Exact WKB analysis for eigenvalue problems of ordinary differential equations

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博　士　論　文

Exact WKB analysis for eigenvalue problems of ordinary differential equations

(常微分方程式の固有値問題に対する完全WKB解析)

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Contents

Introduction

In this thesis, we report about exact WKB analysis for eigenvalue problems of ordinary differential equations. Firstly, in Part I, we study exact WKB analysis of eigenvalue problems for an ordinary differential equation arising from the mathematical model of mesons. This is the main part of this thesis. Section I.1 - Section I.9 and Section I.A will be published in [Sh2]. In Section I.B, we give a summary of physical background of the eigenvalue problem discussed in Part I.

Secondly, we give a brief overview on exact WKB analysis and eigenvalue problems in Part II. In this part, we clarify the position of Part I and Part III of this thesis. We also give an introductory explanation on exact WKB analysis and eigenvalue problems through a simple example in Section II.2.

Finally, we give an introduction to nonlinear eigenvalue problems for a certain first order equation and present our tentative results on the problem (published in [Sh1]) in Part III. As is explained in Section II.1, studying the so-called 'nonlinear eigenvalue problems' is one of the new attempts to apply exact WKB analysis to a certain type of nonlinear equations.

Part I
Exact WKB analysis of eigenvalue problems for an ordinary differential equation arising from the mathematical model of mesons

I.1. Introduction to Part I

In [SaSu] Sakai and Sugimoto study a mass of meson, which is a strongly interacting particle of mass intermediate between proton and neutron. The mass of meson is described as an eigenvalue λ for the eigenvalue problem of a second order ordinary differential equation

$$\varphi''(z) + \frac{2z}{1+z^2}\varphi'(z) + \frac{\lambda}{(1+z^2)^{4/3}}\varphi(z) = 0 \qquad \text{(I.1.1)}$$

on the real axis with the boundary condition

$$\varphi(z) \sim O(z^{-1}) \quad (z \to \pm\infty). \tag{I.1.2}$$

The boundary value problem (I.1.1) - (I.1.2) along the real axis determines a series of eigenvalues λ_n ($0 < \lambda_1 < \lambda_2 < \ldots$) and eigenfunctions $\varphi_n(z)$. As is discussed in [SaSu] and [Su] (cf. Sections 3 and 4 in [SaSu], Section 7.4 in [Su]), the eigenvalue of this boundary value problem is proportional to the square of the mass of the meson and the action functional of the gauge field for the meson is described in terms of the eigenfunctions and their first derivatives. Sakai and Sugimoto obtained the first few eigenvalues numerically as $\lambda_1 \sim 0.669$, $\lambda_2 \sim 1.57$ and $\lambda_3 \sim 2.87$ by the shooting method.

In this part, to investigate the behavior of eigenvalues λ_n for large n and, in particular, the asymptotics of λ_n for $n \to \infty$, we study the eigenvalue problem (I.1.1) - (I.1.2) from the viewpoint of the exact WKB analysis. To this end, we introduce a large parameter η into the equation and consider z as a complex variable although the original eigenvalue problem is considered on the real axis. After making a suitable change of variables, we find that the Schrödinger equation in question has not only simple turning points but also simple poles and the so-called "ghost points". (See Section I.6 below for the definition of ghost points.) As was pointed out by Koike ([Ko3],[Ko4]), simple poles and ghost points should be also considered as a kind of turning points of the equation. In particular, the ghost point is first introduced by Koike under the name of a "new turning point" and its important meaning in the exact WKB analysis is clarified in [Ko4]. Although the ghost point has not been used for a concrete physical problem so far, it plays a crucially important role in the computation of the secular equation for the eigenvalue problem (I.1.1) - (I.1.2).

Furthermore, the Stokes graph of (I.1.1), which is an essential ingredient to discuss the connection problem of solutions by using the exact WKB method, has also an intriguing structure. That is, many types of degenerations of Stokes curves simultaneously occur at $\arg\eta = 0$, which corresponds to the original problem (I.1.1) - (I.1.2). Although our ultimate goal is to treat the case of $\arg\eta = 0$, it is fully degenerate and it is difficult to treat this case at present. Thus in this paper we restrict ourselves on the analysis of one particular degeneration of Stokes curves caused by a simple turning point and two simple poles. The main result of this paper is that the secular equation for the eigenvalue problem (I.1.1) - (I.1.2) is described in terms of the connection coefficients at the ghost point modulo exponentially small terms in a region of $\arg\eta$ where only the above kind of degeneration of Stokes curves is relevant.

The plan of this part is as follows: In Section I.2, introducing a large parameter η into the equation and making an appropriate change of variables, we present the precise formulation of our eigenvalues problem. Then in Section I.3 we explain the basic exact WKB theoretic structure of the problem and specify the region of $\arg\eta$ we consider in the paper. The main result is stated in Section I.4. In Section I.5 we examine what kind of configurations of Stokes curves are observed when the degeneration in question is resolved. After recalling explicit forms of connection formulas across Stokes curves in Section I.6, we then compute connection matrices and derive the secular equation in Sections I.7 and I.8 to prove the main result. Concluding remarks are given in Section I.9. In Appendix (Section I.A) we compare the consequence of the main result with some results of numerical computations. In Section I.B, we give a summary of background of physics of the eigenvalue problem in our main part.

I.2. Precise formulation of eigenvalue problems

The equation (I.1.1) has the following power series solutions near the infinity.

Proposition I.2.1 (Proposition 2.1 in [Sh2]) *The equation* (I.1.1) *has the following convergent solutions near the infinity:*

$$\varphi_0(z) = \sum_{n=0}^{\infty} \varphi_{0,n} z^{-2n/3}, \quad \varphi_1(z) = \sum_{n=0}^{\infty} \varphi_{1,n} z^{-1-2n/3}. \tag{I.2.1}$$

Here $\varphi_{0,0} = 1$ and $\varphi_{1,0} = 1$. For $n \geq 1$, $\varphi_{0,n}$ and $\varphi_{1,n}$ are determined by the following recursive equations.

$$\varphi_{0,n} = \frac{1}{2n(2n-3)} \left\{ \sum_{\substack{3m+l=n,\\ m\geq 1,\\ 0\leq l\leq n-1}} (-1)^m 12l\varphi_{0,l} - 9\lambda \sum_{\substack{3m+l=n-1,\\ m,l\geq 0}} \binom{-4/3}{m} \varphi_{0,l} \right\}, \tag{I.2.2}$$

$$\varphi_{1,n} = \frac{1}{2n(2n+3)} \left\{ \sum_{\substack{3m+l=n,\\ m\geq 1,\\ 0\leq l\leq n-1}} (-1)^m 6(2l+3)\varphi_{1,l} - 9\lambda \sum_{\substack{3m+l=n-1,\\ m,l\geq 0}} \binom{-4/3}{m} \varphi_{1,l} \right\}, \tag{I.2.3}$$

where the symbol $\binom{a}{n}$ *means a binomial coefficient for* $n = 0, 1, 2, \ldots$, *i.e.,*

$$\binom{a}{n} = \frac{\Gamma(a+1)}{\Gamma(n+1)\Gamma(a-n+1)}.$$

Similar recursive equations also appear in [SaSu], (4.9).

Proof. First, we apply a change of the independent variable $t = z^{-2/3}$ and a transformation $\varphi(z) = \tilde{\varphi}(z^{-2/3})$ of the unknown function to (I.1.1) to get

$$\tilde{\varphi}''(t) + \frac{5t^3 - 1}{2t(t^3+1)}\tilde{\varphi}'(t) + \frac{9\lambda}{4t(t^3+1)^{4/3}}\tilde{\varphi}(t) = 0.$$

The origin $t = 0$ is a regular singular point of this equation and the characteristic exponents at $t = 0$ are 0 and $3/2$. Therefore the equation has two convergent solutions $\tilde{\varphi}_0(t) = \sum_{n=0}^{\infty} \varphi_{0,n} t^n$ and $\tilde{\varphi}_1(t) = \sum_{n=0}^{\infty} \varphi_{1,n} t^{n+3/2}$ near $t = 0$. Substituting these expansions into the equation, we obtain the relations (I.2.2) - (I.2.3) for the coefficients $\varphi_{0,n}$ and $\varphi_{1,n}$. $\square$

The boundary condition (I.1.2) requires the eigenfunction to behave as a constant multiple of $\varphi_1(z)$ near $z = \infty$. Since we are discussing the eigenvalue problem on the real axis, we need to seek the condition which guarantees that the solution $\varphi_1(z)$ near $z = +\infty$ also becomes a constant multiple of $\varphi_1(z)$ near $z = -\infty$ after the analytic continuation along the real axis. In general, the solution $\varphi_1(z)$ near $z = +\infty$ is analytically continued along the real axis to $c_0(\lambda)\varphi_0(z) + c_1(\lambda)\varphi_1(z)$ near $z = -\infty$ with some $c_0(\lambda)$ and $c_1(\lambda)$ independent of z. Hence, the eigenvalue λ of the problem (I.1.1) - (I.1.2) should satisfy $c_0(\lambda) = 0$. Thus $c_0(\lambda) = 0$ gives a secular equation for the eigenvalue problem (I.1.1) - (I.1.2). That is, if we can solve the connection problem along the real axis, we obtain a secular equation. In what follows, after introducing a large parameter into the equation (I.1.1) in an appropriate way, we apply the exact WKB analysis to solve this connection problem.

First of all, to eliminate the term of first order derivative, we transform an unknown function of (I.1.1) as

$$\varphi(z) = \phi(z) \exp\left(-\frac{1}{2}\int^z \frac{2z'}{1+z'^2}dz'\right) = (1+z^2)^{-1/2}\phi(z). \tag{I.2.4}$$

Then we obtain

$$\phi''(z) - Q(z)\phi(z) = 0, \quad Q(z) = \frac{1}{(1+z^2)^2} - \frac{\lambda}{(1+z^2)^{4/3}}. \tag{I.2.5}$$

Now we introduce a large parameter η into (I.2.5) by replacing $Q(z)$ by $\eta^2 Q(z)$. Furthermore, eigenvalues commonly appear near a critical value of the potential. In the case of (I.2.5) the potential $Q(z)$ has the form

$$Q(z) = \frac{1}{(1+z^2)^{4/3}} \left(\frac{1}{(1+z^2)^{2/3}} - \lambda \right)$$

and hence $\lambda = 1$ is only the unique candidate for the critical value of the potential (more precisely, critical value of $(1+z^2)^{4/3} Q(z)$). Taking this into account, we rescale λ and set $\lambda = 1 + \eta^{-1}\tilde{\lambda}$. Then we obtain

$$\phi''(z) - \eta^2 (Q_0(z) + \eta^{-1} Q_1(z))\phi(z) = 0, \tag{I.2.6}$$

where

$$Q_0(z) = \frac{1}{(1+z^2)^2} - \frac{1}{(1+z^2)^{4/3}}, \quad Q_1(z) = \frac{-\tilde{\lambda}}{(1+z^2)^{4/3}}.$$

One difficulty in discussing the equation (I.2.6) is that the potential functions $Q_0(z)$ and $Q_1(z)$ are multi-valued functions. To cope with this difficulty, we change the independent variable from z to $s = (1+z^2)^{-1/3}$ so that $Q(z,\eta) = Q_0(z) + \eta Q_1(z)$ becomes a rational function. After eliminating terms of first order derivative by

$$\phi = \psi \exp\left(- \int^s \frac{5 - 8s^3}{4s(1-s^3)} ds \right),$$

(I.2.6) becomes

$$\psi''(s) - \eta^2 (q_0(s) + \eta^{-1} q_1(s) + \eta^{-2} q_2(s))\psi(s) = 0, \tag{I.2.7}$$

where

$$q_0(s) = \frac{9(s^2-1)}{4s(1-s^3)}, \; q_1(s) = \frac{-9\tilde{\lambda}}{4s(1-s^3)}, \; q_2(s) = \frac{5 - 64s^3 + 32s^6}{16s^2(1-s^3)^2}.$$

By this change of variable $s = (1+z^2)^{-1/3}$, the real axis in z-plane is changed to a path γ that starts from $s = 0$, goes around $s = 1$ in a counter-clockwise direction once, and returns to $s = 0$. Furthermore (I.2.7) has the following power series solutions around $s = 0$ which correspond to power series solutions given in Proposition I.2.1.

Proposition I.2.2 (Proposition 2.2 in [Sh2]) *The equation* (I.2.7) *has the following solutions:*

$$\psi_0(s,\eta) = \sum_{n=0}^{\infty} \psi_{0,n}(\eta) s^{n-1/4}, \quad \psi_1(s,\eta) = \sum_{n=0}^{\infty} \psi_{1,n}(\eta) s^{n+5/4}. \tag{I.2.8}$$

Here $\psi_{0,0} = 1$ *and* $\psi_{1,0} = 1$*. For* $n \geq 1$*,* $\psi_{0,n}$ *and* $\psi_{1,n}$ *are determined by the following recursive relations.*

$$\left\{(n-\frac{1}{4})(n-\frac{5}{4})\right\} \psi_{0,n} = -\sum_{k=1}^{n} \tilde{q}_{k-2}(\eta)\psi_{0,n-k}, \tag{I.2.9}$$

$$\left\{(n+\frac{1}{4})(n+\frac{5}{4})\right\} \psi_{1,n} = -\sum_{k=1}^{n} \tilde{q}_{k-2}(\eta)\psi_{1,n-k}. \tag{I.2.10}$$

The functions $\tilde{q}_{k-2}(\eta)$ *of* η *in the above equations are determined by*

$$\eta^2(q_0(s) + \eta^{-1}q_1(s) + \eta^{-2}q_2(s)) = \sum_{k=0}^{\infty} \tilde{q}_{k-2}(\eta) s^{k-2}.$$

The solution $\psi_1(s,\eta)$ corresponds to the solution satisfying the boundary condition, i.e., corresponds to $\varphi_1(z)$ in Proposition I.2.1. Therefore, if we denote the analytic continuation of $\psi_1(s,\eta)$ along the path γ by $\tilde{c}_1(\tilde{\lambda})\psi_1(s,\eta) + \tilde{c}_0(\tilde{\lambda})\psi_0(s,\eta)$, a secular equation for $\tilde{\lambda}$ is given by $\tilde{c}_0(\tilde{\lambda}) = 0$. To be more precise, if $\tilde{\lambda}$ satisfies $\tilde{c}_0(\tilde{\lambda}) = 0$, then $\lambda = 1 + \eta^{-1}\tilde{\lambda}$ becomes an eigenvalue of the problem (I.1.1) - (I.1.2). In what follows, by using the exact WKB analysis, we seek $\tilde{c}_0(\tilde{\lambda})$ in the following way:

(i) We first define WKB solutions of (I.2.7) and find a relation between WKB solutions and $\psi_j(s,\eta)$ $(j = 0, 1)$ in Proposition I.2.2.

(ii) We then use the exact WKB method to obtain the analytic continuation of WKB solutions along γ.

I.3. Exact WKB analysis and purpose of Part I

Consider a Schrödinger equation

$$\psi''(s) - \eta^2(q_0(s) + \eta^{-1}q_1(s) + \eta^{-2}q_2(s))\psi(s) = 0. \tag{I.3.1}$$

By setting solutions of (I.3.1) as $\psi = \exp\left(\int^s S ds\right)$, we find

$$S(s,\eta) = \sum_{m=-1}^{\infty} \eta^{-m} S_m(s) = \eta S_{-1}(s) + S_0(s) + \dots$$

satisfies the Riccati equation

$$S(s,\eta)^2 + S'(s,\eta) = \eta^2(q_0(s) + \eta^{-1}q_1(s) + \eta^{-2}q_2(s)).$$

Here $S(s,\eta)$ is determined uniquely and recursively from $S_{-1}(s) = \pm\sqrt{q_0(s)}$. We denote $S(s,\eta)$ for $S_{-1} = \pm\sqrt{q_0(s)}$ by $S^{(\pm)}(s,\eta)$. Throughout this paper, introducing two cuts as in Fig. 1 (i.e., one cut connecting 0 and ω, and the other connecting -1 and ω^2, where $\omega = e^{2\pi i/3}$), we fix the branch of $\sqrt{q_0(s)}$ on this cut plane so that $\sqrt{q_0(s)} \in i\mathbb{R}_+$ holds for $0 < s < 1$.

We now define a WKB solution of (I.3.1) by

$$\psi_\pm(s,\eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}(s,\eta)}} \exp \int_{\hat{s}}^{s} \pm S_{\mathrm{odd}}(s',\eta) ds'. \tag{I.3.2}$$

Here $S_{\mathrm{odd}}(s,\eta) := (S^{(+)}(s,\eta) - S^{(-)}(s,\eta))/2$ and $\hat{s}$ is a fixed point outside Stokes curves defined below.

WKB solutions are not convergent as a power series of η^{-1} in general and in the exact WKB analysis we give them an analytic meaning by the Borel resummation method with respect to η^{-1}. Here we recall the Borel resummation briefly.

Definition I.3.1 (Definition 3.1 in [Sh2], cf. Section 1 in [KT3]) *Let η be a large parameter with $\theta = \arg\eta$ and α be a real number satisfying $\alpha \notin \{0,-1,-2,\cdots\}$. For a power series*

$$f(\eta) = \exp(y_0\eta) \sum_{n=0}^{\infty} f_n \eta^{-n-\alpha} \tag{I.3.3}$$

with y_0 and f_n being constants with respect to η, we define the Borel transform and the Borel sum of f by

$$f_B(y) := \sum_{n=0}^{\infty} \frac{f_n}{\Gamma(\alpha+n)} (y+y_0)^{\alpha+n-1} \tag{I.3.4}$$

(where Γ denotes the gamma function) and

$$F(\eta) := \int_{-y_0}^{\infty e^{-i\theta}} e^{-\eta y} f_B(y) dy \tag{I.3.5}$$

respectively, where the path of integration is $-y_0 + e^{-i\theta}\mathbb{R}_+$. If the Borel sum (I.3.5) *is well-defined, $f(\eta)$ is said to be Borel summable.*

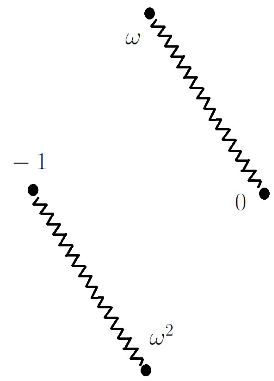

Fig. 1: cut plane for $\sqrt{q_0(s)}$ (Wavy lines designate the branch cuts.)

Remark Throughout this paper, unless otherwise specified, power series of η^{-1} including WKB solutions are interpreted as the Borel sum.

