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State Estimation of Kermack–McKendrick PDE Model With Latent Period and Observation Delay

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***Abstract*—In this paper, we study the problem of estimating the state of the linearized Kermack–McKendrick PDE model in real time. Especially, we assume that the model contains two kinds of delays. One of them is contained in the nonlocal boundary condition, which expresses the latent period of infection. The other is an observation delay, which corresponds to the time needed for counting the number of infected people at an infection elapsed time. The element of time lags can be expressed by a transport equation. As a result, the system with two delays is equivalently written by a 3×3 hyperbolic system. In this paper, we construct observers with three gain functions, using a backstepping method of PDEs. Then, the triple of the designed gain belongs to the domain of the generator governing the state evolution of the error system. Furthermore, based on this fact and the semigroup theory, it is shown that the error system is L^2-stable in the Hilbert space.**

***Index Terms*—Hyperbolic equation, delay, observer, Volterra-Fredholm backstepping transformation, semigroup.**

I. Introduction

In this technical note, we study the problem of estimating the state of the linearized Kermack–McKendrick PDE model in real time. As is well-known, the Kermack–McKendrick model has been used as a model describing progression of infection. Especially, the model is approximated by the linearized equation in early phase of infection [13]. It is very important to estimate the state at an early stage in order to predict the transition of infection in the future. In the field of systems and control theory, the mechanism of estimating the state is called a state observer, or simply an observer. The linearized Kermack–McKendrick model is described by a first-order hyperbolic equation with nonlocal boundary condition. In this note, we assume that the model contains two kinds of delays. One of them is contained in the nonlocal boundary condition, which expresses the latent period of infection. The other is an observation delay, which corresponds to the time needed for counting the number of infected people at an infection elapsed time. In the previous studies, the authors have treated the state estimation problem of the model whose time lag is contained only in the nonlocal boundary consition [18], [19]. Furthermore, in [19] the L^2-stability of the error system was discussed.

For a general class of first-order hyperbolic systems, the observers design problem has been actively studied in recent years (see e.g. [10], [5], [4], [12], [9], [2], [7], [20], [22], [3], [1], [8]. Some of these papers also deal with the stabilization problem by state/output feedback or the output regulation problem). For example, in [10] Di Meglio *et al.* treated a system of $n+1$ coupled hyperbolic equations which consists of n rightward convecting transport equations and one leftward convecting transport equation, and constructed an observer using one sensor on the left boundary of system. In [12], Hu *et al.* extended the design method to a system of $n+m$ coupled hyperbolic equations consisting of n rightward convecting transport equations and m leftward convecting transport equations. Further, in [3] Auriol and Di Meglio considered the same heterodirectional coupled hyperbolic equations, and constructed an observer using sensors on both boundarys of system. In that paper, the dual problem of observers design is defined as a control problem, and the observer gains are obtained by using the gains of the dual controller, where the backstepping using the Fredholm transformation [5] is adoped. Thus, many papers related to this problem have been published. However, the boundary condition is a local one and time lags are not considered.

On the other hand, our settings of the system differ from those papers, that is, the boundary condition is nonlocal and time lags are contained in two parts mentioned above. The element of time lags can be expressed by a transport equation [14], [15]. As a result, the system with two delays is equivalently written by a 3×3 hyperbolic system. In this note, we construct observers with three gain functions $g(x)$, $h(x)$ and $i(x)$, using a backstepping method via the Volterra-Fredholm transformation. Then, we have an interesting result of the triple (g, h, i) belonging to the domain of the generator governing the state evolution of the error system. Furthermore, based on this fact and the semigroup theory [6], [11], [17], it is shown that the error system is L^2-stable in the Hilbert space. Also, in the case where the delay which expresses the latent period of infection is larger than or equal to the observation delay, it is shown that the gain $i(x)$ vanishes.

This technical note is organized as follows: In Section II, we first introduce the linearized Kermack–McKendrick PDE model with two delays under some assumption with respect to the length of infectious interval. Then, we show how to construct a backstepping observer with three gains. In Section III, we show the result with respect to smoothness of the triple of observer gains and prove the L^2-stability of the error system. In Section IV, we give numerical simulation results to show the validity of our design method. Finally, the note is

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concluded in Section V.

II. System Description and Observer Design

A. Kermack–McKendrick Model

Let $L_\dagger$ be the maximum period of infectiousness. Consider the following linearized Kermack–McKendrick model [13]:

$$\begin{cases} u_t(t,x) = -u_x(t,x) - \gamma(x)u(t,x), \ t>0, \ x\in(0,L_\dagger), \\ u(t,0) = S_0 \displaystyle\int_0^{L_\dagger} \beta(x)u(t,x)dx, \quad t>0, \\ u(0,x) = u_0(x), \quad x\in[0,L_\dagger], \end{cases} \tag{1}$$

where $u(t,x)$ denotes the density of infectious population at infection elapsed time x and at time t, $\gamma(x)$ denotes the recovered rate, $\beta(x)$ the transmission coefficient. A positive constant S_0 expresses infectious population at the initial stationary state. We set the following hypothesis:

Hypothesis 1: Let $L_\dagger > 0$ be finite. Further, there exists a constant $L \in (0, L_\dagger)$ such that $\beta(x) = 0$ for all $x \in [L, L_\dagger]$.

Under this hypothesis, we decompose the spatial domain $[0, L_\dagger]$ as $[0, L_\dagger] = [0, L] \cup [L, L_\dagger]$ and express the model (1) as

$$\begin{cases} u_t(t,x) = -u_x(t,x) - \gamma(x)u(t,x), \ t>0, \ x\in(0,L), \\ u(t,0) = S_0 \displaystyle\int_0^{L} \beta(x)u(t,x)dx, \quad t>0, \\ u(0,x) = u_0(x), \quad x\in[0,L], \end{cases} \tag{2}$$

$$\begin{cases} \overline{u}_t(t,x) = -\overline{u}_x(t,x) - \gamma(x)\overline{u}(t,x), \ t>0, \ x\in(L,L_\dagger), \\ \overline{u}(t,L) = u(t,L^-) := \displaystyle\lim_{x\to L-0} u(t,x), \quad t>0, \\ \overline{u}(0,x) = u_0(x), \quad x\in[L,L_\dagger], \end{cases} \tag{3}$$

where we have used the notation u, $\overline{u}$ for u restricted on $[0, L]$, $[L, L_\dagger]$, respectively. The first part (2) dominates the progression of infection, since it contains the transmission coefficient in the boundary condition, whereas the second part (3) does not have its influence. Therefore, estimating the state $u(t,x)$ on $[0,L]$ is very important in order to predict the transition of infection in the future. In the practical point of view, we assume two kinds of delays; one is the latent period of infection and the other is an observation delay.

