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Nonanalytic term in the effective potential at finite temperature for a scalar field on compactified space

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We study nonanalytic terms, which cannot be written in the form of any positive integer power of field-dependent mass squared, in effective potential at finite temperature in one-loop approximation for a real scalar field on the D-dimensional spacetime, $S^1_\tau \times R^{D-(p+1)} \times \prod_{i=1}^{p} S^1_i$. The effective potential can be recast into the integral form in the complex plane by using the integral representation for the modified Bessel function of the second kind and the analytical extension for multiple mode summations. The pole structure of the mode summations is clarified and all the nonanalytic terms are obtained by the residue theorem. We find that the effective potential has a nonanalytic term when the dimension of the flat Euclidean space, $D-(p+1)$ is odd. There appears only one nonanalytic term for the given values of D and p, for which the nonanalytic term exists.

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I. INTRODUCTION

The effective potential at finite temperature has provided a useful tool to study the phenomena of phase transition in quantum field theory. Dolan and Jackiw [1] found that there exists a nonanalytic term, which cannot be written in the form of any positive integer power of field-dependent mass squared, in the effective potential at finite temperature for a scalar field. The nonanalytic term found by them is proportional to three-halves power of the mass squared, and the term turns out to play a crucial role to trigger the first-order phase transition[2,3], for example, in electroweak theories. The magnitude of the term determines the strength of the first order phase transition and thus, it is concerned with the scenario of electroweak baryogenesis [4] as well. Hence, the nonanalytic term in the effective potential is an important quantity.

Quantum field theory with compactified dimensions has been one of the attractive approaches for physics beyond the standard model. Orbifold compactification, for example, provides an attractive framework for gauge-Higgs unification, where the Higgs field is unified into higher-dimensional gauge fields and it is an alternative solution to the gauge hierarchy problem[5,6]. The order of the phase transition in the gauge-Higgs unification at finite temperature has been studied in [7,8], and the first-order phase transition can take place due to the term with three-halves power of the field-dependent mass squared in the effective potential. Compactified dimensions also offer the theoretical framework for studying quantum field theory itself. From a point of view of dimensional reduction [9,10], models with several numbers of S^1 have been investigated. It has been also shown that the quantum field theory with compactified dimensions (at finite temperature) can possess rich phase structures [11,12].

Taking account of the aforementioned studies, it is important and interesting to investigate nonanalytic terms in the effective potential in the presence of extra dimensions at finite temperature. In this paper, we study all the nonanalytic terms for a real scalar field on the D-dimensional spacetime, $S^1_\tau \times R^{D-(p+1)} \times \prod_{i=1}^{p} S^1_i$, where S^1_τ stands for the Euclidean time direction and the spacial directions are compactified on S^1_i. The $R^{D-(p+1)}$ is the $D-(p+1)$-dimensional flat Euclidean space. We assume that the scalar field satisfies the periodic boundary condition for the spacial S^1 direction.

The effective potential contains the modified Bessel function of the second kind accompanied with multiple mode summations. In addition to the integral representation for the modified Bessel function of the second kind given by the inverse Mellin transformation [13,14], we make use of the analytical extension for the mode summations [15] in

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order to recast the effective potential into the integral form in the complex plane and to obtain the nonanalytic terms by the residue theorem.

The analytical extension consists of the products of the gamma, zeta functions and their integrals. We clarify the pole structure of the analytical extension and find that relevant terms in the mode summations for yielding the nonanalytic terms satisfy a recurrence relation, which gives a general form for the relevant terms. Then, all the nonanalytic terms can be obtained by the residue theorem for the poles of the gamma and zeta functions in the general form, depending on the even/odd D and n $(n = 1, 2, \ldots, p+1)$. The positions of the poles that yield the nonanalytic terms turn out not to depend on D.

We find that a nonanalytic term appears in the effective potential when the dimension of the flat Euclidean space, $D-(p+1)$ is odd. There is only one nonanalytic term for the given values of D and p, for which the nonanalytic term exists.

This paper is organized as follows. In the next section we rewrite the effective potential in the integral form in the complex plane and discuss to obtain the nonanalytic terms for the case of $p=1$ explicitly, and we obtain the general form for the relevant terms in the mode summations. We calculate the nonanalytic terms by using the general form in Sec. III and obtain the nonanalytic terms in the effective potential in Sec. IV. The final section is devoted to conclusions and discussions. Some details on the pole structure of the analytical extension in the mode summations are given in Appendix A.

II. EFFECTIVE POTENTIAL IN INTEGRAL FORM AND NONANALYTIC TERMS

We study nonanalytic terms, which cannot be written in the form of any positive integer powers of the field-dependent mass squared, in the effective potential for a real scalar field at finite temperature on the D-dimensional spacetime, $S^1_\tau \times R^{D-(p+1)} \times \prod_{i=1}^p S^1_i$ in one-loop approximation. We employ the Euclidean time formalism for finite temperature field theory and then the Euclidean time direction is compactified on S^1_τ. The spacial p dimensions are compactified on the p numbers of S^1. We denote the circumference of each S^1_i as $L_i (i = 0, 1, \ldots, p)$ and L_0 stands for the inverse temperature T^{-1}.

One needs to evaluate

$$V_{\text{eff}} = (-1)^f \mathcal{N} \frac{1}{2} \left(\prod_{i=0}^{p} \frac{1}{L_i} \sum_{n_i=-\infty}^{\infty} \right) \int \frac{d^{D-(p+1)} p_E}{(2\pi)^{D-(p+1)}} \log \left[p_E^2 + \left(\frac{2\pi}{L_0}\right)^2 (n_0+\eta_0)^2 + \sum_{i=1}^{p} \left(\frac{2\pi}{L_i}\right)^2 (n_i+\eta_i)^2 + M^2(\varphi) \right] \tag{2.1}$$

in order to obtain the effective potential on $S^1_\tau \times R^{D-(p+1)} \times \prod_{i=1}^p S^1_i$ in the one-loop approximation. The $M^2(\varphi)$ is the field-dependent mass squared of the scalar field.[1] The p_E denotes the $D-(p+1)$-dimensional Euclidean momentum. The f is the fermion number, which is 0 (1) for bosons (fermions). The $\mathcal{N}$ is the on shell degrees of freedom. The n_0 denotes the Matsubara mode at finite temperature and the Kaluza-Klein mode $n_i (i = 1, \ldots, p)$ comes from each $S^1_i (i = 1, \cdots p)$. The parameter η_0, which stands for the boundary condition for the S^1_τ direction, is determined by quantum statistics to be 0 ($\frac{1}{2}$) for bosons (fermions). The parameter $\eta_i (i = 1, \ldots, p)$ specifies the boundary condition for the spacial S^1_i direction.

We employ the zeta-function regularization in order to evaluate Eq. (2.1). Let us define

$$I(s) \equiv \left(\prod_{i=0}^{p} \frac{1}{L_i} \sum_{n_i=-\infty}^{\infty} \right) \int \frac{d^{D-(p+1)} p_E}{(2\pi)^{D-(p+1)}} \left[p_E^2 + \left(\frac{2\pi}{L_0}\right)^2 (n_0+\eta_0)^2 + \sum_{i=1}^{p} \left(\frac{2\pi}{L_i}\right)^2 (n_i+\eta_i)^2 + M^2 \right]^{-s}. \tag{2.2}$$

Then, the effective potential can be written as

$$V_{\text{eff}} = (-1)^f \frac{\mathcal{N}}{2} \left(-\frac{d}{ds} I(s) \right) \Big|_{s\to 0}. \tag{2.3}$$

Using the formula

$$A^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} e^{-At} \tag{2.4}$$

and the Poisson summation (A2) in Appendix A with the replacement of $L_0 \to \frac{2\pi}{L_i}$, $m_0 \to n_i$, $n_0 \to m_i$, and $\eta_0 \to \eta_i$, we arrive at

[1]Hereafter, we denote $M(\varphi)$ by M for simplicity.

$$V_{\mathrm{eff}} = (-1)^{f+1}\frac{\mathcal{N}}{2}\frac{\pi^{\frac{D}{2}}}{(2\pi)^D}\sum_{m_0=-\infty}^{\infty}\cdots\sum_{m_p=-\infty}^{\infty}\int_0^\infty dt\, t^{-\frac{D}{2}-1}\mathrm{e}^{-\frac{1}{4t}[(m_0L_0)^2+\cdots+(m_pL_p)^2]-M^2t+2\pi i(m_0\eta_0+\cdots+m_p\eta_p)}. \tag{2.5}$$

It is convenient to separate each summation $\sum_{m_i}$ in Eq. (2.5) into the zero mode ($m_i = 0$) and the nonzero ones ($m_i \neq 0$), and to express Eq. (2.5) into the form

$$V_{\mathrm{eff}} = \sum_{n=0}^{p+1} F^{(n)}, \tag{2.6}$$

where

$$F^{(n)} = \sum_{0\le i_1<i_2<\cdots<i_n\le p} F^{(n)}_{L_{i_1},L_{i_2},\ldots,L_{i_n}}, \tag{2.7}$$

$$F^{(n)}_{L_{i_1},L_{i_2},\ldots,L_{i_n}} = (-1)^{f+1}\frac{\mathcal{N}}{2}\frac{\pi^{\frac{D}{2}}}{(2\pi)^D}\sum_{m_{i_1}=-\infty}^{\infty}{}'\cdots\sum_{m_{i_n}=-\infty}^{\infty}{}'\int_0^\infty dt\, t^{-\frac{D}{2}-1}\mathrm{e}^{-\frac{1}{4t}[(m_{i_1}L_{i_1})^2+\cdots+(m_{i_n}L_{i_n})^2]-M^2t+2\pi i(m_{i_1}\eta_{i_1}+\cdots+m_{i_n}\eta_{i_n})}. \tag{2.8}$$

Here, the prime of the summation $\sum_{m_i=-\infty}^{\prime\infty}$ means that the zero mode ($m_i = 0$) is removed.

The $F^{(0)}$ in Eq. (2.6) corresponds to the contribution from all the zero modes $m_0 = m_1 = \cdots = m_p = 0$ in Eq. (2.5) and is given by[2]

$$F^{(0)} = (-1)^{f+1}\frac{\mathcal{N}}{2}\frac{\pi^{\frac{D}{2}}}{(2\pi)^D}\int_0^\infty dt\, t^{-\frac{D}{2}-1}\mathrm{e}^{-M^2t} = (-1)^{f+1}\frac{\mathcal{N}}{2}\frac{\pi^{\frac{D}{2}}}{(2\pi)^D}\Gamma\left(-\frac{D}{2}\right)(M^2)^{\frac{D}{2}}. \tag{2.9}$$

On the other hand, by using the formula (A3) in Appendix A, $F^{(n)}_{L_{i_1},L_{i_2},\ldots,L_{i_n}}$ $(n \ge 1)$ can be obtained as

$$\begin{aligned} F^{(n)}_{L_{i_1},L_{i_2},\ldots,L_{i_n}} &= (-1)^{f+1}\mathcal{N}\frac{2^n}{(2\pi)^{\frac{D}{2}}}\sum_{m_{i_1}=1}^{\infty}\cdots\sum_{m_{i_n}=1}^{\infty}\left(\frac{M^2}{(m_{i_1}L_{i_1})^2+\cdots+(m_{i_n}L_{i_n})^2}\right)^{\frac{D}{4}} \\ &\quad\times K_{\frac{D}{2}}\left(\sqrt{M^2\{(m_{i_1}L_{i_1})^2+\cdots+(m_{i_n}L_{i_n})^2\}}\right)\cos(2\pi m_{i_1}\eta_{i_1})\cdots\cos(2\pi m_{i_n}\eta_{i_n}). \end{aligned} \tag{2.10}$$

In this paper, we consider a real scalar field ($f = 0, \mathcal{N} = 1$) and take the periodic boundary condition $\eta_j = 0 (j = 1, \ldots, p)$ for the spacial S^1_j direction.

A. Nonanalytic terms for the case of $S^1_\tau \times R^{D-2} \times S^1_1$

Let us first consider that the spacetime is $S^1_\tau \times R^{D-2} \times S^1_1$ and study nonanalytic terms in the effective potential. Although the results concerning the effective potential in this subsection are not new and, in fact, they are given in [9], this is a simple but nontrivial example and is appropriate to present here the analysis on the nonanalytic terms in detail.