As is proved in [KoSc] (cf. Theorem 2.23 in [KT3]), provided that no Stokes curve connects two turning points, WKB solutions are Borel summable in the Stokes regions defined as regions surrounded by Stokes curves. Here a Stokes curve is defined by $\mathrm{Im}\left(e^{i\theta}\int_a^s \sqrt{q_0(s')}ds'\right) = 0$, where θ denotes $\arg\eta$ and a starting point a is a turning point. As was explained in Section I.1, a turning point means a zero of $q_0(s)$, a simple pole of $q_0(s)$ or what they call a ghost point. (See Section I.6 below for the precise definition of ghost points.) Furthermore, the connection formula, that is, a relation between the Borel sums in adjacent Stokes regions is also known. See Section 6 for the details. Thus, using connection formulas in view of Stokes curves, we can study the analytic continuation of WKB solutions and obtain a secular equation for $\tilde{\lambda}$. Stokes curves and connection formulas play a crucially important role in the exact WKB analysis.

In our case (I.2.7), there are five turning points: a simple zero $s = -1$, three simple poles $s = 0, \omega, \omega^2$ where $\omega = e^{2\pi i/3}$, and a ghost point $s = 1$. The Stokes curves of (I.2.7) for $\arg\eta = 0$ are shown in Fig. 2, which is fully degenerate in the sense that all Stokes curves connect turning points. To guarantee the Borel summability of WKB solutions, we need to resolve the degeneration of Stokes curves by perturbing $\arg\eta$. Fig. 3 shows the Stokes curves for $\arg\eta = -\pi/4$, where all the degeneration is resolved. As a matter of fact, when $\arg\eta = -c_R\pi$ with $c_R \sim 0.1337$, a special degenerate configuration of Stokes curves shown in Fig. 4 is observed and Fig. 3 appears in trying to resolve Fig. 4. More details will be discussed in Sections I.4 and I.5.

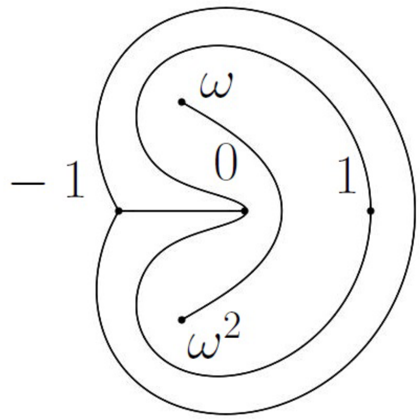

Fig. 2: Stokes curve for $\arg\eta = 0$

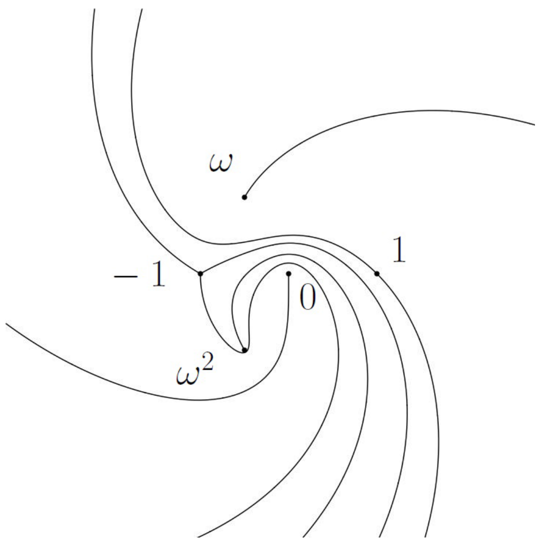

Fig. 3: Stokes curve for $\arg\eta = -\pi/4$

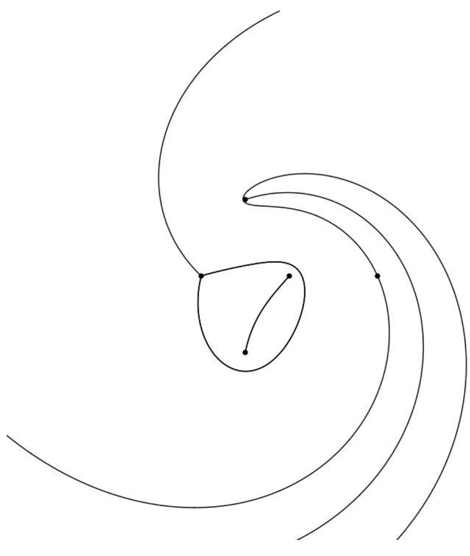

Fig. 4: Stokes curve for $\arg\eta = -c_R\pi$

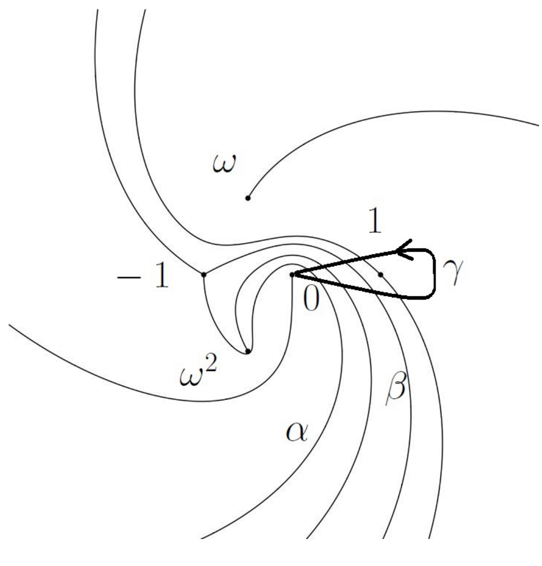

Fig. 5: path γ of the analytic continuation and Stokes curves α, β $(\arg\eta = -\pi/4)$

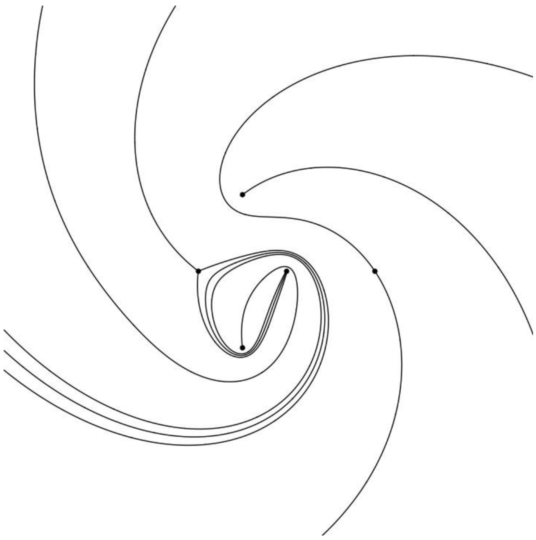

Fig. 6: Stokes curve for $\arg \eta = -3\pi/16$

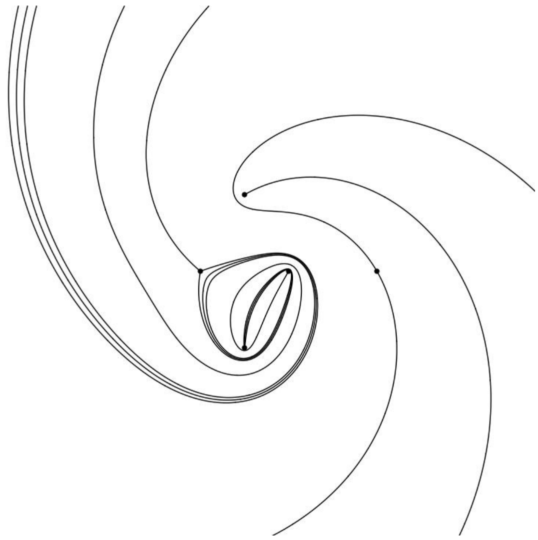

Fig. 7: Stokes curve for $\arg \eta = -\pi/6$

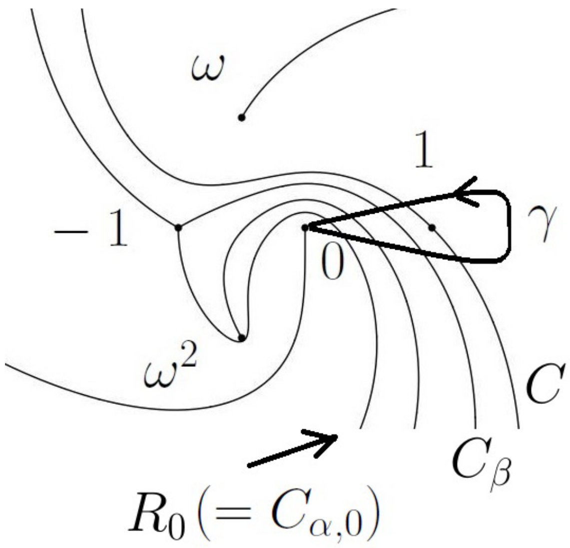

Fig. 8: relevant crossing points of γ with Stokes curves and connection matrices for the proof of Theorem I.7.1

Although our ultimate goal is to discuss the fully degenerate configuration of Fig. 2 for $\arg\eta = 0$, as its first step we study the degenerate configuration of Fig. 4 and restrict our consideration on $-\pi/4 \le \arg\eta < -c_R\pi$ in this paper. Note that Fig. 4 is an interesting degenerate configuration and concerned with one simple turning point $s = -1$ and two simple poles $s = 0, \omega^2$. In [KaKKoT] the transformation theory to a canonical equation (the Mathieu equation in this case) near a triplet of one simple turning point and two simple poles is considered. The purpose of this paper is to discuss the change of configurations of Stokes curves that appear in resolving this degenerate configuration of Fig. 4 and to compute the connection matrices and the secular equation in an explicit manner.

In what follows we compute secular equations and eigenvalues by using the exact WKB analysis when $-\pi/4 \le \arg\eta < -c_R\pi$ and also compare them with the results of numerical calculations in Appendix.

I.4. Main Result

As we refer in Remark after Definition I.3.1, throughout this paper, power series of η^{-1} means the Borel sum.

In order to study the analytic continuation of WKB solutions along γ, we need to observe how γ crosses Stokes curves. We first consider the case where $\arg\eta = -\pi/4$. Let α (resp., β) be the Stokes curve going down (resp., going in the upper-right direction) from the turning point -1 (cf. Fig. 5). As we will see later in Section I.7, the Stokes curves emanating from $s = 0, \omega, \omega^2$ do not give effects on analytic continuations of WKB solutions. Hence we neglect these Stokes curves and focus on the Stokes curves emanating from the turning point $s = -1$ and from the ghost point $s = 1$.

Then we can observe that

(1) γ crosses the curve α first,

(2) γ crosses the curve β next,

(3) γ crosses two Stokes curves emanating from the ghost point $s = 1$,

(4) γ crosses β in the opposite direction of (2), and then

(5) γ crosses α in the opposite direction of (1)

(cf. Fig. 5). Based on this configuration of Stokes curves, we can compute the secular equation by using the connection formulas. We remark that, as $\arg\eta$ increases from $-\pi/4$ to $-c_R\pi$, the configuration of Stokes curves becomes more and more complicated (cf. Section I.5). Nevertheless, we can verify that the secular equation is stable modulo exponentially small terms during $-\pi/4 \le \arg\eta < -c_R\pi$ and obtain the following:

Theorem I.4.1 (Theorem 4.1 in [Sh2]) *When $|\eta|$ is sufficiently large and $-\pi/4 \le \arg\eta < -c_R\pi$, the secular equation is given by $1/(\Gamma(\kappa+3/4)\Gamma(\kappa+1/4))=0$ modulo exponentially small terms, that is,*

$$\kappa \pm \frac{1}{4} + \frac{1}{2} = -n \ (n = 0, 1, 2, \ldots) \tag{I.4.1}$$

holds modulo exponentially small terms, where κ is defined by $-\mathrm{Res}_{s=1} S_{\mathrm{odd}}(s,\eta)$ (and interpreted as the Borel sum, as stated in Remark after Definition I.3.1). Here the branch of $S_{-1}(s) = \sqrt{q_0(s)}$ is determined as explained in Section I.3 (cf. Fig. 1).

The proof of this theorem will be given in Section I.8.

I.5. Configuration of Stokes curves

Figures 6 and 7 are the configuration of Stokes curves for $\arg\eta = -3\pi/16$ and $\arg\eta = -\pi/6$, respectively. The configuration of Stokes curves becomes more complicated when $\arg\eta$ increases from $-\pi/4$ to $-c_R\pi$ in this way. In this section, we investigate the complexity of Stokes curves for $-\pi/4 \le \arg\eta < -c_R\pi$. We first compare Fig. 3 and Fig. 6. We pay attention to the relation between the curve α emanating from $s=-1$ and the Stokes curve emanating from $s=0$. In Fig. 3, α goes over the curve emanating from $s=0$, crosses with [0,1] and flows into the infinity. On the other hand, in Fig. 6, α goes under the curve emanating from $s=0$, return to a neighborhood of $s=-1$ with crossing $[0,1]$, and then flows into the infinity. This difference between Fig. 3 and Fig. 6 implies that the degeneration

(A) the points $s=-1$ and $s=0$ are connected by a Stokes curve

occurs between these two figures, that is, the transition

Fig. 3 ($\arg\eta = -\pi/4$) $\to$ degeneration (A) $\to$ Fig. 6 ($\arg\eta = -3\pi/16$)

is expected to occur between Figs. 3 and 6.

Similarly, the difference between Fig. 6 and Fig. 7 is concerned with the relation of α and the curve emanating from $s=\omega^2$. That is, after passing through a neighborhood of $s=0$, α goes under the curve from $s=\omega^2$ in Fig. 6. On the other hand, in Fig. 7 α goes over the curve from $s=\omega^2$, passing near $s=\omega^2$, and returns to a neighborhood $s=0$. This observation implies that the degeneration

(B) the points $s=-1$ and $s=\omega^2$ are connected by a Stokes curve

occurs between these two figures, that is, the transition

$$\text{Fig. 6 } (\arg\eta = -3\pi/16) \to \text{degeneration (B)} \to \text{Fig. 7 } (\arg\eta = -\pi/6)$$

is expected to occur between Figs. 6 and 7.

As $\arg\eta$ increases from $-\pi/4$ to $-c_R\pi$, the degenerations (A) and (B) are expected to occur alternately. Every time the degeneration (A) or (B) occurs, the intersection number of the curve α and the interval $[0,1]$ increases by one. Thus, if we start from $\arg\eta=-\pi/4$ and assume the degenerations occur $(m-1)$ times, we find that the path γ of analytic continuation crosses the Stokes curves emanating from $s=\pm 1$ in the following manner:

(1) γ first crosses the curve α m times,

(2) γ then crosses the curve β once,

(3) γ crosses two Stokes curves emanating from the ghost point $s=1$,

(4) γ crosses β once in the opposite direction of (2), and then

(5) γ crosses α m times in the opposite direction of (1).

In the range of $-\pi/4 \le \arg\eta < -c_R\pi$, that is, in the transition of the configuration of Stokes curves from Fig. 3 to Fig. 4, these degenerations (A) and (B) occur repeatedly.

Based on these observations, we will compute the secular equation by using the connection formulas in Section I.7.

I.6. Connection formula for WKB solutions

Before computing the secular equation, we recall several known results on the connection formulas for WKB solutions under the assumption that Stokes graphs are not degenerate in this section. First, we describe the connection formula in the case of a simple turning point.

Theorem I.6.1 (Theorem 6.1 in [Sh2]) (Case of simple turning points; cf. [AKKoT], Theorem 2.23 in [KT3])

Suppose that $s=a$ is a simple zero of $q_0(s)$, that is, a simple turning point of (I.3.1). *Let Γ be a Stokes curve $\mathrm{Im}\left(e^{i\theta}\int_a^s\sqrt{q_0(s')}ds'\right)=0$ where $\theta=\arg\eta$ emanating from $s=a$ and let U_1 and U_2 be Stokes regions having Γ as a common boundary.*

Let $\psi_\pm^j$ $(j = 1, 2)$ *be the Borel sums of the WKB solutions* (I.3.2) *with* $\hat{s} = a$ *in the region* U_j*. Then* $\psi_\pm^1$ *is analytically continued to* U_2 *and satisfies one of the following formulas:*

$$\text{If } \operatorname{Re}\left(e^{i\theta}\int_a^s \sqrt{q_0(s')}ds'\right) < 0 \text{ on } \Gamma,\ \psi_+^1 = \psi_+^2,\ \psi_-^1 = \psi_-^2 \pm i\psi_+^2, \quad \text{(I.6.1)}$$

$$\text{If } \operatorname{Re}\left(e^{i\theta}\int_a^s \sqrt{q_0(s')}ds'\right) > 0 \text{ on } \Gamma,\ \psi_+^1 = \psi_+^2 \pm i\psi_-^2,\ \psi_-^1 = \psi_-^2. \quad \text{(I.6.2)}$$

The signs $\pm$ *correspond to the counter-clockwise and clockwise crossing (viewed from the turning point* $s = a$*) of* Γ *with the path of analytic continuation from* U_1 *to* U_2*. If it is counter-clockwise (resp., clockwise) crossing, then the sign is* $+$ *(resp.,* $-$*).*

The formulas (I.6.1) and (I.6.2) are often written as $\psi_+ \mapsto \psi_+$, $\psi_- \mapsto \psi_- \pm i\psi_+$ and $\psi_+ \mapsto \psi_+ \pm i\psi_-$, $\psi_- \mapsto \psi_-$. (We often use this notation in what follows.)

Next, we describe the connection formula in the case of a simple pole.

