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(Citation)
Journal of Differential Equations,347:24-55

(Issue Date)
2023-02-25

(Resource Type)
journal article

(Version)
Accepted Manuscript

LACK OF THE STRICT DISSIPATIVITY AND MODIFICATION FOR THE DISSIPATIVE BRESSE SYSTEM

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Abstract. Recently a lot of physical models which possess a complicated dissipative structure are studied. The key point is to analyze the eigenvalues of the corresponding eigenvalue problem for physical models, and the stability condition introduced in [11, 12] is one of the important tools to get the detailed information of the eigenvalues. Throughout the stability condition, we find that the Bresse system is not strictly dissipative even if it has any frictional damping. In this situation, we introduce the different types of damping effects and analyze the detailed dissipative structure for the Bresse system.

1. Introduction

In the paper, we consider the Bresse system with frictional damping terms

$$\begin{aligned} \rho_1\phi_{tt} - \kappa(\phi_x + \psi + \ell w)_x - \kappa_0\ell(w_x - \ell\phi) + \alpha\phi_t = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\phi_x + \psi + \ell w) + \beta\psi_t = 0, \\ \rho_1 w_{tt} - \kappa_0(w_x - \ell\phi)_x + \kappa\ell(\phi_x + \psi + \ell w) + \gamma w_t = 0, \end{aligned} \tag{1.1}$$

where $\phi = \phi(t,x)$, $\psi = \psi(t,x)$ and $w = w(t,x)$ are unknown scalar function, which denote the longitudinal displacement, the vertical displacement of the beam and the rotation angle of the linear filaments material, respectively. Assume that b, κ and κ_0 are positive constants, ℓ is a non-zero constant, and α, β and γ are non-negative constants. These constants are physically described as $\rho_1 = \rho A_0$, $\rho_2 = \rho_0 I_0$, $\kappa_0 = E_0 A_0$, $\kappa = k' G_0 A_0$, $b = E_0 I_0$ and $\ell = R_0^{-1}$, where ρ_0 is the density, E_0 is the modulus of elasticity, G_0 is the shear modulus, k' is the shear factor, A_0 is the cross-sectional area, I_0 is the second moment of area of the cross section, R_0 is the radius of curvature. Furthermore, the constants α, β, and γ denote the coefficients of the frictional damping. We note that the Bresse system (1.1) is decomposed to the Timoshenko system and the wave equation if ℓ is taken to zero.

The Bresse system is known as the circular problem described in the book of Lagnese–Leugering–Schmidt [7]. In recent years, there are several papers considered the asymptotic behavior of solutions to the Bresse system in a bounded domain. For example, Liu–Rao [8] and Fatori–Rivera [4] studied the decay estimate of the solutions to the thermoelastic Bresse system, and Santos–Almeida Júnior [9], Boussouira–Rivera–Almeida Júnior [1] and Alves–Fatori–Silva–Monteiro [2] analyzed the Bresse system with frictional damping. More precisely, the authors in [9] treated (1.1) with positive α, β and γ, and derived the exponential stability. On the other hand, in [2], (1.1) with positive α and β is considered, and the exponential stability is obtained

Date: February 15, 2023.

under the assumption $\kappa = \kappa_0$. Furthermore, the authors in [1] studied (1.1) with positive β, and the exponential stability is also obtained under the assumption $\rho_1 b = \rho_2 \kappa$ and $\kappa = \kappa_0$.

In a whole space, Ide–Haramoto–Kawashima [5] studied the dissipative structure for the dissipative Timoshenko system in detail and obtained the optimal decay estimate. In [11], the stability condition was introduced to analyze the dissipative structure for the system of linear differential equations. and obtained the weak dissipative structure for the dissipative Bresse system (1.1) with $\alpha = \gamma = 0$. More precisely, the dissipative structure of (1.1) is stated as follows.

Theorem 1.1. *The dissipative Bresse system* (1.1) *does not satisfy Condition* (SC)*. Therefore, this system is not strictly dissipative.*

Remark 1. *Theorem 1.1 tells us that the dissipative structure of* (1.1) *with infinite length is completely different from the one with finite length. Indeed, there is no possibility to get the standard decay estimate in a whole space. However, the authors in* [1, 2, 9] *obtained the exponential stability in a bounded domain under the suitable assumption. These structures come from the corresponding eigenvalues in the middle and high frequency regions.*

Condition (SC) and the definition of the strict dissipativity are mentioned in Section 2. This result tells us that any frictional damping is not enough to derive the strict dissipativity. Therefore, it is worth considering the different types of damping effects. Inspired by this situation, we replace the frictional damping ψ_t with the memory type damping $-g * \psi_{txx}$ in the second equation of (1.1). Precisely, we introduce the following modification for the Bresse system.

$$
\begin{aligned}
\rho_1 \phi_{tt} - \kappa(\phi_x + \psi + \ell w)_x - \kappa_0 \ell (w_x - \ell \phi) + \alpha \phi_t = 0, \\
\rho_2 \psi_{tt} - b g * \psi_{txx} + \kappa(\phi_x + \psi + \ell w) = 0, \\
\rho_1 w_{tt} - \kappa_0 (w_x - \ell \phi)_x + \kappa \ell (\phi_x + \psi + \ell w) + \gamma w_t = 0,
\end{aligned}
\tag{1.2}
$$

where $g(t) = e^{-\delta t}$ with $\delta > 0$. Here $(g * f)$ denotes a convolution defined by

$$
(g * f)(t) = \int_0^t g(t - \tau) f(\tau) d\tau.
$$

Notice that $(g * f')(t) = g(0) f(t) - g(t) f(0) + (g' * f)(t)$, and hence

$$
(g * \psi_{txx})(t) = \psi_{xx}(t) - e^{-\delta t} \psi_{xx}(0) - \delta (g * \psi_{xx})(t).
$$

Namely, ψ_{xx} will be close to $g * \psi_{txx} + \psi_{xx}|_{t=0}$ if δ is sufficiently small.

We also apply the stability condition and obtain the following result for (1.2).

Theorem 1.2. *Let $\alpha \geq 0$, $\gamma \geq 0$ and $\alpha + \gamma > 0$. Then the modified Bresse system* (1.2) *satisfies Condition* (SC)*, and it is strictly dissipative. On the other hand, let $\alpha = \gamma = 0$. Then the modified Bresse system* (1.2) *does not satisfy Condition* (SC)*, and it is not strictly dissipative.*

Theorem 1.1 and Theorem 1.2 are derived by Theorem 2.2 in Section 2. Although Theorem 2.2 gives the polynomial decay estimate of the solution to (1.2), it does not guarantee the optimality of the decay estimates. In order to obtain the sharp decay estimate, we employ the energy method in Fourier space for (1.2) and discuss the optimality for this decay estimate by using the semigroup argument.

The contents of this paper are as follows. We treat the general symmetric hyperbolic system and discuss the dissipative structure in Section 2. In this section, we introduce the stability conditions which are useful conditions to analyze the structure of physical models. Furthermore, we give the proofs of Theorem 1.1 and Theorem 1.2. In Section 3, we focus on the modified Bresse system (1.2) and derive the decay estimate. The important part is to construct the dissipation terms for the energy estimate by the energy methods. Finally, in Section 4, we discuss the optimality of the decay estimate derived in Section 3. To this end, we analyze the behavior of the eigenvalues of the corresponding eigenvalue problem. Then we deduce that the regularity-loss structure is an essential property for (1.2).

In the last of this section, we mention some notations which we use in this paper. Let $\mathcal{F}[f]$ be the Fourier transform of f, which is defined by

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}dx,$$

and the corresponding inverse transform is defined by $\mathcal{F}^{-1}$. For $1 \le \theta \le \infty$, $L^\theta = L^\theta(\mathbb{R})$ denotes the usual Lebesgue space with the norm $\|\cdot\|_{L^\theta}$. $L^1_1 = L^1_1(\mathbb{R})$ means the weighted L^1 space with the norm

$$\|f\|_{L^1_1} := \int_{\mathbb{R}} (1+|x|)|f(x)|dx.$$

For σ be a non-negative integer, $H^\sigma = H^\sigma(\mathbb{R})$ denotes the usual Sobolev space associated with $L^2(\mathbb{R})$ space. The corresponding norm is

$$\|f\|_{H^\sigma} := \Big(\sum_{k=0}^{\sigma} \|\partial_x^k f\|_{L^2}^2\Big)^{\frac{1}{2}}.$$

Finally, we regard every positive constant by the same symbol C or c if no confusion occurs.

2. Application of a general theory to the Bresse system

In order to analyze the dissipative structure for (1.1), we recall the general theory in this section. We study the symmetric hyperbolic system with relaxation

$$A^0u_t + Au_x + Lu = 0, \tag{2.1}$$

where $u = u(t,x)$ over $t > 0$, $x \in \mathbb{R}$ is an unknown vector function, and A^0, A and L are $m \times m$ constant matrices for $m \ge 2$. For the system (2.1), the initial data

$$u(0,x) = u_0(x) \tag{2.2}$$

for $x \in \mathbb{R}$ is assigned. Here we assume the following for the coefficient matrices.

Condition (A): A^0 is real symmetric and positive definite, A is real symmetric, while L is not necessarily real symmetric but $L^\sharp$ is non-negative definite with the non-trivial kernel.

