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Neumann Boundary Stabilization of One-Dimensional Linear Parabolic Systems With Input Delay

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Abstract—**This work addresses the boundary stabilization problem of one-dimensional linear parabolic systems with input delay. Especially, the case of Neumann boundary control is studied. The purpose is to derive the stabilizing controller of predictor type in a Hilbert space, by using a backstepping method combined with the semigroup theory. The use of the semigroup theory makes the proof of continuity of the inverse transformation easy. Also, it is shown that the implementation of the abstract controller is feasible by using a finite number of eigenvalues and eigenfunctions of the system operator.**

Index Terms—**Boundary stabilization, input delay, parabolic system, backstepping, semigroup, predictor.**

I. Introduction

Since early times, the control problem of the system with input delay has been investigated by many researchers (see e.g. the monograph [9] and references therein). The sphere of the control objects treated in [9] ranges from lumped parameter systems to distributed parameter systems, and from linear systems to nonlinear systems. Especially, when one considers the thermal control problem, large time lag exists in the feedback loop, since it is generally difficult to regulate heat quickly. So, it is very important to study the control problem of parabolic systems with input delay.

In this technical note, we study the Neumann boundary stabilization problem of one-dimensional linear parabolic systems with input delay. Since the element of time lag of this type, i.e., $e^{-\tau s}$ with $\tau > 0$ being a time lag, can be equivalently expressed by using a transport equation, the original system can be expressed as a cascade consisting of the transport equation and the parabolic equation with boundary input. In this note, by using a backstepping method of PDEs [7], it is shown that the control law can be constructed by using the solution to a parabolic equation and the solution to a hyperbolic equation. Then, an approach within the existing framework of functional analysis is useful when a target system is designed. Especially, by using the semigroup theory ([17], [13], [4], [10]), we can give an abstract expression of the control law obtained here. Here, we note that, in [11], a result for the case of distributed control with input delay has been given for an unstable reaction-diffusion process, where a backstepping method based on single integral transformation [6] is used and an abstract expression of predictor is given.

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In [8], the control law was designed by a backstepping method based on two kinds of integral transformations for the Dirichlet boundary stabilization problem of an unstable reaction-diffusion process with input delay, but an abstract expression of predictor type was not given for the control law. On the other hand, in this note, we consider the Neumann boundary stabilization problem with input delay and use a backstepping method based on single integral transformation [6]. The purpose is to derive the stabilizing controller of predictor type in a Hilbert space, by using a backstepping method combined with the semigroup theory. Also, the use of the semigroup theory makes the proof of continuity of the inverse transformation easy. Further, it is shown that the abstract controller can be easily implemented by using a finite number of eigenvalues and eigenfunctions of the system operator.

This technical note is organized as follows: In Section 2, we introduce the PDE describing a boundary controlled parabolic system with input delay and formulate it in a Hilbert space. In Section 3, we set a target system and assume the form of integral transformation and control law, and further determine the kernels of the integral transformation as well as the control law so as to convert the original system to the target system. In Section 4, it is shown that the inverse transformation is also continuous. In Section 5, we give a numerical example, and, finally the note is concluded in Section 6.

II. System Description and Formulation

A. System Description

We shall consider the following boundary controlled parabolic system defined on the spatial domain $(0,1)$:

$$\begin{cases} z_t(t,x) = -\mathcal{L}z(t,x), \\ z_x(t,0) = 0, \quad z_x(t,1) = f(t-\tau), \\ z(0,x) = z_0(x), \\ f(\theta) = \phi(\theta), \quad \theta \in [-\tau, 0], \end{cases} \tag{1}$$

where $\mathcal{L}$ is a Sturm-Liouville operator defined by

$$(\mathcal{L}\varphi)(x) = \frac{1}{h(x)}\left(-\frac{d}{dx}\left(a(x)\frac{d\varphi(x)}{dx}\right) + b(x)\varphi(x)\right).$$

In the above, we assume that $h(x)$, $a(x)$, and $b(x)$ are real-valued, sufficiently smooth functions defined on $[0,1]$, and that $h(x) > 0$, $a(x) > 0$. $f(t)$ is the control input and $\tau > 0$ is a time lag. As is well-known, since the element of time lag can be expressed by using the transport equation, system (1) can be expressed as the cascade consisting of the transport equation and the parabolic equation with boundary input:

$$\begin{cases} z_t(t,x) = -\mathcal{L}z(t,x), \\ z_x(t,0) = 0, \quad z_x(t,1) = u(t,0), \\ z(0,x) = z_0(x), \\ u_t(t,x) = \frac{1}{\tau}u_x(t,x), \\ u(t,1) = f(t), \\ u(0,x) = \phi(\tau(x-1)). \end{cases} \tag{2}$$

B. Formulation of the System

Let $L^2_h(0,1)$ be the weighted L^2-space with inner product defined by $\langle f,g\rangle_h = \int_0^1 f(x)g(x)h(x)dx$ for $f,g \in L^2_h(0,1)$, and let us define the operator $A : D(A) \subset L^2_h(0,1) \to L^2_h(0,1)$ by

$$\begin{aligned} &A\varphi = \mathcal{L}\varphi, \quad \varphi \in D(A), \\ &D(A) = \{\, \varphi \in H^2(0,1)\,;\, \varphi'(0) = \varphi'(1) = 0 \,\}. \end{aligned} \tag{3}$$

Then, the operator A is closed and self-adjoint in $L^2_h(0,1)$, and it has compact resolvent and is bounded from below. Therefore, A has the eigenvalues $\{\lambda_i\}_{i=0}^\infty$ such that $-\infty < \lambda_0 < \lambda_1 < \cdots < \lambda_i < \cdots \to \infty$, and the corresponding eigenfunctions $\{\varphi_i\}_{i=0}^\infty$ forms a complete orthonormal system in $L^2_h(0,1)$. Especially, in this note, let $\lambda_0 < 0$ be assumed. Now, we choose a positive number c such that $\lambda_0 + c > 0$ and hereafter fix it.