B. Kermack–McKendrick Model with Two Delays

Let $\tilde{\beta}(x) := S_0\beta(x)$. Consider the following first-order hyperbolic system with observation equation:

$$\begin{cases} u_t(t,x) = -u_x(t,x) - \gamma(x)u(t,x), \ t>0, \ x\in(0,L), \\ u(t,0) = \displaystyle\int_0^{L} \tilde{\beta}(x)u(t-\tau,x)dx, \quad t\geq\tau, \\ u(0,x) = u_0(x), \quad x\in[0,L], \\ Y(t) = u(t-d,L), \quad t\geq d, \quad \text{(observation equation)}, \end{cases} \tag{4}$$

where $\tau \in (0, L)$ denotes the latent period of infection and $d \in (0, L)$ denotes the observation delay (see Fig. 1). We set $u(t, 0) = v_0(t)$ for $t < \tau$ and $Y(t) = w_0(t)$ for $t < d$. We assume that $\gamma \in C[0, L]$, $\tilde{\beta} \in C^1[0, L]$ are non-negative real-valued functions. From Hypothesis 1, note that $\tilde{\beta}(L) = 0$.

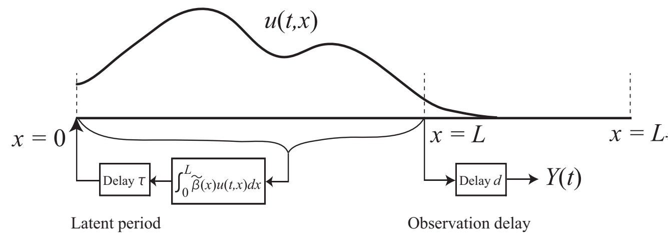

Fig. 1. Kermack–McKendrick model with two delays.

Since the element of time lags can be expressed by a transport equation [14], [15], system (4) with two delays is equivalently written as the following 3×3 hyperbolic system:

$$\begin{cases} u_t(t,x) = -u_x(t,x) - \gamma(x)u(t,x), \ t>0, \ x\in(0,L), \\ u(t,0) = v(t,0), \quad t>0, \\ u(0,x) = u_0(x), \quad x\in[0,L], \\ v_t(t,x) = v_x(t,x), \quad t>0, \ x\in(0,\tau), \\ v(t,\tau) = \displaystyle\int_0^{L} \tilde{\beta}(x)u(t,x)dx, \quad t>0, \\ v(0,x) = v_0(x), \quad x\in[0,\tau], \\ w_t(t,x) = w_x(t,x), \quad t>0, \ x\in(0,d), \\ w(t,d) = u(t,L), \quad t>0, \\ w(0,x) = w_0(x), \quad x\in[0,d], \\ Y(t) = w(t,0), \quad t>0, \quad \text{(observation equation)}. \end{cases} \tag{5}$$

The goal is to design an observer that estimates the state $u(t,x)$ $(0 \leq x \leq L)$ using only the data $Y(t)$.

Remark 1: In [21], Thieme studied semilinear age-structured population dynamics with two delay terms, in which one delay was contained in the governing equation, and the other in the nonlocal boundary condition. In that paper, the equation was formulated as a semilinear evolution equation with a Lipschitz perturbation of nondensely defined operator, and the generalized notion of solution was introduced. On the other hand, the model treated in this note is linear, and one delay is contained in the nonlocal boundary condition, and the other in the observation equation.

C. Backstepping Observer

In order to estimate the states u, v, w of (5), let us consider the following 3×3 hyperbolic system, which is a Luenberger-type observer:

$$\begin{cases} \hat{u}_t(t,x) = -\hat{u}_x(t,x) - \gamma(x)\hat{u}(t,x) \\ \qquad\qquad +g(x)(Y(t) - \hat{w}(t,0)), \ t>0, \ x\in(0,L), \\ \hat{u}(t,0) = \hat{v}(t,0), \quad t>0, \\ \hat{u}(0,x) = \hat{u}_0(x), \quad x\in[0,L], \\ \hat{v}_t(t,x) = \hat{v}_x(t,x) \\ \qquad\qquad +h(x)(Y(t) - \hat{w}(t,0)), \ t>0, \ x\in(0,\tau), \\ \hat{v}(t,\tau) = \displaystyle\int_0^{L} \tilde{\beta}(x)\hat{u}(t,x)dx, \quad t>0, \\ \hat{v}(0,x) = \hat{v}_0(x), \quad x\in[0,\tau], \\ \hat{w}_t(t,x) = \hat{w}_x(t,x) \\ \qquad\qquad +i(x)(Y(t) - \hat{w}(t,0)), \ t>0, \ x\in(0,d), \\ \hat{w}(t,d) = \hat{u}(t,L), \quad t>0, \\ \hat{w}(0,x) = \hat{w}_0(x), \quad x\in[0,d]. \end{cases} \tag{6}$$

Using $Y(t) = w(t,0)$, and introducing the variables $\tilde{u} := u - \hat{u}$, $\tilde{v} := v - \hat{v}$, $\tilde{w} := w - \hat{w}$, we have the error system

$$\left\{\begin{array}{l} \tilde{u}_t(t,x) = -\tilde{u}_x(t,x) - \gamma(x)\tilde{u}(t,x) \\ \qquad\qquad -g(x)\tilde{w}(t,0), \quad t>0,\ x\in(0,L), \\ \tilde{u}(t,0) = \tilde{v}(t,0), \quad t>0, \\ \tilde{u}(0,x) = \tilde{u}_0(x), \quad x\in[0,L], \\ \tilde{v}_t(t,x) = \tilde{v}_x(t,x) - h(x)\tilde{w}(t,0),\ t>0,\ x\in(0,\tau), \\ \tilde{v}(t,\tau) = \displaystyle\int_0^L \tilde{\beta}(x)\tilde{u}(t,x)dx, \quad t>0, \\ \tilde{v}(0,x) = \tilde{v}_0(x), \quad x\in[0,\tau], \\ \tilde{w}_t(t,x) = \tilde{w}_x(t,x) - i(x)\tilde{w}(t,0),\ t>0,\ x\in(0,d), \\ \tilde{w}(t,d) = \tilde{u}(t,L), \quad t>0, \\ \tilde{w}(0,x) = \tilde{w}_0(x), \quad x\in[0,d]. \end{array}\right. \tag{7}$$

We here use the backstepping method [14], [15]. Especially, we use the Volterra-Fredholm integral transformation

$$\zeta(t,x) = \tilde{v}(t,x) - \int_0^L k(x,y)\tilde{u}(t,y)dy - \int_0^d l(x,y)\tilde{w}(t,y)dy, \tag{8}$$

$$\eta(t,x) = \tilde{u}(t,x) - \int_x^L p(x,y)\tilde{u}(t,y)dy - \int_0^d q(x,y)\tilde{w}(t,y)dy, \tag{9}$$

and determine the kernels $k(x,y)$, $l(x,y)$, $p(x,y)$, $q(x,y)$ in (8), (9) and the gains $g(x)$, $h(x)$, $i(x)$ in (7) so as to achieve $\tilde{u}(t,\cdot)\to 0$, $\tilde{v}(t,\cdot)\to 0$, $\tilde{w}(t,\cdot)\to 0$ as $t\to\infty$.