Namely, Condition (A) means that the constant matrices satisfy the followings.

$$\begin{aligned}&(A^j)^\top = A^j \quad (0 \le j \le n),\\ &A^0 > 0, \quad L^\sharp \ge 0 \quad \text{on} \quad \mathbb{R}^m,\\ &\mathrm{Ker}(L) \ne \{0\}.\end{aligned}$$

Here and in the sequel, the superscript $\top$ stands for the transposition, and $X^\sharp$ and $X^\flat$ denote the symmetric and skew-symmetric part of the matrix X, respectively. That is $X^\sharp := (X + X^\top)/2$ and $X^\flat := (X - X^\top)/2$.

Applying the Fourier transform to (2.1), we obtain

$$A^0\hat{u}_t + i\xi A\hat{u} + L\hat{u} = 0, \tag{2.3}$$

and the solution of (2.3) is written as

$$\hat{u}(t,\xi) = e^{t\hat{\Phi}(i\xi)}\hat{u}_0(\xi),$$

where $\hat{u}_0$ is the Fourier transform of the initial data u_0, and

$$\hat{\Phi}(\zeta) := -(A^0)^{-1}(\zeta A + L). \tag{2.4}$$

Then we define the semigroup $e^{t\Phi}$ by the formula

$$e^{t\Phi}\varphi := \mathcal{F}^{-1}[e^{t\hat{\Phi}(i\xi)}\hat{\varphi}(\xi)], \tag{2.5}$$

and the solution to (2.1) is described as $u(t,x) = (e^{t\Phi}u_0)(x)$. We note that the corresponding eigenvalue problem is

$$\lambda\varphi + (A^0)^{-1}(\zeta A + L)\varphi = 0 \tag{2.6}$$

with $\zeta = i\xi$, where $(\lambda, \varphi) \in \mathbb{C} \times \mathbb{C}^m \backslash \{0\}$ is a pair of the eigenvalue and the eigenvector of (2.6).

To discuss the dissipative structure, we define the strict and uniform dissipativity for the system (2.1).

Definition 2.1. (Strict and uniform dissipativity ([13])) (i) *The system* (2.1) *is called strictly dissipative if real parts of all the eigenvalues of* (2.6) *are negative for each* $\xi \in \mathbb{R}\backslash\{0\}$. (ii) *The system* (2.1) *is called uniformly dissipative of the type* (σ_1, σ_2) *if all the eigenvalues* $\lambda(i\xi)$ *of* (2.6) *satisfy*

$$\mathrm{Re}\lambda(i\xi) \le -c\frac{\xi^{2\sigma_1}}{(1+\xi^2)^{\sigma_2}},$$

where c *is a certain positive constant and* (σ_1, σ_2) *is a pair of non-negative integers.*

Ueda [11] introduced the stability conditions for (2.1). Here we use notations that $\mathbb{R}_+ := (0, \infty)$ and $\mathcal{A}(\nu) := (A^0)^{-1}(A - i\nu^{-1}L^\flat)$.

Stability Condition (SC):

$$\mathrm{Ker}(\mu I + \mathcal{A}(\nu)) \cap \mathrm{Ker}(L^\sharp) = \{0\}, \qquad (\mu, \nu) \in \mathbb{R} \times \mathbb{R}_+.$$

Kalman Rank Condition (R):

$$\operatorname{rank}\begin{pmatrix} L^\sharp \\ L^\sharp \mathcal{A}(\nu) \\ \vdots \\ L^\sharp \mathcal{A}(\nu)^{m-1} \end{pmatrix} = m, \qquad \nu \in \mathbb{R}_+.$$

Remark that the Kalman rank condition is first applied to the symmetric hyperbolic system with symmetric relaxation in Beauchard-Zuazua [3], and Ueda [11, 12] extended it for the symmetric hyperbolic system with non-symmetric relaxation. Precisely, the following relation to Stability Condition (SC) and Kalman Rank Condition (R) is derived in [11, 12].

Theorem 2.2. ([11, 12]) *Suppose the system* (2.1) *satisfies Condition* (A). *Then, for the system* (2.1), *the following conditions are equivalent.*
(i) *Stability Condition* (SC) *holds.*
(ii) *Kalman Rank Condition* (R) *holds.*
(iii) *System* (2.1) *is strictly dissipative.*
(iv) *System* (2.1) *is uniformly dissipative of the type* $(m-1, 2(m-1))$.

Corollary 2.3. ([12]) *Suppose the system* (2.1) *satisfies Condition* (A) *and the condition* (iv) *appeared in Theorem 2.2. Then the solution operator in Fourier space satisfies the pointwise estimate* $|e^{t\hat{\Phi}(i\xi)}| \le Ce^{-c\rho(\xi)t}$, *where*

$$\rho(\xi) = \frac{\xi^{2(m-1)}}{(1+\xi^2)^{2(m-1)}}.$$

Furthermore, the corresponding solution to (2.1), (2.2) *satisfies the decay estimate*

$$\|\partial_x^j u(t)\|_{L^2} \le C(1+t)^{-\frac{1+2j}{4(m-1)}}\|u_0\|_{L^1} + C(1+t)^{-\frac{k}{2(m-1)}}\|\partial_x^{j+k}u_0\|_{L^2}$$

for $j \ge 0$ *and* $k \ge 0$. *Here* c *and* C *are certain positive constants.*

To apply Theorem 2.2 to the system (1.1), we introduce new functions that

$$\begin{aligned} &v := \kappa(\phi_x + \psi + \ell w), \qquad s := \phi_t, \qquad z := b\psi_x, \\ &y := \psi_t, \qquad p := \kappa_0(w_x - \ell\phi), \qquad q := w_t, \end{aligned}$$

then (1.1) is rewritten as

$$\begin{aligned} \kappa^{-1}v_t - s_x - y - \ell q &= 0, \\ \rho_1 s_t - v_x - \ell p + \alpha s &= 0, \\ b^{-1}z_t - y_x &= 0, \\ \rho_2 y_t - z_x + v + \beta y &= 0, \\ \kappa_0^{-1}p_t - q_x + \ell s &= 0, \\ \rho_1 q_t - p_x + \ell v + \gamma q &= 0. \end{aligned} \tag{2.7}$$

Namely the system (2.7) is described as (2.1), where $u = (v, s, z, y, p, q)^\top$, and the matrices A^0, A and L are defined by

$$A^0 = \begin{pmatrix} 1/\kappa & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/b & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\kappa_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_1 \end{pmatrix}, \quad A = -\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & -\ell \\ 0 & \alpha & 0 & 0 & -\ell & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \beta & 0 & 0 \\ 0 & \ell & 0 & 0 & 0 & 0 \\ \ell & 0 & 0 & 0 & 0 & \gamma \end{pmatrix}.$$

Here the size of the system is $m = 6$. Notice that the relaxation matrix L is not symmetric, and it is decomposed as $L = L^\sharp + L^\flat$ with

$$L^\sharp = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma \end{pmatrix}, \qquad L^\flat = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & -\ell \\ 0 & 0 & 0 & 0 & -\ell & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ell & 0 & 0 & 0 & 0 \\ \ell & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since these matrices satisfy Condition (A), we can apply Theorem 2.2 to the system (2.7) and get Theorem 1.1.

Proof of Theorem 1.1. It is enough to consider the case $\alpha > 0$, $\beta > 0$ and $\gamma > 0$. If $(\mu, \nu) \in \mathbb{R} \times \mathbb{R}_+$ and $\varphi = (\varphi_1, \cdots, \varphi_6)^\top \in \mathbb{C}^6$ satisfy

$$\text{(2.8)} \qquad \begin{cases} \mu\kappa^{-1}\varphi_1 - \varphi_2 + i\nu^{-1}\varphi_4 + i\nu^{-1}\ell\varphi_6 = 0, \\ \mu\rho_1\varphi_2 - \varphi_1 + i\nu^{-1}\ell\varphi_5 = 0, \\ \mu b^{-1}\varphi_3 - \varphi_4 = 0, \\ \mu\rho_2\varphi_4 - \varphi_3 - i\nu^{-1}\varphi_1 = 0, \\ \mu\kappa_0^{-1}\varphi_5 - \varphi_6 - i\nu^{-1}\ell\varphi_2 = 0, \\ \mu\rho_1\varphi_6 - \varphi_5 - i\nu^{-1}\ell\varphi_1 = 0, \end{cases} \quad \text{and} \quad \begin{cases} \alpha\varphi_2 = 0, \\ \beta\varphi_4 = 0, \\ \gamma\varphi_6 = 0, \end{cases}$$

then (2.8) is rewritten as $\varphi_2 = \varphi_4 = \varphi_6 = 0$, $\mu\varphi_1 = \mu\varphi_3 = \mu\varphi_5 = 0$ and

$$\varphi_1 - i\nu^{-1}\ell\varphi_5 = 0, \qquad \varphi_3 + i\nu^{-1}\varphi_1 = 0, \qquad \varphi_5 + i\nu^{-1}\ell\varphi_1 = 0.$$

Furthermore, these equations give $(1 - \nu^{-2}\ell^2)\varphi_1 = 0$. Therefore, for any $\sigma \in \mathbb{C}$, the vector

$$\varphi = (\sigma, 0, -\frac{1}{|\ell|}i\sigma, 0, -\frac{\ell}{|\ell|}i\sigma, 0)^\top$$

satisfies (2.8) with $(\mu, \nu) = (0, |\ell|)$. This means that the system (2.7) does not satisfy Condition (SC). Hence this completes the proof. □