Let us formulate system (2) in the Hilbert space $L^2_h(0,1)$. Here, we set $A_c = A + c$ and use the variable transformation $\zeta(t) = A_c^{-\frac{1}{4}-\epsilon}z(t,\cdot)$ $(0 < \epsilon < \frac{1}{4})$ [12]. Then, noting the inclusion relation [5]

$$H^2(0,1) \subset H^{\frac{3}{2}-2\epsilon}(0,1) = D(A_c^{\frac{3}{4}-\epsilon}) \subset D(A_c^{\frac{1}{4}+\epsilon}), \tag{4}$$

system (2) is expressed as[1]

$$\begin{cases} \dot{\zeta}(t) = -A\zeta(t) + Bu(t,0), \\ \zeta(0) = A_c^{-\frac{1}{4}-\epsilon}z_0 =: \zeta_0, \\ u_t(t,x) = \frac{1}{\tau}u_x(t,x), \\ u(t,1) = f(t), \\ u(0,x) = \phi(\tau(x-1)), \end{cases} \tag{5}$$

where $B : \mathbf{R} \to L^2_h(0,1)$ is the bounded operator defined by

$$Bv = a(1)A_c^{\frac{3}{4}-\epsilon}\psi v, \quad v \in \mathbf{R}, \tag{6}$$

and, $\psi \in H^2(0,1)$ is the unique solution of the boundary value problem

$$(\mathcal{L}+c)\psi = 0, \quad \psi'(0) = 0, \quad \psi'(1) = \tfrac{1}{a(1)}. \tag{7}$$

Hereafter, we assume that $z_0 \in L^2_h(0,1)$, $\phi \in C_r[-\tau,0]$ $(0 < r \le 1)$, where $C_r[-\tau,0]$ denotes the set consisting of Hölder continuous functions with index r. Here, we note that the operator $-A$ generates an analytic semigroup $T_{-A}(t)$ on $L^2_h(0,1)$ and the concrete expression is as follows:

$$T_{-A}(t)\varphi = \sum_{i=0}^\infty e^{-\lambda_i t}\langle\varphi,\varphi_i\rangle_h\varphi_i, \quad \varphi \in L^2_h(0,1). \tag{8}$$

Since the growth bound of $T_{-A}(t)$ is equal to $-\lambda_0 > 0$ under the assumption, system (5), namely (2) is unstable. The purpose of this note is to construct the control law to stabilize system (5) and to give an abstract expression of the controller.

III. Construction of Control Law

A. Backstepping Method

First of all, by using a function $k \in L^2_h(0,1)$, we define an operator $K : L^2_h(0,1) \to \mathbf{R}$ by $K\varphi = \langle k,\varphi\rangle_h$, $\varphi \in L^2_h(0,1)$. If the function $k \in L^2_h(0,1)$ is chosen so that the operator $-A + BKA_c^{\frac{1}{4}+\epsilon}$ generates an exponentially stable analytic semigroup, it makes possible to consider the following system as a target system[2]:

$$\begin{cases} \dot{\zeta}(t) = (-A + BKA_c^{\frac{1}{4}+\epsilon})\zeta(t) + Bw(t,0), \\ \zeta(0) = \zeta_0, \\ w_t(t,x) = \frac{1}{\tau}w_x(t,x), \\ w(t,1) = 0, \\ w(0,x) = w_0(x). \end{cases} \tag{9}$$

The following theorem assures that such a choice of k is actually possible.

Theorem 1: For a given $\omega > 0$, let the integer n be chosen such that $\omega < \lambda_{n+1}$. Then, there exists a function $k \in L^2_h(0,1)$ such that the operator $-A + BKA_c^{\frac{1}{4}+\epsilon}$ generates an analytic semigroup $T_{-A+BKA_c^{1/4+\epsilon}}(t)$ with growth bound $-\omega$, where $K\varphi = \langle k,\varphi\rangle_h$, $\varphi \in L^2_h(0,1)$. Especially, it is possible to choose the function k within the domain of A, $D(A)$.

Proof: By using the orthogonal projection operator $P_n\varphi = \sum_{i=0}^n \langle\varphi,\varphi_i\rangle_h\varphi_i$, $\varphi \in L^2_h(0,1)$, we decompose the state variable $\zeta(t)$ and the space $L^2_h(0,1)$ of system (9) as follows:

$$\begin{aligned} &\zeta(t) = \zeta_1(t) + \zeta_2(t), \\ &\zeta_1(t) := P_n\zeta(t), \quad \zeta_2(t) := (I-P_n)\zeta(t), \\ &L^2_h(0,1) = \underbrace{P_nL^2_h(0,1)}_{\dim = n+1} \oplus \underbrace{(I-P_n)L^2_h(0,1)}_{\dim=\infty}. \end{aligned}$$

Then, the operators A, B, and K are expressed as follows (see e.g. [1]):

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad K = \begin{bmatrix} K_1 & K_2 \end{bmatrix},$$

where $A_1 := P_nAP_n$, $B_1 := P_nB$, $K_1 := KP_n$, $A_2 := (I-P_n)A(I-P_n)$, $B_2 := (I-P_n)B$, $K_2 := K(I-P_n)$. In the above, the operator A_2 is unbounded, whereas all the other operators are bounded[3]. Hereafter, we identify the finite-dimensional Hilbert space $P_nL^2_h(0,1)$ with the Euclidean space $\mathbf{R}^{n+1}$ with respect to the basis $\{\varphi_0, \varphi_1, \ldots, \varphi_n\}$. In

[1] See e.g. [14] for the derivation. From the process of derivation, it is clear that (2) is equivalent to (5). In the case of Dirichlet boundary control with input delay, the variable transformation $\zeta(t) = A_c^{-\frac{3}{4}-\epsilon}z(t,\cdot)$ $(0 < \epsilon < \frac{1}{4})$ is used [10], where $A\varphi = \mathcal{L}\varphi$, $\varphi \in D(A) = H^2(0,1) \cap H^1_0(0,1)$.