The effective potential on $S^1_\tau \times R^{D-2} \times S^1_1$ takes the form

$$V_{\mathrm{eff}} = F^{(0)} + F^{(1)} + F^{(2)}, \tag{2.11}$$

where $F^{(0)}$ is given by Eq. (2.9) and

$$\begin{aligned} F^{(1)} &= F^{(1)}_{L_0} + F^{(1)}_{L_1} \\ &= -\frac{2}{(2\pi)^{\frac{D}{2}}}\sum_{m_0=1}^{\infty}\left(\frac{M^2}{m_0^2L_0^2}\right)^{\frac{D}{4}}K_{\frac{D}{2}}\left(\sqrt{M^2(m_0L_0)^2}\right) - \frac{2}{(2\pi)^{\frac{D}{2}}}\sum_{m_1=1}^{\infty}\left(\frac{M^2}{m_1^2L_1^2}\right)^{\frac{D}{4}}K_{\frac{D}{2}}\left(\sqrt{M^2(m_1L_1)^2}\right), \end{aligned} \tag{2.12}$$

[2]It must be understood that $F^{(0)}$ is regularized by the dimensional regularization for $D =$ even.

$$F^{(2)} = F^{(2)}_{L_0,L_1} = -\frac{2^2}{(2\pi)^{\frac{D}{2}}}\sum_{m_0=1}^{\infty}\sum_{m_1=1}^{\infty}\left(\frac{M^2}{(m_0L_0)^2+(m_1L_1)^2}\right)^{\frac{D}{4}}K_{\frac{D}{2}}\left(\sqrt{M^2\{(m_0L_0)^2+(m_1L_1)^2\}}\right), \quad (2.13)$$

which follow from Eq. (2.10). Here, the $K_{\frac{D}{2}}(x)$ is the modified Bessel function of the second kind. Let us note that $F^{(0)}+F^{(1)}_{L_0}$ is the effective potential at finite temperature without the compactified spacial dimension.

For our purpose, let us use the integral representation for the modified Bessel function of the second kind [13] in the complex plane,[3]

$$K_\nu(x) = \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} dt\Gamma(t)\Gamma(t-\nu)\left(\frac{x}{2}\right)^{-2t+\nu}. \quad (2.14)$$

The constant c should be understood to be a point located on the real axis which is greater than all the poles of the gamma functions in the integrand. Then, we deform the integration path in such a way that it encloses all the poles in the integrand and we can perform the t integration by the residue theorem.

If we apply Eq. (2.14) to the first term of Eq. (2.12), we have

$$F^{(1)}_{L_0} = -\frac{2}{(2\pi)^{\frac{D}{2}}}\left(\frac{M^2}{2}\right)^{\frac{D}{2}}\frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} dt\Gamma(t) \times\Gamma\left(t-\frac{D}{2}\right)\zeta(2t)\left(\frac{ML_0}{2}\right)^{-2t}. \quad (2.15)$$

The zeta function $\zeta(2t)$ is the consequence of the single mode summation with respect to m_0. One can obtain all the terms in the effective potential in terms of M [2,16] by the residue integral for all the poles in the integrand.

Once we obtain the integral form like Eq. (2.15), it is easy to find the nonanalytic terms, which cannot be written in the form of any positive integer powers of M^2. The pole at $t=\frac{D}{2}-n(n=0,1,2,\ldots)$ of $\Gamma(t-\frac{D}{2})$ yields the mass dependence on $(M^2)^{\frac{D}{2}}M^{-2t}=(M^2)^n$, so that the residue integral for the pole does not produce nonanalytic terms. It turns out that any poles depending on D do not produce nonanalytic terms. This observation is crucially used throughout our discussions.

Moreover, we find that the mass dependence on $(M^2)^{\frac{D}{2}}M^{-2t}$ in Eq. (2.15) tells us that for $D=$ even (odd), half-odd integer (integer) values of t can yield nonanalytic terms. It follows that a nonanalytic term in Eq. (2.15) arises from either a pole of $\Gamma(t)$ at $t=0$ for $D=$ odd or that of $\zeta(2t)$ at $t=\frac{1}{2}$ for $D=$ even, whose value of the pole is independent of D, as stated above. One might think that the poles of $\Gamma(t)$ at $t=-n(n=1,2,\ldots)$ could produce nonanalytic terms for $D=$ odd. This is not, however, the case because of the property $\zeta(2t)=0$ for $t=-n(n=1,2,\ldots)$. This observation is also used throughout our discussions.

For $D=$ even, the residue theorem for the pole $t=\frac{1}{2}$ of $\zeta(2t)$ gives us the nonanalytic term given by

$$F^{(1)}_{L_0}\Big|_{\text{n.a.}} = -\frac{(-1)^{\frac{D}{2}}}{2^{\frac{D}{2}}\pi^{\frac{D-2}{2}}(D-1)!!}\frac{M^{D-1}}{L_0}, \quad (2.16)$$

where we have used

$$\Gamma\left(\frac{1-D}{2}\right)\Big|_{D=\text{even}} = \frac{(-1)^{\frac{D}{2}}2^{\frac{D}{2}}}{(D-1)!!}\sqrt{\pi}. \quad (2.17)$$

The abbreviation denoted by "n.a." in Eq. (2.16) means *nonanalytic terms*. Equation (2.16) is the famous term found by Dolan and Jackiw [1] for $D=4$. The other poles of the gamma functions in the integrand and Eq. (2.9) for $D=$ even do not yield nonanalytic terms, so that the Dolan-Jackiw term (2.16) is the only possible nonanalytic term in the effective potential for $D=4$ and $p=0$.

For $D=$ odd, on the other hand, the pole $t=0$ of $\Gamma(t)$ in Eq. (2.15) gives us

$$F^{(1)}_{L_0}\Big|_{\text{n.a.}} = \frac{(-1)^{\frac{D+1}{2}}}{2^{\frac{D+1}{2}}\pi^{\frac{D-1}{2}}D!!}M^D, \quad (2.18)$$

where we have used

$$\Gamma\left(-\frac{D}{2}\right)\Big|_{D=\text{odd}} = \frac{(-1)^{\frac{D+1}{2}}2^{\frac{D+1}{2}}}{D!!}\sqrt{\pi}. \quad (2.19)$$

Equation (2.18), however, cancels the nonanalytic term of Eq. (2.9). Thus, there is no nonanalytic term in the effective potential for $D=$ odd and $p=0$.

Likewise, one can evaluate the second term of Eq. (2.12) and obtains the nonanalytic term as

$$F^{(1)}_{L_1}\Big|_{\text{n.a.}} = \begin{cases} -\frac{(-1)^{\frac{D}{2}}}{2^{\frac{D}{2}}\pi^{\frac{D-2}{2}}(D-1)!!}\frac{M^{D-1}}{L_1} & \text{for } D=\text{even}, \\ \frac{(-1)^{\frac{D+1}{2}}}{2^{\frac{D+1}{2}}\pi^{\frac{D-1}{2}}D!!}M^D & \text{for } D=\text{odd}. \end{cases} \quad (2.20)$$

[3] This is the inverse Mellin transformation for the $K_\nu(x)$ [14].

Let us next proceed to study nonanalytic terms in $F^{(2)} = F^{(2)}_{L_0,L_1}$, which is rewritten in the integral form by using Eq. (2.14) as

$$F^{(2)}_{L_0,L_1} = -\frac{2^2}{(2\pi)^{\frac{D}{2}}}\left(\frac{M^2}{2}\right)^{\frac{D}{2}}\frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} dt\Gamma\left(t-\frac{D}{2}\right)\left(\frac{M}{2}\right)^{-2t} \times\Gamma(t)\sum_{m_0=1}^{\infty}\sum_{m_1=1}^{\infty}\{(m_0L_0)^2+(m_1L_1)^2\}^{-t}. \quad (2.21)$$

One must evaluate the double summations

$$\Gamma(t)\sum_{m_0=1}^{\infty}\sum_{m_1=1}^{\infty}\{(m_0L_0)^2+(m_1L_1)^2\}^{-t}, \quad (2.22)$$

which is known to have an analytical extension [15], as shortly discussed in Appendix A, given by [see Eq. (A4)]

$$\begin{aligned}\Gamma(t)&\sum_{m_0=1}^{\infty}\sum_{m_1=1}^{\infty}\{(m_0L_0)^2+(m_1L_1)^2\}^{-t}\\ &= -\frac{1}{2}\frac{1}{L_1^{2t}}\Gamma(t)\zeta(2t) + \frac{\sqrt{\pi}}{2}\frac{1}{L_0L_1^{2t-1}}\Gamma\left(t-\frac{1}{2}\right)\zeta(2t-1)\\ &\quad+\frac{1}{\sqrt{\pi}}\left(\frac{\pi}{L_0}\right)^{2t}\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} dt_1\Gamma\left(t_1-t+\frac{1}{2}\right)\zeta(2t_1-2t+1)\Gamma(t_1)\zeta(2t_1)\left(\pi\frac{L_1}{L_0}\right)^{-2t_1},\end{aligned} \quad (2.23)$$

where we have first summed over m_0 and then m_1.

Inserting (2.23) into (2.21) we have

$$\begin{aligned}F^{(2)}_{L_0,L_1} &= \frac{1}{(2\pi)^{\frac{D}{2}}}\left(\frac{M^2}{2}\right)^{\frac{D}{2}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dt\Gamma\left(t-\frac{D}{2}\right)\Gamma(t)\zeta(2t)\left(\frac{ML_1}{2}\right)^{-2t}\\ &\quad-\frac{1}{(2\pi)^{\frac{D}{2}}}\sqrt{\pi}\left(\frac{M^2}{2}\right)^{\frac{D}{2}}\frac{L_1}{L_0}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dt\Gamma\left(t-\frac{D}{2}\right)\Gamma\left(t-\frac{1}{2}\right)\zeta(2t-1)\left(\frac{ML_1}{2}\right)^{-2t}\\ &\quad-\frac{2}{(2\pi)^{\frac{D}{2}}}\left(\frac{M^2}{2}\right)^{\frac{D}{2}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dt\Gamma\left(t-\frac{D}{2}\right)\left(\frac{M}{2}\right)^{-2t}\frac{1}{\sqrt{\pi}}\left(\frac{\pi}{L_0}\right)^{2t}\\ &\quad\times\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} dt_1\Gamma\left(t_1-t+\frac{1}{2}\right)\zeta(2t_1-2t+1)\Gamma(t_1)\zeta(2t_1)\left(\pi\frac{L_1}{L_0}\right)^{-2t_1}.\end{aligned} \quad (2.24)$$

We note again that the poles of $\Gamma(t-\frac{D}{2})$ do not yield nonanalytic terms. This is because the residue integral with the poles of $\Gamma(t-\frac{D}{2})$ at $t=\frac{D}{2}-n (n=0,1,2,\ldots)$ gives the mass dependence on $(M^2)^{\frac{D}{2}}M^{-2t}=(M^2)^n$, which are not nonanalytic terms for $n=0,1,2,\ldots$. Thus, the poles, whose positions depend on D, do not give nonanalytic terms, as mentioned before.

We find that nonanalytic terms in Eq. (2.24) can arise only from the poles other than those of $\Gamma(t-\frac{D}{2})$. In fact, the nonanalytic terms are produced by the poles of $t=0$ for $D=$ odd and $t=\frac{1}{2}$ for $D=$ even ($t=\frac{1}{2}$ for $D=$ even and $t=1$ for $D=$ odd) in the first (second) term of Eq. (2.24).

On the other hand, there is no pole that contributes to the residue integral with respect to t in the third term of Eq. (2.24), other than those of $\Gamma(t-\frac{D}{2})$, as shown in Sec. A.1 of Appendix A. This implies that the third term in Eq. (2.24) is irrelevant for the analyses of the nonanalytic terms because the poles of $\Gamma(t-\frac{D}{2})$ do not produce any nonanalytic terms, as stressed above. This is a crucial observation for our study on the nonanalytic terms in the effective potential and plays a central role in the discussions.

The nonanalytic terms in $F^{(2)}_{L_0,L_1}$ turn out to come from the first and the second terms in Eq. (2.24) and we obtain

$$F^{(2)}_{L_0,L_1}\Big|_{\text{n.a.}} = \frac{(-1)^{\frac{D}{2}}}{2^{\frac{D}{2}}\pi^{\frac{D-2}{2}}(D-1)!!}M^{D-1}\left(\frac{1}{L_1}+\frac{1}{L_0}\right) \quad (2.25)$$

for $D=$ even. The first (second) term in Eq. (2.25) comes from the pole $t=\frac{1}{2}$ of $\zeta(2t)$ ($t=\frac{1}{2}$ of $\Gamma(t-\frac{1}{2})$) in the first (second) term of Eq. (2.24). On the other hand, for $D=$ odd, we obtain the nonanalytic terms as

$$F^{(2)}_{L_0,L_1}\Big|_{\text{n.a.}} = -\frac{(-1)^{\frac{D+1}{2}}}{2^{\frac{D+1}{2}}\pi^{\frac{D-1}{2}}D!!}M^D - \frac{(-1)^{\frac{D-1}{2}}}{2^{\frac{D-1}{2}}\pi^{\frac{D-3}{2}}(D-2)!!}\frac{M^{D-2}}{L_0L_1}. \quad (2.26)$$

The first (second) term of Eq. (2.26) comes from the pole $t = 0$ of $\Gamma(t)$ ($t = 1$ of $\zeta(2t-1)$) in the first (second) term of Eq. (2.24).