In the rest of this section, we prove Theorem 1.2. To this end, we also introduce new functions that

$$
\begin{aligned}
&v := \kappa(\phi_x + \psi + \ell w), \qquad s := \phi_t, \qquad z := bg * \psi_{tx},\\
&y := \psi_t, \qquad p := \kappa_0(w_x - \ell\phi), \qquad q := w_t,
\end{aligned}
$$

and obtain

$$
\begin{aligned}
\kappa^{-1}v_t - s_x - y - \ell q &= 0,\\
\rho_1 s_t - v_x - \ell p + \alpha s &= 0,\\
b^{-1}z_t - y_x + \delta b^{-1} z &= 0,\\
\rho_2 y_t - z_x + v &= 0,\\
\kappa_0^{-1}p_t - q_x + \ell s &= 0,\\
\rho_1 q_t - p_x + \ell v + \gamma q &= 0.
\end{aligned} \tag{2.9}
$$

Then the system (2.9) is also described as (2.1), where $u = (v, s, z, y, p, q)^\top$, and the matrices A^0, A and L are defined by

$$
A^0 = \begin{pmatrix} 1/\kappa & 0 & 0 & 0 & 0 & 0\\ 0 & \rho_1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1/b & 0 & 0 & 0\\ 0 & 0 & 0 & \rho_2 & 0 & 0\\ 0 & 0 & 0 & 0 & 1/\kappa_0 & 0\\ 0 & 0 & 0 & 0 & 0 & \rho_1 \end{pmatrix}, \quad A = -\begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{pmatrix}, \quad L = \begin{pmatrix} 0&0&0&-1&0&-\ell\\ 0&\alpha&0&0&-\ell&0\\ 0&0&\delta/b&0&0&0\\ 1&0&0&0&0&0\\ 0&\ell&0&0&0&0\\ \ell&0&0&0&0&\gamma \end{pmatrix}.
$$

Since these matrices also satisfy Condition (A), we can prove Theorem 1.2 by applying Theorem 2.2 to (2.9).

Proof of Theorem 1.2. Firstly, we consider the case $\alpha > 0$ and $\gamma = 0$. If $(\mu, \nu) \in \mathbb{R} \times \mathbb{R}_+$ and $\varphi = (\varphi_1, \cdots, \varphi_6)^\top \in \mathbb{C}^6$ satisfy

$$
\begin{cases}
\mu\kappa^{-1}\varphi_1 - \varphi_2 + i\nu^{-1}\varphi_4 + i\nu^{-1}\ell\varphi_6 = 0,\\
\mu\rho_1\varphi_2 - \varphi_1 + i\nu^{-1}\ell\varphi_5 = 0,\\
\mu b^{-1}\varphi_3 - \varphi_4 = 0,\\
\mu\rho_2\varphi_4 - \varphi_3 - i\nu^{-1}\varphi_1 = 0,\\
\mu\kappa_0^{-1}\varphi_5 - \varphi_6 - i\nu^{-1}\ell\varphi_2 = 0,\\
\mu\rho_1\varphi_6 - \varphi_5 - i\nu^{-1}\ell\varphi_1 = 0,
\end{cases}
\quad \text{and} \quad
\begin{cases}
\alpha\varphi_2 = 0,\\
\delta b^{-1}\varphi_3 = 0,
\end{cases} \tag{2.10}
$$

then it is easy to get $\varphi = 0$, and we complete the proof of the strict dissipativity in the case $\alpha > 0$ and $\gamma = 0$. On the other hand, we can obtain the same result using the same argument in the case $\alpha = 0$ and $\gamma > 0$.

Secondly, we consider the case $\alpha = \gamma = 0$. Then (2.10) with $\alpha = 0$ gives $\varphi_1 = \varphi_3 = \varphi_4 = \varphi_5 = 0$, $\varphi_2 = i\nu^{-1}\ell\varphi_6$ and

$$
\mu\varphi_6 = 0, \qquad (1 - \nu^{-2}\ell^2)\varphi_6 = 0.
$$

Therefore, for any $\sigma \in \mathbb{C}$, the vector

$$\varphi = (0, \frac{\ell}{|\ell|} i\sigma, 0, 0, 0, \sigma)^\top$$

satisfies (2.10) with $(\mu, \nu) = (0, |\ell|)$. This means that the system (2.9) with $\alpha = \gamma = 0$ does not satisfy Condition (SC). Hence this completes the proof. □

For $\alpha + \gamma > 0$, it follows from Theorem 2.2 and Corollary 2.3 that the solution operator of (2.9) in Fourier space is estimated as $|e^{t\hat{\Phi}(i\xi)}| \le Ce^{-c\rho(\xi)t}$ with

$$\rho(\xi) = \frac{\xi^{10}}{(1+\xi^2)^{10}}.$$

However, this estimate might not be a sharp estimate. Therefore we focus on the case $\alpha > 0$ and shall derive the sharp decay estimate by the energy method in Section 3 and investigate the optimality of the decay estimate in Section 4.

3. Decay estimate

In this section, we derive the decay estimate of the solution to (2.9). To this end, we suppose the initial data for (1.2) that

$$\begin{aligned} &\phi(0,x) = \phi_0(x), \quad \phi_t(0,x) = \phi_1(x), \quad \psi(0,x) = \psi_0(x), \quad \psi_t(0,x) = \psi_1(x), \\ &w(0,x) = w_0(x), \quad w_t(0,x) = w_1(x), \end{aligned}$$

where $\phi_0 = \phi_0(x)$, $\phi_1 = \phi_1(x)$, $\psi_0 = \psi_0(x)$, $\psi_1 = \psi_1(x)$, $w_0 = w_0(x)$ and $w_1 = w_1(x)$ are known function. From these initial data, we also suppose the initial data for (2.9) that

$$\begin{aligned} &v(0,x) = v_0(x), \quad s(0,x) = s_0(x), \quad z(0,x) = z_0(x), \\ &y(0,x) = y_0(x), \quad p(0,x) = p_0(x), \quad q(0,x) = q_0(x), \end{aligned} \tag{3.1}$$

and $u(0,x) = u_0(x) := (v_0, s_0, z_0, y_0, p_0, q_0)^\top(x)$, where

$$\begin{aligned} &v_0 := \kappa(\phi_{0x} + \psi_0 + \ell w_0), \qquad s_0 := \phi_1, \qquad z_0 := 0 \\ &y_0 := \psi_1, \qquad p_0 := \kappa_0(w_{0x} - \ell\phi_0), \qquad q_0 := w_1. \end{aligned}$$

Then we derive the following decay estimates of the solution to (2.9), (3.1) for the Cauchy problem.

Theorem 3.1. *Suppose the coefficients of* (2.9) *satisfy* $\alpha > 0$ *and* $\gamma > 0$. *Then the solution operator of* (2.9) *in Fourier space satisfies the pointwise estimate* $|e^{t\hat{\Phi}(i\xi)}| \le Ce^{-c\rho(\xi)t}$, *where*

$$\rho(\xi) = \frac{\ell^2 + \xi^2}{(1+\ell^2)(1+\ell^2+\xi^2)}. \tag{3.2}$$

Here c and C are certain positive constants, which do not depend on ℓ.

Theorem 3.2. *Suppose the coefficients of* (2.9) *satisfy* $\alpha > 0$ *and* $\gamma = 0$. *Then the solution operator of* (2.9) *in Fourier space satisfies the pointwise estimate* $|e^{t\hat{\Phi}(i\xi)}| \leq Ce^{-c\rho(\xi)t}$, *where*

$$\rho(\xi) = \begin{cases} \dfrac{\ell^2\xi^2}{(1+\ell^2)(1+\ell^2+\xi^2)^2}, & \kappa \neq \kappa_0, \\ \dfrac{\ell^2\xi^2}{(1+\ell^2)^2(1+\ell^2+\xi^2)}, & \kappa = \kappa_0. \end{cases} \tag{3.3}$$

Here c *and* C *are certain positive constants, which do not depend on* ℓ.

Remark 2. *Theorem 3.2 tells us that the modified Bresse system* (1.2) *has the regularity-loss structure which denotes the weak dissipative structure in high frequency region in Fourier space.*

Proof of Theorem 3.1. We apply the Fourier transform to (2.9) and obtain

$$\begin{aligned} \kappa^{-1}\hat{v}_t - i\xi\hat{s} - \hat{y} - \ell\hat{q} = 0, \\ \rho_1\hat{s}_t - i\xi\hat{v} - \ell\hat{p} + \alpha\hat{s} = 0, \\ b^{-1}\hat{z}_t - i\xi\hat{y} + \delta b^{-1}\hat{z} = 0, \\ \rho_2\hat{y}_t - i\xi\hat{z} + \hat{v} = 0, \\ \kappa_0^{-1}\hat{p}_t - i\xi\hat{q} + \ell\hat{s} = 0, \\ \rho_1\hat{q}_t - i\xi\hat{p} + \ell\hat{v} + \gamma\hat{q} = 0. \end{aligned} \tag{3.4}$$

To obtain the standard energy equality, we multiply (3.4) by $(\bar{\hat{v}}, \bar{\hat{s}}, \bar{\hat{z}}, \bar{\hat{y}}, \bar{\hat{p}}, \bar{\hat{q}})$. Then we obtain

$$\frac{1}{2}\partial_t\mathcal{E} + \alpha|\hat{s}|^2 + \frac{\delta}{b}|\hat{z}|^2 + \gamma|\hat{q}|^2 = 0, \tag{3.5}$$

where

$$\mathcal{E} := \frac{1}{\kappa}|\hat{v}|^2 + \rho_1|\hat{s}|^2 + \frac{1}{b}|\hat{z}|^2 + \rho_2|\hat{y}|^2 + \frac{1}{\kappa_0}|\hat{p}|^2 + \rho_1|\hat{q}|^2.$$

