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Metric discrepancy results for subsequences of geometric progressions

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Abstract—In the previous work, we proved the law of the iterated logarithm for a subsequence of geometric progression $\{\theta^k x\}$ and showed that the speed of convergence toward the uniform distribution is faster than that of the original sequence. It is natural to ask if the speed becomes faster again if we take a subsequence of the subsequence. In this note, we give a negative answer to this question by giving a counterexample.

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1. INTRODUCTION

The following discrepancies are frequently used to measure the speed of convergence of the empirical distribution of a sequence $\{x_k\}$ toward the uniform distribution:

$$D_N(\{x_k\}) = \sup_{0\le a'<a\le 1}\left|\frac{1}{N}\sum_{k=1}^N \widetilde{\mathbf{1}}_{a',a}(x_k)\right|; \quad D_N^*(\{x_k\}) = \sup_{0\le a\le 1}\left|\frac{1}{N}\sum_{k=1}^N \widetilde{\mathbf{1}}_{0,a}(x_k)\right|;$$

where $\widetilde{\mathbf{1}}_{a',a}(x) = \mathbf{1}_{[a',a)}(\langle x\rangle) - (a-a')$, $\langle x\rangle$ denotes the fractional part $x-[\,x\,]$ of x, and $\mathbf{1}_{[a',a)}$ denotes the indicator function of $[\,a',a)$.

For uniform distributed independent random variables $\{U_k\}$, Chung-Smirnov theorem [2, 14] claims

$$\varlimsup_{N\to\infty}\frac{ND_N^*(\{U_k\})}{\sqrt{2N\log\log N}} = \frac{ND_N(\{U_k\})}{\sqrt{2N\log\log N}} = \frac{1}{2} \quad \text{a.s.}$$

Philipp [13] modified the method of Takahashi [15] and proved the bounded law of the iterated logarithm

$$1/4\sqrt{2} \le \varlimsup_{N\to\infty}\frac{ND_N^*(\{n_kx\})}{\sqrt{2N\log\log N}} \le \varlimsup_{N\to\infty}\frac{ND_N(\{n_kx\})}{\sqrt{2N\log\log N}} \le C(q) < \infty, \quad \text{a.e.}$$

for $\{n_k\}$ satisfying Hadamard's gap condition $n_{k+1}/n_k > q > 1$. Aistleitner [1] proved the lower bound can be replaced by $1/2 - 8q^{-1/4}$ and the upper bound by $1/2 + 6q^{-1/4}$. It shows that the limsup becomes nearer to $1/2$ if the sequence $\{n_k\}$ grows faster.

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If a sequence $\{n_k\}$ of real numbers grows very fast and satisfies a stronger gap condition

$$\lim_{k\to\infty} n_{k+1}/n_k = \infty, \tag{1}$$

we can prove (see Proposition in [5]) that

$$\varlimsup_{N\to\infty} \frac{ND_N^*(\{n_k x\})}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N(\{n_k x\})}{\sqrt{2N\log\log N}} = \frac{1}{2} \quad \text{a.e.}, \tag{2}$$

which coincides with Chung-Smirnov theorem and shows that $\{\langle n_k x\rangle\}$ is almost independent.

For geometric progressions, it was proved in [3, 4] that the above limsups equal to a constant and the exact law of the iterated logarithm holds. Actually the following results were proved. For all real numbers $\theta > 1$, there exists a constant Σ_θ such that

$$\varlimsup_{N\to\infty} \frac{ND_N^*(\{\theta^k x\})}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N(\{\theta^k x\})}{\sqrt{2N\log\log N}} = \Sigma_\theta \quad \text{a.e.} \tag{3}$$

We have $\Sigma_\theta = 1/2$ if θ satisfies the condition

$$\theta^r \notin \mathbf{Q} \quad \text{for all} \quad r \in \mathbf{N}, \tag{4}$$

and $\Sigma_\theta > 1/2$ otherwise. For concrete evaluation of Σ_θ in this case, see [3, 6–8, 11, 12].

In our previous work [9], we proved the following. Denote by $\mathcal{N}$ the set of all strictly increasing sequences of natural numbers. For any $\theta > 1$ and $\{m(k)\} \in \mathcal{N}$, there exists a constant $\Sigma_{\theta,\{m(k)\}} \geq 1/2$ such that

$$\varlimsup_{N\to\infty} \frac{ND_N^*(\{\theta^{m(k)} x\})}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N(\{\theta^{m(k)} x\})}{\sqrt{2N\log\log N}} = \Sigma_{\theta,\{m(k)\}} \quad \text{a.e.} \tag{5}$$

We have

$$\left\{ \Sigma_{\theta,\{m(k)\}} \;\middle|\; \{m(k)\} \in \mathcal{N} \right\} = [\,1/2, \Sigma_\theta\,]. \tag{6}$$

By this result we see that exact law of the iterated logarithm holds for any subsequence of geometric progression, and see that the constant appearing there decreases if we take a subsequence. Hence it is very natural to ask if the constant decreases again if we take a subsequence of the subsequence. Surprisingly, the answer is negative. Now, we are in a position to state our result.

Theorem 1. *We can construct a subsequence $\{n_k\}$ of $\{\theta^k\}$ satisfying the following properties: It holds that*

$$\varlimsup_{N\to\infty} \frac{ND_N^*(\{n_k x\})}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N(\{n_k x\})}{\sqrt{2N\log\log N}} = \frac{1}{2} \quad \textit{a.e.}, \tag{7}$$

and for any $\Sigma \in [\,1/2, \Sigma_\theta\,]$, there exists a subsequence $\{\nu_k\}$ of $\{n_k\}$ such that

$$\varlimsup_{N\to\infty} \frac{ND_N^*(\{\nu_k x\})}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N(\{\nu_k x\})}{\sqrt{2N\log\log N}} = \Sigma \quad \textit{a.e.} \tag{8}$$

By the same method of construction, we can prove a result concerning with subsequences of Erdős-Fortet sequence $\{2^k - 1\}$. In [10], it was proved that

$$\varlimsup_{N\to\infty} \frac{ND_N^*(\{(2^k-1)x\})}{\sqrt{2N\log\log N}} = \Sigma_{\mathrm{EF}}(x) \quad \text{a.e.},$$

where $\Sigma_{\rm EF}(x)$ is a non-constant function.

