbb{X}^+$ generated by (1.9) is defined by*

$$\Upsilon(t)\phi = (u_1(\cdot, t; \phi), u_2(\cdot, t; \phi), v_1(\cdot, t; \phi), v_2(\cdot, t; \phi))^{\mathrm{T}}, \quad \forall\, x \in \bar{\Omega}, \quad t \geq 0.$$

Moreover, $\Upsilon(t) : \mathbb{X}^+ \to \mathbb{X}^+$ is point dissipative and system (1.9) has a connected global attractor in $\mathbb{X}^+$.

Remark 2. We first prove that the model (1.9) has a unique nonnegative solution $u(\cdot, t; \phi)$ defined on $t \in [0, \tau_{max})$, which is a consequence of simple application of the results in [19] and [34, Theorem 3.1 in Chapter 7]. Then, we confirm the existence of a unique global solution and define $\Upsilon(t) : \mathbb{X}^+ \to \mathbb{X}^+$ as the solution semiflow generated by (1.9). The proof of the point dissipativity of $\Upsilon(t)$ is not trivial since the diffusion rates D_S and D_I are distinct. On account of the absence of diffusion term in v_1 and v_2-equation of (1.9), we also need to verify the asymptotic smoothness of $\Upsilon(t)$ by utilizing the so-called κ-contraction condition. Hence the conditions in Theorem 2.4.6 in [11] are fulfilled, and (1.9) has a connected global attractor.

Under the assumption (**B**1), system (1.9) possesses a unique DFSS, denoted by $(u_1^P(x), 0, 0, 0)$, where $u_1^P(x)$ is the unique PSS of (1.8). In order to study the global dynamics of solutions of (1.9), our starting point is to define BRN, $\Re_0$, of (1.9), identified by the spectral radius of NGO as proceeded in [46, Section 3]. Linearizing (1.9) at DFSS and denoting

$$\mathscr{B} = \begin{pmatrix} D_I\Delta + f_{u_2}(\cdot, u_1^P(\cdot), 0) - (d(\cdot) + \gamma(\cdot) + m) & g_{v_1}(\cdot, u_1^P(\cdot), 0) & h_{v_2}(\cdot, u_1^P(\cdot), 0) \\ \xi(\cdot) & -\delta_1(\cdot) & 0 \\ 0 & \delta_1(\cdot) & -\delta_2(\cdot) \end{pmatrix} \tag{1.11}$$

and $\bar{u} = (u_2, v_1, v_2)^T$, we get the following linear cooperative subsystem:

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} = \mathscr{B}\bar{u}, & (x, t) \in \Omega \times (0, \infty), \\ \dfrac{\partial u_2}{\partial \vartheta} = 0, & (x, t) \in \partial\Omega \times (0, \infty), \\ \bar{u}(x, 0) = \bar{u}^0(x), & x \in \Omega. \end{cases} \tag{1.12}$$

Denote by $\bar{\Upsilon}(t)$ and $\tilde{\Upsilon}(t)$ the semigroups generated by $\mathscr{B}$ and

$$B = \begin{pmatrix} D_I\Delta - (d(\cdot) + \gamma(\cdot) + m) & 0 & 0 \\ \xi(\cdot) & -\delta_1(\cdot) & 0 \\ 0 & \delta_1(\cdot) & -\delta_2(\cdot) \end{pmatrix},$$

respectively. It is readily seen that $\mathscr{B} = B + F$ with

$$F = \begin{pmatrix} f_{u_2}(\cdot, u_1^P(\cdot), 0) & g_{v_1}(\cdot, u_1^P(\cdot), 0) & h_{v_2}(\cdot, u_1^P(\cdot), 0) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, we introduce the NGO,

$$\mathscr{L}\phi(x) = \int_0^\infty F(x)\tilde{\Upsilon}(t)\phi(x)dt = F(x)\int_0^\infty \tilde{\Upsilon}(t)\phi(x)dt,\ \ \phi \in C(\bar{\Omega}, \mathbb{R}^3),\quad x \in \bar{\Omega}, \tag{1.13}$$

where $\mathscr{L}$ belongs to $C(\bar{\Omega}, \mathbb{R}^3_+)$ mapping the initial infection distribution ϕ to the total new infected distribution during the infection period. By [46], the BRN of (1.9), is defined by the spectral radius of $\mathscr{L}$, that is,

$$\Re_0 = r(\mathscr{L}) = \sup\{|\lambda|, \lambda \in \sigma(\mathscr{L})\}, \tag{1.14}$$

where $r(\mathscr{L})$ and $\sigma(\mathscr{L})$ are respectively the spectral radius and spectrum of $\mathscr{L}$. As those in [46, Lemma 2.3(d)] and [48, Lemma 3.1, 3.2], the following result indicates that BRN, $\Re_0$, is closely related to the principal eigenvalues of eigenvalue problems. For convenience, we let

$$H(x) = f_{u_2}(x, u_1^P(x), 0) + \frac{\xi(x)[\delta_2(x)g_{v_1}(x, u_1^P(x), 0) + \delta_1(x)h_{v_2}(x, u_1^P(x), 0)]}{\delta_1(x)\delta_2(x)}. \tag{1.15}$$

Theorem 1.2. *Let $\tilde{\lambda}_0$ and η^0 be the principal eigenvalues of*

$$\begin{cases} -D_I\Delta\psi + (d(x)+\gamma(x)+m)\psi = \tilde{\lambda}H(x)\psi, & x \in \Omega, \\ \dfrac{\partial\psi}{\partial\vartheta} = 0, & x \in \partial\Omega, \\ \psi \in C^2(\Omega, \mathbb{R}) \cap C^1(\bar{\Omega}, \mathbb{R}), \end{cases} \tag{1.16}$$

and

$$\begin{cases} D_I\Delta\varphi - (d(x)+\gamma(x)+m)\varphi + H(x)\varphi = \eta\varphi, & x \in \Omega, \\ \dfrac{\partial\varphi}{\partial\vartheta} = 0, & x \in \partial\Omega, \\ \varphi \in C^2(\Omega, \mathbb{R}) \cap C^1(\bar{\Omega}, \mathbb{R}), \end{cases} \tag{1.17}$$

respectively. We then have

(i) $\Re_0 - 1$ *has the same sign as* $s(\mathscr{B}) = \sup\{\mathrm{Re}\lambda, \lambda \in \sigma(\mathscr{B})\}$*, the spectral bound of* $\mathscr{B}$*;*
(ii) $\Re_0 = 1/\tilde{\lambda}_0$*;*
(iii) $\Re_0 - 1$ *and* $s(\mathscr{B})$ *have the same sign as* η^0*;*
(iv) *If* $\Re_0 \geq 1$*, then* $s(\mathscr{B})$ *is the principal eigenvalue of*

$$\begin{cases} \lambda\begin{pmatrix}\psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = \mathscr{B}\begin{pmatrix}\psi_2\\ \psi_3\\ \psi_4\end{pmatrix}, & x \in \Omega, \\ \dfrac{\partial\psi_2}{\partial\vartheta} = 0, & x \in \partial\Omega, \\ \psi_2 \in C^2(\Omega, \mathbb{R}) \cap C^1(\bar{\Omega}, \mathbb{R}),\ \ \psi_3, \psi_4 \in C(\bar{\Omega}, \mathbb{R}). \end{cases} \tag{1.18}$$

(v) *If* $\delta_1(x) \equiv \delta_1$ *and* $\delta_2(x) \equiv \delta_2$*, then* $s(\mathscr{B})$ *is the principal eigenvalue of* (1.18) *with a strongly positive eigenfunction.*

Remark 3. By (ii) of Theorem 1.2 and variational formula, $\Re_0$ can be expressed by

$$\Re_0 = \frac{1}{\tilde{\lambda}_0} = \sup_{\psi \in H^1(\Omega), \psi \neq 0} \frac{\int_\Omega H(x)\psi^2 dx}{\int_\Omega [D_I|\nabla\psi|^2 + (d(x)+\gamma(x)+m)\psi^2]dx}. \tag{1.19}$$

In terms of $\Re_0$, our main results on disease extinction and persistence will be stated in the following three cases: $\Re_0 < 1$, $\Re_0 > 1$ and $\Re_0 = 1$.

Theorem 1.3. *Let (**B**1)-(**B**3) hold. If $\Re_0 < 1$, then the DFSS $(u_1^P(x), 0, 0, 0)$ is GAS, i.e.,*

$$\lim_{t\to\infty} \|u(\cdot, t; \phi) - (u_1^P(\cdot), 0, 0, 0)\|_{\mathbb{X}} = 0, \text{ uniformly for all } \phi \in \mathbb{X}^+.$$

Theorem 1.4. *Let (**B**1)-(**B**3) hold. If $\Re_0 > 1$, then there exists $\sigma > 0$ such that for any $\phi \in \mathbb{X}^+$ with $\phi_2(\cdot) \not\equiv 0$ or $\phi_3(\cdot) \not\equiv 0$ or $\phi_4(\cdot) \not\equiv 0$, we have*

$$\liminf_{t\to\infty} \tilde{z}(x, t; \phi) \geq \sigma, \text{ uniform for all } x \in \Omega, \tag{1.20}$$

where $\tilde{z} \in \{u_1, u_2, v_1, v_2\}$. That is, (1.9) is uniformly persistent. Moreover, (1.9) with (1.10) admits at least a PSS in $\mathbb{X}^+$.

We now make the following additional assumption:

(B4) $f_{u_2}(\cdot, u_1, 0)$, $g_{v_1}(\cdot, u_1, 0)$ and $h_{v_2}(\cdot, u_1, 0)$ are Lipschitz continuous on u_1. There exists a positive constant $K_3 > 0$ such that $g(\cdot, u_1, v_1) < K_3 u_1 v_1$ and $h(\cdot, u_1, v_2) < K_3 u_1 v_2$.

Theorem 1.5. *Let (**B**1)-(**B**4) hold. If $\Re_0 = 1$, then DFSS $(u_1^P(x), 0, 0, 0)$ is GAS.*

We now present our results in the case that all model parameters remain strictly positive constants. In this case, model (1.9) is converted to the following one,

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = D_S \Delta u_1 + N(u_1) - f(u_1, u_2) - g(u_1, v_1) - h(u_1, v_2), & (x, t) \in \Omega \times (0, \infty), \\ \dfrac{\partial u_2}{\partial t} = D_I \Delta u_2 + f(u_1, u_2) + g(u_1, v_1) + h(u_1, v_2) - (d + \gamma + m)u_2, & (x, t) \in \Omega \times (0, \infty), \\ \dfrac{\partial v_1}{\partial t} = \xi u_2 - \delta_1 v_1, & (x, t) \in \Omega \times (0, \infty), \\ \dfrac{\partial v_2}{\partial t} = \delta_1 v_1 - \delta_2 v_2, & (x, t) \in \Omega \times (0, \infty), \end{cases} \tag{1.21}$$

with boundary and initial conditions (1.10). Clearly, (1.21) has a DFSS $E_0^{(1.21)} = (u_1^c, 0, 0, 0)$, where u_1^c is a positive constant. As in Remark 3, $\Re_0$ can be calculated as

$$[\Re_0] = \left(f_{u_2}(u_1^c, 0) + \frac{\xi[\delta_2 g_{v_1}(u_1^c, 0) + \delta_1 h_{v_2}(u_1^c, 0)]}{\delta_1 \delta_2} \right) / (d + \gamma + m).$$

According to Theorems 1.3, 1.4, and 1.5, we directly have:

Corollary 1.1. *Let (**B**1)-(**B**4) hold. If $[\Re_0] \leq 1$, then $E_0^{(1.21)}$ is GAS, while if $[\Re_0] > 1$, then system (1.21) is uniformly persistent.*

Let $E^* = (u_1^*, u_2^*, v_1^*, v_2^*)$ be the constant PSS of (1.21). The global stability of E^* requires the following additional condition:

(B5) $f_{u_1}(u_1^c, 0) = g_{u_1}(u_1^c, 0) = h_{u_1}(u_1^c, 0) = 0$. For any $u_1 > 0$, $v_1 > 0$ and $v_2 > 0$,

$$\left(\frac{v_1}{v_1^*} - \frac{f(u_1^*, u_2^*) g(u_1, v_1)}{f(u_1, u_2^*) g(u_1^*, v_1^*)} \right) \left(\frac{f(u_1, u_2^*) g(u_1^*, v_1^*)}{f(u_1^*, u_2^*) g(u_1, v_1)} - 1 \right) \leq 0,$$

and

$$\left(\frac{v_2}{v_2^*} - \frac{f(u_1^*, u_2^*) h(u_1, v_2)}{f(u_1, u_2^*) h(u_1^*, v_2^*)} \right) \left(\frac{f(u_1, u_2^*) h(u_1^*, v_2^*)}{f(u_1^*, u_2^*) h(u_1, v_2)} - 1 \right) \leq 0.$$

Theorem 1.6. *Let (**B**1)-(**B**3) and (**B**5) hold. If $[\Re_0] > 1$, then the system (1.21) has the unique constant PSS, and it is GAS.*

For example, if f is a bilinear incidence rate and g and h are saturation incidence rates, then (**B**5) is satisfied. In fact, if $f(u_1,u_2)=\beta u_1u_2$, $g(u_1,v_1)=\alpha_1u_1v_1/(1+k_1v_1)$ and $h(u_1,v_2)=\alpha_1u_1v_2/(1+k_2v_2)$ $(\beta,\alpha_1,\alpha_2,k_1,k_2>0)$, we then have, for any $u_1>0$, $v_1>0$ and $v_2>0$,

$$\begin{aligned}&\left(\frac{v_1}{v_1^*}-\frac{f(u_1^*,u_2^*)g(u_1,v_1)}{f(u_1,u_2^*)g(u_1^*,v_1^*)}\right)\left(\frac{f(u_1,u_2^*)g(u_1^*,v_1^*)}{f(u_1^*,u_2^*)g(u_1,v_1)}-1\right)=\left(\frac{v_1}{v_1^*}-\frac{v_1}{v_1^*}\frac{1+k_1v_1^*}{1+k_1v_1}\right)\left(\frac{v_1^*}{v_1}\frac{1+k_1v_1}{1+k_1v_1^*}-1\right)\\&=\frac{k_1}{v_1^*}(v_1-v_1^*)\left(\frac{v_1^*}{1+k_1v_1^*}-\frac{v_1}{1+k_1v_1}\right)\ \le 0,\end{aligned}$$

and

$$\left(\frac{v_2}{v_2^*}-\frac{f(u_1^*,u_2^*)h(u_1,v_2)}{f(u_1,u_2^*)h(u_1^*,v_2^*)}\right)\left(\frac{f(u_1,u_2^*)h(u_1^*,v_2^*)}{f(u_1^*,u_2^*)h(u_1,v_2)}-1\right)=\frac{k_2}{v_2^*}(v_2-v_2^*)\left(\frac{v_2^*}{1+k_2v_2^*}-\frac{v_2}{1+k_2v_2}\right)\ \le 0.$$

Note that the last inequalities can be obtained from the monotonicity of $x/(1+k_ix)$, $i=1,2$.

The rest of this paper is organized in the following plan. In Section 2, we summarize the well-posedness of (1.9) and provide the proof of Theorem 1.1. Section 3 is spent on investigating the relationship between $\Re_0$ and the principal eigenvalues of eigenvalue problems, and completing the proof of Theorem 1.2. In Section 4, we mainly provide the proof of Theorem 1.3, which indicates that the DFSS $(u_1^P(x),0,0,0)$ is GAS, meaning that disease would go extinct under the condition $\Re_0<1$. The uniform persistence of system (1.9) is achieved through the proof of Theorem 1.4 in Section 5. Section 6 is devoted to studying the critical case that $\Re_0=1$, which confirms that the DFSS $(u_1^P(x),0,0,0)$ is GAS in this critical case. The proof of Theorem 1.6 is given in Section 7. In this part, we confirm the existence and uniqueness of E^* and its global stability under some additional assumptions when all model parameters remain constant. Section 8 is devoted to the numerical simulation that supports the theoretical results. The paper ends with a brief discussion in Section 9.

2. The well-posedness of system (1.9): Proof of Theorem 1.1

Theorem 1.1 will be shown by the following lemmas. According to [34, Theorem 3.1 in Chapter 7], we first prove the results concerning the local solution of (1.9) with (1.10) on $\mathbb{X}^+$.