Theorem I.6.2 (Theorem 6.2 in [Sh2]) (Case of simple poles; cf. Theorem 2.1 in [Ko3], Theorem 1 in [Ko5])
For (I.3.1), assume that $q_0(s) = \tilde{q}_0(s)/(s-a)$, $q_1(s) = \tilde{q}_1(s)/(s-a)$ *and* $q_2(s) = \tilde{q}_2(s)/(s-a)^2$ *where* $\tilde{q}_j(s)$ $(j = 0, 1, 2)$ *is holomorphic in a neighborhood* $U \subset \mathbb{C}$ *of* $s = a$*. Furthermore, we assume* $\tilde{q}_0(a) \neq 0$*. We denote the Stokes curve* $\operatorname{Im}\left(e^{i\theta}\int_a^s \sqrt{\tilde{q}_0(s')/(s'-a)}ds'\right) = 0$ $(\theta = \arg\eta)$ *emanating from* $s = a$ *by* Γ*. Then, when we cross* Γ *in a counter-clockwise manner (viewed from* $s = a$*), we have one of the following formulas for the Borel sums of the WKB solutions (I.3.2) with* $\hat{s} = a$*:*

$$\psi_+ \mapsto \psi_+ + 2i\cos\left(\pi\sqrt{1+4\tilde{q}_2(a)}\right)\psi_-, \quad \psi_- \mapsto \psi_- \qquad \text{(I.6.3)}$$

if $\operatorname{Re}\left(e^{i\theta}\int_a^s \sqrt{\tilde{q}_0(s')/(s'-a)}ds'\right) > 0$ *on* Γ*, or*

$$\psi_+ \mapsto \psi_+, \quad \psi_- \mapsto \psi_- + 2i\cos\left(\pi\sqrt{1+4\tilde{q}_2(a)}\right)\psi_+ \qquad \text{(I.6.4)}$$

if $\operatorname{Re}\left(e^{i\theta}\int_a^s \sqrt{\tilde{q}_0(s')/(s'-a)}ds'\right) < 0$ *on* Γ*.*

Finally, we consider a ghost point. A ghost point introduced in [Ko4] is a point where $q_0(s)$ is holomorphic, while $q_1(s)$ has a simple pole and $q_2(s)$ has a double pole there. As WKB solutions are singular at a ghost point due to the singularity of q_1 and q_2, a ghost point should be considered to be a starting point of Stokes curves. To describe the connection formula in the case of a ghost point, we need some preparations. Let $s = a$ be a ghost point and $\psi_\pm$ be WKB solutions defined by

$$\psi_\pm(s,\eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}(s,\eta)}} \exp\left(\pm\eta \int_a^s \sqrt{q_0(s')}ds'\right) \\ \times \exp \int_{\hat{s}}^s \pm\left(S_{\mathrm{odd}}(s',\eta) - \eta\sqrt{q_0(s')}\right) ds'. \qquad \text{(I.6.5)}$$

Here $\hat{s} \neq a$ is a fixed point outside Stokes curves in a neighborhood of the ghost point a. As was proved in [Ko4], the WKB theoretical formal transformation to the canonical equation, i.e., the equation (I.3.1) with $q_0(s) = 1/4$, $q_1(s) = b/s$ and $q_2(s) = c/s^2$ with b and c being some appropriate formal series of η^{-1} with constant coefficients, can be constructed in a neighborhood of $s = a$. In fact, the formal transformation series $\tilde{s} = \tilde{s}(s,\eta) = \tilde{s_0}(s) + \eta^{-1}\tilde{s_1}(s) + \ldots$ is constructed so that it satisfies

$$q_0(s) + \eta^{-1}q_1(s) + \eta^{-2}q_2(s) \\ = \left(\frac{\partial\tilde{s}(s,\eta)}{\partial s}\right)^2 \left(\frac{1}{4} + \eta^{-1}\frac{b}{\tilde{s}} + \eta^{-2}\frac{c}{\tilde{s}^2}\right) - \frac{1}{2}\eta^{-2}\{\tilde{s}(s,\eta);s\} \qquad \text{(I.6.6)}$$

for $\{\tilde{s}(s,\eta);s\} = \tilde{s}'''/\tilde{s}' - (3/2)(\tilde{s}''/\tilde{s}')^2$, $' = \partial/\partial s$ where b and c are determined respectively by $b = \mathrm{Res}_{s=a}S_{\mathrm{odd}}(s,\eta)$ and $c = \tilde{q_2}(a)$ with $\tilde{q_2}(s) = (s-a)^2 q_2(s)$. As suitably normalized WKB solutions of the canonical equation, we take

$$\psi_\pm^{(\mathrm{can})}(s,\eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}^{(\mathrm{can})}(s,\eta)}} s^{\pm b} \exp\left(\pm\frac{1}{2}\eta s\right) \\ \times \exp \int_\infty^s \pm\left(S_{\mathrm{odd}}^{(\mathrm{can})}(s',\eta) - \frac{1}{2}\eta - \frac{b}{s}\right) ds'. \qquad \text{(I.6.7)}$$

We remark that the transformation series formally satisfies

$$S_{\mathrm{odd}}(s,\eta) = \frac{\partial\tilde{s}(s,\eta)}{\partial s} S_{\mathrm{odd}}^{(\mathrm{can})}(\tilde{s}(s,\eta),\eta).$$

Furthermore, with some appropriate formal series $C_\pm = C_{\pm,0} + \eta^{-1}C_{\pm,1} + \cdots$ with constant coefficients

$$\psi_\pm(s,\eta) = C_\pm \left(\frac{\partial\tilde{s}(s,\eta)}{\partial s}\right)^{-1/2} \psi_\pm^{(\mathrm{can})}(\tilde{s}(s,\eta),\eta) \qquad \text{(I.6.8)}$$

also holds.

Under these preparations, we have the following theorem.

Theorem I.6.3 (Theorem 6.3 in [Sh2], Case of ghost points; cf. Section 4 in [Ko4])

For (I.3.1), assume that $q_0(s) = \tilde{q}_0(s)$*,* $q_1(s) = \tilde{q}_1(s)/(s-a)$ *and* $q_2(s) = \tilde{q}_2(s)/(s-a)^2$ *where* $\tilde{q}_j(s)$ $(j = 0, 1, 2)$ *is holomorphic in a neighborhood* $U \subset \mathbb{C}$ *of* $s = a$*. Furthermore, we assume* $\tilde{q}_j(a) \neq 0$ $(j = 0, 1, 2)$*. We denote the Stokes curve* $\mathrm{Im}\left(e^{i\theta}\int_a^s \sqrt{\tilde{q}_0(s')}ds'\right) = 0$ $(\theta = \arg\eta)$ *emanating from* $s = a$ *by* Γ*. Let* $\psi_\pm$ *be WKB solutions* (I.6.5) *(*$\hat{s} \in U \setminus \{a\}$ *is a fixed point). Then, when crossing* Γ *in a counter-clockwise manner (viewed from* $s = a$*), we have one of the following formulas:*

$$\psi_+ \mapsto \psi_+ + \frac{2i\pi}{\Gamma(\kappa+\mu+1/2)\Gamma(\kappa-\mu+1/2)}\frac{C_+}{C_-}\eta^{2\kappa}\psi_-, \quad \psi_- \mapsto \psi_- \tag{I.6.9}$$

if $\mathrm{Re}\left(e^{i\theta}\int_a^s \sqrt{\tilde{q}_0(s')}ds'\right) > 0$ *on* Γ*, or*

$$\psi_+ \mapsto \psi_+, \quad \psi_- \mapsto \psi_- + \frac{2i\pi e^{2\pi i\kappa}}{\Gamma(-\kappa+\mu+1/2)\Gamma(-\kappa-\mu+1/2)}\frac{C_-}{C_+}\eta^{-2\kappa}\psi_+ \tag{I.6.10}$$

if $\mathrm{Re}\left(e^{i\theta}\int_a^s \sqrt{\tilde{q}_0(s')}ds'\right) < 0$ *on* Γ*, where* $\kappa = -\mathrm{Res}_{s=a}S_{\mathrm{odd}}(s,\eta)$*,* $\mu = \sqrt{1/4 + \tilde{q}_2(a)}$ *and* $C_\pm$ *are infinite series determined by* (I.6.8)*. (In the formulas* (I.6.9) *and* (I.6.10) *these series and WKB solutions are interpreted as their Borel sums.)*

I.7. Calculation of connection matrices for $\arg\eta = -\pi/4$

In this section, using the connection formulas explained in the previous section, we compute the analytic continuation of WKB solutions of (I.2.7) through the path γ for $\arg\eta = -\pi/4$. Let $M_\pm$ be

$$M_+ := \frac{2i\pi}{\Gamma(\kappa+3/4)\Gamma(\kappa+1/4)}\frac{C_+}{C_-}\eta^{2\kappa}, \tag{I.7.1}$$

$$M_- := \frac{2i\pi e^{-2\pi i\kappa}}{\Gamma(-\kappa+3/4)\Gamma(-\kappa+1/4)}\frac{C_-}{C_+}\eta^{-2\kappa}. \tag{I.7.2}$$

Here $\kappa = \kappa_0 + \eta^{-1}\kappa_1 + \cdots = -\mathrm{Res}_{s=1}S_{\mathrm{odd}}(s,\eta)$ and $C_\pm = C_{\pm,0} + \eta^{-1}C_{\pm,1} + \ldots$ is an infinite series satisfying (I.6.8). The top term $C_{\pm,0}$ is explicitly given by

$$C_{\pm,0} = \exp\left[\pm\int_{\hat{s}}^{1}\left(\frac{q_1(s')}{2\sqrt{q_0(s')}} + \kappa_0\frac{x_0{}'(s')}{x_0(s')}\right)ds'\right]x_0(\hat{s})^{\pm\kappa_0}, \qquad \text{(I.7.3)}$$

where $x_0(s) = 2\int_1^s \sqrt{9(s^2-1)/(4s(1-s^3))}ds$.

Since in the case of (I.2.7) we find $\mu = \sqrt{((s-1)^2 q_2(s))\,|_{s=1} + 1/4} = 1/4$ holds at the ghost point $s = 1$, the constant $M_\pm$ is nothing but the constant that appears in Theorem I.6.3, that is, in our case (I.6.9) and (I.6.10) are written as

$$\psi_+ \mapsto \psi_+ + M_+\psi_-, \quad \psi_- \mapsto \psi_- \qquad \text{(I.7.4)}$$

and

$$\psi_+ \mapsto \psi_+, \quad \psi_- \mapsto \psi_- + M_-\psi_+. \qquad \text{(I.7.5)}$$

Under these notations, we describe the connection formula for the case of $\arg\eta = -\pi/4$.

Theorem I.7.1 (Theorem 7.1 in [Sh2]; Connection formula for $\arg\eta = -\pi/4$**)**

We set

$$V_0 = \exp\int_{\mathrm{A}} S_{\mathrm{odd}}(s',\eta)ds',\ V_1 = \exp\int_{-1}^{0} S_{\mathrm{odd}}(s',\eta)ds',$$
$$V_2 = \exp\eta\int_0^1 \sqrt{q_0(s')}ds' \qquad \text{(I.7.6)}$$

where the branch of $\sqrt{q_0(s)}$ is determined as explained in Section I.3 (cf. Fig. 1) and $R = \exp\left(2\pi i\ \mathrm{Res}_{s=1}S_{\mathrm{odd}}(s,\eta)\right)$. Here A is a path from -1 to 0 which goes under the point $\omega^2(=\exp(4\pi i/3))$. The paths of integrals V_1 and V_2 are the intervals $[-1,0]$ and $[0,1]$. Further, we define matrices C, R_0 and C_β by $C := \begin{pmatrix} 1 & V_2^2 M_+ \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ V_2^{-2}M_- & 1\end{pmatrix}\begin{pmatrix} R & 0 \\ 0 & R^{-1}\end{pmatrix}, R_0 := \begin{pmatrix} 1 & iV_0^2 \\ 0 & 1\end{pmatrix}$ and $C_\beta = \begin{pmatrix} 1 & iV_1^2 \\ 0 & 1\end{pmatrix}$. (Here, as stated in Remark after Definition 3.1, all infinite series are interpreted as the Borel sum.) Let $\psi_\pm^{(0)}$ be WKB solutions of (I.2.7) *normalized as* (I.3.2) *with $\hat{s} = 0$. Then the analytic continuation of WKB solutions $\begin{pmatrix}\psi_+^{(0)} \\ \psi_-^{(0)}\end{pmatrix}$ along the path γ is described by*

$$R_0 C_\beta C C_\beta^{-1} R_0^{-1}\begin{pmatrix}\psi_+^{(0)} \\ \psi_-^{(0)}\end{pmatrix}. \qquad \text{(I.7.7)}$$

Proof. We first remark that the Stokes multiplier $2i\cos\left(\pi\sqrt{1+4\tilde{q}_2(a)}\right)$ in Theorem I.6.2 is zero for simple poles $a = 0, \omega, \omega^2$ in the case of (I.2.7). This means that Stokes phenomenon does not occur when γ crosses the Stokes curves emanating from $s = 0, \omega, \omega^2$.

Thus, as already explained in Section I.4, we can neglect the Stokes curves emanating from $s = 0, \omega, \omega^2$ and take account only of the crossing points of the path γ and Stokes curves emanating from $s = \pm 1$. For each crossing point, we apply Theorem I.6.1 and Theorem I.6.3.

The first crossing point is a crossing point of γ and α (cf. Fig. 8). Let $\psi_{\pm}^{(-1)}(s,\eta)$ be WKB solutions defined by (I.3.2) with $\hat{s} = -1$, where the path of integral is defined as in Fig. 10. By Theorem I.6.1, we have the relation

$$\begin{pmatrix} \psi_+^{(-1)} \\ \psi_-^{(-1)} \end{pmatrix} \mapsto \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_+^{(-1)} \\ \psi_-^{(-1)} \end{pmatrix} \tag{I.7.8}$$

when crossing α along γ. We remark that $\mathrm{Re}\left(e^{i\theta}\int_{-1}^{s}\sqrt{q_0(s')}ds'\right) > 0$ ($\theta = \arg\eta$) holds on α according to Fig. 9. Here Fig. 9 shows whether $\mathrm{Re}\left(e^{i\theta}\int_{a}^{s}\sqrt{q_0(s')}ds'\right) >$ 0 or < 0 holds on each Stokes curve; (+) means $\mathrm{Re}\left(e^{i\theta}\int_{a}^{s}\sqrt{q_0(s')}ds'\right) > 0$ holds and (−) means $\mathrm{Re}\left(e^{i\theta}\int_{a}^{s}\sqrt{q_0(s')}ds'\right) < 0$ holds. Since

$$\frac{\psi_+^{(-1)}}{\psi_+^{(0)}} = \exp\int_{-A} S_{\mathrm{odd}}(s',\eta)ds' \tag{I.7.9}$$

holds(cf. Fig. 11), we have

$$\frac{\psi_{\pm}^{(-1)}}{\psi_{\pm}^{(0)}} = V_0^{\mp 1}. \tag{I.7.10}$$

By (I.7.8) and (I.7.10), the analytic continuation of $\psi_{\pm}^{(0)}$ for the first crossing point is expressed as

$$\begin{pmatrix} \psi_+^{(0)} \\ \psi_-^{(0)} \end{pmatrix} \mapsto R_0 \begin{pmatrix} \psi_+^{(0)} \\ \psi_-^{(0)} \end{pmatrix}. \tag{I.7.11}$$

The second crossing point is a crossing point of γ and β (cf. Fig. 8). This time $\psi_{\pm}^{(-1)}(s,\eta)$ is defined by (I.3.2) with $\hat{s} = -1$ and the path of integral is defined as in Fig. 12. By Theorem I.6.1, we have the relation (I.7.8) similarly. In this case we have

$$\frac{\psi_{\pm}^{(-1)}}{\psi_{\pm}^{(0)}} = V_1^{\mp 1} \tag{I.7.12}$$

(cf. Fig. 13). By (I.7.8) and (I.7.12), the analytic continuation of $\psi_\pm^{(0)}$ for the second crossing point is expressed as

$$\begin{pmatrix} \psi_+^{(0)} \\ \psi_-^{(0)} \end{pmatrix} \mapsto C_\beta \begin{pmatrix} \psi_+^{(0)} \\ \psi_-^{(0)} \end{pmatrix}. \tag{I.7.13}$$

Next we consider the crossing point of γ with the two Stokes curves emanating from the ghost point $s = 1$. Let $\psi_\pm^{(1)}(s,\eta)$ be WKB solutions defined by (I.6.5) with $a = 1$ and $\hat{s} = 0$. By Theorem I.6.3, we have the relation

$$\begin{pmatrix} \psi_+^{(1)} \\ \psi_-^{(1)} \end{pmatrix} \mapsto \begin{pmatrix} 1 & M_+ \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_+^{(1)} \\ \psi_-^{(1)} \end{pmatrix} \tag{I.7.14}$$

at the crossing point with the first Stokes curve. Since

$$\frac{\psi_\pm^{(1)}}{\psi_\pm^{(0)}} = \exp\left(\mp\eta \int_0^1 \sqrt{q_0(s')}ds'\right) = V_2^{\mp 1} \tag{I.7.15}$$

holds, by (I.7.14) and (I.7.15) the analytic continuation of $\psi_\pm^{(0)}$ for this crossing point is expressed as

$$\begin{pmatrix} \psi_+^{(0)} \\ \psi_-^{(0)} \end{pmatrix} \mapsto \begin{pmatrix} 1 & V_2^2 M_+ \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_+^{(0)} \\ \psi_-^{(0)} \end{pmatrix}. \tag{I.7.16}$$

Similarly, at the crossing point with the second Stokes curve emanating from $s = 1$ we have

$$\begin{pmatrix} \psi_+^{(1)} \\ \psi_-^{(1)} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ M_- & 1 \end{pmatrix} \begin{pmatrix} \psi_+^{(1)} \\ \psi_-^{(1)} \end{pmatrix} \tag{I.7.17}$$

and

$$\begin{pmatrix} \psi_+^{(0)} \\ \psi_-^{(0)} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ V_2^{-2} M_- & 1 \end{pmatrix} \begin{pmatrix} \psi_+^{(0)} \\ \psi_-^{(0)} \end{pmatrix}. \tag{I.7.18}$$

Furthermore, as $S_{\text{odd}}(s,\eta)$ has the residue at $s = 1$, after crossing two Stokes curves emanating from $s = 1$ we need to multiply the matrix

$$\begin{pmatrix} R & 0 \\ 0 & R^{-1} \end{pmatrix}. \tag{I.7.19}$$

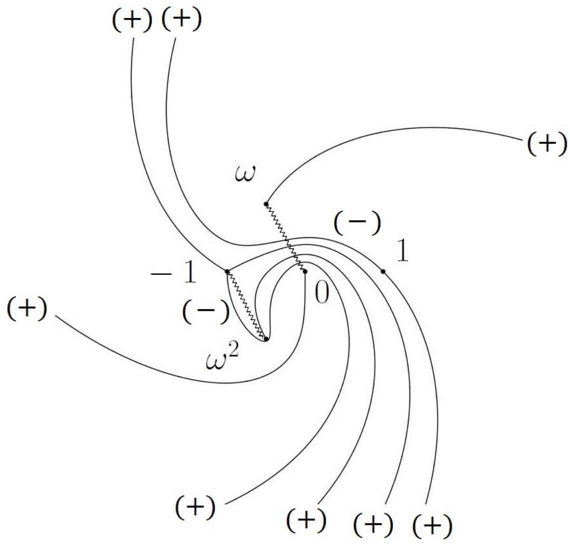

Fig. 9: signature of Stokes curves for $\arg\eta = -\pi/4$

Thus the analytic continuation of $\psi_{\pm}^{(0)}$ around the ghost point $s = 1$ is expressed as

$$\begin{pmatrix} \psi_+^{(0)} \\ \psi_-^{(0)} \end{pmatrix} \mapsto C \begin{pmatrix} \psi_+^{(0)} \\ \psi_-^{(0)} \end{pmatrix}. \tag{I.7.20}$$

Finally, when returning from $s = 1$ to $s = 0$, γ crosses the Stokes curves β and α in the opposite direction. Therefore the connection matrices become the inverse of (I.7.13) and (I.7.11), i.e., the analytic continuation of $\psi_{\pm}^{(0)}$ from $s = 1$ to $s = 0$ is expressed as

$$\begin{pmatrix} \psi_+^{(0)} \\ \psi_-^{(0)} \end{pmatrix} \mapsto C_\beta^{-1} \begin{pmatrix} \psi_+^{(0)} \\ \psi_-^{(0)} \end{pmatrix} \mapsto C_\beta^{-1} R_0^{-1} \begin{pmatrix} \psi_+^{(0)} \\ \psi_-^{(0)} \end{pmatrix}. \tag{I.7.21}$$

Combination of (I.7.11), (I.7.13), (I.7.20) and (I.7.21) completes the proof of Theorem I.7.1. □

I.8. Proof of main result

As we explained in Section I.5, the degenerations (A) and (B) occur alternately and the configuration of Stokes curves becomes more and more

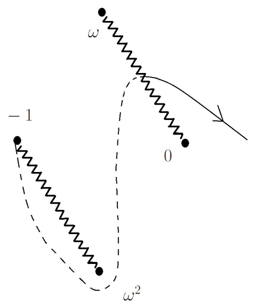

Fig. 10: path of integration for $\psi_{\pm}^{(-1)}$ in (I.7.8)

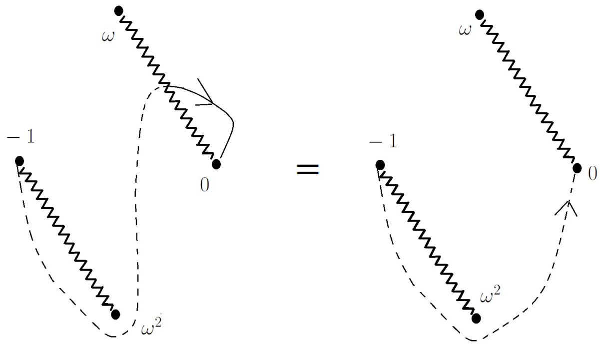

Fig. 11: path of integration for $\psi_{+}^{(-1)}/\psi_{+}^{(0)}$ in (I.7.9), which equals the opposite path of A.