[2] In the case of distributed control [11], the operator $-A + BKA_c^{\frac{1}{4}+\epsilon}$ is replaced by $-A + BK$. Thus, in the case of Neumann boundary control, the additional operator $A_c^{\frac{1}{4}+\epsilon}$ appears under the influence of the variable transformation $\zeta(t) = A_c^{-\frac{1}{4}-\epsilon}z(t,\cdot)$ $(0 < \epsilon < \frac{1}{4})$.

[3] The projections have been widely used in the field of distributed parameter systems. For example, Byrnes *et al.* solved the output regulation problem for a class of infinite-dimensional systems [2]. Christofides and Daoutidis applied approximate inertial manifolds to the stabilization problem of semilinear distributed parameter systems [3].

this way, each element in $P_n L_h^2(0,1)$ is identified with an $(n+1)$-dimensional vector, and the operators A_1, B_1, and K_1 are identified with matrices with appropriate size.

Here, let us set $K_2 = 0$. Then, the operator $-A+BKA_c^{\frac{1}{4}+\epsilon}$ of system (9) becomes

$$-A + BKA_c^{\frac{1}{4}+\epsilon} = \begin{bmatrix} -A_1 + B_1K_1(A_1+c)^{\frac{1}{4}+\epsilon} & 0 \\ B_2K_1(A_1+c)^{\frac{1}{4}+\epsilon} & -A_2 \end{bmatrix}.$$

Since the pair $(-A_1, B_1)$ is controllable (see Appendix A), we can choose a matrix $\tilde{K}_1$ such that the matrix $-A_1 + B_1\tilde{K}_1$ becomes Hurwitz (e.g. [19]). Especially, it is possible to choose the matrix $K_1 := \tilde{K}_1(A_1+c)^{-\frac{1}{4}-\epsilon}$ such as

$$\sigma(-A_1 + B_1K_1(A_1+c)^{\frac{1}{4}+\epsilon}) \subset \{\lambda \in \mathbf{C}; \mathrm{Re}(\lambda) < -\omega\}.$$

Here, noting that $-A_2$ generates an analytic semigroup $T_{-A_2}(t)$ with norm bound $\|T_{-A_2}(t)\| \le e^{-\lambda_{n+1}t}$, $t \ge 0$, we see that the operator $-A + BKA_c^{\frac{1}{4}+\epsilon}$ generates an analytic semigroup $T_{-A+BKA_c^{1/4+\epsilon}}(t)$ with norm bound $\|T_{-A+BKA_c^{1/4+\epsilon}}(t)\| \le me^{-\omega t}$, $t \ge 0$, where m is some positive constant.

From the above discussion, by expressing the matrix K_1 as $K_1 = [k_0, \cdots, k_n]$, the function $k(x)$ can be constructed as

$$k(x) = \sum_{i=0}^{\infty} \langle k, \varphi_i \rangle_h \varphi_i(x) = \sum_{i=0}^{n} k_i\varphi_i(x). \qquad (10)$$

Hence, it follows that $k \in D(A)$ since each φ_i $(0 \le i \le n)$ belongs to $D(A)$. ∎

Remark 1: For the operator $A_c^{\frac{1}{4}+\epsilon}T_{-A+BKA_c^{1/4+\epsilon}}(t)$, the following norm estimate holds:

$$\|A_c^{\frac{1}{4}+\epsilon}T_{-A+BKA_c^{1/4+\epsilon}}(t)\| \le (\alpha + t^{-\frac{1}{4}-\epsilon})e^{-\omega t}, \quad t > 0, \qquad (11)$$

where α is some positive constant.

Based on Theorem 1 and Remark 1, we have the following stability result for system (9):

Theorem 2: Let $\zeta_0 \in D(A_c^{\frac{1}{4}+\epsilon})$ and $w_0 \in C_r[0,1]$ $(0 < r \le 1)$. Suppose that the function $k \in D(A)$ is chosen as stated in Theorem 1. Then, system (9) is asymptotically stable in the sense of norm $(\|w(t,\cdot)\|_h^2 + \|A_c^{\frac{1}{4}+\epsilon}\zeta(t)\|_h^2)^{\frac{1}{2}}$, where $\|\cdot\|_h$ denotes the norm of $L_h^2(0,1)$.

Proof: Note that the subsystem

$$\begin{cases} w_t(t,x) = \frac{1}{\tau}w_x(t,x), \\ w(t,1) = 0, \\ w(0,x) = w_0(x) \end{cases}$$

is exponentially stable with any decay rate in the sense of norm $\|w(t,\cdot)\|_h$, since the solution $w(t,x)$ vanishes after $t = \tau$. Also, note that $w(\cdot,0) \in C_r[0,\tau]$ by the assumption $w_0 \in C_r[0,1]$. Therefore, the first equation of system (9) has a unique solution

$$\zeta(t) = T_{-A+BKA_c^{1/4+\epsilon}}(t)\zeta_0 + \int_0^t T_{-A+BKA_c^{1/4+\epsilon}}(t-s)Bw(s,0)ds, \quad 0 \le t \le \tau,$$

and

$$\zeta(t) = T_{-A+BKA_c^{1/4+\epsilon}}(t-\tau)\zeta(\tau), \quad \tau < t.$$

Note that $\zeta(t) \in D(A)$ for each $t > 0$ and $\zeta \in C([0,\infty); L_h^2(0,1))$. Here, by using (11), we have the following estimate for sufficiently large $t\,(>\tau)$:

$$\begin{aligned} \|A_c^{\frac{1}{4}+\epsilon}\zeta(t)\|_h &\le \|A_c^{\frac{1}{4}+\epsilon}T_{-A+BKA_c^{1/4+\epsilon}}(t-\tau)\|\|\zeta(\tau)\|_h \\ &\le \{\alpha + (t-\tau)^{-\frac{1}{4}-\epsilon}\}e^{-\omega(t-\tau)}\|\zeta(\tau)\|_h. \end{aligned}$$

Therefore, we see that system (9) becomes asymptotically stable in the sense of norm $(\|w(t,\cdot)\|_h^2 + \|A_c^{\frac{1}{4}+\epsilon}\zeta(t)\|_h^2)^{\frac{1}{2}}$. ∎