It is important to construct the dissipation terms for all components. To this end, we prove the useful equalities. Multiplying the first and second equations in (3.4) by $-\rho_1\kappa i\xi\bar{\hat{s}}$ and $i\xi\bar{\hat{v}}$, respectively, and taking the real part, we have

$$\begin{aligned} &-\rho_1\xi\partial_t\mathrm{Re}(i\hat{v}\bar{\hat{s}}) + \xi^2(|\hat{v}|^2 - \rho_1\kappa|\hat{s}|^2) \\ &-\rho_1\kappa\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}}) - \rho_1\kappa\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}}) + \ell\xi\mathrm{Re}(i\hat{v}\bar{\hat{p}}) - \alpha\xi\mathrm{Re}(i\hat{v}\bar{\hat{s}}) = 0. \end{aligned} \tag{3.6}$$

Similarly, the third and fourth equations in (3.4) give

$$\rho_2\xi\partial_t\mathrm{Re}(i\hat{z}\bar{\hat{y}}) + \xi^2(\rho_2 b|\hat{y}|^2 - |\hat{z}|^2) + \rho_2\delta\xi\mathrm{Re}(i\hat{z}\bar{\hat{y}}) - \xi\mathrm{Re}(i\hat{v}\bar{\hat{z}}) = 0, \tag{3.7}$$

and the fifth and sixth equations in (3.4) give

$$\begin{aligned} &-\rho_1\xi\partial_t\mathrm{Re}(i\hat{p}\bar{\hat{q}}) + \xi^2(|\hat{p}|^2 - \rho_1\kappa_0|\hat{q}|^2) \\ &-\rho_1\kappa_0\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}}) + \ell\xi\mathrm{Re}(i\hat{v}\bar{\hat{p}}) - \gamma\xi\mathrm{Re}(i\hat{p}\bar{\hat{q}}) = 0. \end{aligned} \tag{3.8}$$

Furthermore, the first and fourth equations in (3.4) give

$$\rho_2\partial_t\mathrm{Re}(\hat{v}\bar{\hat{y}}) + |\hat{v}|^2 - \rho_2\kappa|\hat{y}|^2 + \xi\mathrm{Re}(i\hat{v}\bar{\hat{z}}) - \rho_2\kappa\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}}) - \rho_2\kappa\ell\mathrm{Re}(\hat{y}\bar{\hat{q}}) = 0, \tag{3.9}$$

the first and sixth equations in (3.4) give

$$\begin{aligned}(3.10)\qquad &\rho_1\ell\partial_t\mathrm{Re}(\hat{v}\bar{\hat{q}})+\ell^2(|\hat{v}|^2-\rho_1\kappa|\hat{q}|^2)\\ &+\ell\xi\mathrm{Re}(i\hat{v}\bar{\hat{p}})-\rho_1\kappa\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}})-\rho_1\kappa\ell\mathrm{Re}(\hat{y}\bar{\hat{q}})+\gamma\ell\mathrm{Re}(\hat{v}\bar{\hat{q}})=0,\end{aligned}$$

and the second and fifth equation in (3.4) give

$$\begin{aligned}(3.11)\qquad &-\rho_1\ell\partial_t\mathrm{Re}(\hat{s}\bar{\hat{p}})+\ell^2(|\hat{p}|^2-\rho_1\kappa_0|\hat{s}|^2)\\ &+\ell\xi\mathrm{Re}(i\hat{v}\bar{\hat{p}})-\alpha\ell\mathrm{Re}(\hat{s}\bar{\hat{p}})-\rho_1\kappa_0\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}})=0.\end{aligned}$$

To handle the interaction terms, we also need the following equations.

$$(3.12)\qquad \rho_1\partial_t\mathrm{Re}(\hat{s}\bar{\hat{z}})-\xi\mathrm{Re}(i\hat{v}\bar{\hat{z}})-\ell\mathrm{Re}(\hat{z}\bar{\hat{p}})+(\rho_1\delta+\alpha)\mathrm{Re}(\hat{s}\bar{\hat{z}})+\rho_1 b\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}})=0,$$

$$\begin{aligned}(3.13)\qquad &\rho_1\xi\partial_t\mathrm{Re}(i\hat{z}\bar{\hat{q}})-\ell\xi\mathrm{Re}(i\hat{v}\bar{\hat{z}})+(\rho_1\delta+\gamma)\xi\mathrm{Re}(i\hat{z}\bar{\hat{q}})\\ &-\xi^2\mathrm{Re}(\hat{z}\bar{\hat{p}})+\rho_1 b\xi^2\mathrm{Re}(\hat{y}\bar{\hat{q}})=0,\end{aligned}$$

$$\begin{aligned}(3.14)\qquad &\rho_2\xi\partial_t\mathrm{Re}(i\hat{y}\bar{\hat{p}})+\xi^2\mathrm{Re}(\hat{z}\bar{\hat{p}})+\xi\mathrm{Re}(i\hat{v}\bar{\hat{p}})\\ &-\rho_2\kappa_0\xi^2\mathrm{Re}(\hat{y}\bar{\hat{q}})-\rho_2\kappa_0\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}})=0,\end{aligned}$$

$$\partial_t\mathrm{Re}(\hat{z}\bar{\hat{p}})-b\xi\mathrm{Re}(i\hat{y}\bar{\hat{p}})+\kappa_0\xi\mathrm{Re}(i\hat{z}\bar{\hat{q}})+\delta\mathrm{Re}(\hat{z}\bar{\hat{p}})+\kappa_0\ell\mathrm{Re}(\hat{s}\bar{\hat{z}})=0.$$

Then, combining (3.13) and (3.14), we get

$$\begin{aligned}(3.15)\qquad &\rho_1\rho_2\xi\partial_t\{\kappa_0\mathrm{Re}(i\hat{z}\bar{\hat{q}})+b\mathrm{Re}(i\hat{y}\bar{\hat{p}})\}+(\rho_1 b-\rho_2\kappa_0)\xi^2\mathrm{Re}(\hat{z}\bar{\hat{p}})+\rho_1 b\xi\mathrm{Re}(i\hat{v}\bar{\hat{p}})\\ &-\rho_2\kappa_0\ell\xi\mathrm{Re}(i\hat{v}\bar{\hat{z}})+\rho_2\kappa_0(\rho_1\delta+\gamma)\xi\mathrm{Re}(i\hat{z}\bar{\hat{q}})-\rho_1\rho_2 b\kappa_0\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}})=0.\end{aligned}$$

Similarly, combining (3.15) and (3.12), we also get

$$\begin{aligned}&\rho_1\rho_2\ell\xi\partial_t\{\kappa_0\mathrm{Re}(i\hat{z}\bar{\hat{q}})+b\mathrm{Re}(i\hat{y}\bar{\hat{p}})\}+\rho_1(\rho_1 b-\rho_2\kappa_0)\xi^2\partial_t\mathrm{Re}(\hat{s}\bar{\hat{z}})\\ &-\{(\rho_1 b-\rho_2\kappa_0)\xi^2+\rho_2\kappa_0\ell^2\}\xi\mathrm{Re}(i\hat{v}\bar{\hat{z}})+\rho_1 b\ell\xi\mathrm{Re}(i\hat{v}\bar{\hat{p}})\\ &+\rho_2\kappa_0(\rho_1\delta+\gamma)\ell\xi\mathrm{Re}(i\hat{z}\bar{\hat{q}})+(\rho_1\delta+\alpha)(\rho_1 b-\rho_2\kappa_0)\xi^2\mathrm{Re}(\hat{s}\bar{\hat{z}})\\ &+\rho_1 b\{(\rho_1 b-\rho_2\kappa_0)\xi^2-\rho_2\kappa_0\ell^2\}\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}})=0.\end{aligned}$$

Using the above equations, we construct the energy estimates.

Because of (3.5) with $\alpha>0$ and $\gamma>0$, our purpose is to construct the dissipation term for $\hat{v}$, $\hat{y}$ and $\hat{p}$. Here, to control $\mathrm{Re}(i\hat{v}\bar{\hat{p}})$ appeared in (3.6) and (3.8), we employ (3.14). Making the combination of (3.6), (3.8) and (3.14) we obtain

$$\begin{aligned}&-\xi\partial_t\{\rho_1\mathrm{Re}(i\hat{v}\bar{\hat{s}})+\rho_2\ell\mathrm{Re}(i\hat{y}\bar{\hat{p}})\}+\xi^2(|\hat{v}|^2-\rho_1\kappa|\hat{s}|^2)-\rho_1\kappa\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}})\\ &-\alpha\xi\mathrm{Re}(i\hat{v}\bar{\hat{s}})+(\rho_2\kappa_0\ell^2-\rho_1\kappa)\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}})-\ell\xi^2\mathrm{Re}(\hat{z}\bar{\hat{p}})+\rho_2\kappa_0\ell\xi^2\mathrm{Re}(\hat{y}\bar{\hat{q}})=0\end{aligned}$$

and

$$\begin{aligned}(3.16)\qquad &-\xi\partial_t\{\rho_1\mathrm{Re}(i\hat{p}\bar{\hat{q}})+\rho_2\ell\mathrm{Re}(i\hat{y}\bar{\hat{p}})\}+\xi^2(|\hat{p}|^2-\rho_1\kappa_0|\hat{q}|^2)-\rho_1\kappa_0\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}})\\ &-\gamma\xi\mathrm{Re}(i\hat{p}\bar{\hat{q}})-\ell\xi^2\mathrm{Re}(\hat{z}\bar{\hat{p}})+\rho_2\kappa_0\ell\xi^2\mathrm{Re}(\hat{y}\bar{\hat{q}})+\rho_2\kappa_0\ell^2\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}})=0.\end{aligned}$$