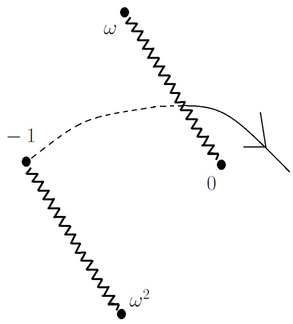

Fig. 12: path of integration for $\psi_{\pm}^{(-1)}$ at the second crossing point

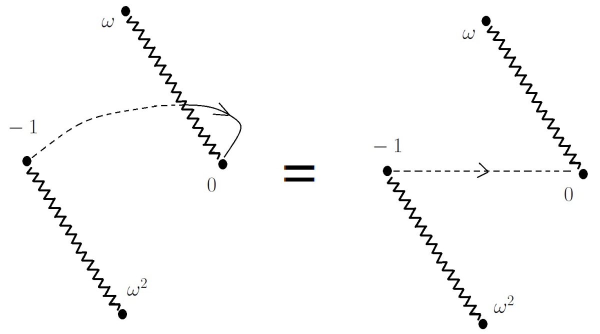

Fig. 13: path of integration for $\psi_{+}^{(-1)}/\psi_{+}^{(0)}$ in (I.7.12)

complicated as $\arg\eta$ increases from $-\pi/4$ to $-c_R\pi$. Although this observation is based on several figures of Stokes curves (cf. Figs. 6, 7, 14 and 15) and is not proved fully rigorously, it is quite reasonable to expect this observation is true. In this section, admitting this observation for Stokes curves is true, we compute the connection matrices for $-\pi/4 \le \arg\eta < -c_R\pi$.

Theorem I.8.1 (Theorem 8.1 in [Sh2]; Connection formula for $-\pi/4 \le \arg\eta < -c_R\pi$)

We use the same notation as in Theorem I.7.1. Furthermore, we set

$$V_3 = \exp\int_{\mathrm{A}'} S_{\mathrm{odd}}(s',\eta)ds', \tag{I.8.1}$$

where A' *is a path which starts from $s=0$, goes around the point $s=\omega^2$ in a clockwise direction, and returns to $s=0$. We fix the branch of $\sqrt{q_0(s)}$ as defined in Section I.3 (cf. Fig. 1). For $k=0,1,2,\ldots$, let $L_k = \begin{pmatrix} 1 & 0 \\ -i(V_0V_3^{-k})^2 & 1 \end{pmatrix}$, $R_k = \begin{pmatrix} 1 & i(V_0V_3^{-k})^2 \\ 0 & 1 \end{pmatrix}$, where V_0 and V_3 are interpreted as their Borel sums. For $m=1,2,\ldots$, we also define a matrix $C_{\alpha,m}$ by*

$$\begin{aligned} &C_{\alpha,m} = L_{(m/2)-1}R_{m/2}\cdots L_1R_{m-2}L_0R_{m-1} \ (m\text{: even}),\\ &C_{\alpha,m} = R_{(m-1)/2}L_{(m-3)/2}R_{(m+1)/2}\cdots L_1R_{m-2}L_0R_{m-1} \ (m\text{: odd}), \end{aligned} \tag{I.8.2}$$

e.g. $C_{\alpha,1}=R_0,\ C_{\alpha,2}=L_0R_1, C_{\alpha,3}=R_1L_0R_2, C_{\alpha,4}=L_1R_2L_0R_3$, etc. Then, for $-\pi/4 \le \arg\eta < -c_R\pi$, when the Stokes curve α emanating from $s=-1$ crosses the interval $\{s|0<s<1\}$ m times, the analytic continuation of $\begin{pmatrix} \psi_+^{(0)} \\ \psi_-^{(0)} \end{pmatrix}$ along the path γ is described by

$$C_{\alpha,m}C_\beta C C_\beta^{-1} C_{\alpha,m}^{-1} \begin{pmatrix} \psi_+^{(0)} \\ \psi_-^{(0)} \end{pmatrix}, \tag{I.8.3}$$

where $\psi_\pm^{(0)}$ are WKB solutions of (I.2.7) normalized as (I.3.2) with $\hat{s}=0$.

Proof. As we explained in Section I.5, when the degenerations (A) and (B) occur $(m-1)$ times, the Stokes curve α crosses the interval $[0,1]$ m times. In this situation the path γ of analytic continuation crosses the Stokes curves emanating from $s=\pm1$ in the following manner:

(1) γ first crosses the curve α m times,

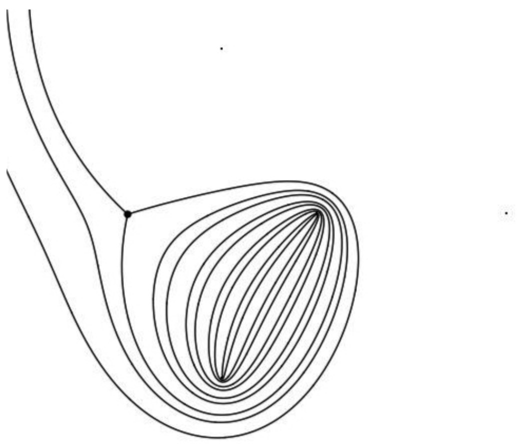

Fig. 14: Stokes curve α for $m = 8$ (even)

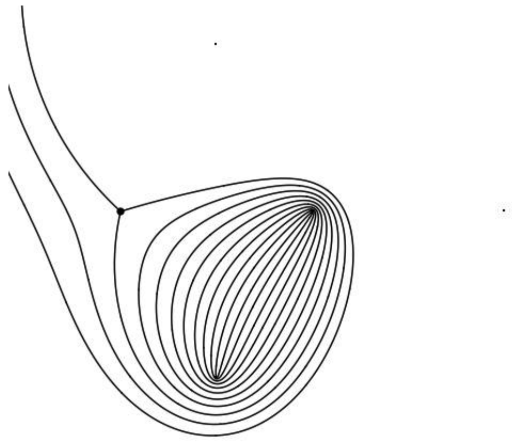

Fig. 15: Stokes curve α for $m = 9$ (odd)

(2) γ then crosses the curve β once,

(3) γ crosses two Stokes curves emanating from the ghost point $s = 1$,

(4) γ crosses β once in the opposite direction of (2), and then

(5) γ crosses α m times in the opposite direction of (1).

Here, as in the case of $\arg \eta = -\pi/4$, we neglect the Stokes curves emanating from the simple poles $s = 0, \omega, \omega^2$ as the Stokes multiplier vanishes there according to Theorem I.6.2.

We now compute a connection matrix for WKB solutions at each crossing point of γ with a Stokes curve. For the steps (2), (3) and (4) the computation is completely the same as that for $\arg \eta = -\pi/4$. Hence we only compute the connection matrices for the steps (1) and (5), that is, the connection matrices at the m crossing points of γ and α in what follows.

The configuration of the Stokes curve α emanating from -1 is illustrated in Figs. 14 and 15. In the case of $m = 2k$ (even), the curve α first approaches to 0 with rotating around 0 and ω^2 in a counter-clockwise direction k times, and consequently crossing the interval $[0, 1]$ from lower side to upper side also k times. Then α goes from upper side of 0 to lower side of ω^2 with passing through an intermediate point between 0 and ω^2. Finally, α rotates around 0 and ω^2 in a clockwise direction with crossing $[0, 1]$ from upper side to lower side k times. On the other hand, in the case of $m = 2k+1$ (odd), the curve α goes up to ω^2 with rotating around 0 and ω^2 in a counter-clockwise direction and crossing $[0,1]$ from lower side to upper side k times. Then, after turning around ω^2, α goes from lower side of ω^2 to upper side of 0 with passing between 0 and ω^2. Finally, α rotates around 0 and ω^2 in a clockwise direction with crossing $[0,1]$ from upper side to lower side $(k+1)$ times.

Now let $\psi_{\pm}^{(-1)}(s,\eta)$ be WKB solutions defined by (I.3.2) with $\hat{s}=-1$. It follows from Theorem I.6.1 that at a crossing point of γ and α where α crosses γ (or equivalently the interval [0,1]) from lower side to upper side we have

$$\begin{pmatrix} \psi_{+}^{(-1)} \\ \psi_{-}^{(-1)} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \begin{pmatrix} \psi_{+}^{(-1)} \\ \psi_{-}^{(-1)} \end{pmatrix}, \tag{I.8.4}$$

since $\operatorname{Re}\left(e^{i\theta}\int_a^s \sqrt{q_0(s')}ds'\right) < 0$ ($\theta = \arg\eta$) holds at such a crossing point. On the other hand, at a crossing point of γ and α where α crosses γ from upper side to lower side, we have

$$\begin{pmatrix} \psi_{+}^{(-1)} \\ \psi_{-}^{(-1)} \end{pmatrix} \mapsto \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_{+}^{(-1)} \\ \psi_{-}^{(-1)} \end{pmatrix}, \tag{I.8.5}$$

since $\operatorname{Re}\left(e^{i\theta}\int_a^s \sqrt{q_0(s')}ds'\right) > 0$ holds there. However, to determine the connection matrix for WKB solutions $\psi_{\pm}^{(0)}$, we have to take into account the explicit form of a relation between $\psi_{\pm}^{(0)}$ and $\psi_{\pm}^{(-1)}$, in particular, the path of integration for $\psi_{\pm}^{(-1)}$ at each crossing point.

Firstly, we consider crossing points of γ and α where α crosses γ from lower side to upper side. For the second crossing point on $[0,1](\simeq\gamma)$ from the right the path of integration for $\psi_{\pm}^{(-1)}(s,\eta)$ is a path from -1 to s near the crossing point with passing under the point $s=\omega^2$. Hence we have

$$\frac{\psi_{\pm}^{(-1)}}{\psi_{\pm}^{(0)}} = \exp \pm \int_{\mathrm{A}} S_{\mathrm{odd}}(s',\eta)ds' = V_0^{\pm 1}. \tag{I.8.6}$$

From this relation and (I.8.4), we get

$$\begin{pmatrix} \psi_{+}^{(0)} \\ \psi_{-}^{(0)} \end{pmatrix} \mapsto L_0 \begin{pmatrix} \psi_{+}^{(0)} \\ \psi_{-}^{(0)} \end{pmatrix}. \tag{I.8.7}$$

For the fourth crossing point on $[0,1]$ from the right the path of integration for $\psi_{\pm}^{(-1)}(s,\eta)$ is a path going from -1 to s with passing under ω^2 and then rotating around 0 and ω^2 once. Therefore, we have

$$\frac{\psi_{\pm}^{(-1)}}{\psi_{\pm}^{(0)}} = \exp \pm \int_{\mathrm{A}-\mathrm{A}'} S_{\mathrm{odd}}(s',\eta)ds' = (V_0 V_3^{-1})^{\pm 1} \tag{I.8.8}$$

and we get

$$\begin{pmatrix} \psi_+^{(0)} \\ \psi_-^{(0)} \end{pmatrix} \mapsto L_1 \begin{pmatrix} \psi_+^{(0)} \\ \psi_-^{(0)} \end{pmatrix}. \tag{I.8.9}$$

Similarly, we find that the connection matrix is L_{l-1} at the $(2l)$-th crossing point on $[0, 1]$ from the right for a positive integer l.

Secondly, we consider crossing points of γ and α where α crosses γ from upper side to lower side. We first consider the second crossing point on $[0, 1]$ from the left in the case of $m = 2k$ (even). As we have confirmed above, at the first crossing point from the left the connection matrix is L_{k-1} and

$$\frac{\psi_\pm^{(-1)}}{\psi_\pm^{(0)}} = \exp \pm \int_{\mathrm{A}-(k-1)\mathrm{A}'} S_{\mathrm{odd}}(s', \eta) ds' = (V_0 V_3^{-k+1})^{\pm 1}. \tag{I.8.10}$$

At the second crossing point from the left a path going around ω^2 in a clockwise direction should be added to (I.8.10). Taking the effects of the branch cuts into account, we find that the path of integration for $\psi_\pm^{(-1)}/\psi_\pm^{(0)}$ is $-\{A - (k-1)A' - A'\} = -A + kA'$ at the second crossing point from the left. Therefore we obtain that the connection matrix there is R_k. Similarly, at the $(2l+2)$-th crossing point on $[0, 1]$ from the left, the path of integration for $\psi_\pm^{(-1)}/\psi_\pm^{(0)}$ is $-A + (k + l)A'$ and the connection matrix is R_{k+l}. On the other hand, in the case of $m = 2k + 1$ (odd), it has already been confirmed that at the second crossing point from the left the connection matrix is L_{k-1} and

$$\frac{\psi_\pm^{(-1)}}{\psi_\pm^{(0)}} = \exp \pm \int_{\mathrm{A}-(k-1)\mathrm{A}'} S_{\mathrm{odd}}(s', \eta) ds' = (V_0 V_3^{-k+1})^{\pm 1}. \tag{I.8.11}$$

At the first crossing point on $[0, 1]$ from the left, a path going around ω^2 in a counter-clockwise direction should be added to (I.8.10) and, taking the effect of the branch cuts into account, we find that the path of integration for $\psi_\pm^{(-1)}/\psi_\pm^{(0)}$ is $-\{A - (k-1)A'\} + A' = -A + kA'$. Therefore, we obtain that the connection matrix there is R_k. Similarly, at the $(2l + 1)$-th crossing point from the left, the path of integration for $\psi_\pm^{(-1)}/\psi_\pm^{(0)}$ is $-A + (k + l)A'$ and the connection matrix is R_{k+l}.

Thus, if we define $C_{\alpha,m}$ by (I.8.2), the connection matrix from 0 to 1 along γ is $C_{\alpha,m}$. The connection matrix from 1 to 0 along γ for the step (5) is given by $C_{\alpha,m}^{-1}$. This completes the proof of (I.8.2). □

Using Theorem I.8.1, we can obtain a secular equation for the eigenvalue $\lambda = 1 + \eta^{-1}\tilde{\lambda}$.

Theorem I.8.2 (Theorem 8.2 in[Sh2], secular equation) *In terms of $C_{\alpha,m}, C_\beta$ and C in Theorem I.8.1, the secular equation for the eigenvalue $\lambda = 1 + \eta^{-1}\tilde{\lambda}$ can be written as*

$$\begin{pmatrix} 1 & 1 \end{pmatrix} C_{\alpha,m} C_\beta C C_\beta^{-1} C_{\alpha,m}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0. \tag{I.8.12}$$

Proof. In order to find relations between WKB solutions $\psi_\pm(s,\eta)$ and $\psi_j(s,\eta)$ $(j = 0,1)$ in Proposition I.2.2, we consider the monodromy around $s = 0$.

Let γ_0 be a path going around $s = 0$ in a counter-clockwise direction. Since no Stokes phenomenon occurs with $\psi_\pm$ on the unique Stokes curve emanating from $s = 0$ and

$$\sqrt{\eta S_{\text{odd},-1}} = \sqrt{\eta}\left(\frac{9(s^2-1)}{4s(1-s^3)}\right)^{1/4} \sim (\text{Const.})\sqrt{\eta} s^{-1/4}$$

near $s = 0$, the analytic continuation of $\psi_\pm$ along γ_0 is $\exp((1/4)\cdot 2\pi i)\psi_\mp = i\psi_\mp$. On the other hand, the analytic continuation of $\psi_0(s,\eta) \sim s^{-1/4}$ and $\psi_1(s,\eta) \sim s^{5/4}$ along γ_0 are $\exp((-1/4)2\pi i)\psi_0(s,\eta) = -i\psi_0(s,\eta)$ and $\exp((5/4)2\pi i)\psi_1(s,\eta) = i\psi_1(s,\eta)$, respectively. Therefore, if

$$\psi_1(s,\eta) = c_+\psi_+(s,\eta) + c_-\psi_-(s,\eta) \tag{I.8.13}$$

holds with some constants $c_\pm$, analytic continuation of both sides along γ_0 becomes

$$i\psi_1(s,\eta) = ic_+\psi_-(s,\eta) + ic_-\psi_+(s,\eta). \tag{I.8.14}$$

Then, comparing the right-hand side of (I.8.14) with the right-hand side of (I.8.13) multiplied by i, we find $c_+ = c_-$. By a similar argument we also find $\psi_0(s,\eta) = c'_+\psi_+(s,\eta) + c'_-\psi_-(s,\eta)$ implies $c'_+ = -c'_-$. Therefore we obtain

$$\psi_1(s,\eta) = c(\psi_+(s,\eta) + \psi_-(s,\eta)), \quad \psi_0(s,\eta) = c'(\psi_+(s,\eta) - \psi_-(s,\eta)), \tag{I.8.15}$$

where c and c' are some constants independent of s.