Lemma 2.1. *For any $\phi\in\mathbb{X}^+$, there exists a positive constant $\tau_{max}=\tau_{max}(\phi)>0$ such that problem (1.9)–(1.10) has the unique nonnegative noncontinuable mild solution $u(\cdot,t)=u(\cdot,t;\phi)$, $u(\cdot,0;\phi)=\phi$, defined on $[0,\tau_{max})$. In particular, $u(\cdot,t;\phi)\in\mathbb{X}^+$ is the classical solution to problem (1.9)–(1.10), defined on $[0,\tau_{max})$.*

Proof. Let us define linear operators A_i, $i=1,2,3,4$ by

$$A_1:=D_S\Delta,\quad A_2:=D_I\Delta-(d(\cdot)+\gamma(\cdot)+m),\quad A_3:=-\delta_1(\cdot),\quad A_4:=-\delta_2(\cdot).$$

For $i=1,2,3,4$, let $\{T_i(t)\}_{t\ge0}$ be the C_0-semigroups generated by A_i with suitable domains. For $\phi\in\mathbb{X}^+$ and $t\ge0$, let $\mathscr{T}(t)\phi:=(T_1(t)\phi_1,T_2(t)\phi_2,T_3(t)\phi_3,T_4(t)\phi_4)^{\mathrm{T}}$. One can then see that $\{\mathscr{T}(t)\}_{t\ge0}$ is also a C_0-semigroup (see also [24, Section 1.2]) on $\mathbb{X}^+$ to itself. The mild solution $u=(u_1,u_2,v_1,v_2)^{\mathrm{T}}$ to problem (1.9) with (1.10) can be written as

$$u(t)=\mathscr{T}(t)\phi+\int_0^t\mathscr{T}(t-s)\mathscr{F}(u(\cdot,s))ds,\quad u(0)=\phi=\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix}=\begin{pmatrix}u_1^0\\u_2^0\\v_1^0\\v_2^0\end{pmatrix}\in\mathbb{X}^+,$$

where

$$\mathscr{F}(\phi):=\begin{pmatrix}\mathscr{F}_1(\phi)\\\mathscr{F}_2(\phi)\\\mathscr{F}_3(\phi)\\\mathscr{F}_4(\phi)\end{pmatrix}=\begin{pmatrix}N(\cdot,\phi_1)-f(\cdot,\phi_1,\phi_2)-g(\cdot,\phi_1,\phi_3)-h(\cdot,\phi_1,\phi_4)\\f(\cdot,\phi_1,\phi_2)+g(\cdot,\phi_1,\phi_3)+h(\cdot,\phi_1,\phi_4)\\\xi(\cdot)\phi_2\\\delta_1(\cdot)\phi_3\end{pmatrix},\quad\phi\in\mathbb{X}^+.$$

One can easily check that $\mathscr{F}$ is a Lipschitz function on $\mathbb{X}^+$. Denote $c_m = \min\{\hat{c} \geq 0 : \min_{x\in\bar{\Omega}}[\mathscr{F}_1(\phi)(x)+\hat{c}\phi_1(x)] \geq 0\}$. Note that, for any $\phi \in \mathbb{X}^+$, c_m is finite because

$$\mathscr{F}_1(\phi) \geq -(K_1\|\phi\|_{\mathbb{X}} + 2K_2)\,\phi_1,$$

by virtue of assumptions (**B**1) and (**B**3). For any $\phi \in \mathbb{X}^+$, $x \in \bar{\Omega}$ and $\ell \geq 0$, we then have

$$\phi(x)+\ell\mathscr{F}(\phi)(x) = \begin{pmatrix} \phi_1 + \ell(N(\cdot,\phi_1) - f(\cdot,\phi_1,\phi_2) - g(\cdot,\phi_1,\phi_3) - h(\cdot,\phi_1,\phi_4)) \\ \phi_2 + \ell(f(\cdot,\phi_1,\phi_2) + g(\cdot,\phi_1,\phi_3) + h(\cdot,\phi_1,\phi_4)) \\ \phi_3 + \ell\xi(\cdot)\phi_2 \\ \phi_4 + \ell\delta_1(\cdot)\phi_3 \end{pmatrix} \geq \begin{pmatrix} \phi_1[1-\ell c_m] \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}.$$

Therefore, we obtain

$$\lim_{\ell\to 0^+} \frac{1}{\ell}\mathrm{dist}(\phi + \ell\mathscr{F}(\phi), \mathbb{X}^+) = 0 \quad \forall\phi \in \mathbb{X}^+.$$

We then see from [34, Theorem 3.1 in Chapter 7] that problem (1.9)–(1.10) has the unique desired solution $u(\cdot,t;\phi)$ on $[0,\tau_{max})$, where $0 < \tau_{max} \leq \infty$. This completes the proof of Lemma 2.1.

We next prove the following lemma on the existence of a global solution and a global attractor.

Lemma 2.2.

(i) *The solution $u(\cdot,t;\phi)$ to problem (1.9)–(1.10) with $\phi \in \mathbb{X}^+$ is global. More strongly, $u(\cdot,t;\phi)$ is ultimately uniformly bounded.*

(ii) *The solution to problem (1.9)–(1.10) induces a semiflow $\Upsilon(t) : \mathbb{X}^+ \to \mathbb{X}^+$ $(t \geq 0)$, and it admits a connected global attractor on $\mathbb{X}^+$.*

Proof. We first prove (i). Let $u(\cdot,t;\phi)$ be the solution of (1.9). Then we prove $u(\cdot,t;\phi)$ is ultimately bounded. The proof is not trivial since $D_S \neq D_I$. Therefore, we will prove (i) step by step.

Step 1. $u_1(\cdot,t)$ is ultimately bounded. From the u_1-equation of (1.9), it is readily seen that $u_1(\cdot,t)$ satisfies $\frac{\partial u_1}{\partial t} \leq D_S\Delta u_1 + N(x,u_1)$, that is, u_1 is a subsolution of

$$\begin{cases} \dfrac{\partial \bar{u}_1}{\partial t} = D_S\Delta\bar{u}_1 + N(x,\bar{u}_1), & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial \bar{u}_1}{\partial \vartheta} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ \bar{u}_1(x,0) = u_1^0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

It then follows from assumption (**B**1) that (2.1) has a unique PSS $u_1^P(x)$, which is GAS. Combining with the comparison principle, we have

$$\limsup_{t\to\infty} u_1(x,t) \leq u_1^P(x), \text{ uniformly for } x \in \bar{\Omega}. \tag{2.2}$$

Accordingly, $u_1(x,t)$ is ultimately bounded, i.e.,

$$\limsup_{t\to\infty} \|u_1(\cdot,t)\|_\infty \leq \|u_1^P(\cdot)\|_\infty := \mathbf{M}_0. \tag{2.3}$$

Step 2. $u_2(\cdot,t)$ is ultimately bounded. We can obtain $u(x,t) \in \mathbb{X}^+$ by using the standard comparison argument and maximum principle. Hence, $u_1(x,t) > 0$ for all $(x,t) \in \Omega \times (0,\infty)$. This together with divergence theorem, u_1 and u_2 equation imply that

$$\begin{aligned} \frac{\partial}{\partial t}\int_\Omega (u_1+u_2)\mathrm{d}x &= \int_\Omega N(x,u_1)\mathrm{d}x - \int_\Omega (d(x)+\gamma(x)+m)u_2\mathrm{d}x \\ &\leq \int_\Omega N(x,0)\mathrm{d}x + \int_\Omega d(x)u_1\mathrm{d}x - \int_\Omega d(x)(u_1+u_2)\mathrm{d}x \\ &\leq N^+|\Omega| + d^+\mathbf{M}_0|\Omega| - d^-\int_\Omega (u_1+u_2)\mathrm{d}x, \end{aligned}$$

where $N^+ := \max_{x\in\bar\Omega} N(x,0)$ and $|\Omega|$ is the volume of Ω. It then follows that

$$\limsup_{t\to\infty} \|u_2(\cdot,t)\|_1 \le \limsup_{t\to\infty}(\|u_1(\cdot,t)\|_1 + \|u_2(\cdot,t)\|_1) \le \mathbf{M}_1 := (N^+ + d^+\mathbf{M}_0)|\Omega|/d^-, \tag{2.4}$$

where $\|\cdot\|_p$ $(1\le p<\infty)$ denotes the L^p-norm. Note that $\mathbf{M}_1$ is independent of $\phi\in\mathbb{X}^+$.

We now claim the L^{2^k} bounded estimate of u_2, i.e., for any $k\ge 0$, $k\in\mathbb{Z}$,

$$\limsup_{t\to\infty} \|u_2(\cdot,t)\|_{2^k} \le \mathbf{M}_{2^k}, \tag{2.5}$$

where $\mathbf{M}_{2^k} > 0$ is independent of $\phi\in\mathbb{X}^+$. In what follows, we shall utilize the method of mathematical induction to verify (2.5). Clearly, (2.4) satisfies the case for $k=0$. We now assume that (2.5) holds for $k-1\ge 0$, i.e., there exists $\mathbf{M}_{2^{k-1}} > 0$ such that, for any $\phi\in\mathbb{X}^+$,

$$\limsup_{t\to\infty} \|u_2(\cdot,t)\|_{2^{k-1}} \le \mathbf{M}_{2^{k-1}}. \tag{2.6}$$

Multiplying the u_2-equation of (1.9) by $u_2^{2^k-1}$ and then integrating the obtained equation yield

$$\begin{aligned}\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega u_2^{2^k}\mathrm{d}x = {}& D_I\int_\Omega u_2^{2^k-1}\Delta u_2 \mathrm{d}x + \int_\Omega (f(x,u_1,u_2)+g(x,u_1,v_1)+h(x,u_1,v_2))u_2^{2^k-1}\mathrm{d}x\\ &- \int_\Omega (d(x)+\gamma(x)+m)u_2^{2^k}\mathrm{d}x.\end{aligned} \tag{2.7}$$

Note that

$$\begin{aligned}D_I\int_\Omega u_2^{2^k-1}\Delta u_2\mathrm{d}x \le -D_I\int_\Omega \nabla u_2\cdot\nabla u_2^{2^k-1}\mathrm{d}x = {}& -(2^k-1)D_I\int_\Omega(\nabla u_2\cdot\nabla u_2)u_2^{2^k-2}\mathrm{d}x\\ = {}& -\frac{2^k-1}{2^{2k-2}}D_I\int_\Omega \left|\nabla u_2^{2^{k-1}}\right|^2\mathrm{d}x.\end{aligned}$$

Then (2.7) can be rewritten as

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega u_2^{2^k}\mathrm{d}x \le -E_k\int_\Omega \left|\nabla u_2^{2^{k-1}}\right|^2\mathrm{d}x + \int_\Omega (f(x,u_1,u_2)+g(x,u_1,v_1)+h(x,u_1,v_2))u_2^{2^k-1}\mathrm{d}x,$$

where $E_k = \frac{2^k-1}{2^{2k-2}}D_I$. Applying (**B**3), we get

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega u_2^{2^k}\mathrm{d}x \le -E_k\int_\Omega \left|\nabla u_2^{2^{k-1}}\right|^2\mathrm{d}x + \int_\Omega (K_1u_1u_2 + 2K_2u_1)u_2^{2^k-1}\mathrm{d}x. \tag{2.8}$$

Further from (2.3), we know that

$$\int_\Omega u_1u_2^{2^k}dx \le (\mathbf{M}_0+1)\int_\Omega u_2^{2^k}dx, \quad \text{for} \quad t\ge t_1,$$

and

$$\int_\Omega u_1u_2^{2^k-1}dx \le (\mathbf{M}_0+1)\int_\Omega u_2^{2^k-1}dx, \quad \text{for} \quad t\ge t_1. \tag{2.9}$$

for some $t_1>0$. We now use Young's inequality

$$ab \le \epsilon a^p + C_\epsilon b^q,$$

where $a,b,\epsilon,q>0$, $p>1$, $C_\epsilon = (\epsilon p)^{-q/p}q^{-1}$ and $p^{-1}+q^{-1}=1$. By setting $p=2^k$ and $q=2^k/(2^k-1)$, and applying Young's inequality, we get the following estimation concerning (2.9):

$$\int_\Omega u_2^{2^k-1}dx \le \epsilon\int_\Omega 1^{2^k}dx + C_\epsilon\int_\Omega u_2^{2^k}dx.$$

Hence, (2.8) will be converted to

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega u_2^{2^k}dx \le -E_k\int_\Omega \left|\nabla u_2^{2^{k-1}}\right|^2dx + C_k\int_\Omega u_2^{2^k}dx + D_k, \quad t\ge t_1, \tag{2.10}$$

where $C_k = (K_1+2K_2C_\epsilon)(\mathbf{M}_0+1)$ and $D_k = 2K_2(\mathbf{M}_0+1)\epsilon|\Omega|$. Without loss of generality, we can assume that $t_1=0$ by taking the solution at time t_1 as the initial condition. Let $\zeta = u_2^{2^{k-1}}$ and $\epsilon_1 = \min(E_k/(2C_k), 1/2)$. As in [1, Proof of Theorem 3.1], by Gagliardo–Nirenberg's inequality and Young's inequality, we obtain the following inequality:

$$\|\zeta\|_2^2 \le \epsilon_1\|\nabla\zeta\|_2^2 + C_{\epsilon_1}\|\zeta\|_1^2,$$

where $C_{\epsilon_1} = (\text{constant})\epsilon_1^{-n/2}$. We then have that

$$-\frac{E_k}{2C_k}\|\nabla\zeta\|_2^2 \le -\epsilon_1\|\nabla\zeta\|_2^2 \le -\|\zeta\|_2^2 + C_{\epsilon_1}\|\zeta\|_1^2,$$

and hence,

$$-E_k\int_\Omega |\nabla u_2^{2^{k-1}}|^2 dx \le -2C_k\int_\Omega u_2^{2^k}dx + 2C_kC_{\epsilon_1}\left(\int_\Omega u_2^{2^{k-1}}dx\right)^2.$$

Hence, we can estimate (2.10) as follows:

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega u_2^{2^k}dx \le -C_k\int_\Omega u_2^{2^k}dx + 2C_kC_{\epsilon_1}\left(\int_\Omega u_2^{2^{k-1}}dx\right)^2 + D_k. \tag{2.11}$$

Here, (2.6) implies

$$\limsup_{t\to\infty}\int_\Omega u_2^{2^{k-1}}dx \le \mathbf{M}_{2^{k-1}}^{2^{k-1}}$$

and hence, (2.11) implies

$$\limsup_{t\to\infty}\|u_2(\cdot,t)\|_{2^k} \le \sqrt[2^k]{\frac{2C_kC_{\epsilon_1}\mathbf{M}_{2^{k-1}}^{2^k} + D_k}{C_k}} =: \mathbf{M}_{2^k}.$$

Thus, (2.5) holds for any $k \ge 0$, $k \in \mathbb{Z}$.

According to the continuous embedding from $L^q(\Omega)$ to $L^p(\Omega)$, $q \ge p \ge 1$, (2.5) implies that, for every $p > 1$, there exists a constant $\mathbf{M}_p > 0$ such that, for any $\phi \in \mathbb{X}^+$,

$$\limsup_{t\to\infty}\|u_2(\cdot,t)\|_p \le \mathbf{M}_p. \tag{2.12}$$

As in the proof of [48, Lemma 2.4], we directly have

$$\limsup_{t\to\infty}\|u_2(\cdot,t)\|_\infty \le \mathbf{M}_\infty, \tag{2.13}$$

where $\mathbf{M}_\infty > 0$ is independent of $\phi \in \mathbb{X}^+$. In fact, by (2.3) and (2.12), there exists a $t_m > 0$ such that

$$\|u_1(\cdot,t)\|_\infty \le \mathbf{M}_0 + 1 \quad \text{and} \quad \|u_2(\cdot,t)\|_p \le \mathbf{M}_p + 1$$

for all $t > t_m - 1$. Let $p > n/2$ and $a > n/(2p)$ so that Y_a is continuously embedded in $C(\bar{\Omega})$, where $Y_a,\ 0 < a < 1$ is the fractional power space equipped with the graph norm. By assumption (**B**3), as in the proof of [48, Lemma 2.4], we have that, for all $t > t_m$,

$$\begin{aligned}
\|A_2^a u_2(\cdot,p)\|_p \le& \|A_2^a T_2(1)u_2(t-1)\|_p + \int_{t-1}^t \|T_2(t-s)\left[K_1u_1(\cdot,s)u_2(\cdot,s) + 2K_2u_1(\cdot,s)\right]\|_p ds\\
\le& \mathbf{M}_a\|u_2(\cdot,t-1)\|_p + (\mathbf{M}_0+1)\mathbf{M}_a\int_{t-1}^t \frac{K_1(\mathbf{M}_p+1)+2K_2}{(t-s)^a}ds\\
\le& \mathbf{M}_a(\mathbf{M}_p+1) + \frac{(\mathbf{M}_0+1)\mathbf{M}_a\left[K_1(\mathbf{M}_p+1)+2K_2\right]}{1-a}
\end{aligned}$$

where $\mathbf{M}_a$ is a positive constant. The inequality (2.13) then follows from the fact that Y_a is continuously embedded in $C(\bar{\Omega})$.

Step 3. $v_1(\cdot,t)$ and $v_2(\cdot,t)$ are ultimately bounded. By (2.13) and the third equation of (1.9), we have

$$\limsup_{t\to\infty}\|v_1(\cdot,t)\|_\infty \le \frac{\xi^+\mathbf{M}_\infty}{\delta_1^-} =: \mathbf{M}_{v_1}.$$

Moreover, by the fourth equation of (1.9), we have

$$\limsup_{t\to\infty} \|v_2(\cdot,t)\|_\infty \le \frac{\delta_1^+ \mathbf{M}_{v_1}}{\delta_2^-} =: \mathbf{M}_{v_2}.$$

Consequently, we obtain

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{\mathbb{X}} \le \max(\mathbf{M}_0, \mathbf{M}_\infty, \mathbf{M}_{v_1}, \mathbf{M}_{v_2}).$$

This completes the proof of (i).