Collecting the terms we have obtained above, we find that for $D =$ even the effective potential has no nonanalytic term, i.e.,

$$V_{\rm eff}|_{\rm n.a.} = (F^{(0)} + F^{(1)}_{L_0} + F^{(1)}_{L_1} + F^{(2)}_{L_0,L_1})|_{\rm n.a.} = 0 \quad \text{for } D = \text{even}, \tag{2.27}$$

although $F^{(1)}_{L_0}, F^{(1)}_{L_1}$, and $F^{(2)}_{L_0,L_1}$ contain the nonanalytic terms. For $D =$ odd, all of $F^{(0)}, F^{(1)}_{L_0}, F^{(1)}_{L_1}$, and $F^{(2)}_{L_0,L_1}$ have the nonanalytic terms, but some of them cancel each other. Then, the result is given by

$$V_{\rm eff}|_{\rm n.a.} = (F^{(0)} + F^{(1)}_{L_0} + F^{(1)}_{L_1} + F^{(2)}_{L_0,L_1})|_{\rm n.a.} = -\frac{(-1)^{\frac{D-1}{2}}}{2^{\frac{D-1}{2}}\pi^{\frac{D-3}{2}}(D-2)!!}\frac{M^{D-2}}{L_0L_1} \quad \text{for } D = \text{odd}. \tag{2.28}$$

We conclude that the effective potential for the scalar field on $S^1_\tau \times R^{D-2} \times S^1_1$ does not have nonanalytic terms for $D =$ even, while for $D =$ odd it has the nonanalytic term given by Eq. (2.28).

B. General form for relevant terms in mode summation

In this subsection we investigate $F^{(n)}_{L_0,L_1,\ldots,L_{n-1}}$ ($n = 1, 2, \ldots, p+1$) and rewrite it in a tractable form to obtain the nonanalytic terms. By use of the formula (2.14), Eq. (2.10) with $f = 0, \mathcal{N} = 1$ and $\eta_i = 0$ can be written as

$$\begin{aligned} F^{(n)}_{L_0,L_1,\ldots,L_{n-1}} &= -\frac{2^n}{(2\pi)^{\frac{D}{2}}}\left(\frac{M^2}{2}\right)^{\frac{D}{2}}\frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} dt\Gamma\left(t-\frac{D}{2}\right)\left(\frac{M}{2}\right)^{-2t} \\ &\quad\times \Gamma(t)\sum_{m_0=1}^{\infty}\sum_{m_1=1}^{\infty}\cdots\sum_{m_{n-1}=1}^{\infty}\{(m_0L_0)^2 + (m_1L_1)^2 + \cdots (m_{n-1}L_{n-1})^2\}^{-t} \\ &= -\frac{2^n}{(2\pi)^{\frac{D}{2}}}\left(\frac{M^2}{2}\right)^{\frac{D}{2}}\frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} dt\Gamma\left(t-\frac{D}{2}\right)\left(\frac{M}{2}\right)^{-2t} S^{(n)}(t; L_0, L_1, \ldots, L_{n-1}), \end{aligned} \tag{2.29}$$

where we have defined

$$S^{(n)}(t; L_0, L_1, \ldots, L_{n-1}) \equiv \Gamma(t)\sum_{m_0=1}^{\infty}\sum_{m_1=1}^{\infty}\cdots\sum_{m_{n-1}=1}^{\infty}\{(m_0L_0)^2 + (m_1L_1)^2 + \cdots (m_{n-1}L_{n-1})^2\}^{-t}. \tag{2.30}$$

Inserting the formula (A8) into Eq. (2.29), we have

$$\begin{aligned} F^{(n)}_{L_0,L_1,\ldots,L_{n-1}} &= -\frac{2^n}{(2\pi)^{\frac{D}{2}}}\left(\frac{M^2}{2}\right)^{\frac{D}{2}}\frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} dt\Gamma\left(t-\frac{D}{2}\right)\left(\frac{M}{2}\right)^{-2t} \\ &\quad\times\left\{-\frac{1}{2}S^{(n-1)}(t; L_1, L_2, \ldots, L_{n-1}) + \frac{\sqrt{\pi}}{2L_0}S^{(n-1)}\left(t-\frac{1}{2}; L_1, L_2, \ldots, L_{n-1}\right)\right. \\ &\quad\left. + \frac{1}{\sqrt{\pi}}\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} dt_1\Gamma\left(t_1 - t + \frac{1}{2}\right)\zeta(2t_1 - 2t + 1)\left(\frac{\pi}{L_0}\right)^{-2t_1+2t} S^{(n-1)}(t_1; L_1, L_2, \ldots, L_{n-1})\right\}. \end{aligned} \tag{2.31}$$

Since we are interested in the nonanalytic terms of the effective potential, Eq. (2.31) can be expressed as

$$\begin{aligned} F^{(n)}_{L_0,L_1,\ldots,L_{n-1}}\Big|_{\rm n.a.} &= -\frac{2^n}{(2\pi)^{\frac{D}{2}}}\left(\frac{M^2}{2}\right)^{\frac{D}{2}}\frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} dt\Gamma\left(t-\frac{D}{2}\right)\left(\frac{M}{2}\right)^{-2t} \\ &\quad\times\left\{-\frac{1}{2}S^{(n-1)}(t; L_1, L_2, \ldots, L_{n-1}) + \frac{\sqrt{\pi}}{2L_0}S^{(n-1)}\left(t-\frac{1}{2}; L_1, L_2, \ldots, L_{n-1}\right)\right\}\Big|_{\rm n.a.}, \end{aligned} \tag{2.32}$$

where we have dropped the third term in Eq. (2.31) because it is irrelevant to obtain the nonanalytic terms in $F^{(n)}_{L_0,L_1,\ldots,L_{n-1}}$ due to the fact that only the poles of $\Gamma(t-\frac{D}{2})$ contribute to the residue integral with respect to t in the third term of Eq. (2.31), as shown in Sec. A.2 of Appendix A, and those of $\Gamma(t-\frac{D}{2})$ do not produce any nonanalytic terms, as stressed before. This is a crucial observation in our analysis.

It follows from Eq. (2.29) and Eq. (2.32) that as long as our considerations are restricted to the nonanalytic terms, we can use the recurrence relation

$$S^{(n)}(t; L_0, L_1, \ldots, L_{n-1}) = -\frac{1}{2} S^{(n-1)}(t; L_1, L_2, \ldots, L_{n-1}) + \frac{\sqrt{\pi}}{2L_0} S^{(n-1)}\left(t - \frac{1}{2}; L_1, L_2, \ldots, L_{n-1}\right) \tag{2.33}$$

in Eq. (2.32). The recurrence relation (2.33) is easily solved as

$$S^{(n)}(t; L_0, L_1, \ldots, L_{n-1}) = \frac{(-1)^{n-1}}{2^{n-1}} \sum_{\ell=0}^{n-1} (-1)^\ell \pi^{\frac{\ell}{2}} \sum_{0 \le i_1 < i_2 < \cdots < i_\ell \le n-2} (L_{i_1} L_{i_2} \cdots L_{i_\ell})^{-1} S^{(1)}\left(t - \frac{\ell}{2}; L_{n-1}\right), \tag{2.34}$$

in terms of $S^{(1)}(t - \frac{\ell}{2}; L_{n-1})$. By direct calculations, $S^{(1)}(t; L)$ is found to be

$$S^{(1)}(t; L) = \Gamma(t) \sum_{m=1}^{\infty} \{(mL)^2\}^{-t} = \frac{\Gamma(t)\zeta(2t)}{L^{2t}}. \tag{2.35}$$

Thus, we have found that

$$\begin{aligned} F^{(n)}_{L_0, L_1, \ldots, L_{n-1}}\Big|_{\text{n.a.}} &= \frac{(-1)^n}{(2\pi)^{\frac{D}{2}}} \left(\frac{M^2}{2}\right)^{\frac{D}{2}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma\left(t - \frac{D}{2}\right) \left(\frac{ML_{n-1}}{2}\right)^{-2t} \\ &\quad \times \sum_{\ell=0}^{n-1} (-1)^\ell \pi^{\frac{\ell}{2}} \sum_{0 \le i_1 < i_2 < \cdots < i_\ell \le n-2} (L_{i_1} L_{i_2} \cdots L_{i_\ell})^{-1} (L_{n-1})^\ell \Gamma\left(t - \frac{\ell}{2}\right) \zeta(2t - \ell)\Big|_{\text{n.a.}}, \end{aligned} \tag{2.36}$$

which will be used in the next section.

III. NONANALYTIC TERMS IN $F^{(n)}_{L_0, L_1, \ldots, L_{n-1}}$

We are ready to calculate the non-analytic terms by using Eq. (2.36) for each case of even/odd D and n. We note again that the mass dependence of $(M^2)^{\frac{D}{2}-t}$ in Eq. (2.36) tells us that $t =$ half-odd integer (integer) poles of $\Gamma(t - \frac{\ell}{2})$ and $\zeta(2t - \ell)$ yield the nonanalytic terms for $D =$ even(odd).

A. $(D, n) = (\text{even}, \text{even})$

Since $D =$ even in the present case, one needs the poles which are half-odd integers in the integrand of Eq. (2.36) in order to have the nonanalytic terms, that is, the poles of $\Gamma(t - \frac{\ell}{2})$ for $\ell =$ odd and those of the $\zeta(2t - \ell)$ for $\ell =$ even. Hence, it is convenient to separate the summation over ℓ into $\ell =$ odd and $\ell =$ even, as follows:

$$\begin{aligned} F^{(n)}_{L_0, L_1, \ldots, L_{n-1}}\Big|_{\text{n.a.}} &= \frac{1}{(2\pi)^{\frac{D}{2}}} \left(\frac{M^2}{2}\right)^{\frac{D}{2}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma\left(t - \frac{D}{2}\right) \left(\frac{ML_{n-1}}{2}\right)^{-2t} \\ &\quad \times \Bigg(\sum_{j=1}^{\frac{n}{2}} (-1)^{2j-1} \pi^{\frac{2j-1}{2}} \sum_{0 \le i_1 < i_2 < \cdots < i_{2j-1} \le n-2} (L_{i_1} L_{i_2} \cdots L_{i_{2j-1}})^{-1} (L_{n-1})^{2j-1} \Gamma\left(t - \frac{2j-1}{2}\right) \zeta(2t - (2j-1)) \\ &\quad + \sum_{j=0}^{\frac{n}{2}-1} (-1)^{2j} \pi^j \sum_{0 \le i_1 < i_2 < \cdots < i_{2j} \le n-2} (L_{i_1} L_{i_2} \cdots L_{i_{2j}})^{-1} (L_{n-1})^{2j} \Gamma(t - j) \zeta(2t - 2j) \Bigg)\Bigg|_{\text{n.a.}}. \end{aligned} \tag{3.1}$$

The residue integral for the first term in Eq. (3.1) is performed by the pole $t = \frac{2j-1}{2}$ of the gamma function, which yields

$$\sum_{j=1}^{\frac{n}{2}} \frac{(-1)^{\frac{D}{2}} (-1)^{j-1} M^{D-(2j-1)}}{2^{\frac{D-2(j-1)}{2}} \pi^{\frac{D-2j}{2}} (D - (2j-1))!!} \sum_{0 \le i_1 < i_2 < \cdots < i_{2j-1} \le n-2} (L_{i_1} L_{i_2} \cdots L_{i_{2j-1}})^{-1}. \tag{3.2}$$

The pole $t = \frac{2j+1}{2}$ of the zeta function in the second term of Eq. (3.1) gives us

$$\sum_{j=0}^{\frac{n}{2}-1} \frac{(-1)^{\frac{D}{2}}(-1)^j M^{D-(2j+1)}}{2^{\frac{D-2j}{2}}\pi^{\frac{D-2(j+1)}{2}}(D-(2j+1))!!} \sum_{0\le i_1<i_2<\cdots<i_{2j}\le n-2} (L_{i_1}L_{i_2}\cdots L_{i_{2j}}L_{n-1})^{-1}$$
$$= \sum_{j=1}^{\frac{n}{2}} \frac{(-1)^{\frac{D}{2}}(-1)^{j-1} M^{D-(2j-1)}}{2^{\frac{D-2(j-1)}{2}}\pi^{\frac{D-2j}{2}}(D-(2j-1))!!} \sum_{0\le i_1<i_2<\cdots<i_{2j-2}\le n-2} (L_{i_1}L_{i_2}\cdots L_{i_{2j-2}}L_{n-1})^{-1}. \tag{3.3}$$