We set

$$C_{\alpha,m} C_\beta C C_\beta^{-1} C_{\alpha,m}^{-1} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}. \tag{I.8.16}$$

Then analytic continuation of $\psi_\pm(s,\eta)$ is written as

$$\begin{pmatrix} \psi_+(s,\eta) \\ \psi_-(s,\eta) \end{pmatrix} \mapsto \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} \psi_+(s,\eta) \\ \psi_-(s,\eta) \end{pmatrix}. \tag{I.8.17}$$

By (I.8.15) and (I.8.17) the analytic continuation of $\psi_1(s,\eta)$ along γ is written as

$$\begin{aligned}\psi_1 = c(\psi_+ + \psi_-) &\mapsto c(c_{11}+c_{21})\psi_+ + c(c_{12}+c_{22})\psi_- \\ &= (1/2)(c_{11}+c_{12}+c_{21}+c_{22})\psi_1 + (c/(2c'))(c_{11}-c_{12}+c_{21}-c_{22})\psi_0.\end{aligned}$$

Therefore, a secular equation for $\lambda = 1+\eta^{-1}\tilde{\lambda}$ can be described as

$$c_{11}-c_{12}+c_{21}-c_{22}=0.$$

The left-hand side is written as

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and hence we obtain the conclusion (I.8.12). □

Corollary I.8.3 (Corollary 8.3 in [Sh2]) *When* $-\pi/4 \le \arg\eta < -c_R\pi$ *and* $|\eta|$ *is sufficiently large,* (I.8.12) *is given by* $M_+ = 0$ *modulo exponentially small terms.*

Proof. When $|\eta|$ is large, V_0, V_1 and ${V_3}^{-1}$ defined by (I.7.6) and (I.8.1) are expressed as

$$\begin{aligned} V_0 &= \exp\left(\eta\int_{\mathrm{A}}\sqrt{q_0(s')}ds'\right)(1+O(\eta^{-1})), \\ V_1 &= \exp\left(\eta\int_{-1}^{0}\sqrt{q_0(s')}ds'\right)(1+O(\eta^{-1})), \\ V_3^{-1} &= \exp\left(-\eta\int_{\mathrm{A}'}\sqrt{q_0(s')}ds'\right)(1+O(\eta^{-1})). \end{aligned} \tag{I.8.18}$$

By numerical calculations, we have $\int_{\mathrm{A}}\sqrt{q_0(s')}ds' \sim -4.71-3.23i$, $\int_{-1}^{0}\sqrt{q_0(s')}ds' \sim -2.53$ and $-\int_{\mathrm{A}'}\sqrt{q_0(s')}ds' \sim -7.25-3.24i$. These mean that $\mathrm{Re}\left(\eta\int_{\mathrm{A}}\sqrt{q_0(s')}ds'\right)$, $\mathrm{Re}\left(\eta\int_{-1}^{0}\sqrt{q_0(s')}ds'\right)$ and $\mathrm{Re}\left(-\eta\int_{\mathrm{A}'}\sqrt{q_0(s')}ds'\right)$ are negative for $-\pi/4 \le \arg\eta < 0$ and hence we find that V_0, V_1 and ${V_3}^{-1}$ are exponentially small terms when $|\eta|$ is large. Therefore, $C_{\alpha,m}C_\beta$ and $C_\beta^{-1}C_{\alpha,m}^{-1} = (C_{\alpha,m}C_\beta)^{-1}$ are written as

$$\begin{pmatrix} 1+o(e) & o(e) \\ o(e) & 1+o(e) \end{pmatrix},$$

where $o(e)$ means exponentially small terms when $|\eta|$ is large. From this and (I.8.12), we obtain

$$\begin{pmatrix} 1 & 1 \end{pmatrix} C \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \tag{I.8.19}$$

by neglecting exponentially small terms.

On the other hand, $V_2 = \exp\left(\eta \int_0^1 \sqrt{q_0(s')}ds'\right)$ is an exponentially large term when $-\pi/4 \leq \arg\eta < -c_R\pi$ and $|\eta|$ is large because $\int_0^1 \sqrt{q_0(s')}ds' \sim 2.85i$ is obtained by numerical calculations. Since the functions R, M_+ and M_- have no exponential terms with respect to η, we can then rewrite (I.8.19) as

$$RV_2^2 M_+(1 + o(e)) = 0. \tag{I.8.20}$$

As R and V_2 are non-zero, we have $M_+ = 0$. $\square$

Main result (Theorem I.4.1) immediately follows from Corollary I.8.3: In (I.7.1), C_+/C_- is non-zero and hence we conclude $1/(\Gamma(\kappa+3/4)\Gamma(\kappa+1/4)) = 0$. Therefore (I.4.1) is proved.

I.9. Concluding remarks

In this paper, we study the secular equation for $\lambda = 1 + \eta^{-1}\tilde{\lambda}$ in the region $-\pi/4 \leq \arg\eta < -c_R\pi$. The degeneration of Stokes curves at $\arg\eta = -c_R\pi$ causes many changes of the configuration of Stokes curves when $\arg\eta$ varies from $-\pi/4$ to $-c_R\pi$ and consequently the connection formula and the secular equation becomes very much complicated (Theorems I.8.1 and I.8.2). Nevertheless, interestingly enough, the secular equation is stable modulo exponentially small terms (Theorem I.4.1) in spite of these complicated changes of the configuration of Stokes curves.

As is explained in Section I.3, the configuration of Stokes curves at $\arg\eta = 0$ is a much more degenerate configuration. It is a future problem to investigate whether the relation (I.4.1) does still hold or not when $\arg\eta$ approaches 0. Studying the structure of eigenvalues is also an important problem.

I.A. Appendix on numerical experiments

In this Appendix, we compare the results of numerical calculations of eigenvalues with our main result (I.4.1).

To this end, as we have not yet succeeded in treating the case of $\arg\eta = 0$, we need to obtain the eigenvalues numerically also for $\arg\eta \neq 0$. In [SaSu], the numerical calculations of eigenvalues are given in the case where $\eta = 1$, that is, $\arg\eta = 0$. We numerically compute eigenvalues for $\arg\eta \neq 0$ by using essentially the same method (shooting method) as in [SaSu] in Section I.A.1, and discuss how the numerical results obtained in Section I.A.1 is compatible with our main result (I.4.1) in Section I.A.2.

I.A.1. Numerical calculation of eigenvalues

For the equation (I.2.5) with the boundary condition $\phi(z) \sim O(z^0)$, we can construct the power series solution

$$\phi_1(z) = 1 + \phi_{1,1}z^{-2/3} + \phi_{1,2}z^{-4/3} + \phi_{1,3}z^{-2} + \phi_{1,4}z^{-8/3} + \cdots \qquad \text{(I.A.1)}$$

at $z = \infty$ similarly as Proposition I.2.1. If λ is an eigenvalue, the corresponding ϕ satisfies $\phi(z) \sim O(z^0)(z \to \pm\infty)$.

We use the shooting method to find approximate eigenvalues. Here we define $g(z)$ by

$$g(z) := 1 + \phi_{1,1}z^{-2/3} + \phi_{1,2}z^{-4/3} + \phi_{1,3}z^{-2} + \phi_{1,4}z^{-8/3}. \qquad \text{(I.A.2)}$$

For a fixed λ and the initial condition given by $g(10^6)$ and $g'(10^6)$, we numerically solve (I.2.5) to $-\infty$. To be more specific, we plot the difference $\phi(-5{\cdot}10^6) - \phi(-10^7)$ (Fig. 16). If λ is an eigenvalue, the corresponding ϕ satisfies $\phi(z) \sim O(z^0)(z \to -\infty)$ and hence this difference $\phi(-5\cdot10^6) - \phi(-10^7)$ is expected to be close to zero. Therefore, we pick up the value of λ satisfying $\phi(-5\cdot10^6) - \phi(-10^7) = 0$ and regard it as an eigenvalue. The list of eigenvalues calculated by Mathematica in this manner is given in Table I.A.1. (The values λ_1, λ_2 and λ_3 coincide with the result in [SaSu] and [Su].)

n	λ_n	n	λ_n
1	0.6693	7	11.797
2	1.5688	8	14.957
3	2.8743	9	18.490
4	4.5461	10	22.394
5	6.5912	11	26.670
6	9.0080		

Table 1: List of eigenvalues λ_n ($\eta = 1$)

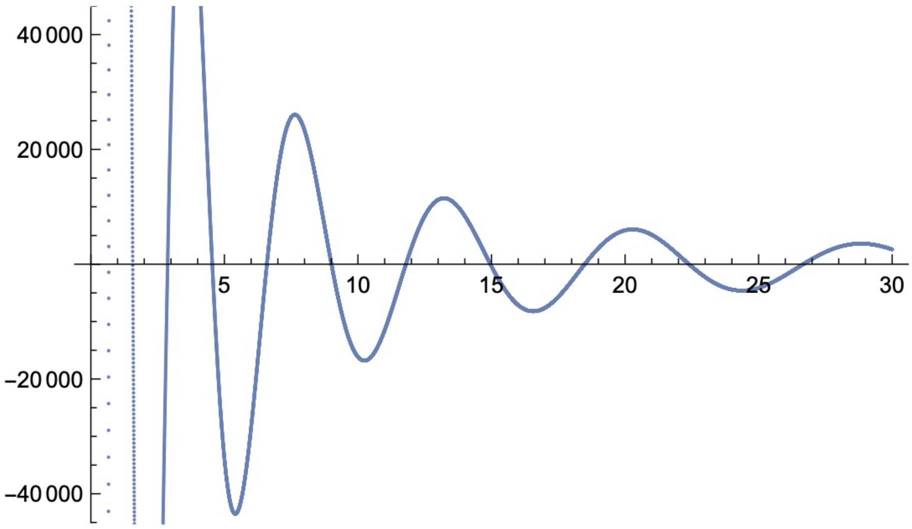

Fig. 16: graph of λ and the difference $\phi(-5 \cdot 10^6) - \phi(-10^7)$

Furthermore, we also compute eigenvalues λ satisfying $\phi(-5 \cdot 10^6) - \phi(-10^7) = 0$ for the equation

$$\phi''(z) - \eta^2 Q(z)\phi(z) = 0, \ Q(z) = \frac{1}{(1+z^2)^2} - \frac{\lambda}{(1+z^2)^{4/3}}, \tag{I.A.3}$$

which is obtained by replacing $Q(z)$ of (I.2.5) by $\eta^2 Q(z)$ with $\eta = e^{ki\pi/8}$ ($k \in \mathbb{Z}$, corresponding to (I.2.6) with $\lambda = 1 + \eta^{-1}\tilde{\lambda}$). For (I.A.3) the solutions are complex-valued and hence eigenvalues are also complex. To compute complex eigenvalues, we plot values of λ satisfying $\mathrm{Re}(\phi(-5 \cdot 10^6) - \phi(-10^7)) = 0$ and those satisfying $\mathrm{Im}(\phi(-5 \cdot 10^6) - \phi(-10^7)) = 0$ on the complex plane. We then adopt intersections of these two curves as eigenvalues (cf. Fig. 17 and Fig. 18).

The programs for graphs in Figs. 16-18 are obtainable from `http://www.math.kobe-u.ac.jp/OpenXM/Math/WKB-eigen/`

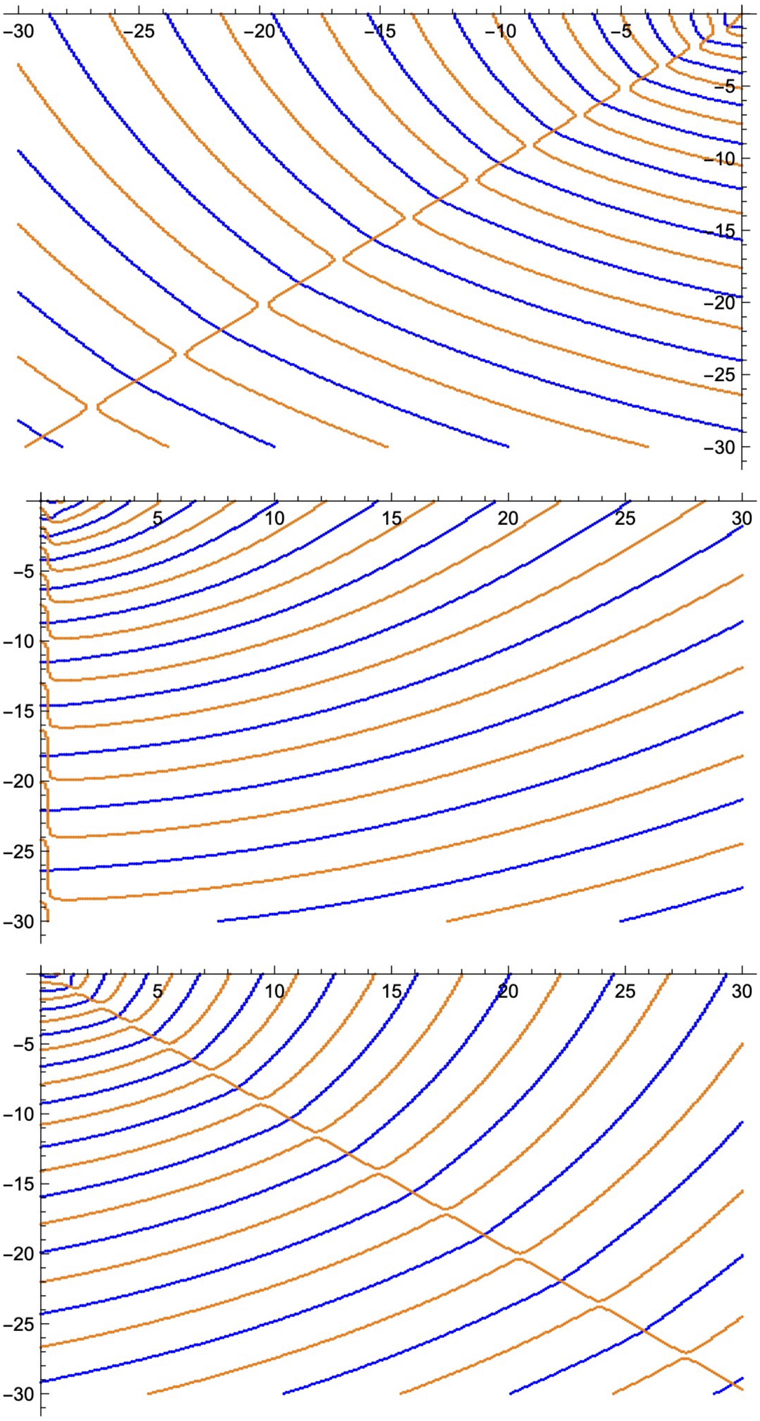

Fig. 17: graph of λ satisfying $\mathrm{Re}(\phi(-5\cdot 10^6)-\phi(-10^7))=0$ (blue curves) and those satisfying $\mathrm{Im}(\phi(-5\cdot 10^6)-\phi(-10^7))=0$ (orange curves) for $\arg\eta = 3\pi/8, \pi/4, \pi/8$

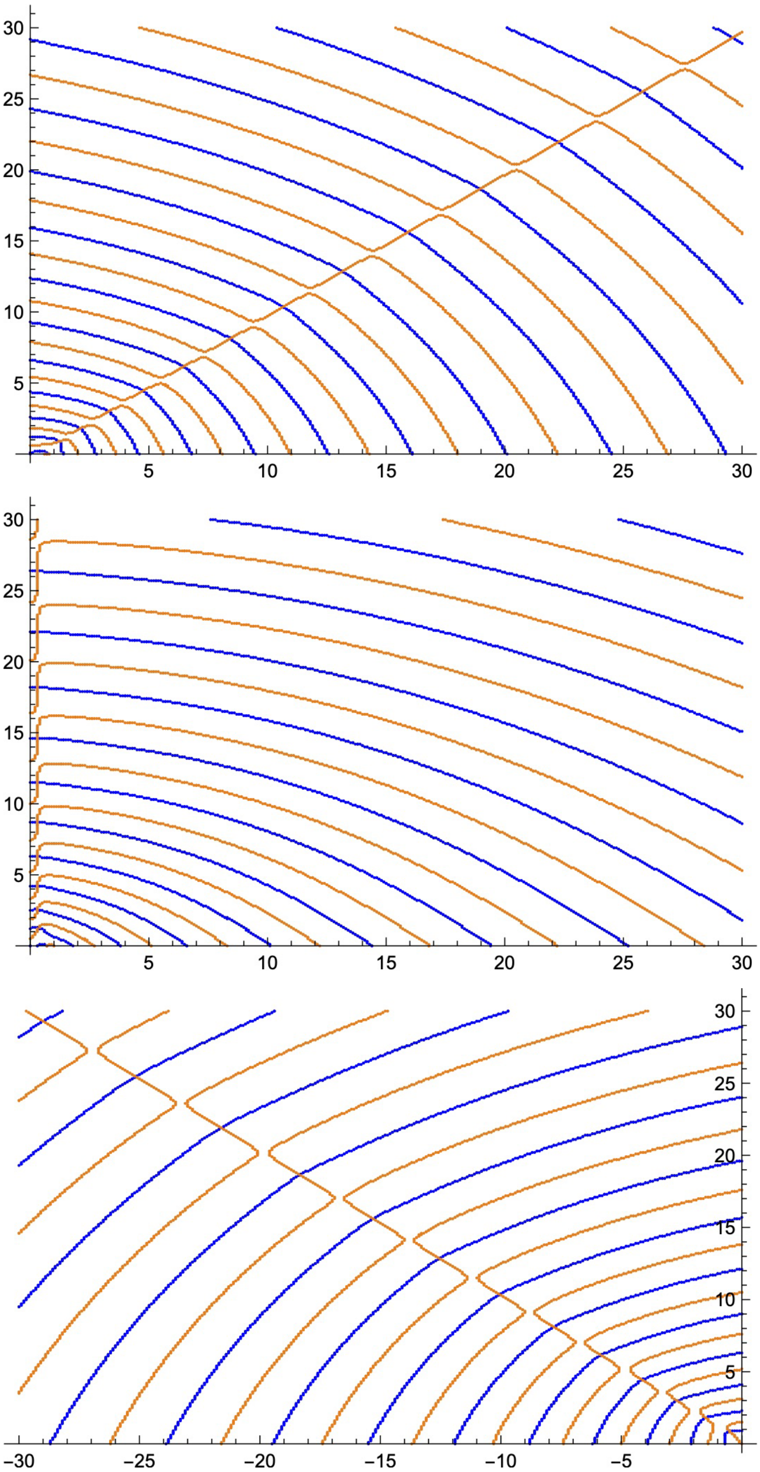

Fig. 18: graph of λ satisfying $\mathrm{Re}(\phi(-5\cdot 10^6)-\phi(-10^7))=0$ (blue curves) and those satisfying $\mathrm{Im}(\phi(-5\cdot 10^6)-\phi(-10^7))=0$ (orange curves) for $\arg\eta = -\pi/8, -\pi/4, -3\pi/8$

I.A.2. Comparison between the main result and numerical calculations

From Figs. 17 and 18 we can observe that the eigenvalues λ appear asymptotically on the line $\arg\lambda = -2\arg\eta$ as $|\lambda| \to \infty$. In this subsection we discuss how this observation derived from numerical calculations is compatible with our main result (I.4.1).