We next prove (ii). It follows from [34, Theorem 3.1 (d) in Chapter 7] that the solution u of problem (1.9)–(1.10) induces a semiflow $\Upsilon(t) : \mathbb{X}^+ \to \mathbb{X}^+$ ($t \ge 0$). As v_1 and v_2-equations of (1.9) do not have the diffusion terms, $\Upsilon(t)$ lacks the compactness. To overcome this issue, we show the asymptotic smoothness of $\Upsilon(t)$ by applying [11, Lemma 2.3.4]. For this purpose, we define the following Kuratowski measure of non-compactness:

$$\kappa(\mathbb{B}) := \inf\{r \in \mathbb{R} : \mathbb{B} \text{ has a finite cover of diameter less than } r\}, \quad \forall \mathbb{B} \subset \mathbb{X}^+. \tag{2.14}$$

Define the right-hand side of the v_1 and v_2-equations of (1.9) by

$$G_1(u_2, v_1) = \xi(\cdot)u_2 - \delta_1(\cdot)v_1, \text{ and } G_2(v_1, v_2) = \delta_1(\cdot)v_1 - \delta_2(\cdot)v_2.$$

It then follows that

$$\frac{\partial G_1(u_2, v_1)}{\partial v_1} = -\delta_1(\cdot) \le -\delta_1^-, \text{ and } \frac{\partial G_2(v_1, v_2)}{\partial v_2} = -\delta_2(\cdot) \le -\delta_2^-.$$

Following [48, Lemma 2.6], we first express $\Upsilon(t)$ as $\Upsilon(t) = \Upsilon_1(t) + \Upsilon_2(t)$, where

$$\Upsilon_1(t)\phi = \left\{u_1(\cdot,t;\phi), u_2(\cdot,t;\phi), \int_0^t e^{-\delta_1(\cdot)(t-s)}\xi(\cdot)u_2(\cdot,s;\phi)ds, \int_0^t e^{-\delta_2(\cdot)(t-s)}\delta_1(\cdot)v_1(\cdot,s;\phi)ds\right\}, \ t \ge 0,$$

and

$$\Upsilon_2(t)\phi = \left\{0, 0, e^{-\delta_1(\cdot)t}\phi_3, e^{-\delta_2(\cdot)t}\phi_4\right\}, \ t \ge 0.$$

With the help of [48, Lemma 2.5], one can see that

$$\mathscr{S}_1 = \left\{\int_0^t e^{-\delta_1(\cdot)(t-s)}\xi(\cdot)u_2(\cdot,s;\phi)\mathrm{d}s\right\} \text{ and } \mathscr{S}_2 = \left\{\int_0^t e^{-\delta_2(\cdot)(t-s)}\delta_1(\cdot)v_1(\cdot,s;\phi)ds\right\}$$

are precompact for any $t > 0$. Therefore, for all $t > 0$, $\kappa(\Upsilon_1(t)\mathbb{B}) = 0$. Moreover, $\Upsilon_2(t)$ can be estimated as

$$\|\Upsilon_2(t)\|_{\text{op}} = \sup_{\phi\in\mathbb{X}, \|\phi\|_{\mathbb{X}}\ne 0} \frac{\|\Upsilon_2(t)\phi\|_{\mathbb{X}}}{\|\phi\|_{\mathbb{X}}} \le \mathrm{e}^{-\delta^- t} \sup_{\phi\in\mathbb{X}, \|\phi\|_{\mathbb{X}}\ne 0} \frac{\|\phi\|_{\mathbb{X}}}{\|\phi\|_{\mathbb{X}}} = \mathrm{e}^{-\delta^- t},$$

where $\|\cdot\|_{\text{op}}$ is the operator norm and $\delta^- := \min_{\bar{\Omega}}\{\delta_1^-, \delta_2^-\} > 0$. Then, for $t > 0$, we obtain

$$\kappa(\Upsilon(t)\mathbb{B}) \le \kappa(\Upsilon_1(t)\mathbb{B}) + \kappa(\Upsilon_2(t)\mathbb{B}) \le 0 + \|\Upsilon_2(t)\|_{\text{op}}\,\kappa(\mathbb{B}) \le \mathrm{e}^{-\delta^- t}\kappa(\mathbb{B}),$$

where $\mathrm{e}^{-\delta^- t}$ is called the contraction function. Therefore, for all $t > 0$, $\Upsilon(t)$ is a conditionally κ-contraction on $\mathbb{X}^+$. Thus, the asymptotic smoothness of $\Upsilon(t)$ follows from [11, Lemma 2.3.4]. Furthermore, the point dissipativeness of $\Upsilon(t)$ follows from the assertion in (i). Hence, [11, Theorem 2.4.6] guarantees the existence of a connected global attractor in $\mathbb{X}^+$. This proves (ii).

Proof of Theorem 1.1. Theorem 1.1 directly follows from Lemmas 2.1 and 2.2.

3. The basic reproduction number: Proof of Theorem 1.2

Proof of Theorem 1.2. We first prove (i). For convenience, we set $B := \text{diag}(D_I\Delta, 0, 0) - V$. It is easy to see that for all $x \in \Omega$, both B and $-V$ are cooperative, ensuring that $\tilde{\Upsilon}(t)C(\bar{\Omega}, \mathbb{R}_+^3) \subseteq C(\bar{\Omega}, \mathbb{R}_+^3)$. According to [38, Theorem 3.12], $\mathscr{B}$ and B are resolvent-positive operators. Let $\mathbb{I}$ be the identity operator. Hence,

$$(\lambda\mathbb{I} - B)^{-1}\psi = \int_0^\infty e^{-\lambda t}\tilde{\Upsilon}(t)\psi dt, \quad \forall\, \lambda > s(B), \ \psi \in \mathbb{X}^+. \tag{3.1}$$

It is check to see that the exponential growth bound $\omega(\tilde{\Upsilon}) = \lim_{t\to\infty} \ln \|\tilde{\Upsilon}(t)\|_{\mathrm{op}}/t$ of $\tilde{\Upsilon}$ is negative. Thus, we obtain $s(B) \le \omega(\tilde{\Upsilon}) < 0$. Choose $\lambda = 0$ in (3.1) such that

$$-B^{-1}\psi = \int_0^\infty \tilde{\Upsilon}(t)\psi dt, \quad \forall\, \psi \in \mathbb{X}^+.$$

The above result together with (1.13) implies $\mathscr{L} = -FB^{-1}$. Thus, the assertion (i) immediately follows from [38, Theorem 3.5].

We next prove (ii). Similar to [46, Theorem 3.3], we define

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \text{ and } V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix},$$

where

$$F_{11} := f_{u_2}(\cdot, u_1^P(\cdot), 0),\ \ F_{12} := (g_{v_1}(\cdot, u_1^P(\cdot), 0), h_{v_2}(\cdot, u_1^P(\cdot), 0)),\ \ F_{21} := \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ \ F_{22} := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$V_{11} := d(\cdot) + \gamma(\cdot) + m,\ \ V_{12} := (0, 0),\ \ V_{21} := \begin{pmatrix} -\xi(\cdot) \\ 0 \end{pmatrix} \text{ and } V_{22} := \begin{pmatrix} \delta_1(\cdot) & 0 \\ -\delta_1(\cdot) & \delta_2(\cdot) \end{pmatrix}.$$

Then $\Re_0 = r(-FB^{-1}) = r(-B^{-1}F) = r(-B_1^{-1}F_2)$, where $B_1 := D_I\Delta - (V_{11} - V_{12}V_{22}^{-1}V_{21}) = D_I\Delta - (d(\cdot) + \gamma(\cdot) + m)$, and

$$\begin{aligned} F_2 :=& F_{11} - F_{12}V_{22}^{-1}V_{21} \\ =& f_{u_2}(\cdot, u_1^P(\cdot), 0) - (g_{v_1}(\cdot, u_1^P(\cdot), 0), h_{v_2}(\cdot, u_1^P(\cdot), 0)) \begin{pmatrix} \delta_1(\cdot) & 0 \\ -\delta_1(\cdot) & \delta_2(\cdot) \end{pmatrix}^{-1} \begin{pmatrix} -\xi(\cdot) \\ 0 \end{pmatrix} = H(\cdot), \end{aligned}$$

where $H(\cdot)$ is defined in (1.15). Then for $\psi \in C^2(\Omega, \mathbb{R}^n) \cap C^1(\bar{\Omega}, \mathbb{R}^n)$,

$$-B_1^{-1}F_2\psi = -[D_I\Delta - (d(\cdot) + \gamma(\cdot) + m)]^{-1}H(x)\psi.$$

Therefore,

$$\Re_0 = r(-[D_I\Delta - (d(\cdot) + \gamma(\cdot) + m)]^{-1}H(\cdot)).$$

Thus, $\Re_0$ satisfies

$$[-[D_I\Delta - (d(\cdot) + \gamma(\cdot) + m)]^{-1}H(\cdot)]\psi = \Re_0\psi,\ \ \psi \in C^2(\Omega, \mathbb{R}^n) \cap C^1(\bar{\Omega}, \mathbb{R}^n),$$

i.e.,

$$D_I\Delta\psi - (d(\cdot) + \gamma(\cdot) + m)\psi + H(\cdot)\frac{1}{\Re_0}\psi = 0,\ \ \psi \in C^2(\Omega, \mathbb{R}^n) \cap C^1(\bar{\Omega}, \mathbb{R}^n), \tag{3.2}$$

This proves (ii).

We begin to prove (iii). Actually, problem (1.17) has a least eigenvalue η^0 associated with a positive eigenfunction φ^*, i.e.,

$$D_I\Delta\varphi^* - (d(\cdot) + \gamma(\cdot) + m)\varphi^* + H(\cdot)\varphi^* = \eta^0\varphi^*,\ \text{ for } x \in \Omega \text{ and } \frac{\partial\varphi^*}{\partial\vartheta} = 0,\ \text{ for } x \in \partial\Omega. \tag{3.3}$$

Similar to the proof of [2, Lemma 2.3(d)], and thanks to (3.2) with $\frac{\partial\psi}{\partial\vartheta} = 0$, for $x \in \partial\Omega$, We respectively multiply (3.3) by ψ and (3.2) by φ^*, then subtract the resulting equations and integrate it over Ω yielding

$$\left(1 - \frac{1}{\Re_0}\right)\int_\Omega H(x)\psi\varphi^* \mathrm{d}x = \eta^0 \int_\Omega \psi\varphi^* \mathrm{d}x.$$

It then follows that $1 - \frac{1}{\Re_0}$ and η^0 have the same sign. This proves (iii).

We next prove (iv). We now pay our attention to the problem (1.18). Let

$$\tilde{G}(u_2, v_1, v_2) = f_{u_2}(x, u_1^P(x), 0)u_2 + g_{v_1}(x, u_1^P(x), 0)v_1 + h_{v_2}(x, u_1^P(x), 0)v_2$$

and $(\phi_2,\ \phi_3,\ \phi_4) \in C(\bar{\Omega}, \mathbb{R}^3)$. This together with (1.12) implies that

$$\begin{cases} u_2(\cdot, t; \phi) = T_2(t)\phi_2 + \displaystyle\int_0^t T_2(t-s)\tilde{G}(u_2(\cdot, s; \phi), v_1(\cdot, s; \phi), v_2(\cdot, s; \phi))\mathrm{d}s, \\ v_1(\cdot, t; \phi) = T_3(t)\phi_3 + \displaystyle\int_0^t T_3(t-s)\xi(\cdot)u_2(\cdot, s; \phi)\mathrm{d}s, \\ v_2(\cdot, t; \phi) = T_4(t)\phi_4 + \displaystyle\int_0^t T_4(t-s)\delta_1(\cdot)v_1(\cdot, s; \phi)\mathrm{d}s, \end{cases}$$

It is readily seen that $\bar{\Upsilon}(t) = \bar{\Upsilon}_2(t) + \bar{\Upsilon}_3(t)$, where

$$\bar{\Upsilon}_2(t)\phi = (0, T_3(t)\phi_3, T_4(t)\phi_4), \quad \phi = (\phi_2, \phi_3, \phi_4) \in C(\bar{\Omega}, \mathbb{R}^3), \tag{3.4}$$

and

$$\bar{\Upsilon}_3(t)\phi = \left(u_2(\cdot, t; \phi), \int_0^t T_3(t-s)\xi(\cdot)u_2(\cdot, s; \phi)\mathrm{d}s, \int_0^t T_4(t-s)\delta_1(\cdot)v_1(\cdot, s; \phi)\mathrm{d}s\right), \ \phi = (\phi_2, \phi_3, \phi_4) \in C(\bar{\Omega}, \mathbb{R}^3).$$

We apply the result in [48, Lemma 2.5] to show that $\bar{\Upsilon}_3(t)$ is compact. On account of (ii) in Lemma 2.2 and (3.4), we obtain that

$$\kappa(\bar{\Upsilon}(t)\mathbb{B}) \ \leq e^{-\delta^- t}\kappa(\mathbb{B}), \quad t > 0, \tag{3.5}$$

for any bounded set $\mathbb{B}$ in $C(\bar{\Omega})$. Let us define the measure of non-compactness of operator $\mathscr{L}$ on $\mathbb{X}^+$ by

$$\alpha(\mathscr{L}) := \inf_{\mathbb{B}\subset\mathbb{X}^+ \text{ is bounded}} \{\varepsilon > 0 : \kappa(\mathscr{L}\mathbb{B}) \leq \varepsilon\kappa(\mathbb{B})\}.$$

Let $\omega_{ess}(\bar{\Upsilon}) := \lim_{t\to\infty} \ln \alpha(\bar{\Upsilon}(t))/t$ be the essential growth bound of $\bar{\Upsilon}$. We then see from (3.5) that $\omega_{ess}(\bar{\Upsilon}) \leq -\delta^-$, and the essential spectral radius $r_e(\bar{\Upsilon}(t))$ of $\bar{\Upsilon}(t)$ satisfies

$$r_e(\bar{\Upsilon}(t)) \leq e^{-\delta^- t} < 1, \ t > 0.$$

Note that the exponential growth bound $\omega(\bar{\Upsilon}) = \lim_{t\to\infty} \ln \|\bar{\Upsilon}(t)\|_{\mathrm{op}}/t$ of $\bar{\Upsilon}$ is defined by

$$\omega(\bar{\Upsilon}) := \inf\left\{\tilde{\omega} \in \mathbb{R} : \text{there exists a } M \geq 1 \text{ such that } \|\bar{\Upsilon}(t)\|_{\mathrm{op}} \leq Me^{\tilde{\omega}t} \text{ for all } t \geq 0\right\}$$

and it satisfies that

$$\omega(\bar{\Upsilon}) = \max\{s(\mathscr{B}), \omega_{ess}(\bar{\Upsilon})\}.$$

It follows from Theorem 1.2 that $s(\mathscr{B}) \geq 0$ for $\Re_0 \geq 1$. Hence, $r(\bar{\Upsilon}(t)) = e^{s(\mathscr{B})t} \geq 1,\ \forall t > 0$, which implies that $r_e(\bar{\Upsilon}(t)) < r(\bar{\Upsilon}(t))$. Applying the generalized Krein–Rutman Theorem [23], we have

$$\bar{\Upsilon}(t)\psi^0 = r(\bar{\Upsilon}(t))\psi^0 = e^{s(\mathscr{B})t}\psi^0, \ t > 0, \tag{3.6}$$

where $\psi^0 \in C(\bar{\Omega}, \mathbb{R}^+_3)$. (iv) is proved by differentiating both sides of (3.6).

Finally, we prove (v). By letting $\hbar_1 = u_2,\ \hbar_2 = (v_1, v_2)^T$, we rewrite (1.12) as

$$\begin{cases} \begin{pmatrix} \frac{\partial\hbar_1}{\partial t} \\ \frac{\partial\hbar_2}{\partial t} \end{pmatrix} = \tilde{\mathscr{B}}\begin{pmatrix} \hbar_1 \\ \hbar_2 \end{pmatrix}, & (x, t) \in \Omega \times (0, \infty), \\ \dfrac{\partial\hbar_1}{\partial\vartheta} = 0, & (x, t) \in \partial\Omega \times (0, \infty), \\ \hbar_1 \in C^2(\Omega, \mathbb{R}) \cap C^1(\bar{\Omega}, \mathbb{R}),\ \hbar_2 \in C(\bar{\Omega}, \mathbb{R}^2), \end{cases}$$

where

$$\tilde{\mathscr{B}} = \begin{pmatrix} L_1 + \tilde{\mathscr{B}}_{11}(\cdot) & \tilde{\mathscr{B}}_{12}(\cdot) \\ \tilde{\mathscr{B}}_{21}(\cdot) & \tilde{\mathscr{B}}_{22} \end{pmatrix},$$

$$L_1 = D_I\Delta,\ \tilde{\mathscr{B}}_{11}(\cdot) = f_{u_2}(\cdot, u_1^P(\cdot), 0) - (d(\cdot) + \gamma(\cdot) + m),\ \tilde{\mathscr{B}}_{12}(\cdot) = (g_{v_1}(\cdot, u_1^P(\cdot), 0), h_{v_2}(\cdot, u_1^P(\cdot), 0)),$$

$$\tilde{\mathscr{B}}_{21}(\cdot) = \begin{pmatrix} \xi(\cdot) \\ 0 \end{pmatrix} \text{ and } \tilde{\mathscr{B}}_{22} = \begin{pmatrix} -\delta_1 & 0 \\ \delta_1 & -\delta_2 \end{pmatrix}.$$

Let $\check{\Upsilon}_2(t)$ be a positive C_0-semigroup on $C(\bar{\Omega}, \mathbb{R}^2)$ generated by the resolvent-positive operator $\tilde{\mathscr{B}}_{22}$. By using [38, Theorem 3.12], we have

$$(\lambda\mathbb{I} - \tilde{\mathscr{B}}_{22})^{-1}\psi = \begin{pmatrix} \frac{1}{\lambda+\delta_1} & 0 \\ \frac{\delta_1}{(\lambda+\delta_1)(\lambda+\delta_2)} & \frac{1}{\lambda+\delta_2} \end{pmatrix}\psi = \int_0^\infty e^{-\lambda t}\, \check{\Upsilon}_2(t)\psi dt, \quad \forall\ \lambda > s(\tilde{\mathscr{B}}_{22}),\ \psi \in C(\bar{\Omega}, \mathbb{R}^2),$$

where $s(\tilde{\mathscr{B}}_{22}) = -\min_{\bar{\Omega}}\{\delta_1, \delta_2\} = -\delta^-$. Then, for $\lambda > -\delta^-$, we define

$$\mathscr{L}_\lambda\psi = D_I\Delta\psi - (d(\cdot)+\gamma(\cdot)+m)\psi + f_{u_2}(\cdot, u_1^P(\cdot), 0)\psi + \frac{\xi(\cdot)g_{v_1}(\cdot, u_1^P(\cdot), 0)}{\lambda+\delta_1}\psi + \frac{\delta_1\xi(\cdot)h_{v_2}(\cdot, u_1^P(\cdot), 0)}{(\lambda+\delta_1)(\lambda+\delta_2)}\psi.$$

Setting

$$\mathscr{C}_1 = \min_{x\in\bar{\Omega}}\{f_{u_2}(x, u_1^P(x), 0)\} > 0,\ \mathscr{C}_2 = \min_{x\in\bar{\Omega}}\{\xi(x)g_{v_1}(x, u_1^P(x), 0)\} > 0 \text{ and } \mathscr{C}_3 = \min_{x\in\bar{\Omega}}\{\delta_1\xi(x)h_{v_2}(x, u_1^P(x), 0)\} > 0.$$

Note that

$$\begin{cases} D_I\Delta\varphi - (d(x)+\gamma(x)+m)\varphi = \eta\varphi, & x \in \Omega, \\ \dfrac{\partial\varphi}{\partial\vartheta} = 0, & x \in \partial\Omega, \end{cases}$$

possesses a principle eigenvalue, written by

$$\eta_0 = -\inf\left\{\int_\Omega [D_I|\nabla\varphi|^2 + (d(\cdot)+\gamma(\cdot)+m)\varphi^2]dx \;\middle|\; \int_\Omega \varphi^2 dx = 1,\ \varphi \in H^1(\Omega)\right\},$$

with eigenvector $\varphi_0 \gg 0$. Let

$$\lambda_1 = \frac{1}{2}\left[(\eta_0 - \delta_1 + \mathscr{C}_1) + \sqrt{(\delta_1 + \eta_0 + \mathscr{C}_1)^2 + 4\mathscr{C}_2}\right]$$

be the larger root of

$$\lambda^2 + (\delta_1 - \mathscr{C}_1 - \eta_0)\lambda - [\mathscr{C}_2 + \delta_1(\mathscr{C}_1 + \eta_0)] = 0.$$

Thus, $\lambda_1 > -\delta_1$ and

$$\begin{aligned} \mathscr{L}_{\lambda_1}\varphi^0 &= D_I\Delta\varphi^0 - (d(\cdot)+\gamma(\cdot)+m)\varphi^0 + f_{u_2}(\cdot, u_1^P(\cdot), 0)\varphi^0 + \frac{\xi(\cdot)g_{v_1}(\cdot, u_1^P(\cdot), 0)}{\lambda+\delta_1}\varphi^0 \\ &\geq \left(\eta_0 + \mathscr{C}_1 + \frac{\mathscr{C}_2}{\lambda_1+\delta_1}\right)\varphi^0 = \lambda_1\varphi^0. \end{aligned}$$

Then we focus on the root of

$$\tilde{L}(\lambda) := -\lambda^3 + [\lambda_1 - (\delta_1+\delta_2)]\lambda^2 + [\lambda_1(\delta_1+\delta_2) - \delta_1\delta_2]\lambda + (\lambda_1\delta_1\delta_2 + \mathscr{C}_3) = 0. \tag{3.7}$$

Calculating the derivative of $\tilde{L}(\lambda)$ associated with $\lambda > 0$ yields

$$\begin{aligned} &\tilde{L}'(\lambda) = -3\lambda^2 + 2[\lambda_1 - (\delta_1+\delta_2)]\lambda + [\lambda_1(\delta_1+\delta_2) - \delta_1\delta_2], \\ &\tilde{L}''(\lambda) = -6\lambda + 2[\lambda_1 - (\delta_1+\delta_2)], \\ &\tilde{L}'''(\lambda) = -6. \end{aligned}$$

Actually, one can use similar arguments in [40,41] to show that $\tilde{L}(\lambda) = 0$ has a unique root, denoted by $\lambda_2 > 0$.