The above target system (9) is equivalent to the following system:

$$\begin{cases} z_t(t,x) = \frac{1}{h(x)}\big(\big(a(x)z_x(t,x)\big)_x - b(x)z(t,x)\big), \\ z_x(t,0) = 0, \\ z_x(t,1) = \langle k, z(t,\cdot)\rangle_h + w(t,0), \\ w_t(t,x) = \frac{1}{\tau}w_x(t,x), \\ w(t,1) = 0. \end{cases} \qquad (12)$$

Of course, system (12) is also asymptotically stable. To convert system (2) to the target system (12), we consider the following integral transformation and control law:

$$w(t,x) = u(t,x) - \int_0^x q(x,y)u(t,y)dy - \int_0^1 \gamma(x,y)z(t,y)h(y)dy, \qquad (13)$$

$$f(t) = \int_0^1 q(1,y)u(t,y)dy + \int_0^1 \gamma(1,y)z(t,y)h(y)dy, \qquad (14)$$

where the kernels $q(x,y)$ and $\gamma(x,y)$ are functions which should be designed.

B. Derivation of Kernels

First, differentiating eq. (13) with respect to x and t, we have

$$w_x(t,x) = u_x(t,x) - q(x,x)u(t,x) - \int_0^x q_x(x,y)u(t,y)dy - \int_0^1 \gamma_x(x,y)z(t,y)h(y)dy, \qquad (15)$$

$$\begin{aligned} w_t(t,x) = {} & \frac{1}{\tau}u_x(t,x) - \frac{1}{\tau}q(x,x)u(t,x) + \frac{1}{\tau}q(x,0)u(t,0) \\ & + \frac{1}{\tau}\int_0^x q_y(x,y)u(t,y)dy - a(1)\gamma(x,1)u(t,0) \\ & + a(1)\gamma_y(x,1)z(t,1) - a(0)\gamma_y(x,0)z(t,0) \\ & - \int_0^1 \big(a(y)\gamma_y(x,y)\big)_y z(t,y)dy \\ & + \int_0^1 \gamma(x,y)b(y)z(t,y)dy. \end{aligned} \qquad (16)$$

Here, from $w_t(t,x) - \frac{1}{\tau} w_x(t,x) = 0$, we have

$$\begin{aligned} & w_t(t,x) - \frac{1}{\tau} w_x(t,x) \\ &= \left\{ \frac{1}{\tau} q(x,0) - a(1)\gamma(x,1) \right\} u(t,0) \\ &\quad + \frac{1}{\tau} \int_0^x \{ q_y(x,y) + q_x(x,y) \} u(t,y) dy \\ &\quad + a(1)\gamma_y(x,1) z(t,1) - a(0)\gamma_y(x,0) z(t,0) \\ &\quad + \int_0^1 \Big\{ -\big(a(y)\gamma_y(x,y)\big)_y + b(y)\gamma(x,y) \\ &\qquad + \frac{1}{\tau} h(y) \gamma_x(x,y) \Big\} z(t,y) dy = 0. \end{aligned} \tag{17}$$

In order for eq. (17) to hold for all u and z, $q(x,y)$ and $\gamma(x,y)$ need to satisfy

$$\begin{cases} \gamma_x(x,y) = \frac{\tau}{h(y)} \big(\big(a(y)\gamma_y(x,y)\big)_y - b(y)\gamma(x,y) \big), \\ \gamma_y(x,0) = \gamma_y(x,1) = 0, \end{cases} \tag{18}$$

$$\begin{cases} q_x(x,y) + q_y(x,y) = 0, \\ q(x,0) = \tau a(1) \gamma(x,1). \end{cases} \tag{19}$$

Since eq. (18) is of parabolic type, we need the initial condition to solve it. We can determine it in the following way. Setting $x = 0$ in the integral transformation (13), we have

$$w(t,0) = u(t,0) - \langle \gamma(0,\cdot), z(t,\cdot) \rangle_h, \tag{20}$$

and, by substituting (20) to the third equation of system (12), we have

$$\begin{aligned} z_x(t,1) &= \langle k, z(t,\cdot) \rangle_h + w(t,0) \\ &= \langle k - \gamma(0,\cdot), z(t,\cdot) \rangle_h + u(t,0). \end{aligned} \tag{21}$$

Here, comparing (21) with the corresponding boundary condition of system (2), we obtain

$$\gamma(0,y) = k(y) \tag{22}$$

as the initial condition for eq. (18).

From these, we can solve the kernels $q(x,y)$ and $\gamma(x,y)$ of integral transformation (13) in the following steps:

(i) First, we solve the solution $\gamma(x,y)$ $(0 \le x, y \le 1)$ of parabolic equation (18) under the initial condition (22).

(ii) Next, we solve the solution $q(x,y)$ $(0 \le x \le 1, 0 \le y \le x)$ of hyperbolic equation (19) by using the solution obtained by Step (i).

C. Abstract Expression of Control Law

By using the operator A, we can formulate (18), (22) as follows:

$$\gamma'(x,\cdot) = -\tau A \gamma(x,\cdot), \quad \gamma(0,\cdot) = k. \tag{23}$$

Since the operator $-\tau A$ generates an analytic semigroup $T_{-\tau A}(x)$ on $L^2_h(0,1)$, the solution of (23) is written as

$$\gamma(x,\cdot) = T_{-\tau A}(x) k. \tag{24}$$

By using this expression, eq. (19) becomes

$$\begin{cases} q_x(x,y) + q_y(x,y) = 0, \\ q(x,0) = \tau a(1) (T_{-\tau A}(x)k)(1). \end{cases} \tag{25}$$