First, using (8) and integration by parts, we calculate $\zeta_t(t,x) - \zeta_x(t,x)$ and obtain

$$\begin{aligned} &\zeta_t(t,x) - \zeta_x(t,x) \\ &= \left\{ l(x,0) - h(x) + \int_0^L k(x,y)g(y)dy + \int_0^d l(x,y)i(y)dy \right\}\tilde{w}(t,0) \\ &\quad + \int_0^L \{k_x(x,y) - k_y(x,y) + \gamma(y)k(x,y)\}\tilde{u}(t,y)dy \\ &\quad + \int_0^d \{l_x(x,y) + l_y(x,y)\}\tilde{w}(t,y)dy \\ &\quad + \{k(x,L) - l(x,d)\}\tilde{u}(t,L) - \{k(x,0)\}\tilde{u}(t,0). \end{aligned} \tag{10}$$

Similarly, using (9) and integration by parts, $\eta_t(t,x) + \eta_x(t,x) + \gamma(x)\eta(t,x)$ is calculated as follows:

$$\begin{aligned} &\eta_t(t,x) + \eta_x(t,x) + \gamma(x)\eta(t,x) \\ &= \left\{ q(x,0) - g(x) + \int_x^L p(x,y)g(y)dy + \int_0^d q(x,y)i(y)dy \right\}\tilde{w}(t,0) \\ &\quad + \int_x^L \{-p_x(x,y) - p_y(x,y) + (\gamma(y) - \gamma(x))p(x,y)\}\tilde{u}(t,y)dy \\ &\quad + \int_0^d \{-q_x(x,y) + q_y(x,y) - \gamma(x)q(x,y)\}\tilde{w}(t,y)dy \\ &\quad + \{p(x,L) - q(x,d)\}\tilde{w}(t,d). \end{aligned} \tag{11}$$

Here, we note that:

1) If the terms surrounded by braces of the right-hand side of (10) and (11) are zero, $\zeta_t(t,x) - \zeta_x(t,x) = 0$ and $\eta_t(t,x) + \eta_x(t,x) + \gamma(x)\eta(t,x) = 0$ hold for all $\tilde{u}$, $\tilde{w}$.
2) By (8), if $k(\tau,y) = \tilde{\beta}(y)$ and $l(\tau,y) = 0$ are satisfied, one has $\zeta(t,\tau) = 0$.
3) By (8) and (9), if $k(0,y) = p(0,y)$ and $l(0,y) = q(0,y)$ are satisfied, one has $\eta(t,0) = \zeta(t,0)$.

In addition, we note that, by (9),

$$\eta(t,L) = \tilde{u}(t,L) - \int_0^d q(L,y)\tilde{w}(t,y)dy.$$

Under $\tilde{\beta}(L) = 0$ and the additional assumption $\tilde{\beta}(0) = 0$, we sequentially determine the kernels $k(x,y)$, $l(x,y)$, $p(x,y)$, $q(x,y)$ and the gains $i(x)$, $g(x)$, $h(x)$ according to the following seven steps:

Step 1 – Design of k. Find the solution $k(x,y)$ on $D_k := \{(x,y)\,;\, 0\le x\le\tau, 0\le y\le L\}$ to the following hyperbolic equation:

$$\left\{\begin{array}{l} k_x(x,y) = k_y(x,y) - \gamma(y)k(x,y), \\ k(x,0) = 0, \\ k(\tau,y) = \tilde{\beta}(y). \end{array}\right. \tag{12}$$

Indeed, one can find the solution as follows:

$$k(x,y) = \left\{\begin{array}{ll} 0, & x+y\le\tau, \\ \tilde{\beta}(x+y-\tau)e^{\int_{x+y-\tau}^{y}\gamma(\xi)d\xi}, & x+y>\tau. \end{array}\right. \tag{13}$$

Note that $k\in H^1(D_k)$ under the condition $\tilde{\beta}(0) = 0$.

Step 2 – Design of l "using k, find l". Find the solution $l(x,y)$ on $D_l := \{(x,y)\,;\, 0\le x\le\tau, 0\le y\le d\}$ to the following hyperbolic equation:

$$\left\{\begin{array}{l} l_x(x,y) + l_y(x,y) = 0, \\ l(x,d) = k(x,L), \\ l(\tau,y) = 0. \end{array}\right. \tag{14}$$

Then, one can find the solution as follows:

$$l(x,y) = \left\{\begin{array}{ll} 0, & y\le x+d-\tau, \\ \tilde{\beta}(x-y+L+d-\tau)e^{\int_{x-y+L+d-\tau}^{L}\gamma(\xi)d\xi}, & y>x+d-\tau. \end{array}\right. \tag{15}$$

Note that $l\in H^1(D_l)$ under the condition $\tilde{\beta}(L) = 0$.

Step 3 – Design of p "using k, find p". Find the solution $p(x,y)$ on $D_p := \{(x,y)\,;\, x\le y\le L, 0\le x\le L\}$ to the following hyperbolic equation:

$$\left\{\begin{array}{l} p_x(x,y) = -p_y(x,y) + (\gamma(y) - \gamma(x))p(x,y), \\ p(0,y) = k(0,y). \end{array}\right. \tag{16}$$

In this case, since the solution is expressed as

$$p(x,y) = e^{\int_x^y \gamma(\xi)d\xi - \int_0^{y-x}\gamma(\xi)d\xi}k(0,y-x),$$

one has

$$p(x,y) = \left\{\begin{array}{ll} 0, & y\le x+\tau, \\ \tilde{\beta}(y-x-\tau)e^{\int_x^y \gamma(\xi)d\xi - \int_0^{y-x-\tau}\gamma(\xi)d\xi}, & y>x+\tau. \end{array}\right. \tag{17}$$

Note that $p \in H^1(D_p)$ under the condition $\tilde{\beta}(0) = 0$.

Step 4 – Design of q "using l, p, find q". Find the solution $q(x, y)$ on $D_q := \{(x, y)\,;\, 0 \le x \le L, 0 \le y \le d\}$ to the following hyperbolic equation:

$$\begin{cases} q_x(x,y) = q_y(x,y) - \gamma(x)q(x,y), \\ q(x,d) = p(x,L), \\ q(0,y) = l(0,y). \end{cases} \tag{18}$$

Then, one can find the solution as follows:

$$q(x,y) = \tag{19}$$
$$\begin{cases} l(0, x+y)e^{-\int_0^x \gamma(\xi)d\xi}, & x+y \le d, \\ p(x+y-d, L)e^{-\int_{x+y-d}^x \gamma(\xi)d\xi}, & d < x+y \le L+d-\tau, \\ 0, & L+d-\tau < x+y. \end{cases}$$

Note that $q \in H^1(D_q)$ under the condition $\tilde{\beta}(0) = \tilde{\beta}(L) = 0$.