An application of $\lambda_2 > -\delta^-$ together with

$$\begin{aligned} \mathscr{L}_{\lambda_2}\varphi^0 &= D_I\Delta\varphi^0 - (d(\cdot)+\gamma(\cdot)+m)\varphi^0 + f_{u_2}(\cdot, u_1^P(\cdot), 0)\varphi^0 + \frac{\xi(\cdot)g_{v_1}(\cdot, u_1^P(\cdot), 0)}{\lambda+\delta_1}\varphi^0 + \frac{\delta_1\xi(\cdot)h_{v_2}(\cdot, u_1^P(\cdot), 0)}{(\lambda+\delta_1)(\lambda+\delta_2)}\varphi^0 \\ &\geq \left(\lambda_1 + \frac{\mathscr{C}_3}{(\lambda_2+\delta_1)(\lambda_2+\delta_2)}\right)\varphi^0 = \lambda_2\varphi^0, \end{aligned}$$

yields $\mathscr{L}_{\lambda_2}\varphi^0 \geq \lambda_2\varphi^0$ for every $x \in \Omega$. Therefore, $e^{\lambda_2 t}\varphi^0(x)$ is a subsolution of $u_t = \mathscr{L}_{\lambda_2}u$. On account of [46, Theorem 2.3 (i)], (1.18) has an eigenvalue with geometric multiplicity one and a nonnegative eigenfunction. An application of (1.18) yields the positivity of this eigenfunction. This completes the proof.

4. Disease extinction: Proof of Theorem 1.3

Proof of Theorem 1.3. The local asymptotically stability of E_0 is a consequence of [46, Theorem 3.1]. We next confirm the global attractively of E_0. From the u_1-equation of (1.9), one can get

$$\begin{cases} \dfrac{\partial u_1}{\partial t} \leq D_S \Delta u_1 + N(x,u_1), & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u_1}{\partial \vartheta} = 0, & (x,t) \in \partial\Omega \times (0,\infty). \end{cases}$$

On account of (2.2), for any fixed $\epsilon_0 > 0$, there exists $t_0 > 0$ such that

$$0 \leq u_1(\cdot,t) \leq u_1^P(\cdot) + \epsilon_0, \ \forall\, t \geq t_0, \ x \in \bar{\Omega}.$$

Without loss of generality, we can assume that $t_0 = 0$ by replacing the initial condition by the solution at $t = t_0$. Let

$$\mathscr{B}_{\epsilon_0} = \begin{pmatrix} D_I\Delta + f_{u_2}(\cdot, u_1^P(\cdot)+\epsilon_0, 0) - (d(\cdot)+\gamma(\cdot)+m) & g_{v_1}(\cdot, u_1^P(\cdot)+\epsilon_0, 0) & h_{v_2}(\cdot, u_1^P(\cdot)+\epsilon_0, 0) \\ \xi(\cdot) & -\delta_1(\cdot) & 0 \\ 0 & \delta_1(\cdot) & -\delta_2(\cdot) \end{pmatrix}.$$

By the comparison principle [19], we obtain

$$(u_2, v_1, v_2)(x,t) \leq (\hat{u}_2, \hat{v}_1, \hat{v}_2)(x,t) \text{ on } \bar{\Omega} \times [0,\infty),$$

where $(\hat{u}_2, \hat{v}_1, \hat{v}_2)(x,t)$ satisfies

$$\begin{cases} \begin{pmatrix} \frac{\partial \hat{u}_2}{\partial t} \\ \frac{\partial \hat{v}_1}{\partial t} \\ \frac{\partial \hat{v}_2}{\partial t} \end{pmatrix} = \mathscr{B}_{\epsilon_0} \begin{pmatrix} \hat{u}_2 \\ \hat{v}_1 \\ \hat{v}_2 \end{pmatrix}, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial \hat{u}_2}{\partial \vartheta} = 0, & (x,t) \in \partial\Omega \times (0,\infty). \end{cases} \tag{4.1}$$

Let $\bar{\Upsilon}_{\epsilon_0}(t)$ be the linear semigroup associated with generator $\mathscr{B}_{\epsilon_0}$. It follows from the similar processes of the proof of (ii) of Lemma 2.2 that $\omega_{ess}(\bar{\Upsilon}_{\epsilon_0}) \leq -\delta^-$. Let $\omega_{\epsilon_0} := \omega(\bar{\Upsilon}_{\epsilon_0})$. Note that

$$\omega_{\epsilon_0} = \max\{s(\mathscr{B}_{\epsilon_0}), \omega_{ess}(\bar{\Upsilon}_{\epsilon_0})\}.$$

Consequently, ω_{ϵ_0} has the same sign as $s(\mathscr{B}_{\epsilon_0})$. Furthermore, by (iii) of Theorem 1.2, $s(\mathscr{B}_{\epsilon_0})$ has the same sign as the principal eigenvalue of

$$\begin{cases} D_I \Delta\varphi - (d(x)+\gamma(x)+m(x))\varphi + H_{\epsilon_0}(x)\varphi = \eta\varphi, & x \in \Omega, \\ \dfrac{\partial \varphi}{\partial \vartheta} = 0, & x \in \partial\Omega, \end{cases}$$

denoted by $\eta^0_{\epsilon_0}$, where

$$H_{\epsilon_0}(x) = f_{u_2}(x, u_1^P(x)+\epsilon_0, 0) + \frac{\xi(x)[\delta_2(x) g_{v_1}(x, u_1^P(x)+\epsilon_0, 0) + \delta_1(x) h_{v_2}(x, u_1^P(x)+\epsilon_0, 0)]}{\delta_1(x)\delta_2(x)}.$$

An application of (iii) of Theorem 1.2 together with $\Re_0 < 1$ implies $\eta^0 < 0$. Further from the continuous dependence of $\eta^0_{\epsilon_0}$ on ϵ_0, we can choose $\epsilon_0 > 0$ such that $\eta^0_{\epsilon_0} < 0$. Hence we have $\omega_{\epsilon_0} < 0$. Since there exists a $\tilde{M} \geq 1$ such that

$$\|T_{\epsilon_0}(t)\|_{\text{op}} \leq \tilde{M} e^{\omega_{\epsilon_0} t}, \text{ for all } t \geq 0,$$

we have $(\hat{u}_2, \hat{v}_1, \hat{v}_2) \to (0,0,0)$ uniformly for $x \in \bar{\Omega}$ as $t \to \infty$. This combined with [37, Corollary 4.3] allows us to obtain that $(u_1, u_2, v_1, v_2) \to (u_1^*(x), 0, 0, 0)$ uniformly for $x \in \bar{\Omega}$ as $t \to \infty$. This proves Theorem 1.3.

5. Disease persistence: Proof of Theorem 1.4

Theorem 1.4 will be proved by the following lemmas, step by step.

Lemma 5.1.

(i) *For any $\phi(\cdot) \in \mathbb{X}^+$, we directly obtain, for $(x,t) \in \Omega \times (0,\infty)$,*

$$u_1(x,t;\phi) > 0. \tag{5.1}$$

Further, there exists a $\sigma_1 > 0$ that

$$\liminf_{t\to\infty} u_1(x,t;\phi) \geq \sigma_1, \text{ uniformly for } x \in \Omega. \tag{5.2}$$

(ii) *If $u_2^0(\cdot) \not\equiv 0$ or $v_1^0(\cdot) \not\equiv 0$ or $v_2^0(\cdot) \not\equiv 0$, we directly have, for $(x,t) \in \Omega \times (0,\infty)$,*

$$w(x,t;\phi) > 0, \tag{5.3}$$

where $w \in \{u_2, v_1, v_2\}$.

Proof. *Proof of (i).* If $u_1(\cdot,0) \not\equiv 0$, then by the strong maximum principle [30, Theorem 4], $u_1(x,t) > 0$ for $(x,t) \in \Omega \times (0,\infty)$. If $u_1(\cdot,0) \equiv 0$, then $\frac{\partial u_1(x,0)}{\partial t} = N(x,0) > 0$, which implies that there exists $t_1 > 0$ such that $u_1(x,t) > 0$ for $t \in (0,t_1)$ and $x \in \Omega$. This combined with the strong maximum principle ensures $u_1(x,t) > 0$ for $(x,t) \in \Omega \times (0,\infty)$. If $\liminf_{t\to\infty} u_1(x,t;\phi) = 0$, then there is a sequence $t_n \to \infty$, and under this sequence, $u_1(x,t_n;\phi) \to 0$ and $\partial u_1(x,t_n;\phi)/\partial t = 0$, which contradicts to the first equation of (1.9). Thus, by selecting a sufficient small $\sigma_1 > 0$, (5.2) holds. This proves (i).

Proof of (ii). For the case that $u_2^0(\cdot) \not\equiv 0$, in view of Lemma 2.1, $u_2(x,t) \geq 0, \ \forall\ x \in \Omega, \ t \geq 0$. By the u_2-equation of (1.9), we know that $\frac{\partial u_2}{\partial t} \geq D_I \Delta u_2 - (d(x)+\gamma(x)+m)u_2$, which tells us that u_2 is the upper solution of

$$\begin{cases} \dfrac{\partial \bar{u}_2}{\partial t} = D_I \Delta \bar{u}_2 - (d(x)+\gamma(x)+m)\bar{u}_2, & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial \bar{u}_2}{\partial \vartheta} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ \bar{u}_2(x,0) = u_2(x,0) = u_2^0, & x \in \bar{\Omega}. \end{cases}$$

By the strong maximum principle and the Hopf boundary lemma, there exists $t_2 > 0$ such that for all $x \in \bar{\Omega}$ and $t > t_2$, $\bar{u}_2(x,t) > 0$ valid. Again from the comparison principle, one can get $u_2(\cdot,t) \geq \bar{u}_2(\cdot,t) > 0$. Moreover, for fixed $x \in \bar{\Omega}$, the third and fourth equations of (1.9) give

$$v_1(x,t) = e^{-\delta_1(x)t} v_1^0(x) + \int_0^t e^{-\delta_1(x)(t-s)} \xi(x) u_2(x,s;\phi) \mathrm{d}s,$$

and

$$v_2(x,t) = e^{-\delta_2(x)t} v_2^0(x) + \int_0^t e^{-\delta_2(x)(t-s)} \delta_1(x) e^{-\delta_1(x)t} v_1^0(x) \mathrm{d}s + \int_0^t e^{-\delta_2(x)(t-s)} \delta_1(x) \int_0^s e^{-\delta_1(x)(s-l)} \xi(x) u_2(x,l;\phi) \mathrm{d}l \mathrm{d}s,$$

which in turn implies that for all $x \in \bar{\Omega}$ and $t > t_2$, $v_1, v_2 > 0$ valid. The other two cases can be treated in a similar fashion. This completes the proof.

The following result can be founded in [44, Lemma 3.7], so we omit the proof here.

Lemma 5.2 (*See [44, Lemma 3.7]*)**.** *Let $w \in \{u_2, v_1, v_2\}$ be given. Provided that there exists a $\sigma_2 > 0$ such that*

$$\liminf_{t\to\infty} w(x,t;\phi) \geq \sigma_2, \text{ uniformly for } x \in \Omega.$$

Then there exists a $\tilde{\sigma}_2 > 0$ *that*

$$\liminf_{t\to\infty} \breve{w}(x,t;\phi) \geq \tilde{\sigma}_2, \text{ for } \breve{w} \in \{u_1, u_2, v_1, v_2\}, \text{ uniformly for } x \in \Omega.$$

Proof of Theorem 1.4. We next prove Theorem 1.4 for the case that $\phi_2(\cdot) \not\equiv 0$. The cases that $\phi_3(\cdot) \not\equiv 0$ and $\phi_4(\cdot) \not\equiv 0$ could be treated in a similar fashion. Let

$$\mathbb{W}_0 = \{(\phi_1, \phi_2, \phi_3, \phi_4) \in \mathbb{X}^+ : \ \phi_2(\cdot) \not\equiv 0\},$$

and

$$\partial\mathbb{W}_0 = \mathbb{X}^+ \backslash \mathbb{W}_0 = \{(\phi_1, \phi_2, \phi_3, \phi_4) \in \mathbb{X}^+ : \ \phi_2(\cdot) \equiv 0\}.$$

Then, $\mathbb{X}^+ = \mathbb{W}_0 \cup \partial\mathbb{W}_0$. As defined in Lemma 2.2, $\Upsilon(t) : \mathbb{X}^+ \to \mathbb{X}^+$ is the solution semiflow generated by the solution of (1.9). For any $\phi \in \mathbb{W}_0$, by (ii) of Lemma 5.1, we directly obtain $\Upsilon(t)\mathbb{W}_0 \subseteq \mathbb{W}_0$, for all $t \geq 0$. Let

$$M_\partial := \{\phi \in \partial\mathbb{W}_0 : \ \Upsilon(t)\phi \in \partial\mathbb{W}_0, \forall\ t \geq 0\},$$

and $\omega(\phi) := \cap_{t\geq 0}\overline{\cup_{s\geq t} \Upsilon(s)\phi}$, the omega limit set of$\{\Upsilon(t)\phi : t \geq 0\}$, for $\phi \in \mathbb{X}^+$.

As for $\phi \in M_\partial$, we know that $u_2(x,t;\phi) \equiv 0$, $\forall\ t \geq 0$. From the u_2-equation of (1.9), one knows that $g(x, u_1, v_1) + h(x, u_1, v_2) \equiv 0$. On account of Lemma 5.1, and thus, $u_1(x,t;\phi) > 0$, for $(x,t) \in \Omega \times (0,\infty)$. Further, we know that $v_1(\cdot,t;\phi) \equiv v_2(\cdot,t;\phi) \equiv 0$. Further from u_1-equation of (1.9) and (ii) of Lemma 2.2, one can get $u_1(\cdot,t;\phi) \to u_1^P(\cdot)$, uniformly for $x \in \Omega$ as $t \to \infty$. This yields $\cup_{\phi\in M_\partial}\omega(\phi) = \{DFSS\}$. We then see that $\{DFSS\}$ is an isolated and compact invariant set for Υ restricted in M_∂.