Note that the solution of eq. (25) can be expressed as $q(x,y) = \nu(x-y)$. Therefore, by putting $y = 0$ in it, we have $q(x,0) = \nu(x) = \tau a(1)(T_{-\tau A}(x)k)(1)$, that is,

$$q(x,y) = \nu(x-y) = \tau a(1) (T_{-\tau A}(x-y)k)(1). \tag{26}$$

Accordingly, using (24) and (26), the control law (14) can be expressed as follows:

$$\begin{aligned} f(t) &= \langle q(1,\cdot), u(t,\cdot) \rangle + \langle \gamma(1,\cdot), z(t,\cdot) \rangle_h \\ &= \langle \tau a(1)(T_{-\tau A}(1-\cdot)k)(1), u(t,\cdot) \rangle \\ &\quad + \langle T_{-\tau A}(1)k, z(t,\cdot) \rangle_h \\ &= \langle \tau a(1)(T_{-\tau A}(1-\cdot)k)(1), f(t+\tau(\cdot-1)) \rangle \\ &\quad + \langle T_{-\tau A}(1)k, z(t,\cdot) \rangle_h \\ &= \int_0^1 \tau a(1)(T_{-\tau A}(1-x)k)(1) f(t+\tau(x-1)) dx \\ &\quad + \langle T_{-\tau A}(1)k, z(t,\cdot) \rangle_h, \end{aligned} \tag{27}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $L^2(0,1)$. Here, setting $t + \tau(x-1) = \theta$, it follows that

$$\begin{aligned} f(t) &= \int_{t-\tau}^t a(1)(T_{-\tau A}((t-\theta)/\tau)k)(1) f(\theta) d\theta \\ &\quad + \langle T_{-\tau A}(1)k, z(t,\cdot) \rangle_h. \end{aligned} \tag{28}$$

Moreover, noting that

$$T_{-\tau A}(x/\tau) = T_{-A}(x), \tag{29}$$

we finally obtain

$$\begin{aligned} f(t) &= \int_{t-\tau}^t a(1)(T_{-A}(t-\theta)k)(1) f(\theta) d\theta \\ &\quad + \langle T_{-A}(\tau)k, z(t,\cdot) \rangle_h. \end{aligned} \tag{30}$$

Remark 2: We can give another expression of predictor (30). Note that, by using the similar techniques as in [15]–[16], $\gamma(x,1)$ contained in the second equation of (19) can be formulated as follows: $\gamma(x,1) = \langle A_c^{\frac{3}{4}-\epsilon}\psi, A_c^{\frac{1}{4}+\epsilon}\gamma(x,\cdot) \rangle_h$, where ψ is the solution of the boundary value problem (7). Further noting that $k \in D(A)$, it is done as follows:

$$\begin{aligned} f(t) &= \int_{t-\tau}^t a(1) \langle A_c^{\frac{3}{4}-\epsilon}\psi, A_c^{\frac{1}{4}+\epsilon} T_{-A}(t-\theta)k \rangle_h f(\theta) d\theta \\ &\quad + \langle T_{-A}(\tau)k, z(t,\cdot) \rangle_h \\ &= \int_{t-\tau}^t \langle a(1)\psi, T_{-A}(t-\theta) A_c k \rangle_h f(\theta) d\theta \\ &\quad + \langle T_{-A}(\tau)k, z(t,\cdot) \rangle_h. \end{aligned} \tag{31}$$

Remark 3: It may seem difficult to implement the control law (30) to system (2), because of the abstract expression. But, it is easy since we have constructed the function $k(x)$ such as $k = \sum_{i=0}^n k_i \varphi_i \in D(A)$. By using (8) and (10), we can give the gain functions of (30) as

$$\begin{aligned} a(1)(T_{-A}(t-\theta)k)(1) &= a(1) \sum_{i=0}^n k_i \varphi_i(1) e^{-\lambda_i (t-\theta)}, \\ (T_{-A}(\tau)k)(x) &= \sum_{i=0}^n k_i e^{-\lambda_i \tau} \varphi_i(x). \end{aligned} \tag{32}$$

In this way, we can implement the control law based on a finite number of eigenpairs $\{\lambda_i, \varphi_i\}_{i=0}^{n}$ of the operator A.

Finally, in this section, we show that, if $\phi \in C_r[-\tau, 0]$ $(0 < r \leq 1)$ is satisfied in system (1), it follows that $w_0 \in C_r[0,1]$ as assumed in Theorem 2. This is actually verified as follows: Substituting $t = 0$ to the integral transformation (13), we have

$$w_0(x) = u(0,x) - \int_0^x q(x,y)u(0,y)dy - \langle \gamma(x,\cdot), z_0 \rangle_h. \quad (33)$$

In the above, it is clear that $u(0,\cdot) \in C_r[0,1]$ since $\phi \in C_r[-\tau, 0]$. Also, $\langle \gamma(x,\cdot), z_0 \rangle_h$ is differentiable and its derivative is expressed as $\frac{d}{dx}\langle \gamma(x,\cdot), z_0 \rangle_h = \frac{d}{dx}\langle T_{-\tau A}(x)k, z_0 \rangle_h = \langle -\tau T_{-\tau A}(x)Ak, z_0 \rangle_h$, i.e., $\langle \gamma(x,\cdot), z_0 \rangle_h$ is in $C^1[0,1]$. Furthermore, $\int_0^x q(x,y)u(0,y)dy$ is differentiable and its derivative is expressed as

$$\begin{aligned}
&\frac{d}{dx}\int_0^x q(x,y)u(0,y)dy \\
&= q(x,x)u(0,x) + \int_0^x q_x(x,y)u(0,y)dy \\
&= \tau a(1)k(1)u(0,x) \\
&\quad + \int_0^x \frac{\partial}{\partial x}\tau a(1)(T_{-\tau A}(x-y)k)(1)u(0,y)dy \\
&= \tau a(1)k(1)u(0,x) \\
&\quad + \tau a(1)\int_0^x \frac{\partial}{\partial x}\langle A_c^{\frac{3}{4}-\epsilon}\psi, A_c^{\frac{1}{4}+\epsilon}T_{-\tau A}(x-y)k\rangle_h \\
&\qquad \times u(0,y)dy \\
&= \tau a(1)k(1)u(0,x) \\
&\quad + \tau a(1)\int_0^x \frac{\partial}{\partial x}\langle \psi, T_{-\tau A}(x-y)A_c k\rangle_h u(0,y)dy \\
&= \tau a(1)k(1)u(0,x) \\
&\quad - \tau^2 a(1)\int_0^x \langle \psi, T_{-\tau A}(x-y)AA_c k\rangle_h u(0,y)dy,
\end{aligned}$$

where ψ is the solution of the boundary value problem (7). That is, $\int_0^x q(x,y)u(0,y)dy$ is in $C^1[0,1]$. Therefore, noting that $C^1[0,1] \subset C_1[0,1] \subset C_r[0,1]$ $(0 < r \leq 1)$, from (33) we see that $w_0 \in C_r[0,1]$.