Step 5 – Design of i "using q, find i". Determine the gain $i(x)$ $(0 \le x \le d)$ such that the solution $\tilde{w}$ of the following hyperbolic equation becomes zero for all $t \ge d$:

$$\begin{cases} \tilde{w}_t(t,x) = \tilde{w}_x(t,x) - i(x)\tilde{w}(t,0), \\ \tilde{w}(t,d) = \int_0^d q(L,y)\tilde{w}(t,y)dy, \\ \tilde{w}(0,x) = \tilde{w}_0(x). \end{cases} \tag{20}$$

For the concrete algorithm with respect to $i(x)$, see Remark 2 and Appendix A below.

Step 6 – Design of g "using p, q, i, find g". Find the solution $g(x)$ $(0 \le x \le L)$ to the following integral equation:

$$g(x) = q(x,0) + \int_x^L p(x,y)g(y)dy + \int_0^d q(x,y)i(y)dy. \tag{21}$$

Step 7 – Design of h "using k, l, i, g, find h". Calculate the gain $h(x)$ $(0 \le x \le \tau)$ as follows:

$$h(x) = \int_0^L k(x,y)g(y)dy + \int_0^d l(x,y)i(y)dy + l(x,0). \tag{22}$$

$\diamondsuit$

As a result, we can consider the following system as a target system of the error system (7):

$$\begin{cases} \eta_t(t,x) = -\eta_x(t,x) - \gamma(x)\eta(t,x), \\ \eta(t,0) = \zeta(t,0), \quad x \in [0,L], \\ \tilde{w}_t(t,x) = \tilde{w}_x(t,x) - i(x)\tilde{w}(t,0), \\ \tilde{w}(t,d) = \eta(t,L) + \int_0^d q(L,y)\tilde{w}(t,y)dy, \, x \in [0,d], \\ \zeta_t(t,x) = \zeta_x(t,x), \\ \zeta(t,\tau) = 0, \quad x \in [0,\tau]. \end{cases} \tag{23}$$

In the above target system (23), note that $\zeta(t, \cdot)$ first vanishes at $t = \tau$ and then $\eta(t, \cdot)$ vanishes at $t = L + \tau$. Finally, $\tilde{w}(t, \cdot)$ vanishes at $t = L + \tau + d$. On the other hand, from (8) and (9), we have

$$0 = \tilde{v}(t,x) - \int_0^L k(x,y)\tilde{u}(t,y)dy,$$
$$0 = \tilde{u}(t,x) - \int_x^L p(x,y)\tilde{u}(t,y)dy$$

at $t \ge L + \tau + d$. From the invertibility[1] of the integral transformation $\mathcal{T} : H^1(0, L) \to H^1(0, L)$ defined by $\mathcal{T}\tilde{u}(t, x) = \tilde{u}(t, x) - \int_x^L p(x, y)\tilde{u}(t, y)dy$, it follows that $\tilde{u}(t, \cdot)$ and $\tilde{v}(t, \cdot)$ become zero at $t \ge L+\tau+d$. Based on this fact and the introduction of a Hilbert space $X := L^2(0, L)\times L^2(0, \tau)\times L^2(0, d)$ and its subspace $W := \{(\varphi, \psi, \omega) \in H^1(0, L) \times H^1(0, \tau) \times H^1(0, d); \varphi(0) = \psi(0), \psi(\tau) = \int_0^L \tilde{\beta}(y)\varphi(y)dy, \omega(d) = \varphi(L)\}$, where X has the inner product $\langle \cdot, \cdot \rangle_X$ defined by

$$\langle f, g \rangle_X := \int_0^L f_1(x)\overline{g_1(x)}dx + \int_0^\tau f_2(x)\overline{g_2(x)}dx + \int_0^d f_3(x)\overline{g_3(x)}dx,$$

for $f = (f_1, f_2, f_3) \in X$, $g = (g_1, g_2, g_3) \in X$, we have the following theorem:

Theorem 1: Let $\gamma \in C[0, L]$, $\tilde{\beta} \in C^1[0, L]$ be non-negative[2] functions with $\tilde{\beta}(0) = \tilde{\beta}(L) = 0$. Consider the error system (7) with observer gains $g(x)$, $h(x)$, $i(x)$ designed according to the Steps 1–7. Then, for any initial data $(\tilde{u}_0, \tilde{v}_0, \tilde{w}_0) \in W$, the error system (7) has a unique solution $(\tilde{u}, \tilde{v}, \tilde{w}) \in C([0, \infty); W) \cap C^1([0, \infty); X)$ which becomes zero at $t \ge L + \tau + d$.

Proof: The proof is similar to that of [18, Theorem 3]. So we omit it here. ∎

Remark 2: In Step 5, the gain $i(x)$ can be designed by solving the following equations (see Appendix A for the derivation):

$$\begin{cases} r_x(x,y) + r_y(x,y) = 0, \quad 0 \le y \le x, \ 0 \le x \le d, \\ r(d,y) = q(L,y), \end{cases} \tag{24}$$

$$i(x) = r(x,0) + \int_0^x r(x,y)i(y)dy. \tag{25}$$

From (24), $r(x, y)$ is solved as

$$r(x,y) = q(L, d-x+y). \tag{26}$$

Note that $r \in H^1(D_r)$ under the condition $\tilde{\beta}(0) = \tilde{\beta}(L) = 0$, where $D_r := \{(x, y)\,;\, 0 \le y \le x, 0 \le x \le d\}$. By calculating the integral equation (25), the gain $i(x)$ is solved. It follows from (25), (26) and Fig. 2 that $i(x) \equiv 0$ in the case of $\tau \ge d$.

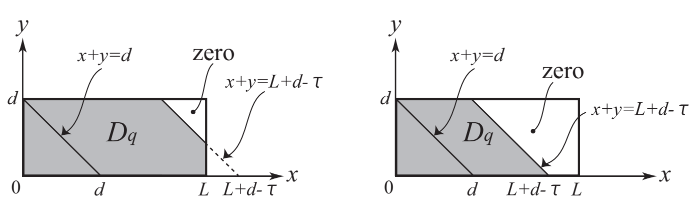

Fig. 2. White zone: the zone where the value of $q(x, y)$ becomes zero. Left: the case $d > \tau$; right: the case $\tau \ge d$.

[1]The result on the invertibility of the Volterra integral transformation is well-known. See e.g. [16].

[2]The condition of non-negativity for γ and $\tilde{\beta}$ is set since the Kermack–McKendrick PDE model is treated in this paper. One can consider the problem without the condition.