Combining the two results (3.2) and (3.3), we obtain

$$F^{(n)}_{L_0,L_1,\ldots,L_{n-1}}\Big|_{\text{n.a.}} = \sum_{j=1}^{\frac{n}{2}} \frac{(-1)^{\frac{D}{2}}(-1)^{j-1} M^{D-(2j-1)}}{2^{\frac{D-2(j-1)}{2}}\pi^{\frac{D-2j}{2}}(D-(2j-1))!!} \sum_{0\le i_1<i_2<\cdots<i_{2j-1}\le n-1} (L_{i_1}L_{i_2}\cdots L_{i_{2j-1}})^{-1}, \tag{3.4}$$

where we have used the relation

$$\sum_{0\le i_1<i_2<\cdots<i_{2j-1}\le n-2} (L_{i_1}L_{i_2}\cdots L_{i_{2j-1}})^{-1} + \sum_{0\le i_1<i_2<\cdots<i_{2j-2}\le n-2} (L_{i_1}L_{i_2}\cdots L_{i_{2j-2}}L_{n-1})^{-1} = \sum_{0\le i_1<i_2<\cdots<i_{2j-1}\le n-1} (L_{i_1}L_{i_2}\cdots L_{i_{2j-1}})^{-1}. \tag{3.5}$$

B. $(D, n) = (\text{even}, \text{odd})$

Likewise the previous case, one needs the poles of $\Gamma(t-\frac{\ell}{2})$ for $\ell = \text{odd}$ and those of $\zeta(2t-\ell)$ for $\ell = \text{even}$ in Eq. (2.36) in order to have the nonanalytic terms for $D = \text{even}$. Then, we write Eq. (2.36) as

$$\begin{aligned} F^{(n)}_{L_0,L_1,\ldots,L_{n-1}}\Big|_{\text{n.a.}} = &-\frac{1}{(2\pi)^{\frac{D}{2}}}\left(\frac{M^2}{2}\right)^{\frac{D}{2}} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dt\,\Gamma\left(t-\frac{D}{2}\right)\left(\frac{ML_{n-1}}{2}\right)^{-2t} \\ &\times \left(\sum_{j=1}^{\frac{n-1}{2}} (-1)^{2j-1}\pi^{\frac{2j-1}{2}} \sum_{0\le i_1<i_2<\cdots<i_{2j-1}\le n-2} (L_{i_1}L_{i_2}\cdots L_{i_{2j-1}})^{-1}(L_{n-1})^{2j-1}\Gamma\left(t-\frac{2j-1}{2}\right)\zeta(2t-(2j-1))\right. \\ &\left.+\sum_{j=1}^{\frac{n+1}{2}} (-1)^{2j-2}\pi^{\frac{2j-2}{2}} \sum_{0\le i_1<i_2<\cdots<i_{2j-2}\le n-2} (L_{i_1}L_{i_2}\cdots L_{i_{2j-2}})^{-1}(L_{n-1})^{2j-2}\Gamma\left(t-\frac{2j-2}{2}\right)\zeta(2t-(2j-2))\right)\Bigg|_{\text{n.a.}}. \end{aligned} \tag{3.6}$$

The residue integral for the pole $t = \frac{2j-1}{2}$ of the gamma function in the first term of Eq. (3.6) yields

$$-\sum_{j=1}^{\frac{n-1}{2}} \frac{(-1)^{\frac{D}{2}}(-1)^{j-1} M^{D-(2j-1)}}{2^{\frac{D-2(j-1)}{2}}\pi^{\frac{D-2j}{2}}(D-(2j-1))!!} \sum_{0\le i_1<i_2<\cdots<i_{2j-1}\le n-2} (L_{i_1}L_{i_2}\cdots L_{i_{2j-1}})^{-1}, \tag{3.7}$$

and the residue integral for the pole $t = \frac{2j-1}{2}$ of the zeta function in the second term in Eq. (3.6) gives

$$-\sum_{j=1}^{\frac{n+1}{2}} \frac{(-1)^{\frac{D}{2}}(-1)^{j-1} M^{D-(2j-1)}}{2^{\frac{D-2(j-1)}{2}}\pi^{\frac{D-2j}{2}}(D-(2j-1))!!} \sum_{0\le i_1<i_2<\cdots<i_{2j-2}\le n-2} (L_{i_1}L_{i_2}\cdots L_{i_{2j-2}}L_{n-1})^{-1}. \tag{3.8}$$

We combine the two results (3.7) and (3.8) to yield

$$F^{(n)}_{L_0,L_1,\ldots,L_{n-1}}\Big|_{\text{n.a.}} = -\sum_{j=1}^{\frac{n+1}{2}} \frac{(-1)^{\frac{D}{2}}(-1)^{j-1}M^{D-(2j-1)}}{2^{\frac{D-2(j-1)}{2}}\pi^{\frac{D-2j}{2}}(D-(2j-1))!!} \sum_{0\le i_1<i_2<\cdots<i_{2j-1}\le n-1} (L_{i_1}L_{i_2}\cdots L_{i_{2j-1}})^{-1}. \tag{3.9}$$

C. $(D,n)=(\text{odd},\text{even})$

The spacetime dimension D is odd in this case, so that one needs the poles which are integers in the integrand of Eq. (2.36) in order to have the nonanalytic terms, that is, the poles of $\Gamma(t-\frac{\ell}{2})$ for $\ell=\text{even}$ and those of $\zeta(2t-\ell)$ for $\ell=\text{odd}$. From Eq. (2.36), we have the same expression as Eq. (3.1). The residue integral for the pole $t=j$ of the zeta function in the first term of Eq. (3.1) yields

$$\sum_{j=1}^{\frac{n}{2}} \frac{(-1)^{\frac{D-1}{2}}(-1)^{j}M^{D-2j}}{2^{\frac{D-(2j-1)}{2}}\pi^{\frac{D-(2j+1)}{2}}(D-2j)!!} \sum_{0\le i_1<i_2<\cdots<i_{2j-1}\le n-2} (L_{i_1}L_{i_2}\cdots L_{i_{2j-1}}L_{n-1})^{-1}. \tag{3.10}$$

For the second term in Eq. (3.1), the pole $t=j$ of the gamma function gives us

$$\sum_{j=0}^{\frac{n}{2}-1} \frac{(-1)^{\frac{D-1}{2}}(-1)^{j}M^{D-2j}}{2^{\frac{D-(2j-1)}{2}}\pi^{\frac{D-(2j+1)}{2}}(D-2j)!!} \sum_{0\le i_1<i_2<\cdots<i_{2j}\le n-2} (L_{i_1}L_{i_2}\cdots L_{i_{2j}})^{-1}. \tag{3.11}$$

We put the two results (3.10) and (3.11) together to yield

$$F^{(n)}_{L_0,L_1,\ldots,L_{n-1}}\Big|_{\text{n.a.}} = \sum_{j=0}^{\frac{n}{2}} \frac{(-1)^{\frac{D-1}{2}}(-1)^{j}M^{D-2j}}{2^{\frac{D-(2j-1)}{2}}\pi^{\frac{D-(2j+1)}{2}}(D-2j)!!} \sum_{0\le i_1<i_2<\cdots<i_{2j}\le n-1} (L_{i_1}L_{i_2}\cdots L_{i_{2j}})^{-1}. \tag{3.12}$$

Let us note that the $j=0$ contribution does not have the dependence on the scale of the S^1, but does the mass scale M alone.

D. $(D,n)=(\text{odd},\text{odd})$

Similar to the previous case, one needs the poles of $\Gamma(t-\frac{\ell}{2})$ for $\ell=\text{even}$ and those of $\zeta(2t-\ell)$ for $\ell=\text{odd}$ in Eq. (2.36) in order to have the nonanalytic terms for $D=\text{odd}$. Then, we write Eq. (2.36) as

$$\begin{aligned} F^{(n)}_{L_0,L_1,\ldots,L_{n-1}}\Big|_{\text{n.a.}} &= -\frac{1}{(2\pi)^{\frac{D}{2}}}\left(\frac{M^2}{2}\right)^{\frac{D}{2}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dt\Gamma\left(t-\frac{D}{2}\right)\left(\frac{ML_{n-1}}{2}\right)^{-2t} \\ &\times\left(\sum_{j=1}^{\frac{n-1}{2}}(-1)^{2j-1}\pi^{\frac{2j-1}{2}}\sum_{0\le i_1<i_2<\cdots<i_{2j-1}\le n-2}(L_{i_1}L_{i_2}\cdots L_{i_{2j-1}})^{-1}(L_{n-1})^{2j-1}\Gamma\left(t-\frac{2j-1}{2}\right)\zeta(2t-(2j-1))\right. \\ &\left.+\sum_{j=0}^{\frac{n-1}{2}}(-1)^{2j}\pi^{j}\sum_{0\le i_1<i_2<\cdots<i_{2j}\le n-2}(L_{i_1}L_{i_2}\cdots L_{i_{2j}})^{-1}(L_{n-1})^{2j}\Gamma(t-j)\zeta(2t-2j)\right)\Bigg|_{\text{n.a.}}. \end{aligned} \tag{3.13}$$

The pole $t=j$ of the zeta function in the first term of Eq. (3.13) gives

$$-\sum_{j=1}^{\frac{n-1}{2}} \frac{(-1)^{\frac{D-1}{2}}(-1)^{j}M^{D-2j}}{2^{\frac{D-(2j-1)}{2}}\pi^{\frac{D-(2j+1)}{2}}(D-2j)!!} \sum_{0\le i_1<i_2<\cdots<i_{2j-1}\le n-2} (L_{i_1}L_{i_2}\cdots L_{i_{2j-1}}L_{n-1})^{-1}. \tag{3.14}$$

The residue integral for the pole $t=j$ of the gamma function in the second term of Eq. (3.13) becomes

$$-\sum_{j=0}^{\frac{n-1}{2}}\frac{(-1)^{\frac{D-1}{2}}(-1)^{j}M^{D-2j}}{2^{\frac{D-(2j-1)}{2}}\pi^{\frac{D-(2j+1)}{2}}(D-2j)!!}\sum_{0\le i_1<i_2<\cdots<i_{2j}\le n-2}(L_{i_1}L_{i_2}\cdots L_{i_{2j}})^{-1}. \tag{3.15}$$

We combine the two results (3.14) and (3.15) to yield

$$F^{(n)}_{L_0,L_1,\ldots,L_{n-1}}\Big|_{\rm n.a.}=-\sum_{j=0}^{\frac{n-1}{2}}\frac{(-1)^{\frac{D-1}{2}}(-1)^{j}M^{D-2j}}{2^{\frac{D-(2j-1)}{2}}\pi^{\frac{D-(2j+1)}{2}}(D-2j)!!}\sum_{0\le i_1<i_2<\cdots<i_{2j}\le n-1}(L_{i_1}L_{i_2}\cdots L_{i_{2j}})^{-1}. \tag{3.16}$$

Let us note again that the $j=0$ contribution does not possess the scale dependence on the S^1, but does the mass scale M.