Theorem 2. *We can construct a subsequence $\{n_k\}$ of $\{2^k-1\}$ satisfying the following properties: It holds that*

$$\varlimsup_{N\to\infty}\frac{ND_N^*(\{n_kx\})}{\sqrt{2N\log\log N}}=\frac{1}{2}\quad a.e.,$$

and there exists a subsequence $\{\nu_k\}$ of $\{n_k\}$ such that

$$\varlimsup_{N\to\infty}\frac{ND_N^*(\{\nu_kx\})}{\sqrt{2N\log\log N}}=\Sigma_{\rm EF}(x)\quad a.e.$$

2. CONSTRUCTION OF THE SEQUENCE

In case of (4), by (6) we see $\Sigma_{\theta,\{m(k)\}}=1/2$ for all $\{m(k)\}$ and there is nothing to be proved. Hereafter, we assume that (4) does not hold. In this case we have $\Sigma_\theta>1/2$.

For simplicity we prove the results only for D_N, since the proof for D_N^* can be done in the same way.

Divide $\mathbf{N}$ into consecutive blocks $\Delta_1,\Delta_1',\Delta_2,\Delta_2',\dots$ satisfying ${}^\#\Delta_i'=i^3$ and ${}^\#\Delta_i=i$. Denote $\Delta=\Delta_1\cup\Delta_2\cup\dots$ and $\Delta'=\Delta_1'\cup\Delta_2'\cup\dots$. By

$$\Big|\sum_{k\le N,k\in\Delta}\widetilde{\mathbf{1}}_{a',a}(x_k)\Big|\le{}^\#(\Delta\cap[\,1,N\,])=:\gamma(N)\sim\sqrt{N},$$

we see

$$\Big|\sum_{k\le N,k\in\Delta'}\widetilde{\mathbf{1}}_{a',a}(x_k)\Big|-\gamma(N)\le\Big|\sum_{k\le N}\widetilde{\mathbf{1}}_{a',a}(x_k)\Big|\le\Big|\sum_{k\le N,k\in\Delta'}\widetilde{\mathbf{1}}_{a',a}(x_k)\Big|+\gamma(N).$$

Hence by taking supremum for a' and a, we have

$$\delta(N)D_{\delta(N)}(\{x_k\}_{k\in\Delta'})-\gamma(N)\le ND_N(\{x_k\})\le\delta(N)D_{\delta(N)}(\{x_k\}_{k\in\Delta'})+\gamma(N),$$

where $\delta(N)={}^\#(\Delta'\cap[\,1,N\,])\sim N$. By noting $\gamma(N)=O(\sqrt{\delta(N)})$, and by dividing by $\sqrt{2\delta(N)\log\log\delta(N)}\sim\sqrt{2N\log\log N}$, we have

$$\varlimsup_{N\to\infty}\frac{\delta(N)D_{\delta(N)}(\{x_k\}_{k\in\Delta'})}{\sqrt{2\delta(N)\log\log\delta(N)}}=\varlimsup_{N\to\infty}\frac{ND_N(\{x_k\})}{\sqrt{2N\log\log N}}.\tag{9}$$

Denote $i^-=\min\Delta_i$, $i^+=\max\Delta_i$, and define $m(k)$ in the following way.

$$m(0)=0,$$
$$m(k)=m(i^--1)+k-i^-+1\qquad(k\in\Delta_i),$$
$$m(k)=m(i^+)+k^2-(i^+)^2\qquad(k\in\Delta_i').$$

If $k,\ k+1\in\Delta_i'$, then $m(k+1)-m(k)=2k+1$. If $k=\max\Delta_{i-1}'=i^--1$ and $l=\min\Delta_i'=i^++1$, by $m(i^+)=m(i^--1)+i^+-i^-+1=m(k)+i$ we have $m(l)=m(i^+)+(i^++1)^2-(i^+)^2=m(k)+2i^++i+1$. Therefore we see that the increment of the sequence $\{m(k)\}_{k\in\Delta'}$ diverges to infinity. Hence $\{\theta^{m(k)}\}_{k\in\Delta'}$ satisfies the condition (1) and (2) claims that

$$\varlimsup_{N\to\infty}\frac{ND_N(\{\theta^{m(k)}x\}_{k\in\Delta'})}{\sqrt{2N\log\log N}}=\frac{1}{2}\quad\text{a.e.}\tag{10}$$

By denoting $\{\theta^{m(k)}\}_{k\in\mathbf{N}}$ by $\{n_k\}$, and by noting (10) and (9), we can prove (7).

To prove the existence of $\{\nu_k\}$, we prepare a Bernoulli i.i.d. $\{X_k\}$ such that $P(X_k = 1) = 1 - P(X_k = 0) = p$, where $p \in (0, 1\,]$.

We now denote $\{\theta^{m(k)} \mid k \in \Delta, X_k = 1\}$ by $\{\nu_k\}$. In the same way as the proof in [9] we see (8) holds for some $\Sigma \in (1/2, \Sigma_\theta\,]$ with probability one. We also see that Σ moves continuously from $1/2$ to Σ_θ if we vary p from 0 to 1.

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