III. L^2-STABILITY OF ERROR SYSTEM (7)

First of all, let us formulate the error system (7) in X. We define the operators $A : D(A) \subset X \to X$, $B : X \to X$, and $E : D(E) \subset X \to X$ as follows:

$$A \begin{bmatrix} \varphi \\ \psi \\ \omega \end{bmatrix} = \begin{bmatrix} -\varphi' \\ \psi' \\ \omega' \end{bmatrix}, \quad \begin{bmatrix} \varphi \\ \psi \\ \omega \end{bmatrix} \in D(A) = W, \tag{27}$$

$$B \begin{bmatrix} \varphi \\ \psi \\ \omega \end{bmatrix} = \begin{bmatrix} -\gamma(\cdot)\varphi \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \varphi \\ \psi \\ \omega \end{bmatrix} \in X, \tag{28}$$

$$E \begin{bmatrix} \varphi \\ \psi \\ \omega \end{bmatrix} = \begin{bmatrix} -g(\cdot)\omega(0) \\ -h(\cdot)\omega(0) \\ -i(\cdot)\omega(0) \end{bmatrix}, \begin{bmatrix} \varphi \\ \psi \\ \omega \end{bmatrix} \in D(E) = W. \tag{29}$$

In the above, W is the same subspace as introduced before Theorem 1. Then, the error system (7) is written as follows:

$$\frac{d}{dt} \begin{bmatrix} \tilde{u}(t,\cdot) \\ \tilde{v}(t,\cdot) \\ \tilde{w}(t,\cdot) \end{bmatrix} = (A+B+E) \begin{bmatrix} \tilde{u}(t,\cdot) \\ \tilde{v}(t,\cdot) \\ \tilde{w}(t,\cdot) \end{bmatrix},$$
$$\begin{bmatrix} \tilde{u}(0,\cdot) \\ \tilde{v}(0,\cdot) \\ \tilde{w}(0,\cdot) \end{bmatrix} = \begin{bmatrix} \tilde{u}_0 \\ \tilde{v}_0 \\ \tilde{w}_0 \end{bmatrix}. \tag{30}$$

Before proceeding to the L^2-stability of the error system (7), we prove that the operator $A+B+E$ generates a C_0-semigroup on X, where A and E are unbounded, and B is bounded. Using Lemmas 1 and 2 below, we first prove that the operator $A+E$ generates a C_0-semigroup on X (Lemma 3). Then, we use the bounded perturbation theorem of semigroups for $A+E$ and B. Finally, we discuss, in Theorem 2, the L^2-stability of the error system (7) based on the result of Theorem 1.

Lemma 1: Under the assumption of Theorem 1, the observer gains $g(x)$, $h(x)$, $i(x)$ designed according to the Steps 1–7 satisfy $(g,h,i) \in D(A) = W$.

Proof: Putting $x = 0$ in (21) and (22), we have

$$g(0) = q(0,0) + \int_0^L p(0,y)g(y)dy + \int_0^d q(0,y)i(y)dy,$$
$$h(0) = \int_0^L k(0,y)g(y)dy + \int_0^d l(0,y)i(y)dy + l(0,0).$$

Noting that $p(0,y) = k(0,y)$ by Step 3 and that $q(0,y) = l(0,y)$ by Step 4, it follows that $g(0) = h(0)$. Also, putting $x = \tau$ in (22) and using $k(\tau,y) = \tilde{\beta}(y)$ in Step 1 and $l(\tau,y) = 0$ in Step 2, we have

$$h(\tau) = \int_0^L k(\tau,y)g(y)dy + \int_0^d l(\tau,y)i(y)dy + l(\tau,0)$$
$$= \int_0^L \tilde{\beta}(y)g(y)dy.$$

Further, putting $x = d$ in (25), and using the second equation of (24), $r(d,y) = q(L,y)$, we obtain

$$i(d) = r(d,0) + \int_0^d r(d,y)i(y)dy$$
$$= q(L,0) + \int_0^d q(L,y)i(y)dy,$$

which implies from (21) that $i(d) = g(L)$. From the construction method of $g(x)$, $h(x)$, $i(x)$ (i.e., (21), (22), (25)), it follows that $(g,h,i) \in H^1(0,L) \times H^1(0,\tau) \times H^1(0,d)$. Thus, $(g,h,i) \in D(A) = W$. ∎

Lemma 2: Under the assumption of Theorem 1, the operator A generates a C_0-semigroup e^{tA} on X.

Proof: From the definitions of the operator A and the inner product of the Hilbert X, for $(\varphi,\psi,\omega) \in D(A) = W$, we have

$$\begin{aligned} &\langle A[\varphi,\psi,\omega]^T, [\varphi,\psi,\omega]^T\rangle_X \\ &= \langle [-\varphi',\psi',\omega']^T, [\varphi,\psi,\omega]^T\rangle_X \\ &= -\int_0^L \varphi'(x)\overline{\varphi(x)}dx + \int_0^\tau \psi'(x)\overline{\psi(x)}dx \\ &\quad + \int_0^d \omega'(x)\overline{\omega(x)}dx, \end{aligned} \tag{31}$$

and

$$\begin{aligned} &\overline{\langle A[\varphi,\psi,\omega]^T, [\varphi,\psi,\omega]^T\rangle_X} \\ &= -\int_0^L \overline{\varphi'(x)}\varphi(x)dx + \int_0^\tau \overline{\psi'(x)}\psi(x)dx \\ &\quad + \int_0^d \overline{\omega'(x)}\omega(x)dx \\ &= \int_0^L \overline{\varphi(x)}\varphi'(x)dx - \int_0^\tau \overline{\psi(x)}\psi'(x)dx \\ &\quad - \int_0^d \overline{\omega(x)}\omega'(x)dx + |\psi(\tau)|^2 - |\omega(0)|^2. \end{aligned} \tag{32}$$

Here, adding the both sides of (31), (32) yields

$$2\mathrm{Re}\langle A[\varphi,\psi,\omega]^T, [\varphi,\psi,\omega]^T\rangle_X = |\psi(\tau)|^2 - |\omega(0)|^2. \tag{33}$$

Further, noting that the inequality

$$|\psi(\tau)| \le \|\tilde{\beta}\|_{L^2}\|\varphi\|_{L^2} \tag{34}$$

holds, we obtain

$$\mathrm{Re}\langle A[\varphi,\psi,\omega]^T, [\varphi,\psi,\omega]^T\rangle_X \le \nu\|[\varphi,\psi,\omega]^T\|_X^2,$$
$$(\varphi,\psi,\omega) \in D(A) = W, \tag{35}$$

where $\nu := \|\tilde{\beta}\|_{L^2}^2/2$. Therefore, the operator $A - \nu I$ is dissipative.

Next, we show that $\mathrm{rg}(\lambda I - A) = X$ holds for a sufficiently large $\lambda > 0$. First, defining a function $\mu(\lambda)$ as

$$\mu(\lambda) := e^{\lambda\tau} - \int_0^L e^{-\lambda x}\tilde{\beta}(x)dx, \tag{36}$$

we see that $\mu(\lambda) \to \infty$ as $\lambda \to \infty$. Hereafter, we choose $\lambda > 0$ sufficiently large and fix such that $\mu(\lambda) \neq 0$. We here show that, for any $(\sigma,\rho,\theta) \in X$, there exists a $(\varphi,\psi,\omega) \in D(A) = W$ such that

$$(\lambda I - A)[\varphi,\psi,\omega]^T = [\sigma,\rho,\theta]^T. \tag{37}$$

From (37), we can find the solution φ, ψ, ω as follows:

$$\varphi(x) = e^{-\lambda x}\varphi(0) + \int_0^x e^{-\lambda(x-y)}\sigma(y)dy, \tag{38}$$

$$\psi(x) = e^{\lambda x}\varphi(0) - \int_0^x e^{\lambda(x-y)}\rho(y)dy, \tag{39}$$

$$\omega(x) = e^{\lambda x}\omega(0) - \int_0^x e^{\lambda(x-y)}\theta(y)dy, \tag{40}$$

where

$$\varphi(0) = \frac{1}{\mu(\lambda)}\int_0^\tau e^{\lambda(\tau-y)}\rho(y)dy + \frac{1}{\mu(\lambda)}\int_0^L \tilde{\beta}(x)\left(\int_0^x e^{-\lambda(x-y)}\sigma(y)dy\right)dx, \quad (41)$$

$$\omega(0) = e^{-\lambda(L+d)}\left(\varphi(0) + \int_0^L e^{\lambda y}\sigma(y)dy + \int_0^d e^{\lambda(L+d-y)}\theta(y)dy\right). \quad (42)$$