We have succeeded in obtaining the nonanalytic terms in $F^{(n)}_{L_0,L_1,\ldots,L_{n-1}}$ as Eqs. (3.4), (3.9), (3.12), and (3.16), depending on even/odd D and n. In order to calculate the nonanalytic terms in Eq. (2.7), it is useful to generalize the expressions of $F^{(n)}_{L_0,L_1,\ldots,L_{n-1}}|_{\rm n.a.}$ to $F^{(n)}_{L_{i_1},\ldots,L_{i_n}}|_{\rm n.a.}$. Then, we have

$$(D,n)=(\text{even},\text{even}):F^{(n)}_{L_{i_1},\ldots,L_{i_n}}\Big|_{\rm n.a.}=\sum_{j=1}^{\frac{n}{2}}\frac{(-1)^{\frac{D}{2}+j-1}M^{D-(2j-1)}}{2^{\frac{D-2(j-1)}{2}}\pi^{\frac{D-2j}{2}}(D-(2j-1))!!}\left\{\sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j-1}\}\\ \in\{i_1,i_2,\ldots,i_n\}}}(L_{\ell_1}L_{\ell_2}\cdots L_{\ell_{2j-1}})^{-1}\right\}, \tag{3.17}$$

$$(D,n)=(\text{even},\text{odd}):F^{(n)}_{L_{i_1},\ldots,L_{i_n}}\Big|_{\rm n.a.}=\sum_{j=1}^{\frac{n+1}{2}}\frac{(-1)^{\frac{D}{2}+j}M^{D-(2j-1)}}{2^{\frac{D-2(j-1)}{2}}\pi^{\frac{D-2j}{2}}(D-(2j-1))!!}\left\{\sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j-1}\}\\ \in\{i_1,i_2,\ldots,i_n\}}}(L_{\ell_1}L_{\ell_2}\cdots L_{\ell_{2j-1}})^{-1}\right\}, \tag{3.18}$$

$$(D,n)=(\text{odd},\text{even}):F^{(n)}_{L_{i_1},\ldots,L_{i_n}}\Big|_{\rm n.a.}=\sum_{j=0}^{\frac{n}{2}}\frac{(-1)^{\frac{D-1}{2}+j}M^{D-2j}}{2^{\frac{D-(2j-1)}{2}}\pi^{\frac{D-(2j+1)}{2}}(D-2j)!!}\left\{\sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j}\}\\ \in\{i_1,i_2,\ldots,i_n\}}}(L_{\ell_1}L_{\ell_2}\cdots L_{\ell_{2j}})^{-1}\right\}, \tag{3.19}$$

$$(D,n)=(\text{odd},\text{odd}):F^{(n)}_{L_{i_1},\ldots,L_{i_n}}\Big|_{\rm n.a.}=\sum_{j=0}^{\frac{n-1}{2}}\frac{(-1)^{\frac{D-1}{2}+j-1}M^{D-2j}}{2^{\frac{D-(2j-1)}{2}}\pi^{\frac{D-(2j+1)}{2}}(D-2j)!!}\left\{\sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j}\}\\ \in\{i_1,i_2,\ldots,i_n\}}}(L_{\ell_1}L_{\ell_2}\cdots L_{\ell_{2j}})^{-1}\right\}, \tag{3.20}$$

where the order of the elements in the set should be understood to be $\ell_1<\ell_2<\cdots<\ell_{2j-1}$ and $i_1<i_2<\cdots<i_{n-1}$, etc. Equations (3.17)–(3.20) will be used in the next section.

IV. NONANALYTIC TERMS IN EFFECTIVE POTENTIAL

Equipped with Eqs. (3.17)–(3.20), let us calculate the nonanalytic terms in the effective potential (2.6). We omit the abbreviation "n.a." used in Secs. II and III, which stands for the nonanalytic terms. One should keep in mind that we are treating the nonanalytic terms in this section. We introduce

$$A^D_j\equiv\frac{(-1)^{\frac{D}{2}+j-1}M^{D-(2j-1)}}{2^{\frac{D-2(j-1)}{2}}\pi^{\frac{D-2j}{2}}(D-(2j-1))!!}, \tag{4.1}$$

$$B^D_j\equiv\frac{(-1)^{\frac{D-1}{2}+j-1}M^{D-2j}}{2^{\frac{D-(2j-1)}{2}}\pi^{\frac{D-(2j+1)}{2}}(D-2j)!!} \tag{4.2}$$

in Eqs. (3.17)–3.20 in order to write them in compact forms.

A. $(D, p+1) = (\text{even}, \text{even})$

Let us write the $F^{(n)}(n \geq 1)$ part of the effective potential in Eq. (2.6) for $(D, p+1) = (\text{even}, \text{even})$ as

$$\begin{aligned}\sum_{n=1}^{p+1} F^{(n)} &= \sum_{k=1}^{\frac{p+1}{2}} \{F^{(2k-1)} + F^{(2k)}\} \\ &= \sum_{k=1}^{\frac{p+1}{2}} \left\{ \sum_{0\leq i_1<i_2<\cdots<i_{2k-1}\leq p} F^{(2k-1)}_{L_{i_1},L_{i_2},\ldots,L_{i_{2k-1}}} + \sum_{0\leq i_1<i_2<\cdots<i_{2k}\leq p} F^{(2k)}_{L_{i_1},L_{i_2},\ldots,L_{i_{2k}}} \right\} \\ &= \sum_{k=1}^{\frac{p+1}{2}} \left\{ \sum_{0\leq i_1<i_2<\cdots<i_{2k-1}\leq p} \sum_{j=1}^{k} (-1) A_j^D \sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j-1}\} \\ \in\{i_1,i_2,\ldots,i_{2k-1}\}}} (L_{\ell_1} L_{\ell_2}\cdots L_{\ell_{2j-1}})^{-1} \right. \\ &\qquad \left. + \sum_{0\leq i_1<i_2<\cdots<i_{2k}\leq p} \sum_{j=1}^{k} A_j^D \sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j-1}\} \\ \in\{i_1,i_2,\ldots,i_{2k}\}}} (L_{\ell_1} L_{\ell_2}\cdots L_{\ell_{2j-1}})^{-1} \right\}, \end{aligned} \tag{4.3}$$

where we have used Eqs. (2.7), (3.17), and (3.18). By changing the order of the summations with respect to j and k, we obtain

$$\sum_{n=1}^{p+1} F^{(n)} = \sum_{j=1}^{\frac{p+1}{2}} \sum_{k=j}^{\frac{p+1}{2}} \left\{ - \sum_{0\leq i_1<i_2<\cdots<i_{2k-1}\leq p} \sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j-1}\} \\ \in\{i_1,i_2,\ldots,i_{2k-1}\}}} + \sum_{0\leq i_1<i_2<\cdots<i_{2k}\leq p} \sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j-1}\} \\ \in\{i_1,i_2,\ldots,i_{2k}\}}} \right\} A_j^D (L_{\ell_1} L_{\ell_2}\cdots L_{\ell_{2j-1}})^{-1}, \tag{4.4}$$

where we have used the formula

$$\sum_{k=1}^{\frac{p+1}{2}} \sum_{j=1}^{k} = \sum_{j=1}^{\frac{p+1}{2}} \sum_{k=j}^{\frac{p+1}{2}}. \tag{4.5}$$

There are ${}_{p+1}C_{2j-1}$ numbers of the independent configurations for $(L_{\ell_1} L_{\ell_2}\cdots L_{\ell_{2j-1}})^{-1}$ for a given value of j, each of which has the same multiplicity in Eq. (4.4) for a fixed value of k. Hence, it is enough to calculate the multiplicity for the configuration with $\ell_1 = 0, \ell_2 = 1, \ldots, \ell_{2j-1} = 2j-2$, that is, $(L_0 L_1 \cdots L_{2j-2})^{-1}$ for each value of $k = j, j+1, \ldots, \frac{p+1}{2}$.

Let us first compute the multiplicity for the case of $k = j$ with fixed j. Aside form the factor A_j^D, we have

$$\left\{ - \sum_{0\leq i_1<i_2<\cdots<i_{2j-1}\leq p} \sum_{\substack{\{0,1,\ldots,2j-2\} \\ \in\{i_1,i_2,\ldots,i_{2j-1}\}}} + \sum_{0\leq i_1<i_2<\cdots<i_{2j}\leq p} \sum_{\substack{\{0,1,\ldots,2j-2\} \\ \in\{i_1,i_2,\ldots,i_{2j}\}}} \right\} (L_0 L_1 \cdots L_{2j-2})^{-1} = \{-1 + {}_{p+1-(2j-1)}C_1\} (L_0 L_1 \cdots L_{2j-2})^{-1}. \tag{4.6}$$

Let us next calculate the multiplicity for the case of $k = j+1$. The result is given by

$$\begin{aligned} &\left\{ - \sum_{0\leq i_1<i_2<\cdots<i_{2j+1}\leq p} \sum_{\substack{\{0,1,\ldots,2j-2\} \\ \in\{i_1,i_2,\ldots,i_{2j+1}\}}} + \sum_{0\leq i_1<i_2<\cdots<i_{2j+2}\leq p} \sum_{\substack{\{0,1,\ldots,2j-2\} \\ \in\{i_1,i_2,\ldots,i_{2j+2}\}}} \right\} (L_0 L_1 \cdots L_{2j-2})^{-1} \\ &\quad = \{-{}_{p+1-(2j-1)}C_2 + {}_{p+1-(2j-1)}C_3\} (L_0 L_1 \cdots L_{2j-2})^{-1}. \end{aligned} \tag{4.7}$$

In the same manner, one can calculate the multiplicity for $k = j+2, j+3, \ldots, \frac{p+1}{2}$. Collecting the terms with $k = j, j+1, \ldots, \frac{p+1}{2}$ and denoting $A \equiv p+1-(2j-1)$, which is odd for the present case, we have

$$\begin{aligned}&\{-(1+{}_AC_2+{}_AC_4+\cdots+{}_AC_{A-1})+({}_AC_1+{}_AC_3+\cdots+{}_AC_A)\}(L_0L_1\cdots L_{2j-2})^{-1}\\&=\left\{-\frac{1}{2}\times 2^A+\frac{1}{2}\times 2^A\right\}(L_0L_1\cdots L_{2j-2})^{-1}=0.\end{aligned} \tag{4.8}$$

This holds for any j satisfying $1\le j\le \frac{p+1}{2}$ and the same conclusion (4.8) holds for the other configurations of $L_{\ell_1}L_{\ell_2}\cdots L_{\ell_{2j-1}}$. Furthermore, Eq. (2.9) is analytic for $D=$ even. Thus, we conclude that $V_{\rm eff}|_{\rm n.a.}=0$, that is, there is no nonanalytic term in the effective potential for $D=$ even and $p+1=$ even.

B. $(D,p+1)=(\text{even},\text{odd})$

We write the $F^{(n)}(n\ge 1)$ part of the effective potential for this case as

$$\begin{aligned}\sum_{n=1}^{p+1}F^{(n)}&=\sum_{k=1}^{\frac{p}{2}}\{F^{(2k-1)}+F^{(2k)}\}+F^{(p+1)}=\sum_{k=1}^{\frac{p}{2}}\left\{\sum_{0\le i_1<i_2<\cdots<i_{2k-1}\le p}\sum_{j=1}^{k}(-1)A_j^D\sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j-1}\}\\\in\{i_1,i_2,\ldots,i_{2k-1}\}}}(L_{\ell_1}L_{\ell_2}\cdots L_{\ell_{2j-1}})^{-1}\right.\\&\left.\quad+\sum_{0\le i_1<i_2<\cdots<i_{2k}\le p}\sum_{j=1}^{k}A_j^D\sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j-1}\}\\\in\{i_1,i_2,\ldots,i_{2k}\}}}(L_{\ell_1}L_{\ell_2}\cdots L_{\ell_{2j-1}})^{-1}\right\}\\&\quad+\sum_{j=1}^{\frac{p+2}{2}}(-1)A_j^D\sum_{0\le\ell_1<\ell_2<\cdots<\ell_{2j-1}\le p}(L_{\ell_1}L_{\ell_2}\cdots L_{\ell_{2j-1}})^{-1},\end{aligned} \tag{4.9}$$

where we have again used Eqs. (2.7), (3.17), and (3.18). The last term in Eq. (4.9) with $j=\frac{p+2}{2}$ yields $-A^D_{\frac{p+2}{2}}(L_0L_1\cdots L_p)^{-1}$. Then, we have

$$\begin{aligned}\sum_{n=1}^{p+1}F^{(n)}&=\sum_{j=1}^{\frac{p}{2}}\left(\sum_{k=j}^{\frac{p}{2}}\left\{-\sum_{0\le i_1<i_2<\cdots<i_{2k-1}\le p}\sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j-1}\}\\\in\{i_1,i_2,\ldots,i_{2k-1}\}}}+\sum_{0\le i_1<i_2<\cdots<i_{2k}\le p}\sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j-1}\}\\\in\{i_1,i_2,\ldots,i_{2k}\}}}\right\}-\sum_{0\le\ell_1<\ell_2<\cdots<\ell_{2j-1}\le p}\right)\\&\quad\times A_j^D(L_{\ell_1}L_{\ell_2}\cdots L_{\ell_{2j-1}})^{-1}-A^D_{\frac{p+2}{2}}(L_0L_1\cdots L_p)^{-1},\end{aligned} \tag{4.10}$$

where we have exchanged the order of the summations with respect to j and k.