We first remark that our numerical results are concerned with the asymptotics for $|\lambda| \to \infty$ and $|\eta|$ being fixed, which is different from the asymptotics for $|\eta| \to \infty$. To discuss this new asymptotics for $|\lambda| \to \infty$ and $|\eta|$ being fixed, we need to change a large parameter from η to $\nu^2 := \eta\tilde{\lambda}$ in the differential equation (I.2.7), that is, we need to rewrite $\eta^2 q(s,\eta) = \eta^2(q_0(s) + \eta^{-1}q_1(s) + \eta^{-2}q_2(s))$ of (I.2.7) as

$$\begin{aligned}\eta^2 q(s,\eta) &= \eta^2\left(\frac{9(s^2-1)}{4s(1-s^3)} + \eta^{-1}\frac{-9\tilde{\lambda}}{4s(1-s^3)} + \eta^{-2}\frac{5-64s^3+32s^6}{16s^2(1-s^3)^2}\right)\\ &= \nu^2\left(\frac{-9}{4s(1-s^3)}\right) + \left(\eta^2\frac{9(s^2-1)}{4s(1-s^3)} + \frac{5-64s^3+32s^6}{16s^2(1-s^3)^2}\right)\\ &=: \nu^2\tilde{q}_0(s) + \tilde{q}_2(s,\eta). \end{aligned} \tag{I.A.4}$$

Since the right-hand side of (I.A.4) is quadratic with respect to ν, the logarithmic derivative of a solution of (I.2.7), i.e., a solution of the corresponding Riccati equation, should have an asymptotic expansion of the form $T_{-1}(s)\nu + T_0(s) + T_1(s)\nu^{-1} + T_2(s)\nu^{-2} + \cdots$ for large ν. In particular, we can expect that $\kappa = -\mathrm{Res}_{s=1}S_{\mathrm{odd}}(s,\eta) = -(2\pi i)^{-1}\oint_{\text{around } s=1} S_{\mathrm{odd}}(s',\eta)ds'$ also has the following asymptotic expansion for large ν:

$$\kappa \sim w_{-1}\nu + w_0 + w_1\nu^{-1} + \cdots, \tag{I.A.5}$$

where $w_{-1}, w_0, \ldots$ are constants independent of s and ν.

We now assume that w_{-1} is a real number and (I.4.1) provides the secular equation modulo exponentially small terms also for large ν, i.e., for large λ. Then, since it follows from (I.4.1) that $\kappa \sim -n$ holds with a large integer n, we obtain

$$\lambda = 1 + \eta^{-1}\tilde{\lambda} = 1 + \eta^{-2}(\eta\tilde{\lambda}) = 1 + \eta^{-2}\nu^2 \sim \eta^{-2}(\kappa/w_{-1})^2 \sim \eta^{-2}n^2/w_{-1}^2 \tag{I.A.6}$$

for large λ. As $n^2/w_{-1}^2 > 0$ by the assumption, this asymptotic relation means that $\arg\lambda = -2\arg\eta$ holds asymptotically, which coincides with the above observation derived from numerical calculations.

It does not seem easy to fix the constant w_{-1}. However, if we admit that (I.4.1) gives the secular equation modulo exponentially small terms also when $\arg\eta \sim 0$, then the relation $\kappa \sim -n$ implies $w_{-1} \sim \kappa/\nu = \kappa/\sqrt{\eta\tilde{\lambda}} \sim -n/\sqrt{\eta\tilde{\lambda}} \in \mathbb{R}$, because the eigenvalue λ and $\tilde{\lambda} = \eta(\lambda - 1)$ are expected to be positive when $\arg\eta = 0$. Hence it is reasonable to assume w_{-1} is a real number. Furthermore, the relation (I.A.6) also means that $w_{-1}^2 \sim \eta^2\lambda_n/n^2$ and our numerical calculations support that $\eta^2\lambda_n/n^2$ tends to a positive constant as $n \to \infty$ (cf. Fig. 19 and Fig. 20). It is an intriguing future problem to compute the exact value of w_{-1}.

I.B. Physical background([SaSu])

We give a summary of background of physics of the eigenvalue problem (I.1.1) - (I.1.2) in this section. In [SaSu], a kind of action functional $S(A)$ of a 1-form

$$A(x^0, x^1, x^2, x^3, z) = \sum_{\mu=0}^{3} A_\mu(x)dx^n + A_z(x)dz$$

plays a crucial role. Here A is defined on a 5-dimensional space whose metric

$$ds^2 = (1+z^2)^{2/3}(-d(x^0)^2 + d(x^1)^2 + d(x^2)^2 + d(x^3)^2) + (1+z^2)^{-2/3}dz^2$$

is different from usual $\mathbb{R}^5$. Then they assume A_μ and A_z can be expanded in terms of complete sets $\{\varphi_n(z)\}$ and $\{y_n(z)\}$:

$$A_\mu(x^0, x^1, x^2, x^3, z) = \sum_{n\geq 1} B_\mu^{(n)}(x^0, x^1, x^2, x^3)\varphi_n(z)$$

$(\mu = 0, 1, 2, 3)$ and

$$A_z(x^0, x^1, x^2, x^3, z) = \sum_{n\geq 0} C^{(n)}(x^0, x^1, x^2, x^3)y_n(z).$$

Then they take $\{\varphi_n(z)\}$ as the solution $\varphi(z) = \varphi_n(z)$ of (I.1.1) with (I.1.2). Here $y_n(z)$'s are taken as $y_n(z) = (d/dz)\varphi_n(z)$ $(n \geq 1)$ and $y_0(z) = (1+z^2)^{-1}$. By the principle of least action, which is written by the coefficient of ϵ in an expansion of $S(A+\epsilon) - S(A)$ with respect to a small ϵ, they find that $\tilde{B}_\rho^{(n)} = B_\rho^{(n)} - (\partial/\partial x_\rho)C_n$ satisfies Klein-Gordon equation. This physically means that the mass of mesons are proportional to the square root of the eigenvalue $\lambda = \lambda_n$ (cf. (7.13) in [SaSu]).

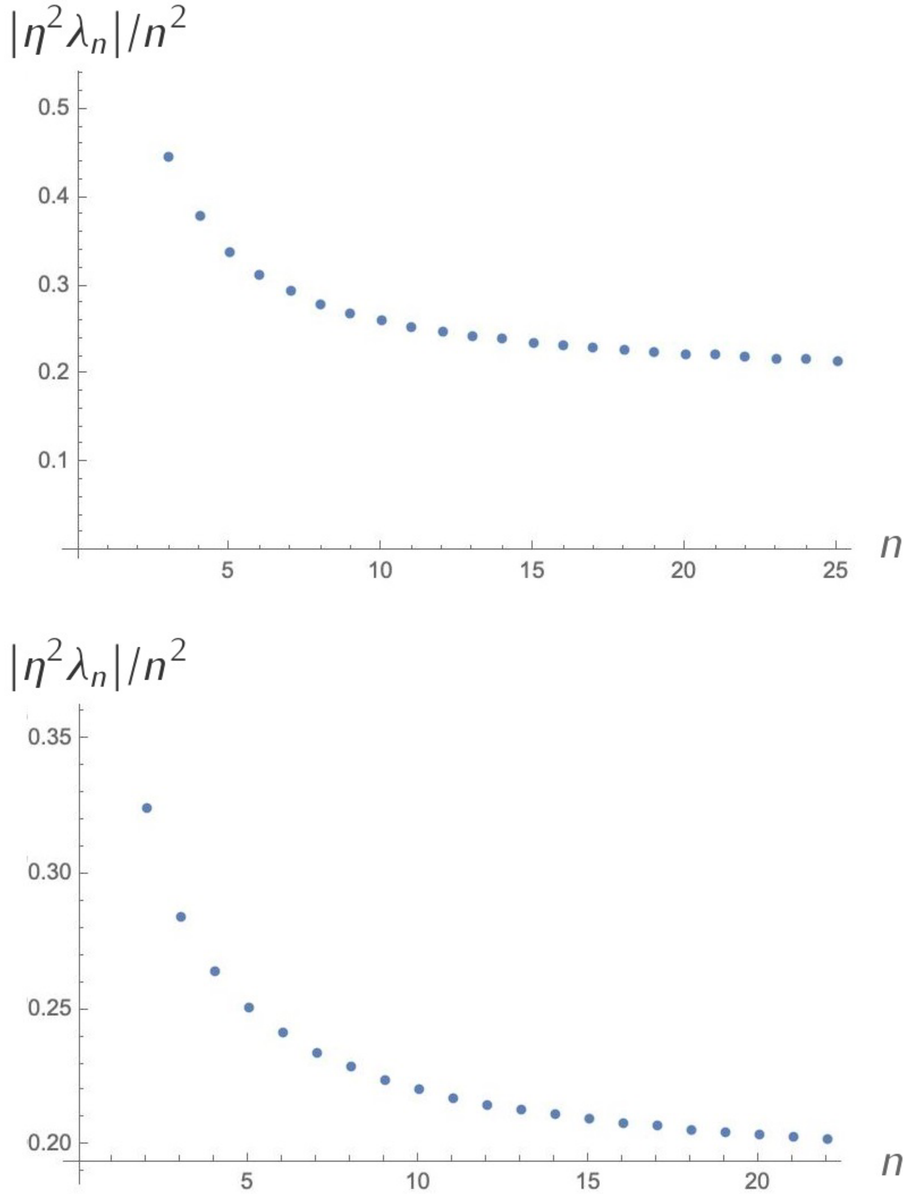

Fig. 19: graph of $|\eta^2\lambda_n|/n^2$ for $\arg\eta = -3\pi/8$ (upper), $\arg\eta = -\pi/4$ (lower)

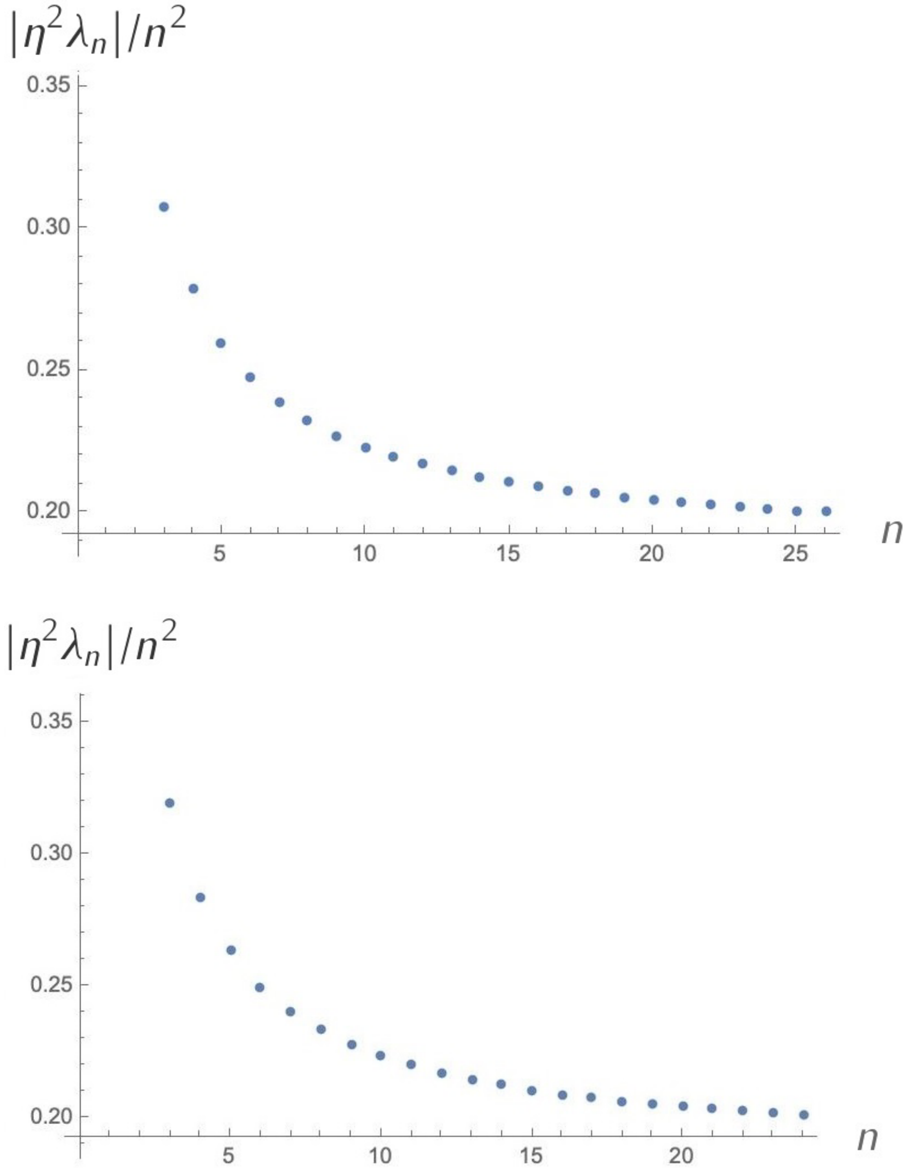

Fig. 20: graph of $|\eta^2\lambda_n|/n^2$ for $\arg\eta = -\pi/8$ (upper), $\arg\eta = 0$ (lower)

The eigenvalue has physical important meanings and the first some terms of eigenvalues for the eigenvalue problem (I.1.1) - (I.1.2) are obtained numerically.

Part II

Overview on exact WKB analysis and eigenvalue problems

II.1. Brief history of exact WKB analysis

We will give a brief overview on exact WKB analysis to clarify the position of the main part of this thesis. We try to convince readers that exact WKB method is useful to study eigenvalue problems.

In quantum mechanics, WKB method is an effective way to analyze the equation (I.3.1) (with $\hbar = 1/\eta$). After a formal solution $\psi(s,\eta) = \exp\int^s(\eta S_{-1}(s) + S_0(s) + \cdots)ds$ is constructed as described in Section I.3, one obtains physically meaningful results by using approximate solutions $\psi(s,\eta) \sim \exp\int^s(\eta S_{-1}(s) + S_0(s))ds$. The name of "WKB" is after three physicists Wentzel, Krammers and Brillouin who developed this method in 1926.

Exact WKB analysis, which is the WKB method based on Borel resummation, is initiated by Voros ([V]). By using Ecalle's "resurgent theory", Delabaere, Dillinger and Pham developed it ([DDP1],[DDP2], [DP]).

On the other hand, Aoki, Kawai and Takei interpreted the method of Voros as a tool for analyzing general differential equations in the complex domain. They succeeded in obtaining connection formulas of WKB solutions by establishing the transformation theory to the Airy equation, whose potential is $q_0(s) = s$ and $q_1(s) = q_2(s) = 0$, near a simple turning point ([AKT], see also Section 2 in [KT3]). They also give a recipe to calculate monodromy matrices of the Fuchsian equation by observing Stokes geometry and repeatedly applying connection formulas (Section 3 in [KT3]).

As to more detailed expositions on exact WKB analysis, we refer to [KT3] and [Kschool]. Several significant results have been obtained so far. For example, Koike and Schäfke proved the Borel summability of WKB solutions under assumptions that there are no degenerate Stokes curves ([KoSc]).

Koike obtained the connection formulas near a simple pole and what they call a ghost point ([Ko3],[Ko4],[Ko5]). Kamimoto and Koike proved the Borel summability of the transformation series to the Airy equation near a simple turning point ([KaKo1]). By these studies, exact WKB analysis for the second-order differential equation is amazingly developed. Recently many researchers try to extend the exact WKB analysis to higher-order equations.

Exact WKB analysis is powerful especially for eigenvalue problems. For example, Koike applied connection formulas of WKB solutions to several concrete eigenvalue problems. In [Ko1], he studied eigenvalues E for

$$\left(-\frac{d^2}{dx^2}+\eta^2(Q(x)-\eta^{-1}E)\right)\psi(x,\eta)=0,\ Q(x)=\frac{1}{4}x^2(1+e^{i\theta}x^{2N}) \quad \text{(II.1.1)}$$

$(N=1,2,\ldots)$ with the boundary condition $\psi(x,\eta)\to 0$ $(x\to\pm\infty)$. This is related to an eigenvalue problem for anharmonic oscillators. The results in [Ko1] are obtained by applying the connection formulas of WKB solutions near simple turning points and double turning points. Furthermore, in [Ko2], he studied Heun's equation

$$\left(-\frac{d^2}{dx^2}+\eta^2(Q(x)-\eta^{-2}Q_2(x))\right)\psi(x,\eta)=0 \quad \text{(II.1.2)}$$

where

$$Q_0(x)=\frac{lx-h}{(x-a_1)(x-a_2)(x-a_3)}, \quad \text{(II.1.3)}$$

$$Q_2(x)=\sum_{j=1}^{3}\frac{\alpha_j(\alpha_j-2)}{4(x-a_j)^2}+\sum_{1\le j<l\le 3}\frac{\alpha_j\alpha_l}{(x-a_j)(x-a_l)} \quad \text{(II.1.4)}$$

under the boundary condition that the solution has the form $\psi(x)=(x-a_1)^{\hat{\beta}_1}(x-a_2)^{\hat{\beta}_2}\tilde{\psi}(x)$ with $\tilde{\psi}(x)$ being holomorphic near $x=a_1$ and $x=a_2$. Then, through the connection formulas of WKB solutions near simple turning points and near simple poles, he obtained the secular equation for h and solved it asymptotically. We note that simple turning points and simple poles are regarded as starting points of Stokes curves, on which the Borel summability of WKB solutions breaks down (see Section I.3).

As observed from the above two examples, the connection formulas of WKB solutions are useful for solving eigenvalue problems. But the connection formulas for WKB solutions near ghost points are not applied yet. Then, it is natural to ask how this type of formula is used. We have shown that connection formulas near ghost points are also useful to solve eigenvalue problems in Part I of this thesis.

Exact WKB analysis for nonlinear equations has also been developed. The Riccati equation, which the logarithmic derivative of a solution of a Schrödinger equation satisfies, is a nonlinear first-order differential equation. The studies for the Schrödinger equations are equivalent to those for the Riccati equations. Aoki, Kawai and Takei applied the exact WKB method also to Painlevé equations, which are second-order nonlinear equations (see [AKT2], [KT1], [KT2] and Section 4 in [KT3]). Here Painlevé equations are regarded as the isomonodormic deformation of Schrödinger equations. Moreover Kamimoto and Koike showed Borel summability of power series solutions (so-called '0-parameter solutions') of second-order nonlinear differential equations including Painlevé equations ([KaKo2]). We are interested in applying the exact WKB analysis to eigenvalue problems of nonlinear equations. Part III is devoted to an attempt in this new research area.