In the sequel, we are ready to show that there exists a $\sigma_0 > 0$ such that

$$\limsup_{t\to\infty} \| \Upsilon(t)\phi - DFSS\|_{\mathbb{X}} \geq \sigma_0, \ \forall\ \phi \in \mathbb{W}_0.$$

We proceed indirectly and suppose that for any $\sigma_0 > 0$, there exists $\phi \in \mathbb{W}_0$ such that

$$\limsup_{t\to\infty} \| \Upsilon(t)\phi - DFSS\|_{\mathbb{X}} < \sigma_0.$$

It follows that there exists $\tilde{t} > 0$ such that for $\forall\ t \geq \tilde{t},\ x \in \bar{\Omega}$, we have

$$u_1^P(x) - \sigma_0 < u_1(x,t;\phi),\ u_2(x,t;\phi) < \sigma_0,\ v_1(x,t;\phi) < \sigma_0 \text{ and } v_2(x,t;\phi) < \sigma_0.$$

With the help of assumptions (**B**2) and (**B**3), we know that the following inequalities hold for all $x \in \bar{\Omega}$:

$$f(\cdot, u_1, u_2) \geq f_{u_2}(\cdot, u_1^P(\cdot) - \sigma_0, \sigma_0)u_2,\ g(\cdot, u_1, v_1) \geq g_{v_1}(\cdot, u_1^P(\cdot) - \sigma_0, \sigma_0)v_1 \text{ and } h(\cdot, u_1, v_2) \geq h_{v_2}(\cdot, u_1^P(\cdot) - \sigma_0, \sigma_0)v_2.$$

An application of the comparison principle [19] gives

$$(u_2, v_1), v_2)(x,t) \geq (\breve{u}_2, \breve{v}_1, \breve{v}_2)(x,t) \ \text{ on } \ \bar{\Omega} \times [\tilde{t}, \infty),$$

where $(\breve{u}_2, \breve{v}_1, \breve{v}_2)$ satisfies

$$\begin{cases} \begin{pmatrix} \frac{\partial \breve{u}_2}{\partial t} \\ \frac{\partial \breve{v}_1}{\partial t} \\ \frac{\partial \breve{v}_2}{\partial t} \end{pmatrix} = \mathscr{B}_{\sigma_0} \begin{pmatrix} \breve{u}_2 \\ \breve{v}_1 \\ \breve{v}_2 \end{pmatrix}, & x \in \Omega,\ t > \tilde{t}, \\ \dfrac{\partial \breve{u}_2}{\partial \vartheta} = 0, & x \in \partial\Omega,\ t > \tilde{t} \end{cases} \tag{5.4}$$

where

$$\mathscr{B}_{\sigma_0} = \begin{pmatrix} D_I\Delta + f_{u_2}(\cdot, u_1^P(\cdot) - \sigma_0, \sigma_0) - (d(\cdot) + \gamma(\cdot) + m) & g_{v_1}(\cdot, u_1^P(\cdot) - \sigma_0, \sigma_0) & h_{v_2}(\cdot, u_1^P(\cdot) - \sigma_0, \sigma_0) \\ \xi(\cdot) & -\delta_1(\cdot) & 0 \\ 0 & \delta_1(\cdot) & -\delta_2(\cdot) \end{pmatrix}.$$

Hence (u_2, v_1, v_2) is actually an upper solution of (5.4). By $\Re_0 > 1$ and (iii) of Theorem 1.2, one knows that $s(\mathscr{B}) > 0$. Therefore, there is a sufficiently small $\sigma_0 > 0$ such that $s(\mathscr{B}_{\sigma_0}) > 0$, where $s(\mathscr{B}_{\sigma_0})$ is the principal eigenvalue of

$$\begin{cases} \lambda \begin{pmatrix} \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \mathscr{B}_{\sigma_0} \begin{pmatrix} \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, & x \in \bar{\Omega}, \\ \dfrac{\partial \psi_2}{\partial \vartheta} = 0, & x \in \partial\Omega, \\ \psi_2 \in C^2(\Omega, \mathbb{R}) \cap C^1(\bar{\Omega}, \mathbb{R}),\ (\psi_3, \psi_4) \in C(\bar{\Omega}, \mathbb{R}^2). \end{cases} \tag{5.5}$$

Let $\psi_{\sigma_0} = (\psi_2^{\sigma_0}(x), \psi_3^{\sigma_0}(x), \psi_4^{\sigma_0}(x))$ be the strongly positive eigenfunction concerning $s(\mathscr{B}_{\sigma_0})$. From (ii) of Lemma 5.1, there is a positive constant $\tilde{\zeta} > 0$ such that

$$(u_2, v_1, v_2)(\cdot, t; \phi) \geq \tilde{\zeta}\psi_{\sigma_0},\ \forall\ x \in \Omega,\ t \geq \tilde{t}.$$

Due to the linear system (5.4) with initial data $(\check{u}_2, \check{v}_1, \check{v}_2)(x, \tilde{t}) = \psi_{\sigma_0}$ and the choice of σ_0, (5.4) admits a unique solution

$$(\check{u}_2, \check{v}_1, \check{v}_2)(\cdot, t; \phi) = \tilde{\zeta} e^{s(\mathscr{B}_{\sigma_0})(t-\tilde{t})}\psi_{\sigma_0},\ \text{for all } t \geq \tilde{t}.$$

The comparison principle enables us to obtain

$$(u_2, v_1, v_2)(\cdot, t; \phi) \geq \tilde{\zeta} e^{s(\mathscr{B}_{\sigma_0})(t-\tilde{t})}\psi_{\sigma_0}, \quad \text{for all } t \geq \tilde{t},$$

which results in u_2, v_1 and v_2 are unbounded since $s(\mathscr{B}_{\sigma_0}) > 0$, a contradiction. This proves $\limsup_{t\to\infty} \| \Upsilon(t)\phi - DFSS\|_{\mathbb{X}} \geq \sigma_0,\ \forall\ \phi \in \mathbb{W}_0$, and so $\{DFSS\}$ is a uniform weak repeller.

To apply [36, Theorem 3], a routine method is to define a generalized distance function $\rho : \mathbb{X}_+ \to \mathbb{R}_+$ for $\Upsilon(t)$ by

$$\rho(\phi) = \min_{x\in\bar{\Omega}} \phi_2(x), \quad \phi \in \mathbb{X}^+.$$

One can easily see that $\rho(\Upsilon(t)\phi) > 0,\ \forall t > 0$ either if $\rho(\phi) > 0,\ \phi \in \mathbb{X}^+$ or if $\rho(\phi) = 0,\ \phi \in \mathbb{W}_0$. Let

$$W^s(DFSS) := \left\{\phi \in \mathbb{X}^+ : \lim_{t\to\infty} \| \Upsilon(t)\phi - DFSS\|_{\mathbb{X}} = 0\right\}$$

be the stable set of $DFSS$. We then see from the above discussions that

- $W^s(DFSS) \cap \rho^{-1}(0, \infty) = \emptyset$.
- $\cup_{\phi\in M_\partial}\omega(\phi) = \{DFSS\}$.
- No subset of $\{DFSS\}$ forms a cycle in $\partial\mathbb{W}_0$.

Hence, by [36, Theorem 3], there exists a positive constant $\varsigma > 0$ such that

$$\min_{\phi\in\mathbb{L}} \rho(\phi) > \varsigma,$$

where $\mathbb{L}$ is an any compact chain transitive set in $\mathbb{X}^+ \setminus \{DFSS\}$. This yields

$$\liminf_{t\to\infty} u_2(x, t; \phi) \geq \sigma,$$

for any $\phi \in \mathbb{W}_0$. Hence, by Lemma 5.2, the uniform persistence results with respect to $(\mathbb{W}_0, \partial\mathbb{W}_0)$ stated in Theorem 1.4 hold. Further by [18, Theorem 4.7], [41, Theorem 2.3] and Lemma 5.1, system (1.9) admits at least a PSS in $\mathbb{W}_0$. This proves Theorem 1.4.

6. A critical case: Proof of Theorem 1.5

Proof of Theorem 1.5. In the critical case $\Re_0 = 1$, the global stability of DFSS will be achieved by the procedure that local stability and global attractivity. We refer the readers to [7] and [48, Lemma 3.11].

Proof of the local stability of E_0. Let $\phi = (u_1^0, u_2^0, v_1^0, v_2^0)$ with $\|\phi - DFSS\|_{\mathbb{X}} \le \varsigma$, where $\varsigma > 0$ is sufficiently small. Define

$$w_1(x,t) = \frac{u_1(x,t)}{u_1^P(x)} - 1 \text{ and } b(t) = \max_{x\in\bar{\Omega}}\{w_1(x,t), 0\}.$$

Noticing $D_S \Delta u_1^P(x) + N(x, u_1^P(x)) = 0$ and by the u_1-equation of (1.9), we have

$$\frac{\partial w_1}{\partial t} - D_S\Delta w_1 - 2D_S\frac{\nabla u_1^P(x)\cdot\nabla w_1}{u_1^P(x)} = -\beta w_1 - \frac{f(x,u_1,u_2)+g(x,u_1,v_1)+h(x,u_1,v_2)}{u_1^P(x)},$$

where $\beta = -\frac{N(x,u_1)-N(x,u_1^P(x))}{u_1-u_1^P(x)} + \frac{N(x,u_1^P(x))}{u_1^P(x)} > 0$ by (B1). Let $\tilde{u}_1^P = \min_{x\in\bar{\Omega}}\{u_1^P(x)\}$. Since $w_1(x,0) \le \varsigma/\tilde{u}_1^P$, we observe from positivity of f, g and h, that $w_1(x,t) \le \varsigma/\tilde{u}_1^P$ for $(x,t) \in \Omega\times(0,\infty)$. Let $\underline{\beta}$ be the minimum of $-\frac{\partial N(x,u_1)}{\partial u_1}$ for $x \in \bar{\Omega}$ and $0 \le u_1 \le \mathbf{M}_0(1+\varsigma/\tilde{u}_1^P)$. Denote by $\tilde{T}_1(t)$ the positive semigroup induced by

$$D_S\Delta + 2D_S\frac{\nabla u_1^P(x)\cdot\nabla}{u_1^P(x)} - \underline{\beta}.$$

Then there exists $r > 0$ that $\|\tilde{T}_1(t)\|_{\text{op}} \le \mathscr{P}_0 e^{-rt}$ for all $t \ge 0$, where $\mathscr{P}_0 > 0$. Solving above equation gives

$$w_1(\cdot,t) \le \tilde{T}_1(t)w_1^0 - \int_0^t \tilde{T}_1(t-s)H(\cdot,s)\mathrm{d}s,$$

where

$$w_1^0 = u_1^0/u_1^P(\cdot) - 1 \text{ and } H(\cdot,s) = \frac{f(\cdot,u_1(\cdot,s),u_2(\cdot,s)) + g(\cdot,u_1(\cdot,s),v_1(\cdot,s)) + h(\cdot,u_1(\cdot,s),v_2(\cdot,s))}{u_1^P(\cdot)}.$$

From the definition of $b(t)$ and the positivity of $\tilde{T}_1(t)$, we then have

$$\begin{aligned} b(t) &\le \max_{x\in\bar{\Omega}}\left\{\tilde{T}_1(t)w_1^0 - \int_0^t \tilde{T}_1(t-s)H(x,s)\mathrm{d}s, 0\right\} \\ &\le \max_{x\in\bar{\Omega}}\{\tilde{T}_1(t)w_1^0, 0\} \le \|\tilde{T}_1(t)w_1^0\|_\infty \\ &\le \mathscr{P}_0 e^{-rt}\left\|\frac{u_1^0}{u_1^P(x)} - 1\right\|_\infty \le \varsigma\frac{\mathscr{P}_0 e^{-rt}}{\tilde{u}_1^P}. \end{aligned}$$

Noticing that, by assumption (**B**2), (u_2, v_1, v_2) satisfies

$$\begin{cases} \dfrac{\partial u_2}{\partial t} \le D_I\Delta u_2 + f_{u_2}(\cdot,u_1^P(\cdot),0)u_2 + g_{v_1}(\cdot,u_1^P(\cdot),0)v_1 + h_{v_2}(\cdot,u_1^P(\cdot),0)v_2 - (d(\cdot)+\gamma(\cdot)+m)u_2 \\ \qquad +(f_{u_2}(\cdot,u_1(\cdot),0) - f_{u_2}(\cdot,u_1^P(\cdot),0))u_2 + (g_{v_1}(\cdot,u_1(\cdot),0) - g_{v_1}(\cdot,u_1^P(\cdot),0))v_1 \\ \qquad +(h_{v_2}(\cdot,u_1(\cdot),0) - h_{v_2}(\cdot,u_1^P(\cdot),0))v_2, & (x,t)\in\Omega\times(0,\infty), \\ \dfrac{\partial v_1}{\partial t} = \xi(\cdot)u_2 - \delta_1(\cdot)v_1, & (x,t)\in\Omega\times(0,\infty), \\ \dfrac{\partial v_2}{\partial t} = \delta_1(\cdot)v_1 - \delta_2(\cdot)v_2, & (x,t)\in\Omega\times(0,\infty). \end{cases}$$

We then have

$$\begin{pmatrix} u_2(\cdot,t) \\ v_1(\cdot,t) \\ v_2(\cdot,t) \end{pmatrix} \le \bar{\Upsilon}(t)\begin{pmatrix} u_2^0 \\ v_1^0 \\ v_2^0 \end{pmatrix} + \int_0^t \bar{\Upsilon}(t-s)\begin{pmatrix} (f_{u_2}(\cdot,u_1(\cdot,s),0) - f_{u_2}(\cdot,u_1^P(\cdot),0))u_2(\cdot,s) \\ +(g_{v_1}(\cdot,u_1(\cdot,s),0) - g_{v_1}(\cdot,u_1^P(\cdot),0))v_1(\cdot,s) \\ +(h_{v_2}(\cdot,u_1(\cdot,s),0) - h_{v_2}(\cdot,u_1^P(\cdot),0))v_2(\cdot,s) \\ 0 \\ 0 \end{pmatrix}\mathrm{d}s.$$

Since $f_{u_2}(x,u_1,0)$, $g_{v_1}(x,u_1,0)$ and $h_{v_2}(x,u_1,0)$ are Lipschitz continuous on u_1, then there exist some $L_1>0$, $L_2>0$ and $L_3>0$ such that

$$\begin{pmatrix} u_2(\cdot,t)\\ v_1(\cdot,t)\\ v_2(\cdot,t)\end{pmatrix} \le \bar{\Upsilon}(t)\begin{pmatrix} u_2^0\\ v_1^0\\ v_2^0\end{pmatrix} + \int_0^t \bar{\Upsilon}(t-s)\begin{pmatrix} L_1\|u_1(\cdot,s)-u_1^P(\cdot)\|_\infty u_2(\cdot,s)+L_2\|u_1(\cdot,s)-u_1^P(\cdot)\|_\infty v_1(\cdot,s)\\ +L_3\|u_1(\cdot,s)-u_1^P(\cdot)\|_\infty v_2(\cdot,s)\\ 0\\ 0\end{pmatrix}\mathrm{d}s.$$

By $\Re_0=1$, (iii) of Theorem 1.2 and $\omega(\bar{\Upsilon}(t))=\max\{s(\mathscr{B}),\omega_{ess}(\bar{\Upsilon}(t))\}$, one can get $\omega(\bar{\Upsilon}(t))=0$, which leads to $\|\bar{\Upsilon}(t)\|_{\mathrm{op}}\le\mathscr{P}$ for $t\ge0$ for some constant $\mathscr{P}>0$. Noticing $b(s)\le\varsigma\frac{\mathscr{P}_0e^{-rs}}{\tilde{u}_1^P}$, we have

$$\begin{aligned}\max\{\|u_2(\cdot,t)\|_\infty,\|v_1(\cdot,t)\|_\infty,\|v_2(\cdot,t)\|_\infty\}\le&\mathscr{P}\max\{\|u_2^0\|_\infty,\|v_1^0\|_\infty,\|v_2^0\|_\infty\}\\ &+\mathscr{P}\bar{L}\mathbf{M}_0\int_0^t b(s)(\|u_2(\cdot,s)\|_\infty+\|v_1(\cdot,s)\|_\infty+\|v_2(\cdot,s)\|_\infty)\mathrm{d}s\\ \le&\mathscr{P}\varsigma+\mathscr{P}_1\varsigma\int_0^t e^{-rs}(\|u_2(\cdot,s)\|_\infty+\|v_1(\cdot,s)\|_\infty+\|v_2(\cdot,s)\|_\infty)\mathrm{d}s,\end{aligned}$$

where $\bar{L}:=\max\{L_1,L_2,L_3\}$ and $\mathscr{P}_1=\mathscr{P}\mathscr{P}_0\bar{L}\mathbf{M}_0/\tilde{u}_1^P$. This yields that

$$\|u_2(\cdot,t)\|_\infty+\|v_1(\cdot,t)\|_\infty+\|v_2(\cdot,t)\|_\infty\le3\mathscr{P}\varsigma+3\mathscr{P}_1\varsigma\int_0^t e^{-rs}(\|u_2(\cdot,s)\|_\infty+\|v_1(\cdot,s)\|_\infty+\|v_2(\cdot,s)\|_\infty)\mathrm{d}s.$$

Then, by Gronwall's inequality, we obtain

$$\|u_2(\cdot,t)\|_\infty+\|v_1(\cdot,t)\|_\infty+\|v_2(\cdot,t)\|_\infty\le3\mathscr{P}\varsigma e^{\int_0^t3\mathscr{P}_1\varsigma e^{-rs}\mathrm{d}s}\le3\mathscr{P}\varsigma e^{\frac{3\mathscr{P}_1\varsigma}{r}}.\tag{6.1}$$

Let $\tilde{K}:=\max(K_1,K_3)$. By the u_1-equation of (1.9), assumptions (**B**3) and (**B**4) and (6.1), we have

$$\frac{\partial u_1}{\partial t}-D_S\Delta u_1>N(x,u_1)-3\tilde{K}\mathscr{P}\varsigma e^{\frac{3\mathscr{P}_1\varsigma}{r}}u_1,$$

for $(x,t)\in\Omega\times(0,\infty)$. Denote by $\hat{u}_1$ the solution of

$$\begin{cases}\dfrac{\partial\hat{u}_1}{\partial t}=D_S\Delta\hat{u}_1+N(x,\hat{u}_1)-3\tilde{K}\mathscr{P}\varsigma e^{\frac{3\mathscr{P}_1\varsigma}{r}}\hat{u}_1, & (x,t)\in\Omega\times(0,\infty),\\ \dfrac{\partial\hat{u}_1}{\partial\vartheta}=0, & (x,t)\in\Omega\times(0,\infty),\\ \hat{u}_1(x,0)=u_1^0(x), & x\in\Omega.\end{cases}\tag{6.2}$$

Then, $u_1(x,t)\ge\hat{u}_1(x,t)$ for $(x,t)\in\Omega\times(0,\infty)$. Let $\hat{u}_1^P(x)$ be the PSS of (6.2) and $\hat{w}(\cdot,t)=\hat{u}_1(\cdot,t)-\hat{u}_1^P(\cdot)$. Then $\hat{w}$ satisfies

$$\begin{cases}\dfrac{\partial\hat{w}}{\partial t}=D_S\Delta\hat{w}-\left(\hat{\beta}+3\tilde{K}\mathscr{P}\varsigma e^{\frac{3\mathscr{P}_1\varsigma}{r}}\right)\hat{w}, & (x,t)\in\Omega\times(0,\infty),\\ \dfrac{\partial\hat{w}}{\partial\vartheta}=0, & (x,t)\in\Omega\times(0,\infty),\\ \hat{w}(\cdot,0)=u_1^0-\hat{u}_1^P, & x\in\Omega,\end{cases}\tag{6.3}$$

where $\hat{\beta}=-\frac{N(x,\hat{u}_1)-N(x,\hat{u}_1^P(x))}{\hat{u}_1-\hat{u}_1^P(x)}>0$ by (**B**1). Since $\hat{w}(x,0)\le\varsigma$, we observe from positiveness of $\hat{\beta}+3\tilde{K}\mathscr{P}\varsigma e^{\frac{3\mathscr{P}_1\varsigma}{r}}$, that $\hat{w}(x,t)\le\varsigma$ for $(x,t)\in\Omega\times(0,\infty)$. Let $\underline{\hat{\beta}}$ be the minimum of $-\frac{\partial N(x,\hat{u}_1)}{\partial\hat{u}_1}$ for $x\in\bar{\Omega}$ and $0\le\hat{u}_1\le\mathbf{M}_0(1+\varsigma)$. Denote by $\hat{T}_1(t)$ the positive semigroup induced by $D_S\Delta-\underline{\hat{\beta}}$. Then $\|\hat{T}_1(t)\|_{\mathrm{op}}\le\mathscr{P}_2e^{-\underline{\hat{\beta}}t}$ for all $t\ge0$, where