Let us count the multiplicity for the configuration with $\ell_1=0,\ell_2=1,\ldots,\ell_{2j-1}=2j-2$. Note that the third term in Eq. (4.10) results one for the configuration. We calculate the multiplicity of the first and the second terms in Eq. (4.10) for $k=j$. Apart from the factor A_j^D, it is calculated as

$$\begin{aligned}&\left\{-\sum_{0\le i_1<i_2<\cdots<i_{2j-1}\le p}\sum_{\substack{\{0,1,\ldots,2j-2\}\\\in\{i_1,i_2,\ldots,i_{2j-1}\}}}+\sum_{0\le i_1<i_2<\cdots<i_{2j}\le p}\sum_{\substack{\{0,1,\ldots,2j-2\}\\\in\{i_1,i_2,\ldots,i_{2j}\}}}\right\}(L_0L_1\cdots L_{2j-2})^{-1}\\&=\{-1+{}_{p+1-(2j-1)}C_1\}(L_0L_1\cdots L_{2j-2})^{-1}.\end{aligned} \tag{4.11}$$

Likewise, for $k=j+1$, we have

$$\begin{aligned}&\left\{-\sum_{0\le i_1<i_2<\cdots<i_{2j+1}\le p}\sum_{\substack{\{0,1,\ldots,2j-2\}\\\in\{i_1,i_2,\ldots,i_{2j+1}\}}}+\sum_{0\le i_1<i_2<\cdots<i_{2j+2}\le p}\sum_{\substack{\{0,1,\ldots,2j-2\}\\\in\{i_1,i_2,\ldots,i_{2j+2}\}}}\right\}(L_0L_1\cdots L_{2j-2})^{-1}\\&=\{-{}_{p+1-(2j-1)}C_2+{}_{p+1-(2j-1)}C_3\}(L_0L_1\cdots L_{2j-2})^{-1}.\end{aligned} \tag{4.12}$$

Similarly, we can compute the multiplicity for $k = j+1, j+2, \ldots, \frac{p}{2}$. Collecting all the terms with $k = j, j+1, \ldots, \frac{p}{2}$ and including the third term in Eq. (4.10), we have

$$\begin{aligned}&\{-(1 + {}_AC_2 + {}_AC_4 + \cdots + {}_AC_{A-2}) + ({}_AC_1 + {}_AC_3 + \cdots + {}_AC_{A-1}) - 1\}(L_0L_1\cdots L_{2j-2})^{-1}\\&\quad = \left\{-\left(\frac{1}{2}\times 2^A - {}_AC_A\right) + \frac{1}{2}\times 2^A - 1\right\}(L_0L_1\cdots L_{2j-2})^{-1} = 0,\end{aligned} \tag{4.13}$$

where $A \equiv p+1-(2j-1)$, which is even in this case. This holds for any value of j between 1 and $\frac{p}{2}$. Hence, the last term in Eq. (4.10) alone is left to yield the nonanalytic term in the effective potential for $D =$ even and $p+1 =$ odd, i.e.,

$$\begin{aligned}V_{\mathrm{eff}}\Big|_{\mathrm{n.a.}} &= -A^D_{\frac{p+2}{2}}(L_0L_1\cdots L_p)^{-1}\\&= \frac{(-1)^{\frac{D+p+2}{2}}}{2^{\frac{D-p}{2}}\pi^{\frac{D-(p+2)}{2}}(D-(p+1))!!}\frac{M^{D-(p+1)}}{L_0L_1\cdots L_p}.\end{aligned} \tag{4.14}$$

C. $(D, p+1) = (\mathrm{odd}, \mathrm{even})$

The $F^{(n)}(n \geq 1)$ part of the effective potential for this case is given by

$$\begin{aligned}\sum_{n=1}^{p+1} F^{(n)} = \sum_{k=1}^{\frac{p+1}{2}}\{F^{(2k-1)} + F^{(2k)}\} = \sum_{k=1}^{\frac{p+1}{2}}\Bigg\{&\sum_{0\leq i_1<i_2<\cdots<i_{2k-1}\leq p}\ \sum_{j=0}^{k-1} B^D_j \sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j}\}\\ \in\{i_1,i_2,\ldots,i_{2k-1}\}}}(L_{\ell_1}L_{\ell_2}\cdots L_{\ell_{2j}})^{-1}\\&+\sum_{0\leq i_1<i_2<\cdots<i_{2k}\leq p}\ \sum_{j=0}^{k}(-1)B^D_j\sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j}\}\\ \in\{i_1,i_2,\ldots,i_{2k}\}}}(L_{\ell_1}L_{\ell_2}\cdots L_{\ell_{2j}})^{-1}\Bigg\},\end{aligned} \tag{4.15}$$

where we have used Eqs. (2.7), (3.19), and (3.20). Separating the $j = k$ contribution from the second term in Eq. (4.15) and exchanging the summations with respect to j and k, we obtain

$$\begin{aligned}\sum_{n=1}^{p+1} F^{(n)} = &\sum_{j=0}^{\frac{p-1}{2}}\sum_{k=j+1}^{\frac{p+1}{2}}\Bigg\{\sum_{0\leq i_1<i_2<\cdots<i_{2k-1}\leq p}\ \sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j}\}\\ \in\{i_1,i_2,\ldots,i_{2k-1}\}}} - \sum_{0\leq i_1<i_2<\cdots<i_{2k}\leq p}\ \sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j}\}\\ \in\{i_1,i_2,\ldots,i_{2k}\}}}\Bigg\}B^D_j(L_{\ell_1}L_{\ell_2}\cdots L_{\ell_{2j}})^{-1}\\&-\sum_{k=1}^{\frac{p-1}{2}}\ \sum_{0\leq \ell_1<\ell_2<\cdots<\ell_{2k}\leq p} B^D_k(L_{\ell_1}L_{\ell_2}\cdots L_{\ell_{2k}})^{-1} - B^D_{\frac{p+1}{2}}(L_0L_1\cdots L_p)^{-1}.\end{aligned} \tag{4.16}$$

Here, the last term in Eq. (4.16) has been separated from the third term and corresponds to the $k = \frac{p+1}{2}$ contribution of it.

Let us first study the $j = 0$ contribution in the first and the second term in Eq. (4.16), which is given by

$$\begin{aligned}\sum_{k=1}^{\frac{p+1}{2}}\{{}_{p+1}C_{2k-1} - {}_{p+1}C_{2k}\}B^D_0 &= \left\{\frac{1}{2}\times 2^{p+1} - \left(\frac{1}{2}\times 2^{p+1} - {}_{p+1}C_0\right)\right\}B^D_0\\&= -\frac{(-1)^{\frac{D-1}{2}}}{2^{\frac{D+1}{2}}\pi^{\frac{D-1}{2}}D!!}M^D.\end{aligned} \tag{4.17}$$

This exactly cancels the contribution coming from the one-loop correction for the zero modes (2.9) for $D =$ odd. Then, rewriting k as j in the third term in Eq. (4.16), we have

$$F^{(0)} + \sum_{n=1}^{p+1} F^{(n)} = \sum_{j=1}^{\frac{p-1}{2}} \left(\sum_{k=j+1}^{\frac{p+1}{2}} \left\{ \sum_{0\le i_1<i_2<\cdots<i_{2k-1}\le p} \sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j}\} \\ \in\{i_1,i_2,\ldots,i_{2k-1}\}}} - \sum_{0\le i_1<i_2<\cdots<i_{2k}\le p} \sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j}\} \\ \in\{i_1,i_2,\ldots,i_{2k}\}}} \right\} - \sum_{0\le\ell_1<\ell_2<\cdots<\ell_{2j}\le p} \right) \times B_j^D (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j}})^{-1} - B^D_{\frac{p+1}{2}} (L_0 L_1 \cdots L_p)^{-1}. \tag{4.18}$$

Let us count the multiplicity for the configuration $\ell_1 = 0, \ell_2 = 1, \ldots, \ell_{2j} = 2j-1$ for fixed j. To this end, we extract all the terms proportional to $L_0 L_1 \cdots L_{2j-1}$ for $k = j+1, j+2, \ldots, \frac{p+1}{2}$ from Eq. (4.18). Apart from the factor B_j^D, the result is

$$\begin{aligned} &\{({}_BC_1 + {}_BC_3 + \cdots + {}_BC_{B-1}) - ({}_BC_2 + {}_BC_4 + \cdots + {}_BC_B) - 1\}(L_0 L_1 \cdots L_{2j-1})^{-1} \\ &= \left\{ \left(\frac{1}{2} \times 2^B \right) - \left(\frac{1}{2} \times 2^B - {}_BC_0 \right) - 1 \right\} (L_0 L_1 \cdots L_{2j-1})^{-1} = 0, \end{aligned} \tag{4.19}$$

where $B = p + 1 - 2j$, which is even in this case. Since Eq. (4.19) holds by any values of $j = 1, 2, \ldots, \frac{p-1}{2}$, only the last term in Eq. (4.18) is left to yield the nonanalytic term in the effective potential,

$$\begin{aligned} V_{\mathrm{eff}}\Big|_{\mathrm{n.a.}} &= -B^D_{\frac{p+1}{2}} (L_0 L_1 \cdots L_p)^{-1} \\ &= \frac{(-1)^{\frac{D+p}{2}}}{2^{\frac{D-p}{2}} \pi^{\frac{D-(p+2)}{2}} (D-(p+1))!!} \frac{M^{D-(p+1)}}{L_0 L_1 \cdots L_p}. \end{aligned} \tag{4.20}$$

D. $(D, p+1) = (\mathrm{odd}, \mathrm{odd})$

Let us proceed the fourth case. The $F^{(n)} (n \ge 1)$ part of the effective potential is given by

$$\begin{aligned} \sum_{n=1}^{p+1} F^{(n)} &= \sum_{k=1}^{\frac{p}{2}} \{F^{(2k-1)} + F^{(2k)}\} + F^{(p+1)} \\ &= \sum_{k=1}^{\frac{p}{2}} \left\{ \sum_{0\le i_1<i_2<\cdots<i_{2k-1}\le p} \sum_{j=0}^{k-1} B_j^D \sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j}\} \\ \in\{i_1,i_2,\ldots,i_{2k-1}\}}} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j}})^{-1} \right. \\ &\quad \left. + \sum_{0\le i_1<i_2<\cdots<i_{2k}\le p} \sum_{j=0}^{k} (-1) B_j^D \sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j}\} \\ \in\{i_1,i_2,\ldots,i_{2k}\}}} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j}})^{-1} \right\} + \sum_{j=0}^{\frac{p}{2}} B_j^D \sum_{0\le\ell_1<\ell_2<\cdots<\ell_{2j}\le p} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j}})^{-1}, \end{aligned} \tag{4.21}$$

where we have again used Eqs. (2.7), (3.19), and (3.20). We write the above equation as

$$\begin{aligned} \sum_{n=1}^{p+1} F^{(n)} = &\sum_{k=1}^{\frac{p}{2}} \sum_{j=0}^{k-1} \left\{ \sum_{0\le i_1<i_2<\cdots<i_{2k-1}\le p} B_j^D \sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j}\} \\ \in\{i_1,i_2,\ldots,i_{2k-1}\}}} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j}})^{-1} + \sum_{0\le i_1<i_2<\cdots<i_{2k}\le p} (-1) B_j^D \sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j}\} \\ \in\{i_1,i_2,\ldots,i_{2k}\}}} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j}})^{-1} \right\} \\ &+ \sum_{k=1}^{\frac{p}{2}} \sum_{0\le i_1<i_2<\cdots<i_{2k}\le p} (-1) B_k^D (L_{i_1} L_{i_2} \cdots L_{i_{2k}})^{-1} + \sum_{j=0}^{\frac{p}{2}} B_j^D \sum_{0\le\ell_1<\ell_2<\cdots<\ell_{2j}\le p} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j}})^{-1}. \end{aligned} \tag{4.22}$$

Note that the third term in Eq. (4.22) cancels the last one in Eq. (4.22) except for the $j = 0$ term, which cancels the one-loop contribution for the zero modes (2.9) for $D = \mathrm{odd}$.

Then, what is left is the first and the second terms in Eq. (4.22). By exchanging the summations with respect to j and k, we obtain

$$F^{(0)}+\sum_{n=1}^{p+1}F^{(n)}=\sum_{j=0}^{\frac{p-2}{2}}\sum_{k=j+1}^{\frac{p}{2}}\left\{\sum_{0\le i_1<i_2<\cdots<i_{2k-1}\le p}\sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j}\}\\ \in\{i_1,i_2,\ldots,i_{2k-1}\}}}-\sum_{0\le i_1<i_2<\cdots<i_{2k}\le p}\sum_{\substack{\{\ell_1,\ell_2,\ldots,\ell_{2j}\}\\ \in\{i_1,i_2,\ldots,i_{2k}\}}}\right\}B_j^D(L_{\ell_1}L_{\ell_2}\cdots L_{\ell_{2j}})^{-1}. \tag{4.23}$$

Let us count the multiplicity for the configuration with $\ell_1=0,\ell_2=1,\ldots,\ell_{2j}=2j-1$ for fixed j by extracting all the terms proportional to $L_0L_1\cdots L_{2j-1}$ for $k=j+1,j+1,\ldots,\frac{p}{2}$ from Eq. (4.23). Apart from the factor B_j^D, the result is given by

$$\begin{aligned}&\{({}_BC_1+{}_BC_3+\cdots+{}_BC_{B-2})-({}_BC_2+{}_BC_4+\cdots+{}_BC_{B-1})\}(L_0L_1\cdots L_{2j-1})^{-1}\\ &\quad=\left\{\left(\frac{1}{2}\times 2^B-{}_BC_B\right)-\left(\frac{1}{2}\times 2^B-{}_BC_0\right)\right\}(L_0L_1\cdots L_{2j-1})^{-1}=0,\end{aligned} \tag{4.24}$$

where $B=p+1-2j$, which is odd in this case. This holds for any j satisfying $0\le j\le\frac{p-2}{2}$, so that we have $V_{\mathrm{eff}}|_{\mathrm{n.a.}}=0$, that is, the effective potential does not possess nonanalytic terms for $D=$ odd and $p+1=$ odd.