On the other hand, combining (3.9) and (3.10), we have

$$\begin{aligned}(3.17)\qquad &\ell\partial_t\{\rho_1\mathrm{Re}(\hat{v}\bar{\hat{q}})-\rho_2\ell\mathrm{Re}(\hat{v}\bar{\hat{y}})\}+\kappa\ell^2(\rho_2|\hat{y}|^2-\rho_1|\hat{q}|^2)+\gamma\ell\mathrm{Re}(\hat{v}\bar{\hat{q}})+\ell\xi\mathrm{Re}(i\hat{v}\bar{\hat{p}})\\ &-\rho_1\kappa\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}})-\ell^2\xi\mathrm{Re}(i\hat{v}\bar{\hat{z}})+\rho_2\kappa\ell^2\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}})+\kappa\ell(\rho_2\ell^2-\rho_1)\mathrm{Re}(\hat{y}\bar{\hat{q}})=0.\end{aligned}$$

Combining the above equations, we will construct the energy inequality with dissipation for all components. However, to get the optimal estimate, we shall proceed further computation. We combine (3.14) and (3.11) and this yields

$$
\begin{aligned}
&-\ell\partial_t\{\rho_1\mathrm{Re}(\hat{s}\bar{\hat{p}})+\rho_2\xi\mathrm{Re}(i\hat{y}\bar{\hat{p}})\}+\ell^2(|\hat{p}|^2-\rho_1\kappa_0|\hat{s}|^2)-\ell\xi^2\mathrm{Re}(\hat{z}\bar{\hat{p}})\\
&-\alpha\ell\mathrm{Re}(\hat{s}\bar{\hat{p}})-\rho_1\kappa_0\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}})+\rho_2\kappa_0\ell\xi^2\mathrm{Re}(\hat{y}\bar{\hat{q}})+\rho_2\kappa_0\ell^2\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}})=0.
\end{aligned}
\tag{3.18}
$$

Furthermore, combining (3.16) and (3.18), we obtain

$$
\begin{aligned}
&-\partial_t\{\rho_1\xi\mathrm{Re}(i\hat{p}\bar{\hat{q}})+\rho_1\ell\mathrm{Re}(\hat{s}\bar{\hat{p}})+2\rho_2\ell\xi\mathrm{Re}(i\hat{y}\bar{\hat{p}})\}+(\ell^2+\xi^2)|\hat{p}|^2\\
&-\rho_1\kappa_0\xi^2|\hat{q}|^2-\rho_1\kappa_0\ell^2|\hat{s}|^2-\gamma\xi\mathrm{Re}(i\hat{p}\bar{\hat{q}})-\alpha\ell\mathrm{Re}(\hat{s}\bar{\hat{p}})-2\ell\xi^2\mathrm{Re}(\hat{z}\bar{\hat{p}})\\
&-2\rho_1\kappa_0\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}})+2\rho_2\kappa_0\ell\xi^2\mathrm{Re}(\hat{y}\bar{\hat{q}})+2\rho_2\kappa_0\ell^2\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}})=0.
\end{aligned}
\tag{3.19}
$$

On the other hand, (3.6) and (3.10) give

$$
\begin{aligned}
&\rho_1\partial_t\{\ell\mathrm{Re}(\hat{v}\bar{\hat{q}})-\xi\mathrm{Re}(i\hat{v}\bar{\hat{s}})\}+(\ell^2+\xi^2)|\hat{v}|^2-\rho_1\kappa\ell^2|\hat{q}|^2-\rho_1\kappa\xi^2|\hat{s}|^2\\
&-\rho_1\kappa\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}})-2\rho_1\kappa\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}})+2\ell\xi\mathrm{Re}(i\hat{v}\bar{\hat{p}})-\alpha\xi\mathrm{Re}(i\hat{v}\bar{\hat{s}})\\
&-\rho_1\kappa\ell\mathrm{Re}(\hat{y}\bar{\hat{q}})+\gamma\ell\mathrm{Re}(\hat{v}\bar{\hat{q}})=0.
\end{aligned}
\tag{3.20}
$$

Similarly, (3.7) and (3.17) also give

$$
\begin{aligned}
&\partial_t\left\{\rho_1\ell\mathrm{Re}(\hat{v}\bar{\hat{q}})-\rho_2\ell^2\mathrm{Re}(\hat{v}\bar{\hat{y}})+\frac{\rho_2\kappa}{b}\xi\mathrm{Re}(i\hat{z}\bar{\hat{y}})\right\}+\rho_2\kappa(\ell^2+\xi^2)|\hat{y}|^2-\frac{\kappa}{b}\xi^2|\hat{z}|^2\\
&-\rho_1\kappa\ell^2|\hat{q}|^2+\gamma\ell\mathrm{Re}(\hat{v}\bar{\hat{q}})+\ell\xi\mathrm{Re}(i\hat{v}\bar{\hat{p}})+\frac{\rho_2\delta\kappa}{b}\xi\mathrm{Re}(i\hat{z}\bar{\hat{y}})-\rho_1\kappa\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}})\\
&-\left(\frac{\kappa}{b}+\ell^2\right)\xi\mathrm{Re}(i\hat{v}\bar{\hat{z}})+\rho_2\kappa\ell^2\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}})+\rho_2\kappa\ell\left(\ell^2-\frac{\rho_1}{\rho_2}\right)\mathrm{Re}(\hat{y}\bar{\hat{q}})=0.
\end{aligned}
\tag{3.21}
$$

Consequently, (3.19), (3.20) and (3.21) lead to

$$
\begin{aligned}
&-\partial_t\{\rho_1\xi\mathrm{Re}(i\hat{p}\bar{\hat{q}})+\rho_1\ell\mathrm{Re}(\hat{s}\bar{\hat{p}})+2\rho_2\ell\xi\mathrm{Re}(i\hat{y}\bar{\hat{p}})\}+\frac{1}{2}(\ell^2+\xi^2)|\hat{p}|^2\\
&\le 4\ell^2\xi^2|\hat{z}|^2+(\gamma^2+\rho_1\kappa_0\xi^2)|\hat{q}|^2+\left(\frac{\alpha^2}{2}+\rho_1\kappa_0\ell^2\right)|\hat{s}|^2\\
&\quad+2\rho_1\kappa_0\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}})-2\rho_2\kappa_0\ell^2\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}})-2\rho_2\kappa_0\ell\xi^2\mathrm{Re}(\hat{y}\bar{\hat{q}}),
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
&\rho_1\partial_t\{\ell\mathrm{Re}(\hat{v}\bar{\hat{q}})-\xi\mathrm{Re}(i\hat{v}\bar{\hat{s}})\}+\frac{1}{2}(\ell^2+\xi^2)|\hat{v}|^2\\
&\le 4\xi^2|\hat{p}|^2+(\gamma^2+\rho_1\kappa\ell^2)|\hat{q}|^2+\left(\frac{\alpha^2}{2}+\rho_1\kappa\xi^2\right)|\hat{s}|^2\\
&\quad+2\rho_1\kappa\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}})+\rho_1\kappa\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}})+\rho_1\kappa\ell\mathrm{Re}(\hat{y}\bar{\hat{q}}),
\end{aligned}
\tag{3.23}
$$

and

$$
\begin{aligned}
&\partial_t\left\{\rho_1\ell\mathrm{Re}(\hat{v}\bar{\hat{q}})-\rho_2\ell^2\mathrm{Re}(\hat{v}\bar{\hat{y}})+\frac{\rho_2\kappa}{b}\xi\mathrm{Re}(i\hat{z}\bar{\hat{y}})\right\}+\frac{\rho_2\kappa}{2}(\ell^2+\xi^2)|\hat{y}|^2\\
&\le\frac{\kappa}{b}\left(\frac{\rho_2\delta^2}{2b}+\xi^2\right)|\hat{z}|^2+\kappa\left\{\rho_1\ell^2+\rho_2\left(\ell^2-\frac{\rho_1}{\rho_2}\right)^2\right\}|\hat{q}|^2+\rho_2\kappa\ell^2\xi^2|\hat{s}|^2\\
&\quad-\gamma\ell\mathrm{Re}(\hat{v}\bar{\hat{q}})-\ell\xi\mathrm{Re}(i\hat{v}\bar{\hat{p}})+\rho_1\kappa\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}})+\left(\frac{\kappa}{b}+\ell^2\right)\xi\mathrm{Re}(i\hat{v}\bar{\hat{z}}).
\end{aligned}
\tag{3.24}
$$