Therefore, for the λ mentioned above, there holds $\mathrm{rg}(\lambda I - A) = X$. Hence, from [11, Corollary II.3.20] (Lumer-Phillips' Theorem), we see that the operator $A-\nu I$ generates a contractive C_0-semigroup on X. Noting that $A = (A-\nu I)+\nu I$ and the operator νI is bounded on X, it follows from the bounded perturbation theorem of semigroups [11, Theorem III.1.3] that the operator A generates a C_0-semigroup on X. ∎

Lemma 3: Under the assumption of Theorem 1, the operator $A+E$ generates a C_0-semigroup $e^{t(A+E)}$ on X.

Proof: We denote by X_1, $D(A)$ with 1-norm with respect to $\lambda I - A$, $\|[\varphi,\psi,\ \omega]^T\|_1 := \|(\lambda I - A)[\varphi,\psi,\omega]^T\|_X$ for a sufficiently large λ (where λ is in the resolvent set of A). Since the gains g, h, i of (29) satisfy $(g,h,i)\in D(A) = W$ from Lemma 1, there hold

$$\|E[\varphi,\psi,\omega]^T\|_1 = \|[g(\cdot),h(\cdot),i(\cdot)]^T\|_1|\omega(0)|,$$
$$|\omega(0)| \le \text{const.}\,\|[\varphi,\psi,\omega]^T\|_1,$$

for any $(\varphi,\psi,\omega)\in D(E) = D(A) = W$. Therefore, the operator E is bounded on X_1. Further, from Lemma 2, it has been shown that A generates a C_0-semigroup on X. Hence, from [11, Corollary III.1.5], the operator $A+E$ generates a C_0-semigroup on X. ∎

Noting that the operator B is bounded on X and using Lemma 3, we have from the bounded perturbation theorem of semigroups [11, Theorem III.1.3] that the operator $A+B+E$ generates a C_0-semigroup $e^{t(A+B+E)}$ on X. Hence, when $\tilde{u}_0\in L^2(0,L)$, $\tilde{v}_0\in L^2(0,\tau)$, $\tilde{w}_0\in L^2(0,d)$, the mild solution of (30) is expressed as

$$[\tilde{u}(t,\cdot),\tilde{v}(t,\cdot),\tilde{w}(t,\cdot)]^T = e^{t(A+B+E)}[\tilde{u}_0,\tilde{v}_0,\tilde{w}_0]^T. \quad (43)$$

Then, we have the following theorem:

Theorem 2: Under the assumption of Theorem 1, consider the error system (7) with observer gains $g(x)$, $h(x)$, $i(x)$ designed according to the Steps 1–7. Then, for any initial data $(\tilde{u}_0,\tilde{v}_0,\tilde{w}_0)\in X$, the error system (7) (i.e. (30)) has a unique solution $(\tilde{u},\tilde{v},\tilde{w})\in C([0,\infty);X)$ which becomes zero at $t\ge L+\tau+d$.

Proof: When an initial data $(\tilde{u}_0,\tilde{v}_0,\tilde{w}_0)$ is arbitrarily chosen from $D(A+B+E)(=D(A)=W)$, it follows from Theorem 1 that $e^{(L+\tau+d)(A+B+E)}[\tilde{u}_0,\tilde{v}_0,\tilde{w}_0]^T = 0$. Noting that $e^{t(A+B+E)} = e^{(t-L-\tau-d)(A+B+E)}e^{(L+\tau+d)(A+B+E)}$ holds for $t\ge L+\tau+d$, we see that the solution becomes zero at $t\ge L+\tau+d$ for any initial data in X, since $e^{(L+\tau+d)(A+B+E)}[\tilde{u}_0,\tilde{v}_0,\tilde{w}_0]^T = 0$ for all $(\tilde{u}_0,\tilde{v}_0,\tilde{w}_0)\in X$ by density of $D(A)$. ∎

Remark 3: From (43), it follows that $\|[\tilde{u}(t,\cdot),\tilde{v}(t,\cdot),\tilde{w}(t,\cdot)]^T\|_X \le M\|[\tilde{u}_0,\tilde{v}_0,\tilde{w}_0]^T\|_X$, where M is a constant given by $M = \sup_{t\in[0,L+\tau+d]}\|e^{t(A+B+E)}\|_{\mathcal{L}(X)}$. It is an interesting but difficult problem to investigate the influence of L, τ and d on M.

IV. Numerical Simulation

In the system (4), that is, in the system (5), we set $\gamma(x)\equiv c$, $\tilde{\beta}(x) = x(L-x)$, where c is a non-negative constant. We assume that $d>\tau$. According the Steps 1–7, we can design the observer gains $i(x)$, $g(x)$, $h(x)$ as follows:

First of all, by Step 1, the kernel $k(x,y)$ is determined as

$$k(x,y) = \begin{cases} 0, & x+y\le\tau, \\ e^{c(\tau-x)}\tilde{\beta}(x+y-\tau), & x+y>\tau. \end{cases}$$

Next, by Step 2, the kernel $l(x,y)$ is determined as

$$l(x,y) = \begin{cases} 0, & y\le x+d-\tau, \\ e^{c(-x+y-d+\tau)}\tilde{\beta}(x-y+L+d-\tau), & y>x+d-\tau. \end{cases}$$

And then, by Step 3, the kernel $p(x,y)$ is determined as

$$p(x,y) = \begin{cases} 0, & y\le x+\tau, \\ e^{c\tau}\tilde{\beta}(y-x-\tau), & y>x+\tau, \end{cases}$$

and, by Step 4, the kernel $q(x,y)$ is determined as

$$q(x,y) = \begin{cases} e^{-cx}l(0,x+y), & x+y\le d, \\ e^{c(y-d)}p(x+y-d,L), & d<x+y\le L+d-\tau, \\ 0, & L+d-\tau<x+y. \end{cases}$$

As for the function $i(x)$ $(0\le x\le d)$ which should be solved in Step 5 (see also Remark 2), we first have

$$r(x,y) = q(L,d-x+y).$$

The integral equation with respect to $i(x)$ can be solved by using the successive approximation method.

$$i_m(x) = r(x,0) + \int_0^x r(x,y)i_{m-1}(y)dy, \quad i_0(x)\equiv C.$$

Further, in Step 6, the integral equation with respect to $g(x)$ $(0\le x\le L)$ can be solved by using the successive approximation method.