IV. Closed-Loop Stability

A. Inverse Integral Transformation

To assure the asymptotical stability of the closed-loop system consisting of system (2) and the control law (30) (or, (31)), we need to show that the inverse transformation of (13) exists and that it is continuous. We assume that the inverse integral transformation from the target system (12) to system (2) is expressed as

$$\begin{aligned}
u(t,x) &= w(t,x) + \int_0^x p(x,y)w(t,y)dy \\
&\quad + \int_0^1 \beta(x,y)z(t,y)h(y)dy, \quad (34)
\end{aligned}$$

where the kernels $p(x,y)$ and $\beta(x,y)$ are functions whose existence should be shown.

B. Derivation of Kernels

By the similar discussion as in Subsection 3.2, we can solve the kernels $p(x,y)$, $\beta(x,y)$ of the inverse transformation (34). The concrete steps are as follows:

(i') First, we solve the solution $\beta(x,y)$ $(0 \leq x, y \leq 1)$ of the following parabolic equation with term $\tau a(1)k(y)\beta(x,1)$:

$$\begin{cases}
\beta_x(x,y) = \frac{\tau}{h(y)}\left(\left(a(y)\beta_y(x,y)\right)_y - b(y)\beta(x,y)\right) \\
\qquad\qquad + \tau a(1)k(y)\beta(x,1), \\
\beta_y(x,0) = \beta_y(x,1) = 0, \\
\beta(0,y) = k(y).
\end{cases} \quad (35)$$

(ii') Next, using the value at $y = 1$ of the solution, we solve the solution $p(x,y)$ $(0 \leq x \leq 1, 0 \leq y \leq x)$ of the following hyperbolic equation:

$$\begin{cases}
p_x(x,y) + p_y(x,y) = 0, \\
p(x,0) = \tau a(1)\beta(x,1).
\end{cases} \quad (36)$$

C. Abstract Expression of Inverse Transformation

Note that, similarly as in Remark 2, $\beta(x,1)$ contained in the first equation of (35) can be formulated as follows:

$$\beta(x,1) = \langle A_c^{\frac{3}{4}-\epsilon}\psi, A_c^{\frac{1}{4}+\epsilon}\beta(x,\cdot)\rangle_h, \quad (37)$$

where ψ is the solution of the boundary value problem (7). Also, note that the adjoint operators $B^* : L_h^2(0,1) \to \mathbf{R}$ and $K^* : \mathbf{R} \to L_h^2(0,1)$ of the bounded operators B, K defined in Subsections 2.2 and 3.1 are expressed as follows:

$$B^*\varphi = \langle a(1)A_c^{\frac{3}{4}-\epsilon}\psi, \varphi\rangle_h, \quad \varphi \in L_h^2(0,1), \quad (38)$$

$$K^*v = kv, \quad v \in \mathbf{R}. \quad (39)$$

By using these adjoint operators, the self-adjoint operator A defined by (3), and the above (37), eq. (35) can be formulated as

$$\begin{cases}
\beta'(x,\cdot) = \tau(-A + K^*B^*A_c^{\frac{1}{4}+\epsilon})\beta(x,\cdot), \\
\beta(0,\cdot) = k.
\end{cases} \quad (40)$$

Since the operator $\tau(-A + K^*B^*A_c^{\frac{1}{4}+\epsilon})$ generates an analytic semigroup $T_{\tau(-A+K^*B^*A_c^{1/4+\epsilon})}(x)$ on $L_h^2(0,1)$, the solution of (40) is written as

$$\beta(x,\cdot) = T_{\tau(-A+K^*B^*A_c^{1/4+\epsilon})}(x)k. \quad (41)$$

By using this expression, eq. (36) becomes

$$\begin{cases}
p_x(x,y) + p_y(x,y) = 0, \\
p(x,0) = \tau a(1)(T_{\tau(-A+K^*B^*A_c^{1/4+\epsilon})}(x)k)(1).
\end{cases} \quad (42)$$

Since the solution of (42) is expressed as $p(x,y) = \mu(x-y)$, it follows by setting $y = 0$ that

$$p(x,0) = \mu(x) = \tau a(1)(T_{\tau(-A+K^*B^*A_c^{1/4+\epsilon})}(x)k)(1),$$

as a result,

$$\begin{aligned}
p(x,y) &= \mu(x-y) \\
&= \tau a(1)(T_{\tau(-A+K^*B^*A_c^{1/4+\epsilon})}(x-y)k)(1). \quad (43)
\end{aligned}$$

Therefore, from (41) and (43), the inverse transformation (34) can be expressed as follows:

$$\begin{aligned} &u(t,x) \\ &= w(t,x) + \int_0^x p(x,y)w(t,y)dy + \langle \beta(x,\cdot), z(t,\cdot)\rangle_h \\ &= w(t,x) + \int_0^x \tau a(1)(T_{\tau(-A+K^*B^*A_c^{1/4+\epsilon})}(x-y)k)(1) \\ &\qquad\qquad \times w(t,y)dy \\ &\quad + \langle T_{\tau(-A+K^*B^*A_c^{1/4+\epsilon})}(x)k, z(t,\cdot)\rangle_h. \end{aligned} \tag{44}$$

Moreover, noting that

$$T_{\tau(-A+K^*B^*A_c^{1/4+\epsilon})}(x/\tau) = T_{-A+K^*B^*A_c^{1/4+\epsilon}}(x), \tag{45}$$

we finally obtain the abstract expression

$$\begin{aligned} u(t,x) &= w(t,x) \\ &\quad + \int_0^x \tau a(1)(T_{-A+K^*B^*A_c^{1/4+\epsilon}}(\tau(x-y))k)(1) \\ &\qquad\qquad \times w(t,y)dy \\ &\quad + \langle T_{-A+K^*B^*A_c^{1/4+\epsilon}}(\tau x)k, z(t,\cdot)\rangle_h. \end{aligned} \tag{46}$$

Using the expression (46) and Theorems 1 and 2, we have the following stability result.