Combining (3.23) and (3.24) and using the Schwarz inequality, we obtain

$$
\begin{aligned}
&\partial_t \left\{ 2\rho_1 \ell \mathrm{Re}(\hat{v}\bar{\hat{q}}) - \rho_2 \ell^2 \mathrm{Re}(\hat{v}\bar{\hat{y}}) - \rho_1 \xi \mathrm{Re}(i\hat{v}\bar{\hat{s}}) + \frac{\rho_2\kappa}{b} \xi \mathrm{Re}(i\hat{z}\bar{\hat{y}}) \right\} \\
&+ \frac{1}{4}(\ell^2+\xi^2)|\hat{v}|^2 + \frac{\rho_2\kappa}{2}(\ell^2+\xi^2)|\hat{y}|^2 \\
&\le 2(\ell^2+2\xi^2)|\hat{p}|^2 + \left\{ 2\left(\frac{\kappa}{b}+\ell^2\right)^2 + \frac{\kappa}{b}\left(\frac{\rho_2\delta^2}{2b}+\xi^2\right)\right\}|\hat{z}|^2 \\
&+ \left(2\gamma^2 + \frac{\rho_1^2\kappa}{\rho_2} + \rho_2\kappa\ell^4\right)|\hat{q}|^2 + \left(\frac{\alpha^2}{2} + \kappa(\rho_1+\rho_2\ell^2)\xi^2\right)|\hat{s}|^2 \\
&+ 3\rho_1\kappa\ell\xi \mathrm{Re}(i\hat{s}\bar{\hat{q}}) + \rho_1\kappa\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}}) + \rho_1\kappa\ell\mathrm{Re}(\hat{y}\bar{\hat{q}}).
\end{aligned} \tag{3.25}
$$

Furthermore we make a combination (3.22) + (3.25) $\times$ 1/16 to obtain

$$
\begin{aligned}
&\partial_t \mathcal{E}_{01} + \frac{1}{64}(\ell^2+\xi^2)|\hat{v}|^2 + \frac{\rho_2\kappa}{32}(\ell^2+\xi^2)|\hat{y}|^2 + \frac{1}{4}(\ell^2+\xi^2)|\hat{p}|^2 \\
&\le \left\{ \frac{1}{8}\left(\frac{\kappa}{b}+\ell^2\right)^2 + \frac{\kappa}{16b}\left(\frac{\rho_2\delta^2}{2b}+\xi^2\right) + 4\ell^2\xi^2\right\}|\hat{z}|^2 \\
&+ \left(\frac{9\gamma^2}{8} + \frac{\rho_1^2\kappa}{16\rho_2} + \frac{\rho_2\kappa}{16}\ell^4 + \rho_1\kappa_0\xi^2\right)|\hat{q}|^2 + \left(\frac{17\alpha^2}{32} + \rho_1\kappa_0\ell^2 + \frac{1}{16}\kappa(\rho_1+\rho_2\ell^2)\xi^2\right)|\hat{s}|^2 \\
&+ \rho_1\left(\frac{3\kappa}{16}+2\kappa_0\right)\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}}) + \left(\frac{\rho_1\kappa}{16} - 2\rho_2\kappa_0\ell^2\right)\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}}) + \left(\frac{\rho_1\kappa}{16} - 2\rho_2\kappa_0\xi^2\right)\ell\mathrm{Re}(\hat{y}\bar{\hat{q}}),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{E}_{01} := &\frac{1}{16}\left\{ 2\rho_1 \ell \mathrm{Re}(\hat{v}\bar{\hat{q}}) - \rho_2 \ell^2 \mathrm{Re}(\hat{v}\bar{\hat{y}}) - \rho_1 \xi \mathrm{Re}(i\hat{v}\bar{\hat{s}}) + \frac{\rho_2\kappa}{b} \xi \mathrm{Re}(i\hat{z}\bar{\hat{y}}) \right\} \\
&- \{\rho_1\xi\mathrm{Re}(i\hat{p}\bar{\hat{q}}) + \rho_1\ell\mathrm{Re}(\hat{s}\bar{\hat{p}}) + 2\rho_2\ell\xi\mathrm{Re}(i\hat{y}\bar{\hat{p}})\}.
\end{aligned}
$$

Thus the Schwarz inequality gives

$$
\begin{aligned}
\frac{\rho_1\kappa}{16}|\xi||\hat{s}||\hat{y}| + \frac{\rho_1\kappa}{16}|\ell||\hat{y}||\hat{q}| &\le \frac{\rho_2\kappa}{128}(\ell^2+\xi^2)|\hat{y}|^2 + \frac{\rho_1^2\kappa}{8\rho_2}(|\hat{s}|^2+|\hat{q}|^2), \\
2\rho_2\kappa_0|\xi||\hat{s}||\hat{y}| &\le \frac{\rho_2\kappa}{128}|\hat{y}|^2 + \frac{128\rho_2\kappa_0^2}{\kappa}\xi^2|\hat{s}|^2, \\
2\rho_2\kappa_0|\ell||\hat{y}||\hat{q}| &\le \frac{\rho_2\kappa}{128}|\hat{y}|^2 + \frac{128\rho_2\kappa_0^2}{\kappa}\ell^2|\hat{q}|^2,
\end{aligned}
$$

and we obtain

$$
\begin{aligned}
&\partial_t \mathcal{E}_{01} + \frac{1}{64}(\ell^2+\xi^2)|\hat{v}|^2 + \frac{\rho_2\kappa}{64}(\ell^2+\xi^2)|\hat{y}|^2 + \frac{1}{4}(\ell^2+\xi^2)|\hat{p}|^2 \\
&\le \left\{ \frac{1}{8}\left(\frac{\kappa}{b}+\ell^2\right)^2 + \frac{\kappa}{16b}\left(\frac{\rho_2\delta^2}{2b}+\xi^2\right) + 4\ell^2\xi^2\right\}|\hat{z}|^2 \\
&+ \left\{\frac{9\gamma^2}{8} + \frac{3\rho_1^2\kappa}{16\rho_2} + \frac{\rho_2\kappa}{16}\ell^4 + 3\rho_1\left(\frac{\kappa}{16}+\kappa_0\right)\xi^2 + \frac{128\rho_2\kappa_0^2}{\kappa}\ell^2\xi^2\right\}|\hat{q}|^2 \\
&+ \left\{\frac{17\alpha^2}{32} + \frac{\rho_1^2\kappa}{8\rho_2} + 3\rho_1\left(\frac{\kappa}{16}+\kappa_0\right)\ell^2 + \frac{\rho_1\kappa}{16}\xi^2 + \rho_2\left(\frac{\kappa}{16}+\frac{128\kappa_0^2}{\kappa}\right)\ell^2\xi^2\right\}|\hat{s}|^2 \\
&\le C_0(1+\ell^2)(1+\ell^2+\xi^2)(|\hat{z}|^2+|\hat{q}|^2+|\hat{s}|^2),
\end{aligned} \tag{3.26}
$$

where C_0 is defined by

$$C_0 := \max\left\{\frac{\kappa}{8b^2}\left(\kappa+\frac{\rho_2\delta^2}{4b}\right), \frac{\kappa}{8b}, 4, \frac{9\gamma^2}{8}+\frac{3\rho_1^2\kappa}{16\rho_2}, 3\rho_1\left(\frac{\kappa}{16}+\kappa_0\right), \frac{17\alpha^2}{32}+\frac{\rho_1^2\kappa}{8\rho_2}, \rho_2\left(\frac{\kappa}{16}+\frac{128\kappa_0^2}{\kappa}\right)\right\}.$$

Finally, calculating (3.5) + (3.26) $\times\, c_0/(2C_0(1+\ell^2)(1+\ell^2+\xi^2))$, we arrive at

$$\begin{aligned}\partial_t\mathcal{E}_0 &+ \frac{c_0}{64C_0}\frac{\ell^2+\xi^2}{(1+\ell^2)(1+\ell^2+\xi^2)}(|\hat{v}|^2+\rho_2 b|\hat{y}|^2+16|\hat{p}|^2)\\ &+\frac{c_0}{2}(|\hat{s}|^2+|\hat{z}|^2+|\hat{q}|^2)\le 0,\end{aligned} \tag{3.27}$$

where $c_0 := \min\{\alpha, \delta/b, \gamma\}$ and

$$\mathcal{E}_0 := \frac{1}{2}\mathcal{E}+\frac{c_0}{2C_0(1+\ell^2)(1+\ell^2+\xi^2)}\mathcal{E}_{01}.$$

Furthermore, there exist the positive constants c_0^* and C_0^* such that $c_0^*|\hat{u}|^2\le\mathcal{E}_0\le C_0^*|\hat{u}|^2$. Therefore, the estimate (3.27) gives

$$\begin{aligned}&|\hat{u}|^2+\frac{c_0}{2c_0^*}\int_0^t\left\{\frac{\tilde{c}_0}{32C_0}\frac{\ell^2+\xi^2}{(1+\ell^2)(1+\ell^2+\xi^2)}(|\hat{v}|^2+|\hat{y}|^2+|\hat{p}|^2)+|\hat{s}|^2+|\hat{z}|^2+|\hat{q}|^2\right\}d\tau\\ &\le\frac{C_0^*}{c_0^*}|\hat{u}_0|^2\end{aligned}$$

and the pointwise estimate (3.2), where $\tilde{c}_0 := \min\{1, \rho_2 b\}$. Hence this completes the proof. □

Proof of Theorem 3.2. In this proof, we utilize the equalities obtained in the proof of Theorem 3.1. We construct the dissipation term for $\hat{v}$ in (3.6). To eliminate $\mathrm{Re}(i\hat{v}\bar{\hat{p}})$, we combine (3.6) and (3.15). Then we obtain

$$\begin{aligned}&-\rho_1\xi\partial_t\{\rho_1 b\mathrm{Re}(i\hat{v}\bar{\hat{s}})+\rho_2\kappa_0\ell\mathrm{Re}(i\hat{z}\bar{\hat{q}})+\rho_2 b\ell\mathrm{Re}(i\hat{y}\bar{\hat{p}})\}+\rho_1 b\xi^2(|\hat{v}|^2-\rho_1\kappa|\hat{s}|^2)\\ &-(\rho_1 b-\rho_2\kappa_0)\ell\xi^2\mathrm{Re}(\hat{z}\bar{\hat{p}})+\rho_2\kappa_0\ell^2\xi\mathrm{Re}(i\hat{v}\bar{\hat{z}})+\rho_1 b(\rho_2 b\kappa_0\ell^2-\rho_1\kappa)\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}})\\ &-\rho_1^2 b\kappa\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}})-\rho_1 b\alpha\xi\mathrm{Re}(i\hat{v}\bar{\hat{s}})-\rho_2\kappa_0(\rho_1\delta+\gamma)\ell\xi\mathrm{Re}(i\hat{z}\bar{\hat{q}})=0.\end{aligned}$$