$$g_n^m(x) = q(x,0) + \int_x^L p(x,y)g_{n-1}^m(y)dy + \int_0^d q(x,y)i_m(y)dy, \quad g_0^m(x)\equiv C.$$

Using $i_m(x)$, $g_n^m(x)$ solved numerically in the Steps 5, 6, we compute, in Step 7, the following equation and determine the gain $h_n^m(x)$ $(0\le x\le\tau)$.

$$h_n^m(x) = \int_0^L k(x,y)g_n^m(y)dy + \int_0^d l(x,y)i_m(y)dy + l(x,0).$$

In a numerical simulation, we set $L=3$, $\tau=0.75$, $d=1.2$, $c=0.2$, $C=1$, and set the spatial mesh width as $\Delta x=\Delta y=0.003$. The iteration of each successive approximation method was set as 4 times. In Fig. 3, the observer gains $i_m(x)$, $g_n^m(x)$, $h_n^m(x)$ were plotted. Then, we could confirm the following facts in this computation:

- $g_4^4(0) = 11.9002$, $h_4^4(0) = 11.9002$, that is, $g_4^4(0) = h_4^4(0)$.
- $h_4^4(0.75) = 18.4819$, $\int_0^3 \tilde{\beta}(y) g_4^4(y) dy = 18.4819$, that is,

$$h_4^4(0.75) = \int_0^3 \tilde{\beta}(y) g_4^4(y) dy.$$

- $i_4(1.2) = 1.0487$, $g_4^4(3) = 1.0487$, that is, $i_4(1.2) = g_4^4(3)$.

This shows that the gains $g(x) = g_4^4(x)$, $h(x) = h_4^4(x)$, and $i(x) = i_4(x)$ satisfy $(g, h, i) \in D(A) = W$ numerically (see Lemma 1).

In Fig. 4, the left column shows the time evolution of $v(t, x)$, $\hat{v}(t, x)$, $\tilde{v}(t, x)$, the center column the time evolution of $w(t, x)$, $\hat{w}(t, x)$, $\tilde{w}(t, x)$, and the right column the time evolution of $u(t, x)$, $\hat{u}(t, x)$, $\tilde{u}(t, x)$. In this case, the Kermack-McKendrick model is unstable as shown in the time evolution of $u(t, x)$. However, we see that the observer (6) estimates the states correctly, since the three error variables $\tilde{v}(t, x)$, $\tilde{w}(t, x)$, $\tilde{u}(t, x)$ converge to zero as time goes to infinity (see the top of Fig. 5 for the time evolution of $\|\tilde{v}(t, \cdot)\|_{L^2}$, $\|\tilde{w}(t, \cdot)\|_{L^2}$, $\|\tilde{u}(t, \cdot)\|_{L^2}$). As initial conditions, we set $v_0(x) \equiv 0$, $w_0(x) \equiv 0$, $u_0(x) = e^{-(x-0.9)^2} + 1$, $\hat{v}_0(x) \equiv 0$, $\hat{w}_0(x) \equiv 0$, $\hat{u}_0(x) \equiv 0$. In this case, note that $(\tilde{u}_0, \tilde{v}_0, \tilde{w}_0) \notin D(A)$, where $\tilde{u}_0(x) = u_0(x) - \hat{u}_0(x)$, $\tilde{v}_0(x) = v_0(x) - \hat{v}_0(x)$, $\tilde{w}_0(x) = w_0(x) - \hat{w}_0(x)$. Fig. 6 shows the time evolution in the case where $(\tilde{u}_0, \tilde{v}_0, \tilde{w}_0) \in D(A)$. As asserted in Theorem 1, the error variables $\tilde{v}(t, x)$, $\tilde{w}(t, x)$, $\tilde{u}(t, x)$ smoothly evolve. Especially, we set $v_0(x) = 7.6501\,x + 1.4449$, $w_0(x) \equiv 1.0122$, $u_0(x) = e^{-(x-0.9)^2} + 1$, $\hat{v}_0(x) \equiv 0$, $\hat{w}_0(x) \equiv 0$, $\hat{u}_0(x) \equiv 0$. To solve the hyperbolic equations numerically, we used the finite difference method and further the Runge-Kutta method of the fourth order for its time integration. A specialistic algorithm for transport equations is found in [23].

Remark 4: We used the successive approximation method to have the gains $i(x)$, $g(x)$. The convergences are very fast as shown in Fig. 3. It shows that $i_m(x)$ fully converges at $m = 4$ and then $g_n^4(x)$ converges at $n = 4$ based on the $i_4(x)$. Fig. 5 shows the effects of this approximation on the convergence of the observer. From Fig. 5 (bottom), we see that the observer is not robust with respect to a small perturbation of gains.

Remark 5: As stated in Remark 2, we have $i(x) \equiv 0$ in the case where the latent period of infection is larger than or equal to the observation delay, that is, $\tau \geq d$. In the case of $\tau \geq d$, we have also numerically verified that the observer designed according to the Steps 1–7 works well.

V. Concluding Remarks

In this note, we proposed the design method of observers for estimating the state of the linearized Kermack–McKendrick PDE model with two kinds of delays. The assertion of Lemma 1 was essential in order to show the L^2-stability of the error system. It is very important to verify whether or not the three observer gains obtained numerically satisfy the necessary condition of Lemma 1, since the design procedure of gains in Steps 1–7 is complicated. In connection with Remark 4, the problem of designing robust observers remains as an urgent issue. In the future, we plan to study observer design problems for the Kermack–McKendrick PDE model with nonlinear boundary condition.

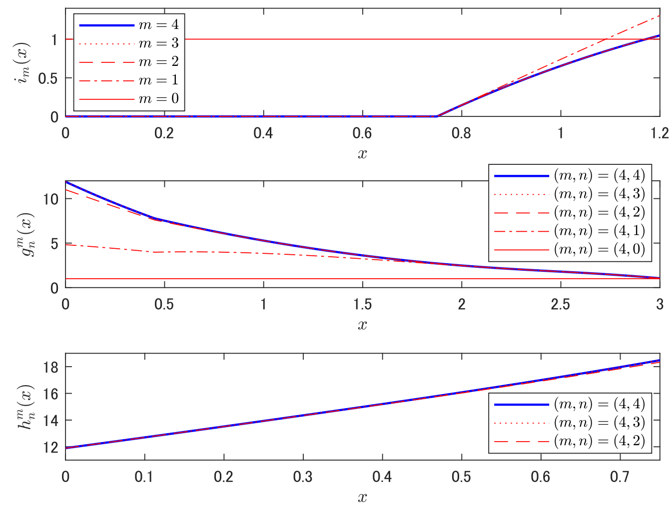

Fig. 3. Convergence of observer gains $i_m(x)$, $g_n^m(x)$, $h_n^m(x)$. Note that $i_m(x)$ first converges at $m = 4$ and then $g_n^4(x)$ converges at $n = 4$. The gains $i_4(x)$ and $g_4^4(x)$ determines the last one $h_4^4(x)$.