We have calculated the nonanalytic terms in the effective potential for any D and p. We have found that there is no nonanalytic term for $(D,p+1)=$ (even, even) and (odd, odd). On the other hand, the nonanalytic term appears for $(D,p+1)=$ (even, odd) and (odd, even), as shown in Eqs. (4.14) and (4.20), respectively. The results are summarized as

$$V_{\mathrm{eff}}\Big|_{\mathrm{n.a.}}=\begin{cases}0 & \text{for } (D,p+1)=(\text{even},\text{even}),(\text{odd},\text{odd}),\\ \frac{(-1)^{\frac{D+p}{2}}(-1)^{p+1}}{2^{\frac{D-p}{2}}\pi^{\frac{D-(p+2)}{2}}(D-(p+1))!!}\frac{M^{D-(p+1)}}{L_0L_1\cdots L_p} & \text{for } (D,p+1)=(\text{even},\text{odd}),(\text{odd},\text{even}).\end{cases} \tag{4.25}$$

The famous Dolan-Jackiw term corresponds to the case of $D=4$, $p=0$ case in Eq. (4.25). We present some results followed from Eq. (4.25) in Table I.

We observe that the nonanalytic term appears in the effective potential when $D-(p+1)=$ odd, which corresponds to the odd uncompactified spacial dimensions. It must be noticed that there is only one nonanalytic term for the given values of D and p, for which the nonanalytic term exists.

TABLE I. Some results on nonanalytic terms in the effective potential on the spacetime $S_\tau^1\times R^{D-(p+1)}\times\prod_{i=1}^p S_i^1$. We use $T(=L_0^{-1})$. The "non" in the table means that there is no nonanalytic term in the effective potential and the "×" stands for the case, where the condition $D\ge p+1$ is not satisfied.

$S_\tau^1\times R^{D-(p+1)}\times\prod_{i=1}^p S_i^1$					
	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$
$D=3$	non	$\frac{1}{2}\frac{MT}{L_1}$	non	×	×
$D=4$	$-\frac{M^3T}{12\pi}$	non	$\frac{1}{2}\frac{MT}{L_1L_2}$	non	×
$D=5$	non	$-\frac{1}{12\pi}\frac{M^3T}{L_1}$	non	$\frac{1}{2}\frac{MT}{L_1L_2L_3}$	non
$D=6$	$\frac{M^5T}{120\pi^2}$	non	$-\frac{1}{12\pi}\frac{M^3T}{L_1L_2}$	non	$\frac{1}{2}\frac{MT}{L_1L_2L_3L_4}$
$D=7$	non	$\frac{1}{120\pi^2}\frac{M^5T}{L_1}$	non	$-\frac{1}{12\pi}\frac{M^3T}{L_1L_2L_3}$	non

V. CONCLUSIONS AND DISCUSSIONS

We have studied the nonanalytic terms in the effective potential for the real scalar field at finite temperature in one-loop approximation on the D-dimensional spacetime, $S_\tau^1\times R^{D-(p+1)}\times\prod_{i=1}^p S_i^1$. The effective potential is given in terms of the modified Bessel function of the second kind accompanied with the multiple mode summations. We have introduced the integral representation for the modified Bessel function of the second kind and have also made use of the analytical extension for the mode summations. The effective potential is recast into the integral form in the complex plane, and the nonanalytic terms are obtained by the residue theorem.

We have clarified the pole structure of the analytical extension for the mode summations and have found the recurrence relation (2.33), from which we have obtained the general form (2.34) for the relevant terms in the mode summations. We have calculated the nonanalytic terms, Eqs. (3.17)–(3.20) by the residue theorem for the poles of the gamma and zeta functions in Eq. (2.34) with Eq. (2.35), depending on the even/odd D and $n(n=1,2,\ldots,p+1)$. The positions of the poles that yield the nonanalytic terms are found to be independent of D. Equipped with them, we have calculated the nonanalytic terms in the effective potential, which is given by Eq. (4.25), including the famous Dolan-Jackiw term. Some explicit results are summarized in Table I.

We have found that the effective potential has the nonanalytic term when the dimension of the flat Euclidean space, $D-(p+1)$ is odd. There is only one nonanalytic term for the given values of D and p, for which the nonanalytic term exists.

There are untouched issues in the paper. We have not discussed the physical origin of such the nonanalytic term in the effective potential. Paper [1] showed that the famous Dolan-Jackiw term had been emerged through the zero mode of the S^1_τ direction, reflecting the infrared dynamics of the theory. It may be important to clarify the physical origin of the nonanalytic term found in this paper. Moreover, it may be important to study the physical implication of the nonanalytic terms on, for example, the phase transition at finite temperature.

For the case of the fermion, the boundary condition for the S^1_τ direction is antiperiodic due to the Fermi statistics, and we have the factor $(-1)^{m_0}$ in the mode summation. Then, the zeta function $\zeta(2t)$ in Eq. (2.15) is replaced by the eta function, $-\eta(2t)$, which does not possess the pole, so that the nonanalytic term does not arise for $D=4$, $p=0$ [1]. It is expected that the analytical extension for the mode summations is modified to yield different nonanalytic terms from those of the scalar field. Moreover, there are degrees of freedom to choose the periodic or antiperiodic boundary condition for each spacial $S^1_i (i=1,2,\ldots,p)$ direction for the fermion (scalar) field. We also expect that the analytical extension for the mode summations is different from that of the case for the periodic boundary condition. Accordingly, we may have a different type of nonanalytic terms. These will be reported elsewhere.

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APPENDIX: MODE SUMMATIONS AND ANALYTICAL EXTENSION

The effective potential contains the modified Bessel function of the second kind accompanied with the mode summations. In addition to the integral representation (2.14) in the text for the modified Bessel function of the second kind, one needs an analytical extension for the mode summations in order to recast the effective potential into the integral form and to obtain the nonanalytic terms by the residue integral. In this appendix we present an analytical extension for the multiple mode summations and clarify its pole structure in the residue integral.

1. Double mode summations

The double summations (2.22) in the text has an analytical extension [15]. In our analysis, the following formula plays a crucial role,

$$\begin{aligned}\Gamma(t)\sum_{m_0=1}^{\infty}\{(m_0L_0)^2+c^2\}^{-t} &= -\frac{1}{2}\frac{\Gamma(t)}{c^{2t}}+\frac{\sqrt{\pi}}{2L_0}\frac{\Gamma(t-\frac{1}{2})}{c^{2(t-\frac{1}{2})}}+\frac{2\pi^t}{L_0^{t+\frac{1}{2}}}\frac{1}{c^{t-\frac{1}{2}}}\sum_{n_0=1}^{\infty}n_0^{t-\frac{1}{2}}K_{t-\frac{1}{2}}\left(\frac{2\pi n_0}{L_0}c\right)\\ &= -\frac{1}{2}\frac{\Gamma(t)}{c^{2t}}+\frac{\sqrt{\pi}}{2L_0}\frac{\Gamma(t-\frac{1}{2})}{c^{2(t-\frac{1}{2})}}\\ &\quad+\frac{1}{\sqrt{\pi}}\left(\frac{\pi}{L_0}\right)^{2t}\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}dt_1\Gamma\left(t_1-t+\frac{1}{2}\right)\zeta(2t_1-2t+1)\Gamma(t_1)\left(c\frac{\pi}{L_0}\right)^{-2t_1},\end{aligned}\tag{A1}$$

where we have used Eq. (2.14) in the last equality. In obtaining Eq. (A1), we have made use of the Poisson summation

$$\sum_{m_0=-\infty}^{\infty}e^{-[(m_0+\eta_0)L_0]^2t}=\sum_{n_0=-\infty}^{\infty}\frac{1}{L_0}\left(\frac{\pi}{t}\right)^{\frac{1}{2}}e^{-\frac{(2\pi n_0)^2}{4L_0^2}\frac{1}{t}+2\pi i n_0\eta_0}\tag{A2}$$

and the formula

$$K_\nu(z)=\frac{1}{2}\left(\frac{z}{2}\right)^{\nu}\int_0^{\infty}dt\,t^{-\nu-1}e^{-\frac{1}{t}(\frac{z}{2})^2-t}.\tag{A3}$$

Setting $c = m_1 L_1$ and taking the summation $\sum_{m_1=1}^{\infty}$, we arrive at

$$\Gamma(t)\sum_{m_0=1}^{\infty}\sum_{m_1=1}^{\infty}\{(m_0L_0)^2+(m_1L_1)^2\}^{-t}=-\frac{1}{2}\frac{1}{L_1^{2t}}\Gamma(t)\zeta(2t)+\frac{\sqrt{\pi}}{2}\frac{1}{L_0L_1^{2t-1}}\Gamma\left(t-\frac{1}{2}\right)\zeta(2t-1)$$
$$+\frac{1}{\sqrt{\pi}}\left(\frac{\pi}{L_0}\right)^{2t}\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}dt_1\,\Gamma\left(t_1-t+\frac{1}{2}\right)\zeta(2t_1-2t+1)\Gamma(t_1)\zeta(2t_1)\left(\pi\frac{L_1}{L_0}\right)^{-2t_1}. \tag{A4}$$

This is Eq. (2.23) in the text.

As we will show below, an important observation is that the third term in Eq. (A4) has the property

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}dt\,f(t)\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}dt_1\,\Gamma\left(t_1-t+\frac{1}{2}\right)\zeta(2t_1-2t+1)\Gamma(t_1)\zeta(2t_1)\left(\pi\frac{L_1}{L_0}\right)^{-2t_1}=0, \tag{A5}$$

where $f(t)$ is any function that has no poles inside the region of the residue integral with respect to t. Equation (A5) implies that the poles coming from the third term of Eq. (A4) do not contribute to the residue integral with respect to t in Eq. (A5). A similar property holds for the multiple mode summations, as we will see in the next subsection.

To show Eq. (A5), we first note that the combination of $\Gamma(t_1-\frac{a}{2})\zeta(2t_1-a)$ has no poles except for the pole of $\Gamma(t_1-\frac{a}{2})$ at $t_1=\frac{a}{2}$ and that of $\zeta(2t_1-a)$ at $t_1=\frac{a}{2}+\frac{1}{2}$. One might think that $\Gamma(t_1-\frac{a}{2})\zeta(2t_1-a)$ could have an infinite number of poles at $t_1=\frac{a}{2}-n(n=1,2,3\cdots)$. This is not, however, the case because of the property $\zeta(-2n)=0$ for $n=1,2,3,\ldots$. This is an important observation, which will be used throughout our analyses.

The t_1 integration on the left-hand side of Eq. (A5) can be performed as the residue integral and the result is

$$\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}dt_1\,\Gamma\left(t_1-t+\frac{1}{2}\right)\zeta\left(2t_1-2t+1)\Gamma(t_1)\zeta(2t_1\right)\left(\pi\frac{L_1}{L_0}\right)^{-2t_1}$$
$$=\zeta(0)\Gamma\left(t-\frac{1}{2}\right)\zeta(2t-1)\left(\pi\frac{L_1}{L_0}\right)^{-2(t-\frac{1}{2})}+\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma(t)\zeta(2t)\left(\pi\frac{L_1}{L_0}\right)^{-2t}$$
$$+\zeta(0)\Gamma\left(\frac{1}{2}-t\right)\zeta(1-2t)+\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma(1-t)\zeta(2-2t)\left(\pi\frac{L_1}{L_0}\right)^{-1}, \tag{A6}$$

which leads to Eq. (A5), as can be confirmed by direct calculations.