$\mathscr{P}_2 > 0$. For (6.3), we have

$$\hat{w}(\cdot,t) \le \hat{T}_1(t)(u_1^0 - \hat{u}_1^P) - \int_0^t \hat{T}_1(t-s)3\tilde{K}\mathscr{P}\varsigma e^{\frac{3\mathscr{P}_1\varsigma}{r}}\hat{w}(\cdot,s)\mathrm{d}s.$$

Therefore, we have

$$\|\hat{w}(\cdot,t)\|_\infty \le \mathscr{P}_2\|u_1^0 - \hat{u}_1^P\|_\infty e^{-\underline{\hat{\beta}}t} + \int_0^t \mathscr{P}_2 e^{-\underline{\hat{\beta}}(t-s)}3\tilde{K}\mathscr{P}\varsigma e^{\frac{3\mathscr{P}_1\varsigma}{r}}\|\hat{w}(\cdot,s)\|_\infty\mathrm{d}s.$$

Let $\mathtt{K} = 3\tilde{K}\mathscr{P}\mathscr{P}_2\varsigma e^{\frac{3\mathscr{P}_1\varsigma}{r}}$. Then, by Gronwall's inequality, we obtain

$$\|\hat{u}_1(\cdot,t) - \hat{u}_1^P(\cdot)\|_\infty = \|\hat{w}(\cdot,t)\|_\infty \le \mathscr{P}_2\|u_1^0 - \hat{u}_1^P\|_\infty e^{\mathtt{K}t-\underline{\hat{\beta}}t}.$$

Choosing $\varsigma > 0$ sufficiently small such that $\mathtt{K} < \underline{\hat{\beta}}/2$, it then gives

$$\|\hat{u}_1(\cdot,t) - \hat{u}_1^P(\cdot)\|_\infty \le \mathscr{P}_2\|u_1^0 - \hat{u}_1^P\|_\infty e^{-\frac{\underline{\hat{\beta}}}{2}t}. \tag{6.4}$$

Now by (6.4), we have

$$\begin{aligned} u_1(\cdot,t) - u_1^P(\cdot) &\ge \hat{u}_1(\cdot,t) - u_1^P(\cdot) = \hat{u}_1(\cdot,t) - \hat{u}_1^P(\cdot) + \hat{u}_1^P(\cdot) - u_1^P(\cdot) \\ &\ge -\mathscr{P}_2\|u_1^0 - \hat{u}_1^P\|_\infty e^{-\frac{\underline{\hat{\beta}}}{2}t} + \hat{u}_1^P - u_1^P \\ &\ge -\mathscr{P}_2(\|u_1^0 - u_1^P\|_\infty + \|u_1^P - \hat{u}_1^P\|_\infty) - \|\hat{u}_1^P - u_1^P\|_\infty \\ &\ge -\mathscr{P}_2\varsigma - (\mathscr{P}_2+1)\|\hat{u}_1^P - u_1^P\|_\infty. \end{aligned} \tag{6.5}$$

On the other hand, due to $b(t) \le \varsigma\frac{\mathscr{P}_0}{\tilde{u}_1^P}$, one knows that

$$u_1(\cdot,t) - u_1^P(\cdot) = u_1^P(\cdot)\left(\frac{u_1(\cdot,t)}{u_1^P(\cdot)} - 1\right) \le \mathbf{M}_0 b(t) \le \frac{\varsigma\mathscr{P}_0\mathbf{M}_0}{\tilde{u}_1^P}. \tag{6.6}$$

Combining (6.5) and (6.6), we immediately get

$$\|u_1(\cdot,t) - u_1^P(\cdot)\|_\infty \le \max\left\{\mathscr{P}_2\varsigma + (\mathscr{P}_2+1)\|\hat{u}_1^P - u_1^P\|_\infty, \frac{\varsigma\mathscr{P}_0\mathbf{M}_0}{\tilde{u}_1^P}\right\}. \tag{6.7}$$

Finally, combining (6.1), (6.7) and $\lim_{\varsigma\to 0}\hat{u}_1^P = u_1^P$, we can choose a sufficiently small ς and a given $\varepsilon_* > 0$ such that for all $t > 0$,

$$\|u_1(\cdot,t) - u_1^P(\cdot)\|_\infty,\ \|u_2(\cdot,t)\|_\infty,\ \|v_1(\cdot,t)\|_\infty \text{ and } \|v_2(\cdot,t)\|_\infty \le \varepsilon_*,$$

proving the local stability of DFSS.

The global attractivity of DFSS can be achieved by [48, Lemma 3.11] after some slight modifications, we omit the proof here.

7. A homogeneous case: Proof of Theorem 1.6

Proof of Theorem 1.6. The equilibrium equations for (1.21) are given by

$$\begin{cases} N(u_1) = f(u_1,u_2) + g(u_1,v_1) + h(u_1,v_2), \\ f(u_1,u_2) + g(u_1,v_1) + h(u_1,v_2) = (d+\gamma+m)u_2, \\ \xi u_2 = \delta_1 v_1, \\ \delta_1 v_1 = \delta_2 v_2. \end{cases} \tag{7.1}$$

According to the first two equations of (7.1), we know that

$$N(u_1) = f(u_1,u_2) + g(u_1,v_1) + h(u_1,v_2) = (d+\gamma+m)u_2. \tag{7.2}$$

For convenience, we define $u_2 = \chi(u_1) = N(u_1)/(d+\gamma+m)$ with $\chi(u_1^c) = 0$, $\chi(0) = u_2^c$, where u_2^c is the unique positive root of the equation $N(0) = (d+\gamma+m)u_2$. Solving u_2, v_1, and v_2 with respect to u_1 allows us to define

$$\Theta(u_1) = f(u_1,\chi(u_1)) + g\left(u_1, \frac{\xi\chi(u_1)}{\delta_1}\right) + h\left(u_1, \frac{\xi\chi(u_1)}{\delta_2}\right) - (d+\gamma+m)\chi(u_1).$$

Clearly, $\Theta(0) = -(d+\gamma+m)\chi(0) < 0$ and $\Theta(u_1^c) = 0$. Note that

$$\begin{aligned}\Theta'(u_1^c) = &\ f_{u_1}(u_1^c, 0) + f_{u_2}(u_1^c, 0)\chi'(u_1^c) + g_{u_1}(u_1^c, 0) + g_{v_1}(u_1^c, 0)\frac{\xi}{\delta_1}\chi'(u_1^c) \\ &+ h_{u_1}(u_1^c, 0) + h_{v_2}(u_1^c, 0)\frac{\xi}{\delta_2}\chi'(u_1^c) - (d+\gamma+m)\chi'(u_1^c) \\ = &\ (d+\gamma+m)\chi'(u_1^c)([\Re_0] - 1) \\ = &\ N'(u_1^c)([\Re_0] - 1).\end{aligned}$$

Since assumption (**B**1) ensures that $N'(u_1^c) < 0$, then $\Theta'(u_1^c) < 0$ when $[\Re_0] > 1$. Hence there exists $u_1^* \in (0, u_1^c)$ such that $\Theta(u_1^*) = 0$. The value of u_2^* is then given by $\chi(u_1^*)$. The third and fourth equations of (7.1) ensure that v_1^* and v_2^* can be uniquely determined. Therefore, E^* exists if $[\Re_0] > 1$.

We next show that $[\Re_0] > 1$ is also a necessary condition for the existence of E^*. With the help of assumption (B2), we know that

$$\frac{\partial \mathscr{W}(x, u_1, 0)}{\partial v} = \lim_{v\to 0^+} \frac{\mathscr{W}(x, u_1, v)}{v} \text{ and } \mathscr{W}(x, u_1, v) \le \frac{\partial \mathscr{W}(x, u_1, 0)}{\partial v} v, \quad x \in \Omega,\ u_1, v \ge 0,$$

for $\mathscr{W} = f,\ g$ and h, respectively. Hence, for $0 < u_1 < u_1^c$ and $u_2 > 0$, we have

$$\begin{aligned}[\Re_0] &> \left(\frac{f(u_1, u_2)}{u_2} + \frac{\xi\left[\delta_2\frac{g(u_1,v_1)}{v_1} + \delta_1\frac{h(u_1,v_2)}{v_2}\right]}{\delta_1\delta_2}\right)/(d+\gamma+m) \\ &= \frac{f(u_1, u_2) + g(u_1, v_1) + h(u_1, v_2)}{(d+\gamma+m)u_2},\end{aligned}$$

which enables us to obtain that $(d+\gamma+m)u_2 > f(u_1, u_2) + g(u_1, v_1) + h(u_1, v_2)$ if $[\Re_0] \le 1$. This brings the contradiction with (7.2). So E^* exists if and only if $[\Re_0] > 1$.

We next pay attention to prove the uniqueness of E^*. We proceed indirectly and suppose that there exists $E^{**} = (u_1^{**}, u_2^{**}, v_1^{**}, v_2^{**})$ satisfying (7.2). Without of loss of generality, let $u_1^{**} < u_1^*$. Under the assumption (**B**1), one can get $N(u_1^{**}) > N(u_1^*)$. This together with the fact that $N(u_1^*) = (d+\gamma+m)u_2^*$ and $N(u_1^{**}) = (d+\gamma+m)u_2^{**}$ yields $u_2^{**} > u_2^*$. An application of assumption (**B**2), together with the third and fourth equations of (7.1), yields

$$\begin{aligned}\frac{f(u_1^*, u_2^{**}) + g(u_1^*, v_1^{**}) + h(u_1^*, v_2^{**})}{u_2^{**}} &= \frac{f(u_1^*, u_2^{**}) + g(u_1^*, \frac{\xi u_2^{**}}{\delta_1}) + h(u_1^*, \frac{\xi u_2^{**}}{\delta_2})}{u_2^{**}} \\ &\le \frac{f(u_1^*, u_2^*) + g(u_1^*, \frac{\xi u_2^*}{\delta_1}) + h(u_1^*, \frac{\xi u_2^*}{\delta_2})}{u_2^*} \\ &= \frac{f(u_1^*, u_2^*) + g(u_1^*, v_1^*) + h(u_1^*, v_2^*)}{u_2^*}.\end{aligned} \tag{7.3}$$

By virtue of $u_1^{**} < u_1^*$, we have

$$\begin{aligned}\frac{f(u_1^{**}, u_2^{**}) + g(u_1^{**}, v_1^{**}) + h(u_1^{**}, v_2^{**})}{u_2^{**}} &= \frac{f(u_1^{**}, u_2^{**}) + g(u_1^{**}, \frac{\xi u_2^{**}}{\delta_1}) + h(u_1^{**}, \frac{\xi u_2^{**}}{\delta_2})}{u_2^{**}} \\ &\le \frac{f(u_1^*, u_2^{**}) + g(u_1^*, \frac{\xi u_2^{**}}{\delta_1}) + h(u_1^*, \frac{\xi u_2^{**}}{\delta_2})}{u_2^{**}}.\end{aligned} \tag{7.4}$$

Combined with (7.3) and (7.4), we obtain that

$$\frac{f(u_1^{**}, u_2^{**}) + g(u_1^{**}, v_1^{**}) + h(u_1^{**}, v_2^{**})}{u_2^{**}} \le \frac{f(u_1^*, u_2^*) + g(u_1^*, v_1^*) + h(u_1^*, v_2^*)}{u_2^*},$$

which leads to a contradiction with (7.2), and thus E^* is a unique positive equilibrium.

Finally, we shall study the global attractivity of E^* by Lyapunov function. Theorem 1.4 has established that $\Upsilon(t) : \mathbb{W}_0 \to \mathbb{W}_0$ has a global attractor. Denote $U(\theta) = \theta - 1 - \ln\theta$. It is really seen that $U(\theta) \ge 0$ for $\theta > 0$, and

$U(\theta)=0$ if and only if $\theta=1$. Inspired by [33], we let

$$\mathbb{V}(t) := \int_\Omega V(u_1,u_2,v_1,v_2)(x,t)\mathrm{d}x,$$

where

$$V := V(u_1,u_2,v_1,v_2) = u_1 - \int_{u_1^*}^{u_1} \frac{f(u_1^*,u_2^*)}{f(\theta,u_2^*)}\mathrm{d}\theta + u_2^* U\left(\frac{u_2}{u_2^*}\right) + \sum_{j=1}^{2} c_j v_j^* U\left(\frac{v_j}{v_j^*}\right),$$

where $(u_1,u_2,v_1,v_2)(x,t)$ is the solution of (1.21) with $\phi\in\mathbb{X}^+$, and

$$c_1 = \frac{g(u_1^*,v_1^*)+h(u_1^*,v_2^*)}{\xi u_2^*},\ \ c_2 = \frac{h(u_1^*,v_2^*)}{\delta_1 v_1^*}. \tag{7.5}$$

It is noted here that Theorem 1.4 ensures that both $\mathbb{V}(t)$ and $V(u_1,u_2,v_1,v_2)$ are well-defined.

For convenience, we assume

$$\begin{aligned}
\mathscr{G}_1 &:= \mathscr{G}_1(u_2,v_1) = \xi u_2 - \delta_1 v_1,\\
\mathscr{G}_2 &:= \mathscr{G}_2(v_1,v_2) = \delta_1 v_1 - \delta_2 v_2\\
\mathscr{G}_3 &:= \mathscr{G}_3(u_1,u_2,v_1,v_2) = N(u_1) - f(u_1,u_2) - g(u_1,v_1) - h(u_1,v_2),\\
\mathscr{G}_4 &:= \mathscr{G}_4(u_1,u_2,v_1,v_2) = f(u_1,u_2) + g(u_1,v_1) + h(u_1,v_2) - (d+\gamma+m)u_2,
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\mathrm{d}\mathbb{V}(t)}{\mathrm{d}t} &= \int_\Omega \left[\left(1-\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}\right)(D_S\Delta u_1) + \left(1-\frac{u_2^*}{u_2}\right)(D_I\Delta u_2)\right]\mathrm{d}x\\
&\quad + \int_\Omega \left[\left(1-\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}\right)\mathscr{G}_3 + \left(1-\frac{u_2^*}{u_2}\right)\mathscr{G}_4 + \sum_{j=1}^{2} c_j\left(1-\frac{v_j^*}{v_j}\right)\mathscr{G}_j\right]\mathrm{d}x.
\end{aligned}$$

By (B2), we immediately have

$$\begin{aligned}
&\int_\Omega \left[\left(1-\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}\right)(D_S\Delta u_1) + \left(1-\frac{u_2^*}{u_2}\right)(D_I\Delta u_2)\right]\mathrm{d}x\\
&= -\int_\Omega \left[D_S(\partial_{u_1} f(u_1,u_2^*))\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)^2}|\nabla u_1|^2 + D_I\frac{u_2^*}{u_2^2}|\nabla u_2|^2\right]\mathrm{d}x \le 0.
\end{aligned} \tag{7.6}$$

Next we shall show that

$$\mathbb{J} := \left(1-\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}\right)\mathscr{G}_3 + \left(1-\frac{u_2^*}{u_2}\right)\mathscr{G}_4 + \sum_{j=1}^{2} c_j\left(1-\frac{v_j^*}{v_j}\right)\mathscr{G}_j \le 0.$$

Notice that the constant steady state $E^*=(u_1^*,u_2^*,v_1^*,v_2^*)$ of (1.21) satisfies

$$\begin{aligned}
&N(u_1^*) = f(u_1^*,u_2^*) + g(u_1^*,v_1^*) + h(u_1^*,v_2^*) = (d+\gamma+m)u_2^*,\\
&\delta_1 = \frac{\xi u_2^*}{v_1^*},\ \text{ and } \delta_2 = \frac{\delta_1 v_1^*}{v_2^*}.
\end{aligned}$$

Direct calculation gives

$$\begin{aligned}
\mathbb{J} :=&\left(1-\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}\right)\mathscr{G}_3+\left(1-\frac{u_2^*}{u_2}\right)\mathscr{G}_4+\sum_{j=1}^{2}c_j\left(1-\frac{v_j^*}{v_j}\right)\mathscr{G}_j\\
=&\left(1-\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}\right)(N(u_1)-N(u_1^*))\\
&+\ f(u_1^*,u_2^*)\left(2-\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}+\frac{f(u_1,u_2)}{f(u_1,u_2^*)}-\frac{u_2}{u_2^*}-\frac{u_2^*f(u_1,u_2)}{u_2f(u_1^*,u_2^*)}\right)\\
&+\ g(u_1^*,v_1^*)\left(2-\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}-\frac{u_2}{u_2^*}+\frac{f(u_1^*,u_2^*)g(u_1,v_1)}{f(u_1,u_2^*)g(u_1^*,v_1^*)}-\frac{u_2^*g(u_1,v_1)}{u_2g(u_1^*,v_1^*)}\right)\\
&+\ h(u_1^*,v_2^*)\left(2-\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}-\frac{u_2}{u_2^*}+\frac{f(u_1^*,u_2^*)h(u_1,v_2)}{f(u_1,u_2^*)h(u_1^*,v_2^*)}-\frac{u_2^*h(u_1,v_2)}{u_2h(u_1^*,v_2^*)}\right)\\
&+\ c_1\xi u_2^*\left(1+\frac{u_2}{u_2^*}-\frac{v_1}{v_1^*}-\frac{v_1^*u_2}{v_1u_2^*}\right)+c_2\delta_1v_1^*\left(1+\frac{v_1}{v_1^*}-\frac{v_2^*v_1}{v_2v_1^*}-\frac{v_2}{v_2^*}\right).
\end{aligned}\tag{7.7}$$

By Assumption (B1) (resp. (B2)), $N(u_1)$ (resp. $f(u_1,u_2)$) is decreasing (resp. increasing) with respect to u_1. Therefore,

$$\left(1-\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}\right)(N(u_1)-N(u_1^*))\le 0 \text{ for all } u_1\in(0,\mathbf{M}_0].$$

Note that

$$\begin{aligned}
&2-\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}-\frac{u_2}{u_2^*}+\frac{f(u_1,u_2)}{f(u_1,u_2^*)}-\frac{u_2^*f(u_1,u_2)}{u_2f(u_1^*,u_2^*)}\\
=&-U\left(\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}\right)-U\left(\frac{u_2^*f(u_1,u_2)}{u_2f(u_1^*,u_2^*)}\right)-U\left(\frac{u_2f(u_1,u_2^*)}{u_2^*f(u_1,u_2)}\right)+\bar{\Theta},
\end{aligned}\tag{7.8}$$

where

$$\bar{\Theta}=\frac{u_2}{u_2^*}\left(\frac{f(u_1,u_2)}{f(u_1,u_2^*)}-1\right)\left(\frac{u_2^*}{u_2}-\frac{f(u_1,u_2^*)}{f(u_1,u_2)}\right).$$

In view of Assumptions (B1) and (B2), $f(u_1,u_2)$ is strictly increasing and concave down with respect to u_2. Hence $\bar{\Theta}\le 0$ in $\mathbb{W}_0$.