II.2. Basic example of the application of connection formulas for WKB solutions to eigenvalue problems

To illustrate how exact WKB analysis is applied to eigenvalue problems, we consider a very simple example in this section, that is, the following Schrödinger equation describing a 1-dimensional harmonic oscillator:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}m\omega^2x^2\right)\tilde{u}(x)=E\tilde{u}(x). \tag{II.2.1}$$

Here m denotes the mass of a particle and ω denotes the angular frequency of the oscillator. It is well known that the energy E takes discrete values

$$E=E_n=\hbar\omega\left(n+\frac{1}{2}\right)\ (n=0,1,2,\ldots) \tag{II.2.2}$$

under the boundary condition

$$\tilde{u}(x)\to 0\ (x\to\pm\infty). \tag{II.2.3}$$

Let us derive this well-known fact by exact WKB method. By the transformation

$$z=x\sqrt{2m\omega},\ \kappa=\frac{E}{\hbar\omega},\ u(z,\eta)=\tilde{u}(x)$$

and $\eta = \hbar^{-1}$, (II.2.1) is transformed to

$$\frac{d^2}{dz^2}u(z,\eta) - \eta^2\left(\frac{z^2}{4} - \eta^{-1}\kappa\right)u(z,\eta) = 0. \tag{II.2.4}$$

We apply exact WKB analysis to (II.2.4). We note that $\eta = \hbar^{-1}$ is a positive large parameter. The WKB solutions are defined as

$$u_\pm(z,\eta) = \frac{1}{\sqrt{T_{\rm odd}}}\left(\eta^{1/2}z\right)^{\mp\kappa}\exp\pm\left\{\eta\int_0^z \frac{z}{2}dz + \int_\infty^z \left(T_{\rm odd} - \eta\frac{z}{2} + \frac{\kappa}{z}\right)dz\right\} \tag{II.2.5}$$

Here $T = T^{(\pm)}(z,\eta) = \sum_{n=-1}^{\infty}\eta^{-n}T_n^{(\pm)}(z)$ is a solution of the associated Riccati equation

$$T^2 + \frac{dT}{dz} = \eta^2\left(\frac{z^2}{4} - \eta^{-1}\kappa\right), \tag{II.2.6}$$

which is determined uniquely from $T_{-1}(z) = \pm z/2$, and $T_{\rm odd}$ is defined as $T_{\rm odd} = (T^{(+)} - T^{(-)})/2 (= \eta(z/2) - (\kappa/z) + \eta^{-1}T_{{\rm odd},1} + \cdots)$.

In this case Stokes curves consist of the real axis and the imaginary axis emanating from a double turning point $z = 0$, and hence, there are four Stokes regions. We set the right-lower region, the right-upper region and the left-upper region respectively as O, I and II. We can define two linearly independent WKB solutions on each region. The connection formulas for these WKB solutions are given as follows.

Theorem II.2.1 (Proposition 5 in [T]) *We denote the Borel sum of WKB solutions $u_\pm(z,\eta)$ in the region X by $u_\pm^{(X)}$ $(X = O, I, II)$. Then $u_\pm^{(O)}$ is analytically continued to the region I and satisfies*

$$u_+^{(O)} = u_+^{(I)} + b_{01}u_-^{(I)},\ u_-^{(O)} = u_-^{(I)}.$$

Furthermore, $u_\pm^{(I)}$ is analytically continued to the region II and satisfies

$$u_+^{(I)} = u_+^{(II)},\ u_-^{(I)} = u_-^{(II)} + b_{12}u_+^{(II)}.$$

Here

$$b_{01} = \frac{i\sqrt{2\pi}}{\Gamma(\kappa + 1/2)},\ b_{12} = \frac{i\sqrt{2\pi}e^{i\pi\kappa}}{\Gamma(-\kappa + 1/2)}.$$

In particular, Theorem II.2.1 implies that the Borel sum of the subdominant WKB solution u_- is well-defined on the positive real axis, i.e., the common

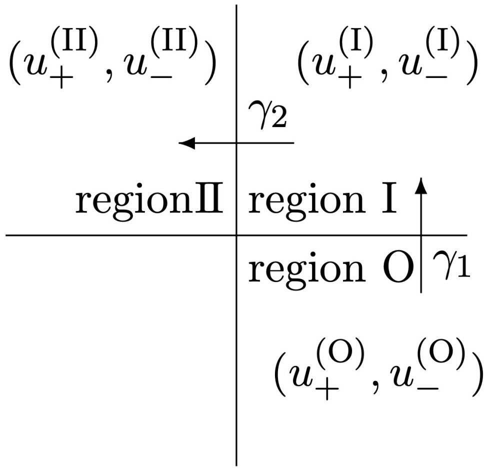

Fig. 21: Stokes curves of (II.2.4) and the path γ_j ($j = 1, 2$) of analytic continuation

boundary of the two regions O and I. Furthermore, $u_-^{(O)} (= u_-^{(I)})$ is analytically continued to the region II along γ_1 and γ_2 (cf. Fig. 21) as

$$u_-^{(O)} = u_-^{(II)} + b_{12} u_+^{(II)}. \tag{II.2.7}$$

Since we have

$$u_\pm(z, \eta) = \sqrt{\frac{2}{\eta z}} \left(\eta^{1/2} z\right)^{\mp\kappa} e^{\pm \eta z^2/4} (1 + O((\eta z^2)^{-1})),$$

$u_-^{(O)} \to 0 (z \to +\infty)$ holds on the positive real axis. Similarly, $u_-^{(II)} \to 0 (z \to -\infty)$ holds on the negative real axis. Therefore, the boundary condition (II.2.3) is satisfied if and only if $b_{12} = 0$, that is,

$$\kappa = \frac{1}{2} + n \ (n = 0, 1, 2, \ldots).$$

Since we set $\kappa = E/(\hbar\omega)$, we obtain (II.2.2) by the exact WKB analysis.

Part III
Nonlinear eigenvalue problems for a certain first order equation

III.1. Introduction to Part III

Bender, Fring and Komijani introduced *nonlinear eigenvalue problems* in [BFK]. We give a summary of [Sh1] in this part which tried to apply exact WKB analysis to the nonlinear eigenvalue problem. What is the nonlinear eigenvalue problem? As a typical example, they studied

$$y'(x) = \cos[\pi x y(x)]. \tag{III.1.1}$$

They pointed out that, for each n, the boundary condition

$$y(x) \sim \frac{m + 1/2}{x} \quad (\text{with } m = 2n - 1) \text{ as } x \to \infty \tag{III.1.2}$$

determines a unique solution of (III.1.1), and proposed to call the initial value $a_n := y(0)$ the corresponding eigenvalue. Based on the complex WKB method with some physically reasonable intuition, they derived an asymptotic behavior of eigenvalues:

$$a_n \sim 2^{5/6}\sqrt{n} \quad \text{as} \quad n \to \infty. \tag{III.1.3}$$

One of our goals is to give a mathematically rigorous proof of the formula (III.1.3) by employing exact WKB analysis (see, e.g., [KT3]). Although we have not succeeded in proving it, we obtain the following partial results.

(1) The Borel summability of the so-called 0-parameter solution of the equation (III.3.3) associated to (III.1.1) is shown.

(2) The solution of (III.1.1) which satisfies the boundary condition (III.1.2) is constructed by an exact WKB theoretic argument.

III.2. Nonlinear eigenvalue problems by Bender et al.

In [BFK], Bender, Fring and Komijani studied

$$y'(x) = \cos[\pi x y(x)] \tag{III.2.1}$$

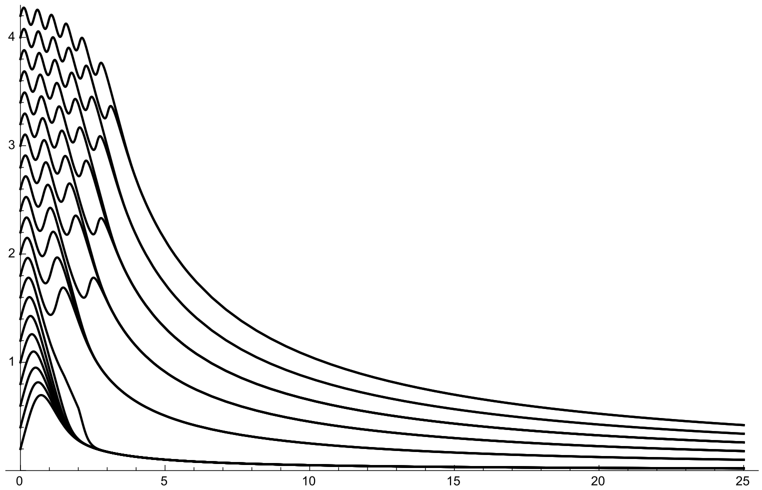

Fig. 22: Graph of the solutions of (III.2.1) with $y(0) = 0.2k$ ($k = 1, 2, \dots 21$).

as one example of nonlinear eigenvalue problems. To this equation we expect that the solution behaves like

$$y(x) \sim \frac{m + 1/2}{x} \tag{III.2.2}$$

with some integer m (so that $y'(x)$ tends to zero), as x tends to the infinity. Fig. 22 shows a result of numerical computations[1] of the initial value problem of (III.2.1). We can see from this figure that each solution approaches to a curve $xy = \text{(const)}$. An interesting observation made by [BFK] is that these asymptotic curves are $\{xy = m + 1/2\}$ with an *even* integer m (cf. Fig. 23). In fact there exists one and only one solution which satisfies (III.2.2) with an *odd* integer m.

In [BFK, §I] it is claimed that

Proposition III.2.1 (Proposition 2.1. in [Sh1]) *For any solution $y(x)$ of* (III.2.1) *with a positive initial value at the origin, there exists an positive integer m such that $y(x)$ has an asymptotic expansion*

$$y(x) \sim \frac{m + 1/2}{x} + \sum_{k=1}^{\infty} \frac{c_k}{x^{2k+1}} \tag{III.2.3}$$

as x tends to $+\infty$ along the positive real axis.

[1]Numerical computations in this section were done by Mathematica.

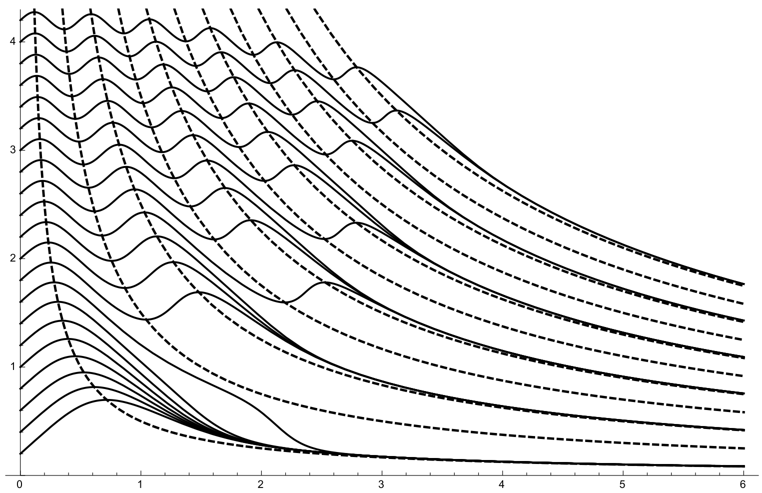

Fig. 23: The solid curves are graph of the solutions of (III.2.1) with $y(0) = 0.2k$ ($k = 1, 2, \ldots 21$). The dashed curves are $y = (m + 1/2)/x$ ($m = 0, 1, \ldots, 10$).

We can determine coefficients of the asymptotic expansion (III.2.3) uniquely and recursively by substituting (III.2.3) into (III.2.1), and comparing both sides degree by degree. First four coefficients are

$$c_1 = \frac{(-1)^m}{\pi}(m + 1/2), \tag{III.2.4}$$

$$c_2 = \frac{3}{\pi^2}(m + 1/2), \tag{III.2.5}$$

$$c_3 = (-1)^m \left[\frac{(m + 1/2)^3}{6\pi} + \frac{15(m + 1/2)}{\pi^3}\right], \tag{III.2.6}$$

$$c_4 = \frac{8(m + 1/2)^3}{3\pi^2} + \frac{105(m + 1/2)}{\pi^4}, \tag{III.2.7}$$

and, in general, we obtain

Proposition III.2.2 (Proposition 2.2. in [Sh1]) *The coefficients c_k in (III.2.3) for $k \geq 1$ are determined by the recursive relations $c_1 = ((-1)^m/\pi)c_0$ and*

$$c_k = -\frac{1}{\pi} \sum_{1 \leq l \leq (k-1)/2} \frac{(-1)^l \pi^{2l+1}}{(2l+1)!} \sum_{\substack{k_1 + \cdots + k_{2l+1} = k \\ k_1, \ldots, k_{2l+1} \geq 1}} c_{k_1} \cdots c_{k_{2l+1}} + \frac{(-1)^m (2k-1)}{\pi} c_{k-1} \tag{III.2.8}$$

for $k \geq 2$ with $c_0 = m + 1/2$.

As Figs. 22 and 23 illustrate, a solution which behaves like

$$y(x) \sim \frac{m+1/2}{x} \quad (x \to +\infty) \tag{III.2.9}$$

with $m = 2n-1$ ($n = 1, 2, \cdots$) plays a special role ([BFK], § I-A):

Proposition III.2.3 (Proposition2.4. in [Sh1]) *For $n = 1, 2, \cdots$, there exists a unique solution $y_n(x)$ of* (III.2.1) *which satisfies* (III.2.9) *with $m = 2n-1$.*

This $y_n(x)$ is called the *n-th separatrix* in [BFK], and $a_n := y_n(0)$ the *n-th eigenvalue*. To study the asymptotic behavior of the eigenvalue a_n, they introduce the scaling of variables (cf. [BFK, §III]):

$$x = \sqrt{2n-1/2}\cdot t, \;\; y(x) = \sqrt{2n-1/2}\cdot z(t). \tag{III.2.10}$$

Then (III.2.1) is transformed to

$$z'(t) = \cos[\lambda t z(t)] \tag{III.2.11}$$

with

$$\lambda = (2n-1/2)\pi. \tag{III.2.12}$$

Because $z(t)$ also depends on λ, we denote it by $z(t,\lambda)$ in the sequel. Bender-Fring-Komijani then claimed that the limit $Z(t) = \lim_{\lambda\to\infty} z(t,\lambda)$ exists, and $Z(0) = 2^{1/3}$ holds. Hence they concluded that

$$a_n = y_n(0) = \sqrt{2n-1/2}\cdot z(0,(2n-1/2)\pi) \sim 2^{1/3}\sqrt{2n} = 2^{5/6}\sqrt{n}. \tag{III.2.13}$$

holds as n tends to the infinity.

Bender-Fring-Komijani compared their results with the usual eigenvalue problem of the Schrödinger equation. See Table 2. Because of this analogy, they call a_n the eigenvalue of the problem.

III.3. Settings for exact WKB analysis

As is mentioned in Introduction, our goal is to prove the asymptotic behavior (III.2.13) of the eigenvalue a_n. Following the conventional notation of exact WKB analysis, we use η instead of λ as a large parameter (i.e., we replace λ with η in (III.2.11)). An immediate consequence of Proposition III.2.1, (III.2.10) and (III.2.12) is

	NEP	Schrödinger
Equation	$y'(x) = \cos[\pi x y(x)]$	$-\psi''(x) + x^4\psi(x) = E\psi(x)$
Boundary condition	$y(x) \sim \frac{2n-1/2}{x}$ $(x \to \infty)$	$\lim_{x\to\pm\infty} \psi(x) = 0$
Eigenvalue	$a_n = y(0)$	$E_n = E$
Asymptotic behavior $(n \to \infty)$	$a_n \sim 2^{5/6}\sqrt{n}$	$E_n \sim 3\Gamma(3/4)\sqrt{\pi}n^{4/3}/\Gamma(1/4)$

Table 2: Nonlinear eigenvalue problems (NEP) and well-known eigenvalue problems of Schrödinger equation.

Proposition III.3.1 (Proposition 3.1. in [Sh1]) *The equation* (III.2.11) *has a formal solution of the form*

$$z(t,\eta) = \frac{1}{t}(1+\eta^{-1}\tilde{u}(t,\eta)) \quad \text{with} \quad \tilde{u}(t,\eta) = \sum_{l=1}^{\infty} \tilde{c}_l(\eta)\frac{1}{t^{2l}}. \tag{III.3.1}$$

Here $\tilde{c}_l(\eta)$ is a polynomial in η^{-1} of degree $l-1$.

Noting this property, we introduce new unknown function $u(t,\eta)$ by

$$z(t,\eta) = \frac{1}{t}(1+\eta^{-1}u(t,\eta)) \tag{III.3.2}$$

to remove the term $1/t$ from $z(t,\eta)$. Then the resulting equation

$$\eta^{-1}\frac{\partial}{\partial t}u(t,\eta) = t\sin u(t,\eta) + \frac{1}{t} + \eta^{-1}\frac{u(t,\eta)}{t} \tag{III.3.3}$$

has a suitable form for the WKB analysis. We now construct the so-called 0-parameter solution (cf. [KT3]) of (III.3.3).

Proposition III.3.2 (Proposition 3.2 in [Sh1]) *The equation* (III.3.3) *has a formal power series solution*

$$\hat{u}(t,\eta) = \sum_{j=0}^{\infty} \eta^{-j}u_j(t) \tag{III.3.4}$$

with respect to η. Here $u_0(t)$ satisfies

$$\sin u_0(t) = -\frac{1}{t^2}, \tag{III.3.5}$$

and $u_j(t)$ $(j \geq 1)$ are (multi-valued) holomorphic functions in $U^* = \mathbb{C} \setminus \{0, \pm 1, \pm i\}$, which are determined uniquely and recursively once we fix a solution of (III.3.5). *Furthermore, for any compact set K in U^*, there exist positive constants A_K, C_K such that*

$$\sup_{t \in K} |u_{j+1}(t)| \leq A_K {C_K}^j j! \ (j = 0, 1, 2, \ldots). \tag{III.3.6}$$

The function $u_j(t)$ has the following expression

$$u_0(t) = -\sin^{-1}\frac{1}{t^2}, \quad u_1(t) = \frac{2}{t^4 - 1} + (-1)^N \frac{1}{\sqrt{t^4 - 1}} \sin^{-1}\frac{1}{t^2}, \tag{III.3.7}$$

$$u_{j+1}(t) = (-1)^N \frac{1}{\sqrt{t^4 - 1}} \{t u_j'(t) - u_j(t) + \Phi_{1,j}(t)\} - \Phi_{2,j}(t) \tag{III.3.8}$$

for

$$\Phi_{1,j}(t) = \sum_{1 \leq k \leq (j+1)/2} \left\{ \frac{(-1)^k}{(2k)!} \sum_{\substack{l_1 + \cdots + l_{2k} = j+1 \\ l_i \neq 0}} u_{l_1}(t) \cdots u_{l_{2k}}(t) \right\} \tag{III.3.9}$$

and

$$\Phi_{2,j}(t) = \sum_{1 \leq k \leq j/2} \left\{ \frac{(-1)^k}{(2k+1)!} \sum_{\substack{l_1 + \cdots + l_{2k+1} = j+1 \\ l_i \neq 0}} u_{l_1}(t) \cdots u_{l_{2k+1}}(t) \right\}. \tag{III.3.10}$$

It follows from (III.3.7) and (III.3.8) that each $u_j(t)$ is holomorphic except at $t \neq 0, \pm 1, \pm i$. It is also holomorphic near the infinity.

By substituting the Taylor expansion of $u_j(t)$ near $t = \infty$ into (III.3.4), $u(t, \eta)$ becomes a double power series of t^{-1} and η^{-1}.