2. Multiple mode summations and pole structure

In this section we generalize the previous analysis of the double mode summations to the multiple ones

$$S^{(n)}(t;L_0,L_1,\ldots,L_{n-1})\equiv\Gamma(t)\sum_{m_0=1}^{\infty}\sum_{m_1=1}^{\infty}\cdots\sum_{m_{n-1}=1}^{\infty}\{(m_0L_0)^2+(m_1L_1)^2+\cdots+(m_{n-1}L_{n-1})^2\}^{-t}. \tag{A7}$$

By use of the formula (A1), $S^{(n)}(t;L_0,L_1,\ldots,L_{n-1})$ can be expressed into a recursive form as

$$S^{(n)}(t;L_0,L_1,\ldots,L_{n-1})=-\frac{1}{2}S^{(n-1)}(t;L_1,L_2,\ldots,L_{n-1})+\frac{\sqrt{\pi}}{2L_0}S^{(n-1)}\left(t-\frac{1}{2};L_1,L_2,\ldots,L_{n-1}\right)$$
$$+\frac{1}{\sqrt{\pi}}\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}dt_1\,\Gamma\left(t_1-t+\frac{1}{2}\right)\zeta(2t_1-2t+1)\left(\frac{\pi}{L_0}\right)^{-2t_1+2t}S^{(n-1)}(t_1;L_1,L_2,\ldots,L_{n-1}), \tag{A8}$$

where we have defined

$$S^{(n-k)}(t;L_k,L_{k+1},\ldots,L_{n-1}) = \Gamma(t)\sum_{m_k=1}^{\infty}\sum_{m_{k+1}=1}^{\infty}\cdots\sum_{m_{n-1}=1}^{\infty}\{(m_kL_k)^2+(m_{k+1}L_{k+1})^2+\cdots+(m_{n-1}L_{n-1})^2\}^{-t} \quad \text{(A9)}$$

for $k=0,1,\ldots,n-1$.

In the following, we shall show that the third term in Eq. (A8) has the property

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}dt\, f(t)\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}dt_1\,\Gamma\left(t_1-t+\frac{1}{2}\right)\zeta(2t_1-2t+1)\left(\frac{\pi}{L_0}\right)^{-2t_1+2t}S^{(n-1)}(t_1;L_1,L_2,\ldots,L_{n-1})=0, \quad \text{(A10)}$$

where $f(t)$ is any function that has no poles inside the region of the residue integral with respect to t. Equation (A10) implies that the poles of the third term in Eq. (A8) with respect to t do not totally contribute to the residue integral of t, although the third term in Eq. (A8) has several poles with respect to t, as we will see below.

By repeatedly using the relation (A8), $S^{(n)}(t;L_0,L_1,\ldots,L_{n-1})$ can be expressed into the form

$$\begin{aligned} S^{(n)}(t;L_0,L_1,\ldots,L_{n-1}) &= (P_1+P_2+P_3)^{n-1}S^{(n)}(t;L_0,L_1,\ldots,L_{n-1}) \\ &= \sum_{j_1=1}^{3}\sum_{j_2=1}^{3}\cdots\sum_{j_{n-1}=1}^{3}P_{j_1}P_{j_2}\cdots P_{j_{n-1}}S^{(n)}(t;L_0,L_1,\ldots,L_{n-1}), \end{aligned} \quad \text{(A11)}$$

where the operation of P_j $(j=1,2,3)$ is defined by

$$P_1S^{(n-k)}(t;L_k,L_{k+1},\ldots,L_{n-1}) = -\frac{1}{2}S^{(n-k-1)}(t;L_{k+1},L_{k+2},\ldots,L_{n-1}), \quad \text{(A12)}$$

$$P_2S^{(n-k)}(t;L_k,L_{k+1},\ldots,L_{n-1}) = \frac{\sqrt{\pi}}{2L_k}S^{(n-k-1)}\left(t-\frac{1}{2};L_{k+1},L_{k+2},\ldots,\quad L_{n-1}\right), \quad \text{(A13)}$$

$$\begin{aligned} &P_3S^{(n-k)}(t;L_k,L_{k+1},\ldots,L_{n-1}) \\ &= \frac{1}{\sqrt{\pi}}\frac{1}{2\pi i}\int_{c_1-\infty}^{c_1+i\infty}dt_1\Gamma\left(t_1-t+\frac{1}{2}\right)\zeta(2t_1-2t+1)\left(\frac{\pi}{L_k}\right)^{-2t_1+2t}S^{(n-k-1)}(t_1;L_{k+1},L_{k+2},\ldots,L_{n-1}). \end{aligned} \quad \text{(A14)}$$

For instance, let us consider the term $(P_1)^{n-\ell-m-1}(P_2)^{\ell}(P_3)^mS^{(n)}(t;L_0,L_1,\ldots,L_{n-1})$, which is explicitly given by

$$\begin{aligned} &(P_1)^{n-\ell-m-1}(P_2)^{\ell}(P_3)^mS^{(n)}(t;L_0,L_1,\ldots,L_{n-1}) \\ &= \frac{1}{\sqrt{\pi}}\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}dt_1\Gamma\left(t_1-t+\frac{1}{2}\right)\zeta(2t_1-2t+1)\left(\frac{\pi}{L_0}\right)^{-2t_1+2t} \\ &\times\frac{1}{\sqrt{\pi}}\frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}dt_2\Gamma\left(t_2-t_1+\frac{1}{2}\right)\zeta(2t_2-2t_1+1)\left(\frac{\pi}{L_1}\right)^{-2t_2+2t_1}\times\cdots \\ &\times\frac{1}{\sqrt{\pi}}\frac{1}{2\pi i}\int_{c_m-i\infty}^{c_m+i\infty}dt_m\Gamma\left(t_m-t_{m-1}+\frac{1}{2}\right)\zeta(2t_m-2t_{m-1}+1)\left(\frac{\pi}{L_{m-1}}\right)^{-2t_m+2t_{m-1}} \\ &\times\left(\frac{\sqrt{\pi}}{2L_m}\right)\left(\frac{\sqrt{\pi}}{2L_{m+1}}\right)\cdots\left(\frac{\sqrt{\pi}}{2L_{\ell+m-1}}\right)\left(-\frac{1}{2}\right)^{n-\ell-m-1}S^{(1)}\left(t_m-\frac{\ell}{2};L_{n-1}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^{n-\ell-m-1}(\sqrt{\pi})^{\ell-m}}{2^{n-m-1}} \frac{(L_{n-1})^{\ell}}{L_m L_{m+1}\cdots L_{\ell+m-1}} \left(\frac{\pi}{L_0}\right)^{2t} \\
&\times \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma\left(t_1 - t + \frac{1}{2}\right)\zeta(2t_1 - 2t + 1)\left(\frac{L_0}{L_1}\right)^{2t_1} \\
&\times \frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty} dt_2 \Gamma\left(t_2 - t_1 + \frac{1}{2}\right)\zeta(2t_2 - 2t_1 + 1)\left(\frac{L_1}{L_2}\right)^{2t_2} \times \cdots \\
&\times \frac{1}{2\pi i}\int_{c_m-i\infty}^{c_m+i\infty} dt_m \Gamma\left(t_m - t_{m-1} + \frac{1}{2}\right)\zeta(2t_m - 2t_{m-1} + 1) \\
&\times \Gamma\left(t_m - \frac{\ell}{2}\right)\zeta(2t_m - \ell)\left(\frac{L_{m-1}}{\pi L_{n-1}}\right)^{2t_m}. \qquad \text{(A15)}
\end{aligned}
$$

In the second equality of Eq. (A15), we have used

$$
S^{(1)}\left(t_m - \frac{\ell}{2}; L_{n-1}\right) = \Gamma\left(t_m - \frac{\ell}{2}\right)\sum_{m_{n-1}=1}^{\infty}(m_{n-1}L_{n-1})^{-2t_m+\ell} = \frac{\Gamma(t_m - \frac{\ell}{2})\zeta(2t_m - \ell)}{(L_{n-1})^{2t_m-\ell}}. \qquad \text{(A16)}
$$

Let $f(t)$ be any function which has no pole inside the region of the residue integral with respect to t. Then, we can show that

$$
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dt\, f(t)[(P_1)^{n-\ell-m-1}(P_2)^{\ell}(P_3)^m S^{(n)}(t; L_0, L_1, \ldots, L_{n-1})] = 0 \qquad \text{(A17)}
$$

if $m \geq 1$.

To show Eq. (A17), we perform the residue integrals of Eq. (A15) with respect to $\{t_m, t_{m-1}, \cdots, t_2\}$ successively by use of the relations

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{c_j-i\infty}^{c_j+i\infty} dt_j g(t_j)\Gamma\left(t_j - t_{j-1} + \frac{1}{2}\right)\zeta(2t_j - 2t_{j-1} + 1)\Gamma\left(t_j - \frac{k}{2}\right)\zeta(2t_j - k) \\
&= g\left(t_{j-1} - \frac{1}{2}\right)\zeta(0)\Gamma\left(t_{j-1} - \frac{k+1}{2}\right)\zeta(2t_{j-1} - (k+1)) + \frac{1}{2}g(t_{j-1})\Gamma\left(\frac{1}{2}\right)\Gamma\left(t_{j-1} - \frac{k}{2}\right)\zeta(2t_{j-1} - k) \\
&\quad + g\left(\frac{k}{2}\right)\zeta(0)\Gamma\left(\frac{k+1}{2} - t_{j-1}\right)\zeta(k + 1 - 2t_{j-1}) + \frac{1}{2}g\left(\frac{k+1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{k+2}{2} - t_{j-1}\right)\zeta(k + 2 - 2t_{j-1}), \qquad \text{(A18)}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{c_j-i\infty}^{c_j+i\infty} dt_j g(t_j)\Gamma\left(t_j - t_{j-1} + \frac{1}{2}\right)\zeta(2t_j - 2t_{j-1} + 1)\Gamma\left(\frac{k}{2} - t_j\right)\zeta(k - 2t_j) \\
&= g\left(t_{j-1} - \frac{1}{2}\right)\zeta(0)\Gamma\left(\frac{k+1}{2} - t_{j-1}\right)\zeta(k + 1 - 2t_{j-1}) + \frac{1}{2}g(t_{j-1})\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{k}{2} - t_{j-1}\right)\zeta(k - 2t_{j-1}) \\
&\quad - g\left(\frac{k}{2}\right)\zeta(0)\Gamma\left(\frac{k+1}{2} - t_{j-1}\right)\zeta(k + 1 - 2t_{j-1}) - \frac{1}{2}g\left(\frac{k-1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{k}{2} - t_{j-1}\right)\zeta(k - 2t_{j-1}), \qquad \text{(A19)}
\end{aligned}
$$

where $g(t_j)$ is any function that has no poles inside the region of the residue integral with respect to t_j. We note that there appears only the combination of the type $\Gamma(\pm(t_{j-1} - \frac{\ell}{2}))\zeta(\pm(2t_{j-1} - \ell))$ for some integer ℓ on the right-hand side of the formulas (A18) and (A19), so that we can perform the residue integrals with respect to $t_m, t_{m-1}, \ldots, t_2$, successively.

After performing the residue integrals with respect to $t_j (j = m, m-1, \ldots, 2)$ by use of the formulas (A18) and (A19), Eq. (A15) can be written as the sum of the terms proportional to the following types of the integrals,

$$
\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} dt_1 g_{\ell}(t_1)\Gamma\left(t_1 - t + \frac{1}{2}\right)\zeta(2t_1 - 2t + 1)\Gamma\left(\pm\left(t_1 - \frac{\ell}{2}\right)\right)\zeta(\pm(2t_1 - \ell)), \qquad \text{(A20)}
$$

where ℓ is some positive integer and $g_\ell(t_1)$ is some function, which has no poles inside the region of the residue integral with respect t_1.

It is now easy to show Eq. (A17), which follows from the relation

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dt\, f(t)\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} dt_1 g_\ell(t_1)\Gamma\left(t_1-t+\frac{1}{2}\right)\zeta(2t_1-2t+1)\Gamma\left(\pm\left(t_1-\frac{\ell}{2}\right)\right)\zeta(\pm(2t_1-\ell))=0 \tag{A21}$$

with Eqs. (A18) and (A19). It should be emphasized that Eq. (A21) does not mean that the integrand of Eq. (A17) has no poles with respect to t. In fact, the integrand of Eq. (A17) has several poles with respect to t and each pole contributes to the residue integral of t, though the sum of their residues totally cancels each other.

We have proved Eq. (A17) for a special order of $P_{j_1}P_{j_2}\cdots P_{j_{n-1}}$. Since $P_jP_kS^{(n-k)}(t;L_k,L_{k+1},\cdots,L_{n-1})$ $(j,k=1,2,3)$ is identical to the opposite order of $P_kP_jS^{(n-k)}(t;L_k,L_{k+1},\ldots,L_{n-1})$ with the exchange of $L_k\leftrightarrow L_{k+1}$ (and with a shift of the integration parameter, if necessary), we generally have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dt\, f(t)[P_{j_1}P_{j_2}\cdots P_{j_{n-1}}S^{(n)}(t;L_0,L_1,\ldots,L_{n-1})]=0 \tag{A22}$$

if some of $j_s(s=1,2,\ldots,n-1)$ take the value of 3. This result immediately leads to Eq. (A10) because Eq. (A10) corresponds to the case of $j_1=3$ in Eq. (A22).

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