Theorem 3: Let $z_0 \in L_h^2(0,1)$ and $\phi \in C_r[-\tau, 0]$ $(0 < r \leq 1)$. Then, the closed-loop system consisting of system (2) and control law (30) (or, (31)) is asymptotically stable in the sense of norm $(\|u(t,\cdot)\|_h^2 + \|z(t,\cdot)\|_h^2)^{\frac{1}{2}}$.

Proof: Noting that the analytic semigroup $T_{-A+BKA_c^{1/4+\epsilon}}(x)$ is exponentially stable and further that $T_{-A+K^*B^*A_c^{1/4+\epsilon}}(x)$ and $T_{-A+BKA_c^{1/4+\epsilon}}(x)$ have the same growth bound, it follows that $T_{-A+K^*B^*A_c^{1/4+\epsilon}}(x)$ is exponentially stable as well. Also, since $k \in D(A)$ by Theorem 1, it is assured that $(T_{-A+K^*B^*A_c^{1/4+\epsilon}}(\tau\xi)k)(1)$ is continuous on $0 \leq \xi \leq 1$. Let us set

$$M_1 = \max_{\xi\in[0,1]} |(T_{-A+K^*B^*A_c^{1/4+\epsilon}}(\tau\xi)k)(1)| \ (< \infty).$$

From (46), we have

$$\begin{aligned} &|u(t,x)| \\ &\leq |w(t,x)| + \int_0^x \tau a(1)|(T_{-A+K^*B^*A_c^{1/4+\epsilon}}(\tau(x-y))k)(1)| \\ &\qquad\qquad \times |w(t,y)|dy \\ &\quad + |\langle T_{-A+K^*B^*A_c^{1/4+\epsilon}}(\tau x)k, z(t,\cdot)\rangle_h| \\ &\leq |w(t,x)| + \tau a(1)M_1 \int_0^1 |w(t,y)|dy \\ &\quad + \|T_{-A+K^*B^*A_c^{1/4+\epsilon}}(\tau x)k\|_h \|z(t,\cdot)\|_h \\ &\leq |w(t,x)| + \tau a(1)M_1\|w(t,\cdot)\| + M_2\|z(t,\cdot)\|_h, \end{aligned} \tag{47}$$

where $\|\cdot\|$ denotes the usual L^2-norm, and

$$M_2 = \max_{\xi\in[0,1]} \|T_{-A+K^*B^*A_c^{1/4+\epsilon}}(\tau\xi)k\|_h.$$

Here, squaring the both sides of (47) and integrating over $[0,1]$ with respect to x, we have

$$\begin{aligned} &\|u(t,\cdot)\|^2 \\ &\leq (3+3\tau^2 a(1)^2 M_1^2)\|w(t,\cdot)\|^2 + 3M_2^2\|z(t,\cdot)\|_h^2. \end{aligned} \tag{48}$$

Furthermore, noting that $\|\cdot\|$ and $\|\cdot\|_h$ are equivalent, i.e., there exist positive constants c_1 and c_2 such that

$$c_1\|v\| \leq \|v\|_h \leq c_2\|v\| \ \text{ for all } v \in L_h^2(0,1) = L^2(0,1), \tag{49}$$

from (48) we obtain

$$\|u(t,\cdot)\|_h \leq C(\|w(t,\cdot)\|_h^2 + \|z(t,\cdot)\|_h^2)^{\frac{1}{2}}, \tag{50}$$

where C is some positive constant. This means that the inverse transformation (46), which maps from $L_h^2(0,1) \times L_h^2(0,1)$ to $L_h^2(0,1)$, is continuous. Also, it follows from (50) that

$$(\|u(t,\cdot)\|_h^2 + \|z(t,\cdot)\|_h^2)^{\frac{1}{2}} \leq C'(\|w(t,\cdot)\|_h^2 + \|z(t,\cdot)\|_h^2)^{\frac{1}{2}}, \tag{51}$$

where C' is some positive constant. Based on the fact stated in Theorem 2, we can conclude the assertion of this theorem. ∎

Remark 4: We can rewrite system (35) as the following parabolic system with boundary feedback loop:

$$\begin{cases} \beta_x(x,y) = \frac{\tau}{h(y)}\big(\big(a(y)\beta_y(x,y)\big)_y - b(y)\beta(x,y)\big) \\ \qquad + \tau k(y)Y(t), \\ \beta_y(x,0) = \beta_y(x,1) = 0, \\ \beta(0,y) = k(y), \\ Y(t) = a(1)\beta(x,1). \end{cases} \tag{52}$$

In this, by defining the unbounded operator $\Gamma : D(A) \to \mathbf{R}$ as $\Gamma\zeta = a(1)\zeta(1)$, $\zeta \in D(A)$, we can express the observation equation of (52) as

$$Y(t) = \Gamma\beta(x,\cdot). \tag{53}$$

On the other hand, by using (37) and (38), we can formulate the observation equation of (52) as

$$Y(t) = B^* A_c^{\frac{1}{4}+\epsilon}\beta(x,\cdot). \tag{54}$$

Here, note that the operator $B^*A_c^{\frac{1}{4}+\epsilon}$ of (54) is the Λ-extension of the operator Γ of (53) (see [18] as for the definition of Λ-extension).