Proposition III.3.3 (Proposition 3.3. in [Sh1]) *The solution $\hat{u}(t, \eta)$ in Proposition III.3.2 coincides with $\tilde{u}(t, \eta)$ in Proposition III.3.1 as a formal power series in t^{-1} and η^{-1}. Here the branch of $u_0(t)$ in $\hat{u}(t, \eta)$ is chosen as the principal value, that is, $N = 0$ in*

$$\sin^{-1}\frac{1}{t^2} = N\pi + (-1)^N \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 4^n (2n+1)} \left(\frac{1}{t^2}\right)^{2n+1} \tag{III.3.11}$$

for $t > 1$.

The leading order part of the linearized equation of (III.3.3) at $u_0(t)$ (i.e., an equation obtained by substituting $u(t,\eta) = u_0(t)+(\Delta u)(t,\eta)$ into (III.3.3) and eliminating both non-linear terms with respect to Δu and the lower order terms with respect to η) is

$$\eta^{-1}(\Delta u)' = t\cos\big(u_0(t)\big)\Delta u = (-1)^N \frac{\sqrt{t^4-1}}{t}\Delta u. \tag{III.3.12}$$

Here we have used

$$\cos u_0(t) = (-1)^N \sqrt{1-\sin^2 u_0(t)} = (-1)^N \frac{\sqrt{t^4-1}}{t^2}. \tag{III.3.13}$$

In analogy with the Riccati equations, we define Stokes geometry of (III.3.3) by

$$Q(t) := \left\{(-1)^N \frac{\sqrt{t^4-1}}{t}\right\}^2 = \frac{t^4-1}{t^2}, \tag{III.3.14}$$

that is, we define a **turning point** as a zero of $Q(t)$ and a **Stokes curves** as a curve emanating from a turning point a satisfying

$$\mathrm{Im}\int_a^t \sqrt{Q(s)}ds = 0. \tag{III.3.15}$$

Note that, in our case, Stokes geometry does not depend on N, i.e., a choice of a branch of $u_0(t)$, as $Q(t)$ does not depend on N. We also define a *Stokes region* as a domain surrounded by Stokes curves. Figure 24 shows the Stokes geometry of (III.3.3). There exist five Stokes regions. This Stokes geometry is degenerate in the sense that there exists Stokes curve which connect two turning points. As in the case of second order linear differential equations, the sign of $\mathrm{Re}\int_a^t \sqrt{Q(s)}ds$ does not change on a Stokes curve emanating from a turning point a (cf. [KT3]).

Let $X = \mathbb{C}\setminus\{0,\pm 1,\pm i\}$ a set of points which are not turning points nor a singular point. Following [KoSc] we introduce two notions. First one is

Definition III.3.4 *For any $t_0 \in X$, the level curve Γ_{t_0} is defined as a curve passing through t_0 and satisfying*

$$\mathrm{Im}\int_{t_0}^t \sqrt{Q(s)}ds = 0. \tag{III.3.16}$$

We also define its positive (resp., negative) component of the level curve Γ_{t_0} by

$$\Gamma_{t_0}^{(+)} := \left\{ t\in\Gamma_{t_0} \,\middle|\, \mathrm{Im}\int_{t_0}^t \sqrt{\frac{s^4-1}{s^2}}ds = 0, \mathrm{Re}\int_{t_0}^t \sqrt{\frac{s^4-1}{s^2}}ds \geq 0 \right\} \tag{III.3.17}$$

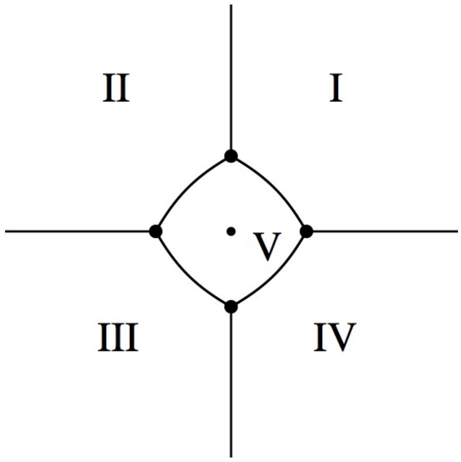

Fig. 24: Stokes curves of (III.3.3).

$$\left(resp., \quad \Gamma_{t_0}^{(-)} := \left\{ t \in \Gamma_{t_0} \,\middle|\, \mathrm{Im} \int_{t_0}^{t} \sqrt{\frac{s^4-1}{s^2}} ds = 0, \mathrm{Re} \int_{t_0}^{t} \sqrt{\frac{s^4-1}{s^2}} ds \leq 0 \right\} \right). \tag{III.3.18}$$

See Figure 25 for examples of level curves of $Q(t)$. (We choose one point from each Stokes regions, and draw a level curve passing through it.)

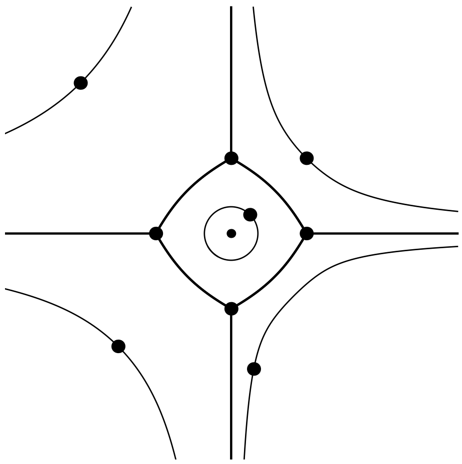

Fig. 25: Stokes curves and level curves of (III.3.3).

The second notion we introduce is

Definition III.3.5 *For a domain $\Omega \subset X$, we define a Stokes closure of Ω by*

$$\hat{\Omega} := \bigcup_{t \in \Omega} \Gamma_t. \tag{III.3.19}$$

We also define the positive (resp., negative) component of $\hat{\Omega}$ *by*

$$\hat{\Omega}^{(+)} := \bigcup_{t\in\Omega} \Gamma_t^{(+)} \quad \left(\textit{resp.,} \quad \hat{\Omega}^{(-)} := \bigcup_{t\in\Omega} \Gamma_t^{(-)} \right). \tag{III.3.20}$$

III.4. Borel summability of the 0-parameter solution

A first result on the Borel summability is

Theorem III.4.1 (Theorem 4.7. in [Sh1]) *In each Stokes region* I,II,III *or* IV *in Figure 24, the formal solution* (III.3.4) *of* (III.3.3) *is Borel summable uniformly.*

To state this theorem more precise, firstly, we fix the branch of $u_0(t)$: we place cuts as in Figure 26 and choose an integer N such that

$$u_0(t) = -\sin^{-1}(1/t^2) \to -N\pi \tag{III.4.1}$$

as t tends to the infinity along the positive real axis. Secondly we define $w(t)$ by

$$\eta^{-1} w(t,\eta) = u(t,\eta) - u_0(t) - \eta^{-1} u_1(t). \tag{III.4.2}$$

Then Theorem III.4.1 follows from

Theorem III.4.2 (Theorem 4.8. in [Sh1]) *Let* Ω *be an open or closed region in* $X = \mathbb{C} \setminus \{0, \pm 1, \pm i\}$.

(I) ***In the case when*** N ***is even***:

We further assume that (i) *the level curve* $\Gamma_{t_0}^{(+)}$ *flows into* ∞ *for each* $t_0 \in \Omega$, *and* (ii) *the* (*usual*) *closure of* $\hat{\Omega}^{(+)}$ *does not contain turning points* $\pm 1, \pm i$. *Then there exist* $\delta, B_1, B_2 > 0$ *such that the Borel transform* $w_B(t,y)$ *satisfies*

$$\left| w_B(t,y) \right| \leq \frac{B_0}{|t|^2} e^{B_1 |y|} \quad ((t,y) \in \hat{\Omega}^{(+)} \times \Sigma(\delta)). \tag{III.4.3}$$

Especially the formal solution (III.3.4) *is Borel summable uniformly in* $\hat{\Omega}^{(+)}$.

(II) ***In the case when N is odd***:

We further assume that (i) the level curve $\Gamma^{(-)}_{t_0}$ flows into ∞ for each $t_0 \in \Omega$, and (ii) the (usual) closure of $\hat{\Omega}^{(-)}$ does not contain turning points $\pm 1, \pm i$. Then there exist $\delta, B_1, B_2 > 0$ such that the Borel transform $w_B(t, y)$ satisfies

$$\left|w_B(t,y)\right| \le \frac{B_0}{|t|^2} e^{B_1|y|} \quad ((t,y) \in \hat{\Omega}^{(-)} \times \Sigma(\delta)). \tag{III.4.4}$$

Especially the formal solution (III.3.4) *is Borel summable uniformly in* $\hat{\Omega}^{(-)}$.

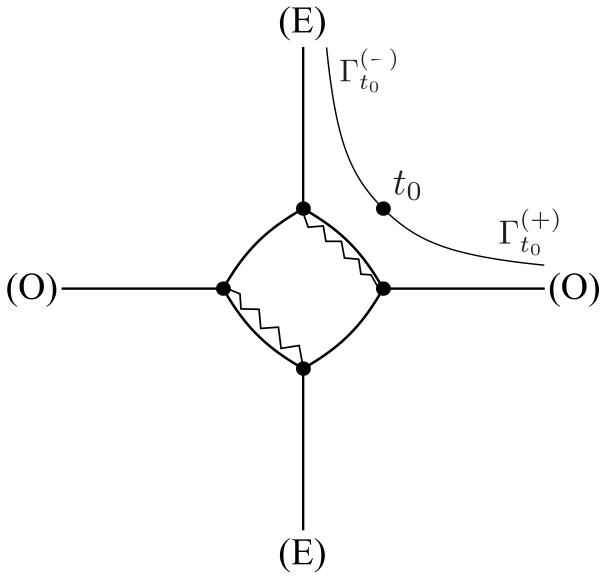

Fig. 26: The level curve Γ_{t_0} and its positive and negative component.

We can show that any compact set K included in Region I, II, III or IV satisfies the assumption of Theorem III.4.2: In fact, let us take a point t_0 from Region I. Then the positive and negative components of the level curve Γ_{t_0} run as shown in Figure 26, i.e., both of them flow into the infinity. Furthermore, because K is a compact set in Region I, its Stokes closure does not contain ± 1, $\pm i$. Thus Theorem III.4.1 follows in this case.

Even in the case of t being on Stokes curves, the formal solution (III.3.4) can be Borel summable: For example when N is even, $\Gamma^{(+)}_{t_0}$ for t_0 on the Stokes curve (O) indicated in Figure 26 flows into the infinity. Hence the Borel summability of (III.3.4) follows from Theorem III.4.2. On the other hand, when t_0 lies on the Stokes curve (E), the corresponding level curve

$\Gamma^{(+)}_{t_0}$ flows into a turning point. Therefore we cannot say anything about the Borel summability in this case from Theorem III.4.2.

III.5. Remarks

We can construct the solution of (III.2.1) satisfying the boundary condition (III.2.2) with an odd integer m from the Borel sum of the 0-parameter solution (III.3.4) of (III.3.3), which we will see below. Note that the solution $y(x)$ of (III.2.1) is transformed as

$$y(x) = \sqrt{2n-1/2}\left\{\frac{\sqrt{2n-1/2}}{x} + \frac{1}{\pi x\sqrt{2n-1/2}}\cdot u\left(\frac{x}{\sqrt{2n-1/2}}, (2n-1/2)\pi\right)\right\} \tag{III.5.1}$$

$$= \frac{2n-1/2}{x} + \frac{1}{\pi x}\cdot u\left(\frac{x}{\sqrt{2n-1/2}}, (2n-1/2)\pi\right)$$

in Section III.2 and (III.3.2).

Theorem III.5.1 (Theorem 5.1 in [Sh1]) *Let $U(t,\eta)$ be the Borel sum of $u(t,\eta)$ defined by* (III.3.4)*. Then there exist $M > 0$ and $R > 0$ such that*

$$y_n(x) = \frac{2n-1/2}{x} + \frac{1}{\pi x}\cdot U\left(\frac{x}{\sqrt{2n-1/2}}, (2n-1/2)\pi\right) \tag{III.5.2}$$

satisfies

$$\left|y_n(x) - \frac{2n-1/2}{x}\right| \le \frac{Mn}{x^3} \quad (x > Rn) \tag{III.5.3}$$

for any sufficiently large positive integer n. Here we choose a branch of $u_0(t)$ as $N = 0$ (cf. (III.4.1)*).*

Theorem III.5.1 and the uniqueness of the solution satisfying $y(x) \sim (2n-1/2)/x$ as $x \to \infty$ guarantee that (III.5.2) should coincide the n-th separatrix of (III.2.1) for sufficiently large n.

The second remark is on the Stokes geometry when $\arg\eta \neq 0$: One way to study the analytic structure of the Borel transform of 0-parameter solutions is to vary $\arg\eta$ from 0, and see what happens for such $\arg\eta$. This method is known as the "Voros' radar method", because rotating $\arg\eta$ corresponds to rotating the path of integration of the Laplace integral to define Borel sum. Stokes curves for $\arg \eta = \theta$ is defined by

$$\mathrm{Im}\left(e^{i\theta}\int_a^t \sqrt{Q(s)}ds\right) = \mathrm{Im}\left(e^{i\theta}\int_a^t \sqrt{\frac{s^4-1}{s^2}}ds\right) = 0, \qquad \text{(III.5.4)}$$

where a is a turning point (i.e., a zero of Q). The level curves and their positive or negative components are defined similarly. See Figure 27 for the Stokes curve when $\theta = k\pi/8$ $(-4 \le k \le 4)$. As these figures show, the degeneration of the Stokes geometry for $\theta = 0$ is resolved for $-\pi/2 \le \theta < 0$ and for $0 < \theta \le \pi/2$ (actually the degeneration of Stokes geometry occurs only when $\theta = 0 \mod \pi$.).

For $\arg \eta \ne 0 \mod \pi$, i.e., when the degeneration is resolved, Theorem III.4.2 gives only a partial answer to the Borel summability of the 0-parameter solutions. To make the argument concrete, we consider the case when $\theta = -\pi/4$. In this case Stokes regions consist of 8 regions (Fig. 28). All of the level curves passing through a point in Regions I, II, III and IV in Fig. 28 flow into the infinity, and we can show the Borel summability of the 0-parameter solutions by the same argument which gives Theorem III.4.2. If we choose a point from Region V, VI, VII or VIII, however, one end of the level curve passing through it flow into the origin, as shown in Fig. 29.

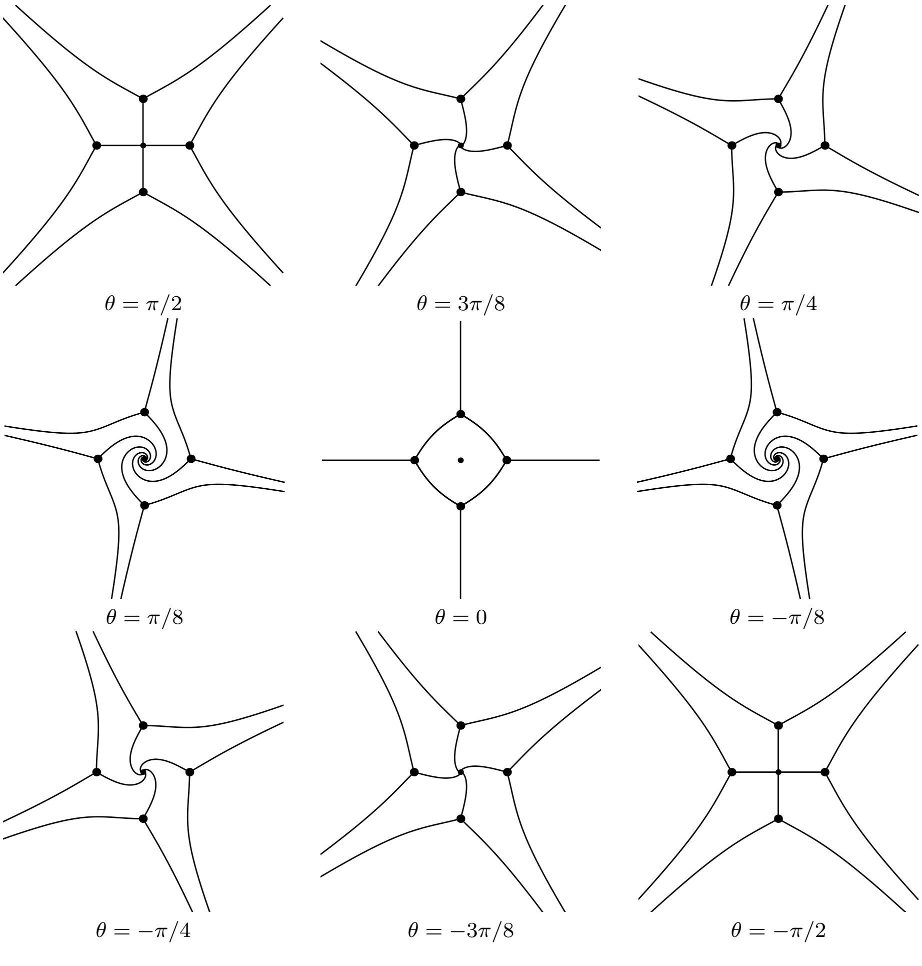

Fig. 27: Stokes curves for $\theta = \pi/2, 3\pi/8, \pi/4, \pi/8, 0, -\pi/8, -\pi/4, -3\pi/8, -\pi/2$ ($\arg \eta = \theta$).

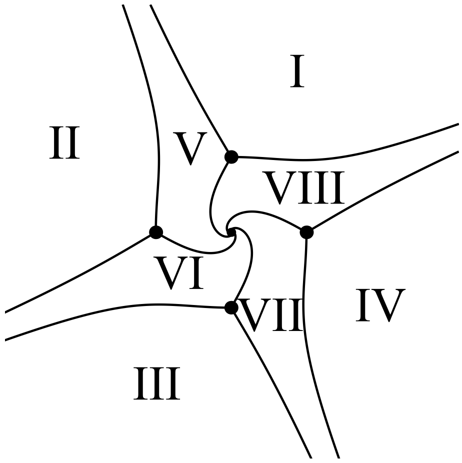

Fig. 28: Stokes regions for $\theta = -\pi/4$.

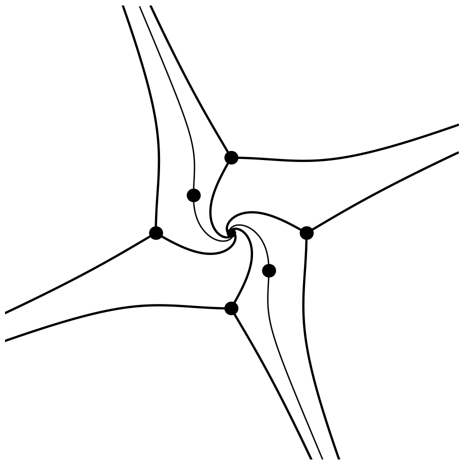

Fig. 29: The level curve for $t_0 \in$V and VII.

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