Therefore, this gives

$$\begin{aligned}&-\rho_1\xi\partial_t\{\rho_1 b\mathrm{Re}(i\hat{v}\bar{\hat{s}})+\rho_2\kappa_0\ell\mathrm{Re}(i\hat{u}_3\bar{\hat{u}}_6)+\rho_2 b\ell\mathrm{Re}(i\hat{y}\bar{\hat{p}})\}+\frac{\rho_1 b}{2}\xi^2|\hat{v}|^2\\ &\le\rho_1 b(\rho_1\kappa\xi^2+\alpha^2)|\hat{s}|^2+\frac{\rho_2^2\kappa_0^2}{\rho_1 b}\ell^4|\hat{z}|^2+(\rho_1 b-\rho_2\kappa_0)\ell\xi^2\mathrm{Re}(\hat{z}\bar{\hat{p}})\\ &-\rho_1 b(\rho_2 b\kappa_0\ell^2-\rho_1\kappa)\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}})+\rho_1^2 b\kappa\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}})+\rho_2\kappa_0(\rho_1\delta+\gamma)\ell\xi\mathrm{Re}(i\hat{z}\bar{\hat{q}}).\end{aligned} \tag{3.28}$$

Furthermore, the equation (3.11) gives

$$-\rho_1\ell\partial_t\mathrm{Re}(\hat{s}\bar{\hat{p}})+\frac{1}{2}\ell^2|\hat{p}|^2\le\xi^2|\hat{v}|^2+(\rho_1\kappa_0\ell^2+\alpha^2)|\hat{s}|^2+\rho_1\kappa_0\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}}). \tag{3.29}$$

On the other hand, for the terms $\hat{p}$ and $\hat{q}$, we use (3.10) and (3.11) and obtain

$$
\begin{aligned}
&-\rho_1\ell\partial_t\{\mathrm{Re}(\hat{s}\bar{\hat{p}})+\mathrm{Re}(\hat{v}\bar{\hat{q}})\}+\ell^2(|\hat{p}|^2+\rho_1\kappa|\hat{q}|^2)-\ell^2(|\hat{v}|^2+\rho_1\kappa_0|\hat{s}|^2)\\
&+\rho_1(\kappa-\kappa_0)\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}})+\rho_1\kappa\ell\mathrm{Re}(\hat{y}\bar{\hat{q}})-\alpha\ell\mathrm{Re}(\hat{s}\bar{\hat{p}})-\gamma\ell\mathrm{Re}(\hat{v}\bar{\hat{q}})=0.
\end{aligned}
$$

Since this equation with $\gamma=0$, we have

$$
\begin{aligned}
&-\rho_1\ell\partial_t\{\mathrm{Re}(\hat{s}\bar{\hat{p}})+\mathrm{Re}(\hat{v}\bar{\hat{q}})\}+\frac{1}{2}\ell^2|\hat{p}|^2+\frac{\rho_1\kappa}{2}\ell^2|\hat{q}|^2\\
&\le\ell^2|\hat{v}|^2+\left(\frac{\rho_1(\kappa-\kappa_0)^2}{\kappa}\xi^2+\rho_1\kappa_0\ell^2+\frac{\alpha^2}{2}\right)|\hat{s}|^2+\rho_1\kappa|\hat{y}|^2.
\end{aligned}
\tag{3.30}
$$

Therefore, we calculate (3.29) $\times$ $(1+\ell^2)$ + (3.30) $\times$ ξ^2, and obtain

$$
\begin{aligned}
&-\rho_1\ell\xi^2\partial_t\{\mathrm{Re}(\hat{s}\bar{\hat{p}})+\mathrm{Re}(\hat{v}\bar{\hat{q}})\}-\rho_1(1+\ell^2)\ell\partial_t\mathrm{Re}(\hat{s}\bar{\hat{p}})\\
&+\frac{1}{2}(1+\ell^2+\xi^2)\ell^2|\hat{p}|^2+\frac{\rho_1\kappa}{2}\ell^2\xi^2|\hat{q}|^2\\
&\le 2(1+\ell^2)\xi^2|\hat{v}|^2+\left(\frac{\alpha^2}{2}+\rho_1\kappa_0\ell^2+\frac{\rho_1(\kappa-\kappa_0)^2}{\kappa}\xi^2\right)\xi^2|\hat{s}|^2+\rho_1\kappa\xi^2|\hat{y}|^2\\
&+(\alpha^2+\rho_1\kappa_0\ell^2)(1+\ell^2)|\hat{s}|^2+\rho_1\kappa_0(1+\ell^2)\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}}).
\end{aligned}
\tag{3.31}
$$

Furthermore, computation of (3.28) $\times$ (ℓ^2+1) + (3.31) $\times$ $\rho_1 b/8$ gives

$$
\begin{aligned}
&-\rho_1(1+\ell^2)\xi\partial_t\{\rho_1 b\mathrm{Re}(i\hat{v}\bar{\hat{s}})+\rho_2\kappa_0\ell\mathrm{Re}(i\hat{z}\bar{\hat{q}})+\rho_2 b\ell\mathrm{Re}(i\hat{y}\bar{\hat{p}})\}-\frac{\rho_1^2 b}{8}\ell\xi^2\partial_t\{\mathrm{Re}(\hat{s}\bar{\hat{p}})+\mathrm{Re}(\hat{v}\bar{\hat{q}})\}\\
&-\frac{\rho_1^2 b}{8}(1+\ell^2)\ell\partial_t\mathrm{Re}(\hat{s}\bar{\hat{p}})+\frac{\rho_1 b}{4}(1+\ell^2)\xi^2|\hat{v}|^2+\frac{\rho_1 b}{16}(1+\ell^2+\xi^2)\ell^2|\hat{p}|^2+\frac{\rho_1^2 b\kappa}{16}\ell^2\xi^2|\hat{q}|^2\\
&\le\frac{\rho_1 b}{8}(1+\ell^2)(9\alpha^2+\rho_1\kappa_0\ell^2+8\rho_1\kappa\xi^2)|\hat{s}|^2+\frac{\rho_1 b}{8}\left(\frac{\alpha^2}{2}+\rho_1\kappa_0\ell^2+\frac{\rho_1(\kappa-\kappa_0)^2}{\kappa}\xi^2\right)\xi^2|\hat{s}|^2\\
&+\frac{\rho_2^2\kappa_0^2}{\rho_1 b}(1+\ell^2)\ell^4|\hat{z}|^2+\frac{\rho_1 b}{8}\rho_1\kappa\xi^2|\hat{y}|^2+\rho_1 b(\rho_1\kappa-\rho_2 b\kappa_0\ell^2)(1+\ell^2)\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}})\\
&+\frac{\rho_1 b}{8}(8\rho_1\kappa+\rho_1\kappa_0)(\ell^2+1)\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}})+\rho_1\rho_2\kappa_0\delta(\ell^2+1)\ell\xi\mathrm{Re}(i\hat{z}\bar{\hat{q}})+(\rho_1 b-\rho_2\kappa_0)(1+\ell^2)\ell\xi^2\mathrm{Re}(\hat{z}\bar{\hat{p}}).
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&\partial_t\mathcal{E}_{11}+\frac{1}{4}(1+\ell^2)\xi^2|\hat{v}|^2+\frac{1}{32}(1+\ell^2+\xi^2)\ell^2|\hat{p}|^2+\frac{\rho_1\kappa}{32}\ell^2\xi^2|\hat{q}|^2\\
&\le C_{11}(1+\ell^2)(1+\ell^2+\xi^2)|\hat{s}|^2+\frac{\rho_1(\kappa-\kappa_0)^2}{8\kappa}\xi^4|\hat{s}|^2+\frac{\rho_1\kappa}{8}\xi^2|\hat{y}|^2\\
&+C_{12}(1+\ell^2)^2(1+\ell^2+\xi^2)|\hat{z}|^2+(\rho_1\kappa-\rho_2 b\kappa_0\ell^2)(1+\ell^2)\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}}),
\end{aligned}
\tag{3.32}
$$

where

$$\mathcal{E}_{11} := -\rho_1(1+\ell^2)\xi\{\rho_1 b\mathrm{Re}(i\hat{v}\bar{\hat{s}}) + \rho_2\kappa_0\ell\mathrm{Re}(i\hat{z}\bar{\hat{q}}) + \rho_2 b\ell\mathrm{Re}(i\hat{y}\bar{\hat{p}})\}$$
$$\quad - \frac{\rho_1^2 b}{8}\ell\xi^2\mathrm{Re}(\hat{v}\bar{\hat{q}}) - \frac{\rho_1^2 b}{8}(1+\ell^2+\xi^2)\ell\mathrm{Re}(\hat{s}\bar{\hat{p}}),$$

$$C_{11} := \frac{1}{8}\max\left\{9\alpha^2 + 2\rho_1\frac{(8\kappa+\kappa_0)^2}{\kappa}, \rho_1\left(\kappa_0 + 2\frac{(8\kappa+\kappa_0)^2}{\kappa}\right), \frac{\alpha^2}{2} + \rho_1(8\kappa+\kappa_0)\right\},$$

$$C_{12} := \frac{\rho_2^2\kappa_0^2}{\rho_1^2 b^2}\max\left\{\frac{16\rho_1\delta^2}{\kappa}, 1, \frac{8(\rho_1 b - \rho_2\kappa_0)^2}{\rho_2^2\kappa_0^2}\right\}.$$