V. Numerical Example: Linear Diffusion System

Consider the system (1) with $h(x) \equiv 1$, $a(x) \equiv 1$, $b(x) \equiv -1$, and $\tau = 0.1$. Then, A has a set of eigenpairs $\{\lambda_i, \varphi_i\}_{i=0}^\infty$ in $L^2(0,1)$, where $\lambda_i = i^2\pi^2 - 1$ $(i \geq 0)$, $\varphi_0(x) \equiv 1$, $\varphi_i(x) = \sqrt{2}\cos i\pi x$ $(i \geq 1)$. In the variable transformation $\zeta(t) = A_c^{-\frac{1}{4}-\epsilon}z(t,\cdot)$, let us set $c = 2$ and $\epsilon = 0.1$. Then, the boundary value problem (7) is solved as $\psi(x) = \cosh x/\sinh 1$.

First, we give $\omega = 5$ and choose an integer n $(n \geq 0)$ as $n = 0$. In fact, the inequality $\omega < \lambda_{n+1}$ holds with $n = 0$. Since the pair $(-A_1, B_1)$ is controllable, we can choose a matrix K_1 such that $\sigma(-A_1 + B_1K_1(A_1+c)^{\frac{1}{4}+\epsilon}) = \{-3\}$. In this case, K_1 is a scalar and it is solved as $K_1 = k_0 = -4$, since $A_1 = -1$ and $B_1 = 1$. Note that, by (10), $k(x) = k_0\varphi_0(x) \equiv -4$. Fig. 1 shows the simulation result of the closed-loop system. Thus, we see that the control law (30) works effectively as a stabilizing controller.

To solve the linear diffusion equation numerically, we used the finite difference method with mesh width $\Delta x = 0.02$, and the Runge-Kutta method of the fourth order with time step $\Delta t = 0.0001$ for its time integration. As initial conditions, we set $z_0(x) = \exp\{-50(x-0.6)^2\}$ and $\phi(\theta) \equiv 0$ for (1).

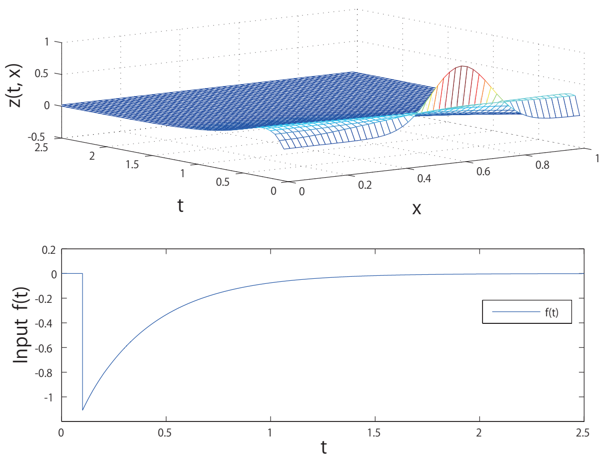

Fig. 1. Evolution of the state $z(t, x)$ and the control input $f(t)$.

VI. Conclusion

In this note, we treated the Neumann boundary stabilization problem of unstable parabolic systems with input delay. After this system was expressed as a cascade consisting of the transport equation and the parabolic equation with boundary input, it was shown that the control law was constructed by using the solution to a parabolic equation and the solution to a hyperbolic equation, based on a backstepping method using single integral transformation. Also, by using the semigroup theory, we gave an abstract expression of the control law and further showed the continuity of the inverse transformation. Especially, the fact that the function $k(x)$ contained in the target system could be designed as $k = k_0\varphi_0 + \cdots + k_n\varphi_n \in D(A)$ is a key point of this work. Based on this fact, we could give a concrete expression for the abstract controller. By using the similar way, it is possible to give the controller of predictor type such as (31) for the Dirichlet boundary stabilization of unstable parabolic systems with input delay.

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Appendix A. Controllability of the Pair $(-A_1, B_1)$

From the definition of the matrix B_1 and the fact that the eigenfunctions $\{\varphi_i\}_{i=0}^{\infty}$ of A forms a complete orthonormal system in $L_h^2(0,1)$, we have $B_1 = [b_0, b_1, \cdots, b_n]^T$, where

$$b_i = a(1)(\lambda_i + c)^{\frac{3}{4}-\epsilon}\langle\psi, \varphi_i\rangle_h, \quad 0 \le i \le n. \tag{A.1}$$

First, let us calculate the value of inner product $\langle\psi, \varphi_i\rangle_h$. Since ψ is the solution of the boundary value problem (7), it satisfies

$$\frac{1}{h(x)}\left(-\frac{d}{dx}\left(a(x)\frac{d\psi(x)}{dx}\right) + b(x)\psi(x)\right) = -c\psi(x). \tag{A.2}$$

Here, multiplying $\varphi_i(x)h(x)$ on both sides of (A.2) and integrating over $[0,1]$, we have

$$-\int_0^1 \left(a(x)\psi_x(x)\right)_x \varphi_i(x)dx + \int_0^1 b(x)\psi(x)\varphi_i(x)dx$$

$$= -c\langle\psi, \varphi_i\rangle_h. \tag{A.3}$$

Moreover, using integration by parts for the first term of the left-hand side of (A.3) and noting that $\psi'(0) = 0$, $\psi'(1) = \frac{1}{a(1)}$, we have $(\lambda_i + c)\langle\psi, \varphi_i\rangle_h = \varphi_i(1)$, which leads to

$$\langle\psi, \varphi_i\rangle_h = \frac{\varphi_i(1)}{\lambda_i + c}, \tag{A.4}$$

since $\lambda_i + c > 0$.

Now, note that $\varphi_i(1) \neq 0$ for all $i \ge 0$, since $h(x)$, $a(x)$, $b(x)$ are sufficiently smooth and $a(1) > 0$. Therefore, from (A.1) and (A.4) as well as $a(1) > 0$, we obtain $b_i = a(1)(\lambda_i + c)^{\frac{3}{4}-\epsilon}\langle\psi, \varphi_i\rangle_h \neq 0$, $0 \le i \le n$. Thus, the controllability of the pair $(-A_1, B_1)$ follows, since each element of the matrix B_1 is not zero.

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