Meanwhile, we see that, by assumption (**B**5),

$$\begin{aligned}
&2-\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}-\frac{u_2}{u_2^*}+\frac{f(u_1^*,u_2^*)g(u_1,v_1)}{f(u_1,u_2^*)g(u_1^*,v_1^*)}-\frac{u_2^*g(u_1,v_1)}{u_2g(u_1^*,v_1^*)}\\
=&\left[2-\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}-\frac{u_2}{u_2^*}-\frac{u_2^*g(u_1,v_1)}{u_2g(u_1^*,v_1^*)}+1-\frac{v_1f(u_1,u_2^*)g(u_1^*,v_1^*)}{v_1^*f(u_1^*,u_2^*)g(u_1,v_1)}+\frac{v_1}{v_1^*}\right]\\
&+\left(\frac{v_1}{v_1^*}-\frac{f(u_1^*,u_2^*)g(u_1,v_1)}{f(u_1,u_2^*)g(u_1^*,v_1^*)}\right)\left(\frac{f(u_1,u_2^*)g(u_1^*,v_1^*)}{f(u_1^*,u_2^*)g(u_1,v_1)}-1\right)\\
\le&\ 3-\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}-\frac{u_2}{u_2^*}-\frac{u_2^*g(u_1,v_1)}{u_2g(u_1^*,v_1^*)}-\frac{v_1f(u_1,u_2^*)g(u_1^*,v_1^*)}{v_1^*f(u_1^*,u_2^*)g(u_1,v_1)}+\frac{v_1}{v_1^*}\\
=&\left(\frac{v_1}{v_1^*}-\frac{u_2}{u_2^*}\right)+\left(1-\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}\right)+\left(1-\frac{u_2^*g(u_1,v_1)}{u_2g(u_1^*,v_1^*)}\right)+\left(1-\frac{v_1f(u_1,u_2^*)g(u_1^*,v_1^*)}{v_1^*f(u_1^*,u_2^*)g(u_1,v_1)}\right)\\
\le&\left(\frac{v_1}{v_1^*}-\frac{u_2}{u_2^*}\right)-\ln\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}-\ln\left(\frac{u_2^*g(u_1,v_1)}{u_2g(u_1^*,v_1^*)}\right)-\ln\left(\frac{v_1f(u_1,u_2^*)g(u_1^*,v_1^*)}{v_1^*f(u_1^*,u_2^*)g(u_1,v_1)}\right)\\
=&\left(\frac{v_1}{v_1^*}-\ln\frac{v_1}{v_1^*}\right)-\left(\frac{u_2}{u_2^*}-\ln\frac{u_2}{u_2^*}\right),
\end{aligned}\tag{7.9}$$

and

$$
\begin{aligned}
&2-\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}-\frac{u_2}{u_2^*}+\frac{f(u_1^*,u_2^*)h(u_1,v_2)}{f(u_1,u_2^*)h(u_1^*,v_2^*)}-\frac{u_2^*h(u_1,v_2)}{u_2h(u_1^*,v_2^*)}\\
=&\left[2-\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}-\frac{u_2}{u_2^*}-\frac{u_2^*h(u_1,v_2)}{u_2h(u_1^*,v_2^*)}+1-\frac{v_2f(u_1,u_2^*)h(u_1^*,v_2^*)}{v_2^*f(u_1^*,u_2^*)h(u_1,v_2)}+\frac{v_2}{v_2^*}\right]\\
&+\left(\frac{v_2}{v_2^*}-\frac{f(u_1^*,u_2^*)h(u_1,v_2)}{f(u_1,u_2^*)h(u_1^*,v_2^*)}\right)\left(\frac{f(u_1,u_2^*)h(u_1^*,v_2^*)}{f(u_1^*,u_2^*)h(u_1,v_2)}-1\right)\\
\leq&3-\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}-\frac{u_2}{u_2^*}-\frac{u_2^*h(u_1,v_2)}{u_2h(u_1^*,v_2^*)}-\frac{v_2f(u_1,u_2^*)h(u_1^*,v_2^*)}{v_2^*f(u_1^*,u_2^*)h(u_1,v_2)}+\frac{v_2}{v_2^*}\\
=&\left(\frac{v_2}{v_2^*}-\frac{u_2}{u_2^*}\right)+\left(1-\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}\right)+\left(1-\frac{u_2^*h(u_1,v_2)}{u_2h(u_1^*,v_2^*)}\right)+\left(1-\frac{v_2f(u_1,u_2^*)h(u_1^*,v_2^*)}{v_2^*f(u_1^*,u_2^*)h(u_1,v_2)}\right)\\
\leq&\left(\frac{v_2}{v_2^*}-\frac{u_2}{u_2^*}\right)-\ln\frac{f(u_1^*,u_2^*)}{f(u_1,u_2^*)}-\ln\left(\frac{u_2^*h(u_1,v_2)}{u_2h(u_1^*,v_2^*)}\right)-\ln\left(\frac{v_2f(u_1,u_2^*)h(u_1^*,v_2^*)}{v_2^*f(u_1^*,u_2^*)h(u_1,v_2)}\right)\\
=&\left(\frac{v_2}{v_2^*}-\ln\frac{v_2}{v_2^*}\right)-\left(\frac{u_2}{u_2^*}-\ln\frac{u_2}{u_2^*}\right),
\end{aligned}
\tag{7.10}
$$

where the inequality $1-\theta\leq-\ln\theta,\theta>0$ is utilized.

One can further check that

$$
\begin{aligned}
&1+\frac{u_2}{u_2^*}-\frac{v_1}{v_1^*}-\frac{v_1^*u_2}{v_1u_2^*}\leq\left(\frac{u_2}{u_2^*}-\ln\frac{u_2}{u_2^*}\right)-\left(\frac{v_1}{v_1^*}-\ln\frac{v_1}{v_1^*}\right),\\
&1+\frac{v_1}{v_1^*}-\frac{v_2^*v_1}{v_2v_1^*}-\frac{v_2}{v_2^*}\leq\left(\frac{v_1}{v_1^*}-\ln\frac{v_1}{v_1^*}\right)-\left(\frac{v_2}{v_2^*}-\ln\frac{v_2}{v_2^*}\right).
\end{aligned}
\tag{7.11}
$$

Applying (7.8), (7.9), (7.10) and (7.11) to (7.7), we have

$$
\begin{aligned}
\mathbb{J}\leq\ &g(u_1^*,v_1^*)\left[\left(\frac{v_1}{v_1^*}-\ln\frac{v_1}{v_1^*}\right)-\left(\frac{u_2}{u_2^*}-\ln\frac{u_2}{u_2^*}\right)\right]+h(u_1^*,v_2^*)\left[\left(\frac{v_2}{v_2^*}-\ln\frac{v_2}{v_2^*}\right)-\left(\frac{u_2}{u_2^*}-\ln\frac{u_2}{u_2^*}\right)\right]\\
&+c_1\xi u_2^*\left[\left(\frac{u_2}{u_2^*}-\ln\frac{u_2}{u_2^*}\right)-\left(\frac{v_1}{v_1^*}-\ln\frac{v_1}{v_1^*}\right)\right]+c_2\delta_1v_1^*\left[\left(\frac{v_1}{v_1^*}-\ln\frac{v_1}{v_1^*}\right)-\left(\frac{v_2}{v_2^*}-\ln\frac{v_2}{v_2^*}\right)\right].
\end{aligned}
$$

With c_1 and c_2 defined in (7.5), one can get $\mathbb{J}\leq 0$. Furthermore, if $\mathbb{J}=0$, there exists a constant k_0 such that

$$u_1=u_1^*,\ u_2=k_0u_2^*,\ v_1=k_0v_1^*\text{ and }v_2=k_0v_2^*.$$

This combined with the first two equations of (1.21) results in $N(u_1^*)-(d+\gamma+m)k_0u_2^*=0$, and hence, $k_0=1$. Hence, together with the arguments as those in [35, Theorem 2.53] and [35, Section 9.9], we finish the proof of Theorem 1.6.

8. Numerical simulation

In this section, we will perform some numerical simulations for system (1.9) to investigate the dynamics of the solutions as some of the parameters are varied. For simplicity, we consider the spatially one-dimensional domain $\Omega=(0,1)\subset\mathbb{R}$. We set the nonlinear functions as follows:

$$N(x,u_1)=\Lambda(x)-\mu u_1,\quad f(x,u_1,u_2)=\beta(x)u_1u_2,\quad g(x,u_1,v_1)=\frac{\alpha_1(x)u_1v_1}{1+v_1},\quad h(x,u_1,v_2)=\frac{\alpha_2(x)u_1v_2}{1+v_2}.$$

The parameters $\Lambda(x)$, $\beta(x)$, $\alpha_1(x)$ and $\alpha_2(x)$ are positive and continuous functions on $\bar{\Omega}$, and μ is a positive constant. We can easily check that these forms of functions satisfy assumptions (**B**1)-(**B**3). In particular, w in (**B**1) is given by

$$w(x,t) = e^{-\mu t}\int_0^1 \Gamma(t,x,y)w_0(y)dy + \int_0^t e^{-\mu s}\int_0^1 \Gamma(s,x,y)\Lambda(y)dyds,$$

where Γ is the fundamental solution to problem $u_t = D_S u_{xx}$, $0 < x < 1$; $u_x = 0$, $x = 0, 1$, which eigenfunction expansion [13, Section 16] is given by

$$\Gamma(t,x,y) = 1 + 2\sum_{n=1}^{\infty}\cos(n\pi x)\cos(n\pi y)e^{-D_S n^2\pi^2 t}.$$

Hence, u_1^P is given by

$$u_1^P(x) = \int_0^\infty e^{-\mu s}\int_0^1 \Gamma(s,x,y)\Lambda(y)dyds, \quad x \in [0,1]. \tag{8.1}$$

Throughout this section, we fix the following parameter values,

$$\Lambda(x) = 1 + \frac{\cos \pi x}{100}, \quad x \in [0,1], \quad \mu = 1, \quad \gamma = 10, \quad d = 0.01, \quad m = 1, \quad \xi = 0.01, \quad \delta_1 = \delta_2 = 10, \tag{8.2}$$

and initial conditions:

$$u_1^0(x) = 0.999, \quad u_2^0(x) = 0.001, \quad v_1^0(x) = v_2^0(x) = 0, \quad x \in [0,1]. \tag{8.3}$$

By (8.1), u_1^P can be calculated as

$$u_1^P(x) = \frac{1}{\mu} + \frac{\cos \pi x}{100\left(\mu + D_S\pi^2\right)}, \quad x \in [0,1].$$

8.1. Computation of the basic reproduction number

To compute the BRN $\Re_0$, we divide $\bar{\Omega} = [0,1]$ into $\mathscr{N} \in \mathbb{N}$, $\mathscr{N} \gg 1$ subintervals with step size $0 < \Delta x = 1/\mathscr{N} \ll 1$ and discretize the eigenvalue problem (1.16). More precisely, we discretize $-D_I\Delta + (d+\gamma+m)$ with the homogeneous Neumann boundary condition as the following $\mathscr{N} \times \mathscr{N}$ matrix:

$$-\begin{pmatrix} -\chi_0 & \chi_0 & 0 & \cdots & 0 \\ \chi_0 & -2\chi_0 & \chi_0 & \cdots & 0 \\ 0 & \chi_0 & -2\chi_0 & & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & & -2\chi_0 & \chi_0 \\ 0 & \cdots & & \chi_0 & -\chi_0 \end{pmatrix} + \mathrm{diag}(d+\gamma+m) =: D_{\mathscr{N}},$$

where $\chi_0 = D_I/\Delta x^2$. We then numerically solve the following eigenvalue problem:

$$\frac{1}{\lambda_{\mathscr{N}}}\psi = D_{\mathscr{N}}^{-1}H_{\mathscr{N}}\psi, \tag{8.4}$$

where

$$\lambda_{\mathscr{N}} \in \mathbb{C}\setminus\{0\}, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{\mathscr{N}} \end{pmatrix} \quad \text{and} \quad H_{\mathscr{N}} := \begin{pmatrix} H(\Delta x) & 0 & \cdots & 0 \\ 0 & H(2\Delta x) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & H(\mathscr{N}\Delta x) \end{pmatrix}.$$

By Theorem 1.2 (ii), the principal eigenvalue $1/\lambda_{\mathscr{N}}$ to the eigenvalue problem (8.4) can approximate the BRN $\Re_0$. For an example of a MATLAB code, see Appendix A.1.

8.2. Computation of the numerical solution

To compute the numerical solution of system (1.9) for $x \in \bar{\Omega} = [0, 1]$ and $t \in [0, t_e]$, $t_e > 0$, we divide $[0, 1]$ into $\mathscr{N}$ subintervals as in Section 8.1 and $[0, t_e]$ into $\mathscr{M} \in \mathbb{N}$, $\mathscr{M} \gg 1$ subintervals with step size $0 < \Delta t = t_e/\mathscr{M} \ll 1$. For $0 \le \mathrm{n} \le \mathscr{N}$ and $0 \le \mathrm{m} \le \mathscr{M}$, let $x^{\mathrm{n}} := \mathrm{n}\Delta x$, $t^{\mathrm{m}} := \mathrm{m}\Delta t$, $u_i^{\mathrm{n,m}} := u_i(x^{\mathrm{n}}, t^{\mathrm{m}})$, $v_i^{\mathrm{n,m}} := v_i(x^{\mathrm{n}}, t^{\mathrm{m}})$, $i = 1, 2$ and

$$
\begin{aligned}
f_1^{\mathrm{n,m}} &:= u_1^{\mathrm{n,m}} + \Delta t\left[N(x^{\mathrm{n}}, u_1^{\mathrm{n,m}}) - f(x^{\mathrm{n}}, u_1^{\mathrm{n,m}}, u_2^{\mathrm{n,m}}) - g(x^{\mathrm{n}}, u_1^{\mathrm{n,m}}, v_1^{\mathrm{n,m}}) - h(x^{\mathrm{n}}, u_1^{\mathrm{n,m}}, v_2^{\mathrm{n,m}})\right], \\
f_2^{\mathrm{n,m}} &:= u_2^{\mathrm{n,m}} + \Delta t\left[f(x^{\mathrm{n}}, u_1^{\mathrm{n,m}}, u_2^{\mathrm{n,m}}) + g(x^{\mathrm{n}}, u_1^{\mathrm{n,m}}, v_1^{\mathrm{n,m}}) + h(x^{\mathrm{n}}, u_1^{\mathrm{n,m}}, v_2^{\mathrm{n,m}}) - (d + \gamma + m)u_2^{\mathrm{n,m}}\right], \\
g_1^{\mathrm{n,m}} &:= v_1^{\mathrm{n,m}} + \Delta t\left[\xi u_2^{\mathrm{n,m}} - \delta_1 v_1^{\mathrm{n,m}}\right], \\
g_2^{\mathrm{n,m}} &:= v_2^{\mathrm{n,m}} + \Delta t\left[\delta_1 v_1^{\mathrm{n,m}} - \delta_2 v_2^{\mathrm{n,m}}\right].
\end{aligned}
$$

By initial conditions (8.3), we know $u_1^{\mathrm{n},0}$, $u_2^{\mathrm{n},0}$, $v_1^{\mathrm{n},0}$ and $v_2^{\mathrm{n},0}$ for all $0 \le \mathrm{n} \le \mathscr{N}$. Inductively, if we know $u_1^{\mathrm{n,m}}$, $u_2^{\mathrm{n,m}}$, $v_1^{\mathrm{n,m}}$ and $v_2^{\mathrm{n,m}}$ for all $0 \le \mathrm{n} \le \mathscr{N}$, then we can compute $f_1^{\mathrm{n,m}}$, $f_2^{\mathrm{n,m}}$, $g_1^{\mathrm{n,m}}$ and $g_2^{\mathrm{n,m}}$ for all $0 \le \mathrm{n} \le \mathscr{N}$, and obtain $u_1^{\mathrm{n,m+1}}$, $u_2^{\mathrm{n,m+1}}$, $v_1^{\mathrm{n,m+1}}$ and $v_2^{\mathrm{n,m+1}}$ for all $0 \le \mathrm{n} \le \mathscr{N}$ in the following way:

$$
\begin{pmatrix} u_i^{1,\mathrm{m}+1} \\ u_i^{2,\mathrm{m}+1} \\ \vdots \\ u_i^{\mathscr{N},\mathrm{m}+1} \end{pmatrix} = \tilde{D}_i^{-1} \begin{pmatrix} f_i^{1,\mathrm{m}} \\ f_i^{2,\mathrm{m}} \\ \vdots \\ f_i^{\mathscr{N},\mathrm{m}} \end{pmatrix}, \quad v_i^{\mathrm{n,m+1}} = g_i^{\mathrm{n,m}}, \quad i = 1, 2,
$$

where

$$
\tilde{D}_i := \begin{pmatrix}
1+\chi_i & -\chi_i & 0 & \cdots & 0 \\
-\chi_i & 1+2\chi_i & -\chi_i & \cdots & 0 \\
0 & -\chi_i & 1+2\chi_i & & \vdots \\
\vdots & & & \ddots & \\
0 & \cdots & & 1+2\chi_i & -\chi_i \\
0 & \cdots & & -\chi_i & 1+\chi_i
\end{pmatrix}, \quad i = 1, 2, \quad \chi_1 := D_S\frac{\Delta t}{\Delta x^2}, \quad \chi_2 := D_I\frac{\Delta t}{\Delta x^2}.
$$

We can then approximate the solution (u_1, u_2, v_1, v_2) of system (1.9) by $(u_1^{\mathrm{n,m}}, u_2^{\mathrm{n,m}}, v_1^{\mathrm{n,m}}, v_2^{\mathrm{n,m}})$, $0 \le \mathrm{n} \le \mathscr{N}$, $0 \le \mathrm{m} \le \mathscr{M}$. For an example of a MATLAB code, see Appendix A.2.