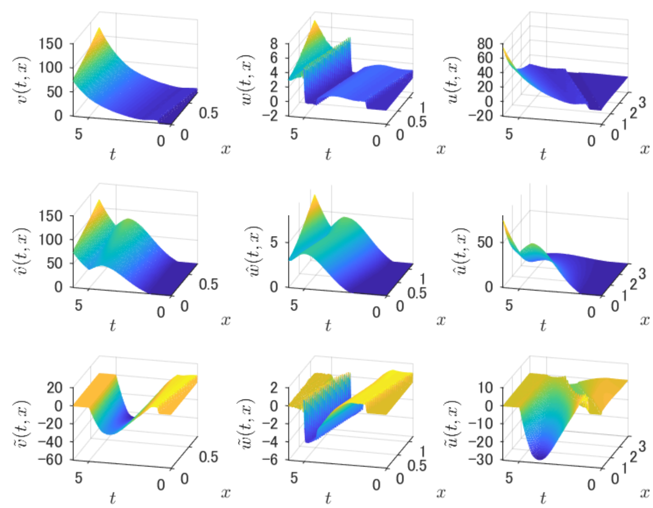

Fig. 4. Time evolution in case of $(\tilde{u}_0, \tilde{v}_0, \tilde{w}_0) \notin D(A)$. Left column: $v(t, x)$, $\hat{v}(t, x)$, $\tilde{v}(t, x)$; center column: $w(t, x)$, $\hat{w}(t, x)$, $\tilde{w}(t, x)$; right column: $u(t, x)$, $\hat{u}(t, x)$, $\tilde{u}(t, x)$, where the gains $i_4(x)$, $g_4^4(x)$, $h_4^4(x)$ were used.

Appendix A. Derivation of (24) and (25)

The goal of this appendix is to derive (24), (25) which is the algorithm for the gain $i(x)$ in Step 5. For system (20), we set the target system and the integral transformation as follows:

- Target system

$$\begin{cases} z_t(t, x) = z_x(t, x), \quad t > 0, \ x \in (0, d), \\ z(t, d) = 0, \quad t > 0, \\ z(0, x) = z_0(x), \quad x \in [0, d]. \end{cases} \tag{44}$$

- Integral transformation

$$z(t, x) = \tilde{w}(t, x) - \int_0^x r(x, y) \tilde{w}(t, y) dy, \quad 0 \leq x \leq d. \tag{45}$$

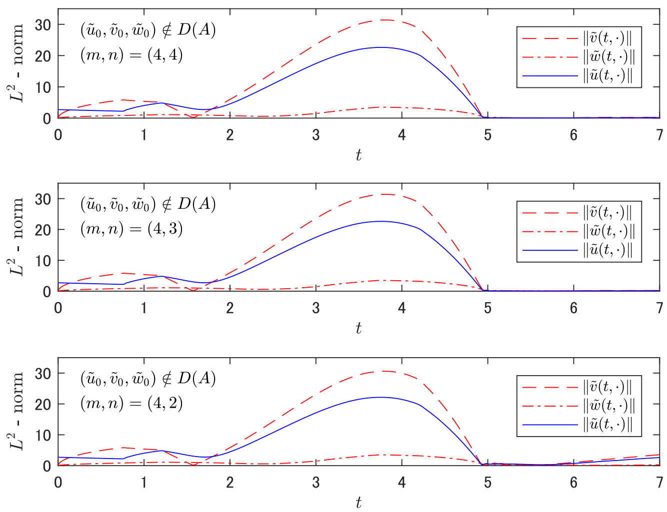

Fig. 5. 2D plot of $\|\tilde{v}(t,\cdot)\|_{L^2}$, $\|\tilde{w}(t,\cdot)\|_{L^2}$, $\|\tilde{u}(t,\cdot)\|_{L^2}$ when the gains $i_m(x)$, $g_n^m(x)$, $h_n^m(x)$ with $(m,n) = (4,4),(4,3),(4,2)$ were used, where $(\tilde{u}_0, \tilde{v}_0, \tilde{w}_0) \notin D(A)$. Since the gains $i_4(x)$, $g_2^4(x)$, $h_2^4(x)$ contain truncation errors, we observe from the bottom figure that the error system is unstable.

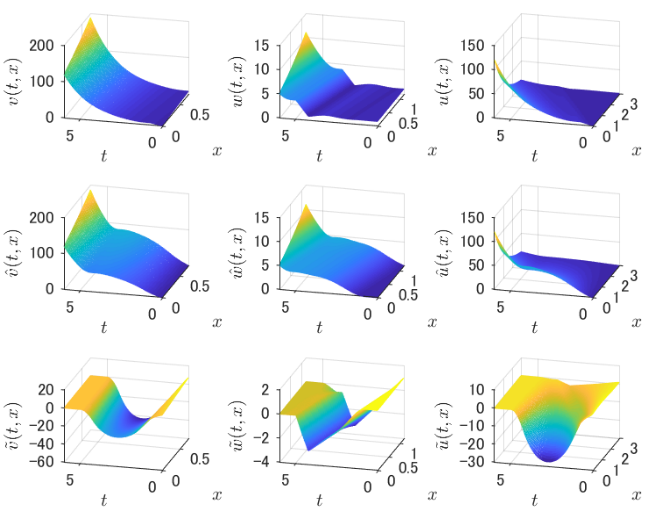

Fig. 6. Time evolution in case of $(\tilde{u}_0, \tilde{v}_0, \tilde{w}_0) \in D(A)$. Left column: $v(t,x)$, $\hat{v}(t,x)$, $\tilde{v}(t,x)$; center column: $w(t,x)$, $\hat{w}(t,x)$, $\tilde{w}(t,x)$; right column: $u(t,x)$, $\hat{u}(t,x)$, $\tilde{u}(t,x)$, where the gains $i_4(x)$, $g_4^4(x)$, $h_4^4(x)$ were used.

Note that the solution z of (44) becomes zero for all $t \geq d$. First, using (20) and integration by parts, we calculate $z_t(t,x) - z_x(t,x)$ as follows:

$$\begin{aligned} &z_t(t,x) - z_x(t,x) \\ &= \left\{ -i(x) + r(x,0) + \int_0^x r(x,y) i(y) dy \right\} \tilde{w}(t,0) \\ &\quad + \int_0^x \{ r_y(x,y) + r_x(x,y) \} \tilde{w}(t,y) dy. \end{aligned} \tag{46}$$

If the terms surrounded by braces of the right-hand side of (46) are zero, $z_t(t,x) - z_x(t,x) = 0$ holds for all $\tilde{w}$. Also, putting $x = d$ in (45) and using the boundary condition of (20), we have

$$z(t,d) = \int_0^d (q(L,y) - r(d,y)) \tilde{w}(t,y) dy. \tag{47}$$

If $q(L,y) = r(d,y)$ is satisfied, $z(t,d) = 0$ holds for all $\tilde{w}$. In this way, we obtain (24) and (25).

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