8.3. Threshold dynamics

We first set, for all $x \in \bar{\Omega}$,

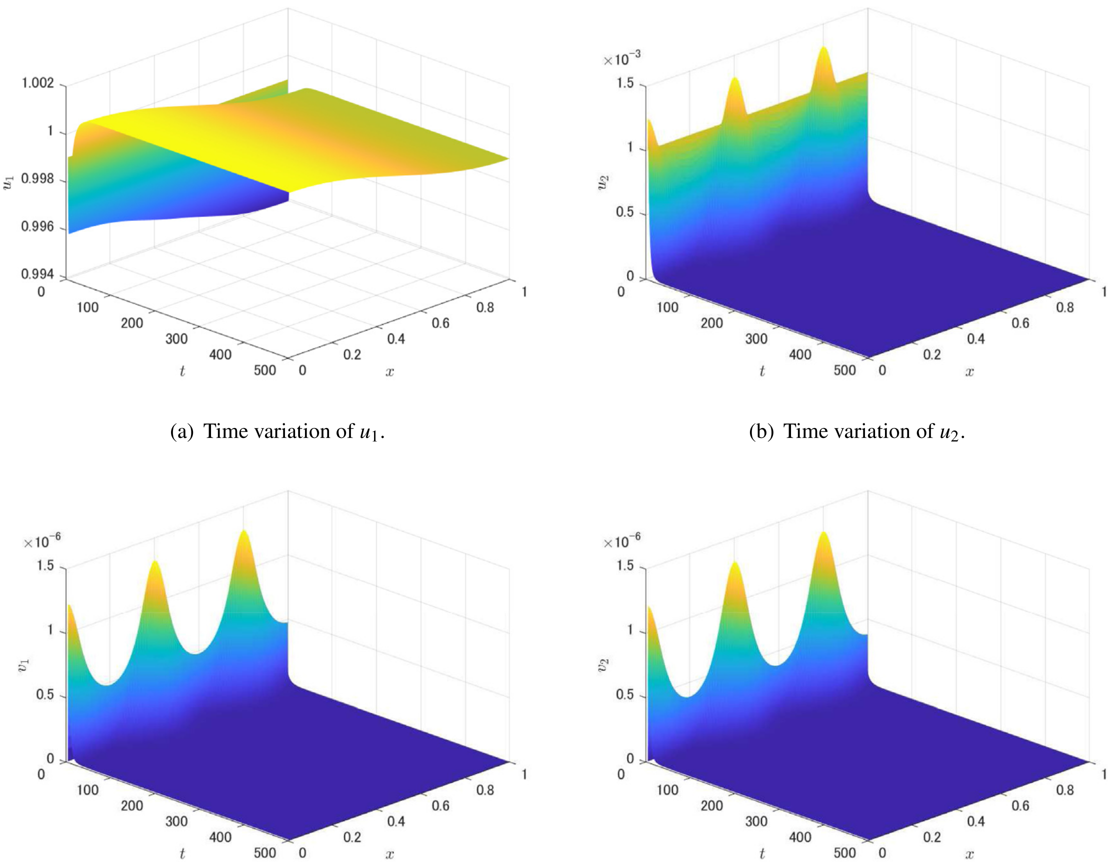
$$
D_S = 1,\ D_I = 0.01,\ \beta(x) = 8.7(1 + 0.5\cos 5\pi x),\ \alpha_1(x) = 2(1 + 0.5\cos 5\pi x),\ \alpha_2(x) = 4(1 + 0.5\cos 5\pi x). \tag{8.5}
$$

In this case, we obtain $\Re_0 \approx 0.9923 < 1$ and Fig. 1 shows that the solution converges to the DFSS as time evolves. This result is consistent with the assertion of Theorem 1.3.

We next set, for all $x \in \bar{\Omega}$,

$$
D_S = 1,\ D_I = 0.01,\ \beta(x) = 8.8(1 + 0.5\cos 5\pi x),\ \alpha_1(x) = 2(1 + 0.5\cos 5\pi x),\ \alpha_2(x) = 4(1 + 0.5\cos 5\pi x). \tag{8.6}
$$

In this case, we obtain $\Re_0 \approx 1.0037 > 1$ and Fig. 2 shows that the system is uniformly persistent and a PSS exists. This result is consistent with the assertion of Theorem 1.4.



(a) Time variation of u_1.

(b) Time variation of u_2.

(c) Time variation of v_1.

(d) Time variation of v_2.

Fig. 1. Time variation of system (1.9) with parameters (8.2), (8.3) and (8.5). The BRN is $\Re_0 \approx 0.9923 < 1$.

8.4. *Sensitivity of the basic reproduction number to the spatial heterogeneity and the diffusion rate*

We now investigate the sensitivity of the BRN $\Re_0$. Fix parameters (8.2) and $(D_S, D_I) = (1, 0.01)$. Let $0 < \upsilon < 1$ represent the spatial heterogeneity of the contact rates and set

$$\beta(x) = 8.7(1 + \upsilon \cos 5\pi x), \quad \alpha_1(x) = 2(1 + \upsilon \cos 5\pi x), \quad \alpha_2(x) = 4(1 + \upsilon \cos 5\pi x), \quad x \in \bar{\Omega}.$$

We then compute $\Re_0$ as in Section 8.1 for each $\upsilon \in (0, 1)$. Fig. 3(a) shows that $\Re_0$ monotonically increases as υ increases. This implies that the spatial heterogeneity could enhance the intensity of epidemic.

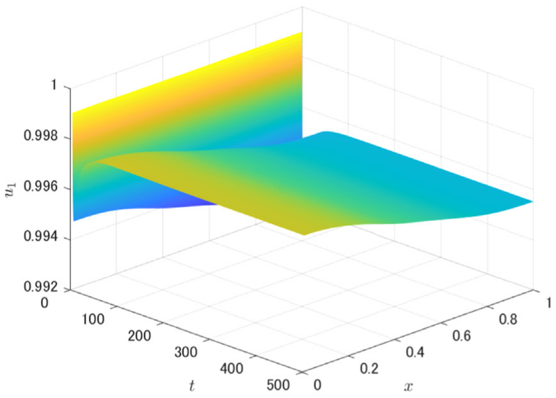
On the other hand, we fix parameters (8.2), $D_S = 1$ and

$$\beta(x) = 8.7(1 + 0.5 \cos 5\pi x), \quad \alpha_1(x) = 2(1 + 0.5 \cos 5\pi x), \quad \alpha_2(x) = 4(1 + 0.5 \cos 5\pi x), \quad x \in \bar{\Omega},$$

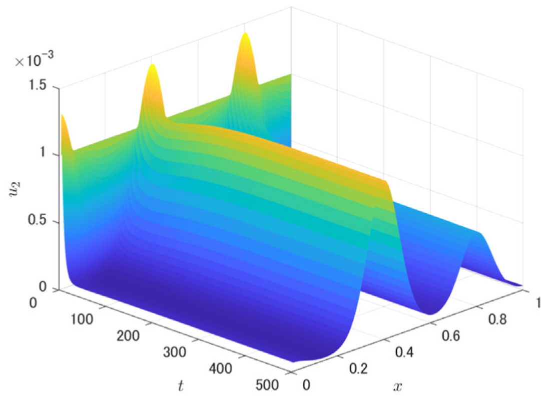
and investigate the sensitivity of $\Re_0$ to the diffusion rate $D_I \in (0, 0.1)$ of infected humans. Fig. 3(b) shows that $\Re_0$ monotonically decreases as D_I increases. This implies that the diffusion effect could reduce the intensity of epidemic.

9. Conclusion and discussion

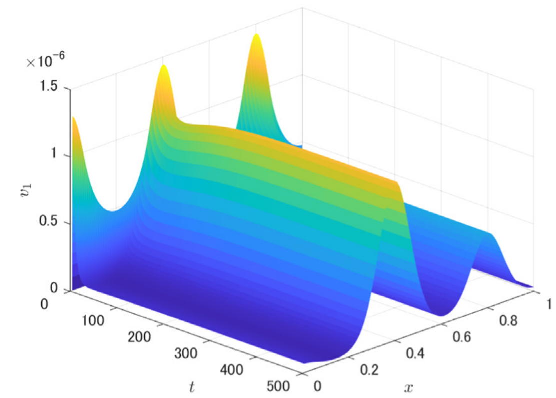
In this paper, we have performed the mathematical analysis of the host–pathogen model (1.9) with diffusion, hyperinfectivity and nonlinear incidence functions. We have defined the BRN $\Re_0$ by the spectral radius of the NGO $\mathscr{L}$, and investigated the relation between $\Re_0$ and the principal eigenvalues linearized at the DFSS $(u_1^P, 0, 0, 0)$.



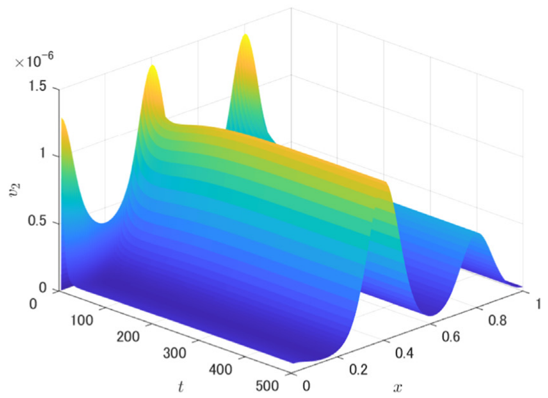
(a) Time variation of u_1.



(b) Time variation of u_2.

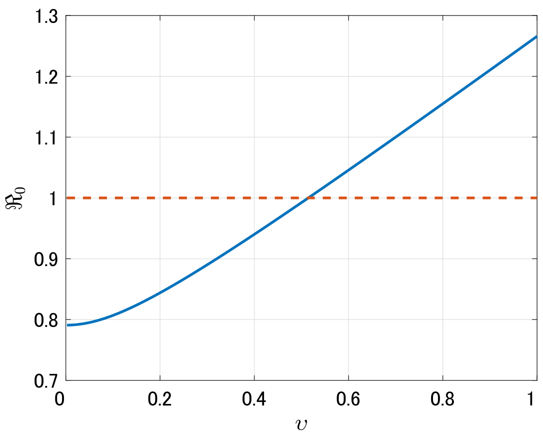


(c) Time variation of v_1.

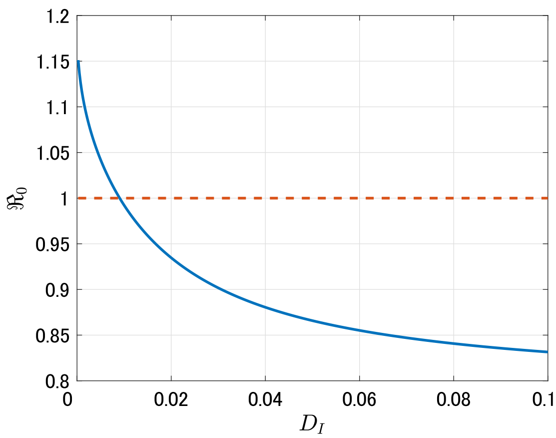


(d) Time variation of v_2.

Fig. 2. Time variation of system (1.9) with parameters (8.2), (8.3) and (8.6). The BRN is $\Re_0 \approx 1.0037 > 1$.



(a) $0 < \upsilon < 1$.



(b) $0 < D_I < 0.1$.

Fig. 3. Sensitivity of the BRN $\Re_0$ to (a) the spatial heterogeneity υ and (b) the diffusion rate D_I of infected humans.

Under several assumptions, we have proved the threshold property of $\Re_0$, that is, if $\Re_0 < 1$, then the DFSS is GAS, whereas if $\Re_0 > 1$, then the system (1.9) is uniformly persistent and there exists a PSS. Furthermore, for the special case where all parameters are strictly positive constants, we have shown that the PSS is GAS if $\Re_0 = [\Re_0] > 1$. Our analysis has included the difficulties especially in the proof of the ultimate boundedness of the solution (Lemma 2.2) and the proof of GAS of the DFSS for the critical case $\Re_0 = 1$ (Theorem 1.5). We have solved them by using the methods of functional analysis and operator semigroups. In Section 8, we have confirmed the validity of our results from the numerical point of view, and obtained two epidemiological suggestions: (i) the spatial heterogeneity could enhance the intensity of epidemic; (ii) the diffusion effect could reduce the intensity of epidemic. As these suggestions were obtained just under a special parameter set, more general analysis on this perspective would be an important future work. The proof of the GAS of the PSS for $\Re_0 > 1$ with non-constant parameters would also be an important future work from the analytical viewpoint.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. MATLAB codes for numerical simulations

A.1. Computation of the basic reproduction number

```
mu=1;
gam=10;
d=0.01;
m=1;
xi=0.01;
del1=10;
del2=10;

DS=1;
DI=0.01;

dx=0.001;xe=1;nx=xe/dx;
for x=1:1:nx
    xx=x*dx;
    u1P(x)=1/mu+cos(pi*xx)/(100*(mu+DS*pi^2));
    bet(x)=8.8*(1+0.5*cos(5*pi*xx));
    alp1(x)=2*(1+0.5*cos(5*pi*xx));
    alp2(x)=4*(1+0.5*cos(5*pi*xx));
    H(x)=bet(x)*u1P(x)+xi*(del2*alp1(x)*u1P(x)+del1*alp2(x)*u1P(x))/(del1*del2);
end

chi0=DI/(dx^2);

D=-diag(2*chi0*ones(nx,1))+diag(chi0*ones(nx-1,1),1)...
    +diag(chi0*ones(nx-1,1),-1);
D(1,1)=-chi0;
D(nx,nx)=-chi0;
D=-D+diag((d+gam+m)*ones(nx,1));

```

```
K=inv(D)*diag(H);
[V1,V2]=eig(K);
for i=1:1:nx
    if sign(V1(:,i)) == ones(nx,1)
        i0=i;
    elseif sign(V1(:,i)) == (-1)*ones(nx,1)
        i0=i;
    end
end

R0=V2(i0,i0)
```

A.2. Computation of the numerical solution

```
mu=1;
gam=10;
d=0.01;
m=1;
xi=0.01;
del1=10;
del2=10;

DS=1;
DI=0.01;

dx=0.01;xe=1;nx=xe/dx;
for x=1:1:nx
    xx=x*dx;
    Lam(x)=1+cos(pi*xx)/100;
    bet(x)=8.8*(1+0.5*cos(5*pi*xx));
    alp1(x)=2*(1+0.5*cos(5*pi*xx));
    alp2(x)=4*(1+0.5*cos(5*pi*xx));

    u1(x,1)=0.999;
    u2(x,1)=0.001;
    v1(x,1)=0;
    v2(x,1)=0;
end

dt=0.01;te=500;nt=te/dt;

chi1=DS*dt/(dx^2);
chi2=DI*dt/(dx^2);

D1=diag((1+2*chi1)*ones(nx,1))+diag((-chi1)*ones(nx-1,1),1)+diag((-chi1)*ones(nx-1,1),-1);
D1(1,1)=1+chi1;
D1(nx,nx)=1+chi1;
ID1=inv(D1);

D2=diag((1+2*chi2)*ones(nx,1))+diag((-chi2)*ones(nx-1,1),1)+diag((-chi2)*ones(nx-1,1),-1);
D2(1,1)=1+chi2;
D2(nx,nx)=1+chi2;
ID2=inv(D2);

for t=1:1:nt
    for x=1:1:nx
        lam=bet(x)*u1(x,t)*u2(x,t)+alp1(x)*u1(x,t)*v1(x,t)/(1+v1(x,t))...
            +alp2(x)*u1(x,t)*v2(x,t)/(1+v2(x,t));
        fu1(x)=u1(x,t)+dt*(Lam(x)-mu*u1(x,t)-lam);
        fu2(x)=u2(x,t)+dt*(lam-(d+gam+m)*u2(x,t));
```

```
            fv1(x)=v1(x,t)+dt*(xi*u2(x,t)-del1*v1(x,t));
            fv2(x)=v2(x,t)+dt*(del1*v1(x,t)-del2*v2(x,t));
        end

        y1=ID1*transpose(fu1);
        y2=ID2*transpose(fu2);

        for x=1:1:nx
            u1(x,t+1)=y1(x);
            u2(x,t+1)=y2(x);
            v1(x,t+1)=fv1(x);
            v2(x,t+1)=fv2(x);
        end
    end

    surf(u2,'Edgecolor','None')
    xlim([0 nt])
    ylim([0 nx])
    xlabel('$t$','Interpreter','Latex')
    ylabel('$x$','Interpreter','Latex')
    zlabel('$u_2$','Interpreter','Latex')
    set(gca,'XTick',0:nt/5:nt,'XTicklabel',0:te/5:te)
    set(gca,'YTick',0:nx/5:nx,'YTicklabel',0:xe/5:xe)
    view([45 30])
    set(gca,'Fontsize',11)
```

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