In the case $\kappa \neq \kappa_0$, to add more dissipation for $\hat{u}_1$, we employ

$$\begin{aligned} &-\rho_1\xi\partial_t\mathrm{Re}(i\hat{v}\bar{\hat{s}}) + \frac{1}{2}\xi^2|\hat{v}|^2 \\ &\le (\alpha^2+\rho_1\kappa\xi^2)|\hat{s}|^2 + \ell^2|\hat{p}|^2 + \rho_1\kappa\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}}) + \rho_1\kappa\ell\xi\mathrm{Re}(i\hat{s}\bar{\hat{q}}) \end{aligned} \tag{3.33}$$

which comes from (3.6). Then computing (3.7) $\times\ \rho_1\kappa(1+\ell^2)(1+\ell^2+\xi^2)/(\rho_2 b)$ + (3.32) + (3.33) $\times\ \xi^2/64$, we get

$$\begin{aligned} &\partial_t\mathcal{E}_{11} + \frac{\rho_1\kappa}{b}(1+\ell^2)(1+\ell^2+\xi^2)\xi\partial_t\mathrm{Re}(i\hat{z}\bar{\hat{y}}) - \frac{\rho_1}{64}\xi^3\partial_t\mathrm{Re}(i\hat{v}\bar{\hat{s}}) \\ &+ \frac{1}{4}(1+\ell^2)\xi^2|\hat{v}|^2 + \frac{1}{128}\xi^4|\hat{v}|^2 + \frac{3\rho_1\kappa}{8}(1+\ell^2)(1+\ell^2+\xi^2)\xi^2|\hat{y}|^2 \\ &+ \frac{1}{64}(1+\ell^2+\xi^2)\ell^2|\hat{p}|^2 + \frac{\rho_1\kappa}{32}\ell^2\xi^2|\hat{q}|^2 \\ &\le \left\{C_{11} + C_{13} + \frac{\rho_1(\kappa-\kappa_0)^2}{8\kappa}\right\}(1+\ell^2+\xi^2)^2|\hat{s}|^2 \\ &+ (C_{12}+C_{14})(1+\ell^2)(1+\ell^2+\xi^2)^2|\hat{z}|^2 + (\rho_1\kappa - \rho_2 b\kappa_0\ell^2)(1+\ell^2)\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}}) \\ &+ \frac{\rho_1\kappa}{\rho_2 b}(1+\ell^2)(1+\ell^2+\xi^2)\xi\mathrm{Re}(i\hat{v}\bar{\hat{z}}) + \frac{\rho_1\kappa}{64}\xi^3\mathrm{Re}(i\hat{s}\bar{\hat{y}}) + \frac{\rho_1\kappa}{64}\ell\xi^3\mathrm{Re}(i\hat{s}\bar{\hat{q}}) \end{aligned}$$

where

$$C_{13} := \frac{1}{64}\max\left\{\alpha^2, \rho_1\kappa\right\}, \qquad C_{14} := \frac{\rho_1\kappa}{\rho_2 b}\max\left\{1, \frac{\rho_2\delta^2}{2b}\right\}.$$

Hence this gives

$$\begin{aligned} &\partial_t\mathcal{E}_{11} + \frac{\rho_1\kappa}{b}(1+\ell^2)(1+\ell^2+\xi^2)\xi\partial_t\mathrm{Re}(i\hat{z}\bar{\hat{y}}) - \frac{\rho_1}{64}\xi^3\partial_t\mathrm{Re}(i\hat{v}\bar{\hat{s}}) \\ &+ \frac{1}{8}(1+\ell^2)\xi^2|\hat{v}|^2 + \frac{1}{128}\xi^4|\hat{v}|^2 + \frac{7\rho_1\kappa}{64}(1+\ell^2)(1+\ell^2+\xi^2)\xi^2|\hat{y}|^2 \\ &+ \frac{1}{64}(1+\ell^2+\xi^2)\ell^2|\hat{p}|^2 + \frac{\rho_1\kappa}{64}\ell^2\xi^2|\hat{q}|^2 \\ &\le \left\{C_{11} + C_{13} + \frac{\rho_1(\kappa-\kappa_0)^2}{8\kappa}\right\}(1+\ell^2+\xi^2)^2|\hat{s}|^2 + \frac{(\rho_1\kappa - \rho_2 b\kappa_0\ell^2)^2}{\rho_1\kappa}|\hat{s}|^2 \\ &+ \frac{\rho_1\kappa}{256}(1+\xi^2)\xi^2|\hat{s}|^2 + \left(C_{12} + C_{14} + \frac{2\rho_1^2\kappa^2}{\rho_2^2 b^2}\right)(1+\ell^2)(1+\ell^2+\xi^2)^2|\hat{z}|^2 \\ &\le C_1(1+\ell^2)(1+\ell^2+\xi^2)^2(|\hat{s}|^2+|\hat{z}|^2), \end{aligned} \tag{3.34}$$

where

$$C_1 := \max\left\{C_{11}+C_{13}+C_{15}+\frac{\rho_1(\kappa-\kappa_0)^2}{8\kappa}+\frac{\rho_1\kappa}{256}, C_{12}+C_{14}+\frac{2\rho_1^2\kappa^2}{\rho_2^2b^2}\right\},$$
$$C_{15} := \max\left\{\rho_1\kappa, \frac{\rho_2^2b^2\kappa_0^2}{\rho_1\kappa}\right\}.$$

Consequently, calculating (3.5) + (3.34) $\times\, c_1/(2C_1(1+\ell^2)(1+\ell^2+\xi^2)^2)$, we arrive at

$$\begin{aligned}&\partial_t\mathcal{E}_1+\frac{c_1}{256C_1}\frac{\xi^2}{(1+\ell^2)(1+\ell^2+\xi^2)}|\hat{v}|^2+\frac{7\rho_1\kappa c_1}{128C_1}\frac{\xi^2}{1+\ell^2+\xi^2}|\hat{y}|^2\\&+\frac{c_1}{128C_1}\frac{\ell^2}{(1+\ell^2)(1+\ell^2+\xi^2)}|\hat{p}|^2+\frac{\rho_1\kappa c_1}{128C_1}\frac{\ell^2\xi^2}{(1+\ell^2)(1+\ell^2+\xi^2)^2}|\hat{q}|^2\\&+\frac{c_1}{2}(|\hat{s}|^2+|\hat{z}|^2)\le 0,\end{aligned}\tag{3.35}$$

where

$$\begin{aligned}\mathcal{E}_1 :=\ &\frac{1}{2}\mathcal{E}+\frac{c_1}{2C_1(1+\ell^2)(1+\ell^2+\xi^2)^2}\mathcal{E}_{11}\\&+\frac{\rho_1\kappa c_1\xi}{2bC_1(1+\ell^2+\xi^2)}\mathrm{Re}(i\hat{z}\bar{\hat{y}})-\frac{\rho_1c_1\xi^3}{128C_1(1+\ell^2)(1+\ell^2+\xi^2)^2}\mathrm{Re}(i\hat{v}\bar{\hat{s}}),\end{aligned}$$

and c_1 is a positive constant taken as $c_1\le\min\{\alpha,\delta/b\}$.

Furthermore, there exist the positive constants c_1^* and C_1^* such that $c_1^*|\hat{u}|^2\le\mathcal{E}_1\le C_1^*|\hat{u}|^2$. Therefore, the estimate (3.35) gives

$$\begin{aligned}&|\hat{u}|^2+\frac{1}{c_1^*}\int_0^t\frac{c_1}{256C_1}\frac{\xi^2}{(1+\ell^2)(1+\ell^2+\xi^2)}|\hat{v}|^2+\frac{7\rho_1\kappa c_1}{128C_1}\frac{\xi^2}{1+\ell^2+\xi^2}|\hat{y}|^2\\&+\frac{c_1}{128C_1}\frac{\ell^2}{(1+\ell^2)(1+\ell^2+\xi^2)}|\hat{p}|^2+\frac{\rho_1\kappa c_1}{128C_1}\frac{\ell^2\xi^2}{(1+\ell^2)(1+\ell^2+\xi^2)^2}|\hat{q}|^2\\&+\frac{c_1}{2}(|\hat{s}|^2+|\hat{z}|^2)d\tau\le\frac{C_1^*}{c_1^*}|\hat{u}_0|^2,\end{aligned}$$

and the pointwise estimate (3.2).

In the case $\kappa=\kappa_0$, calculating (3.7) $\times\,\rho_1\kappa(1+\ell^2)^2/(\rho_2b)$ + (3.32), we obtain

$$\begin{aligned}&\partial_t\mathcal{E}_{11}+\frac{\rho_1\kappa}{b}(1+\ell^2)^2\xi\partial_t\mathrm{Re}(i\hat{z}\bar{\hat{y}})+\frac{1}{4}(1+\ell^2)\xi^2|\hat{v}|^2\\&+\frac{3\rho_1\kappa}{8}(1+\ell^2)^2\xi^2|\hat{y}|^2+\frac{1}{32}(1+\ell^2+\xi^2)\ell^2|\hat{p}|^2+\frac{\rho_1\kappa}{32}\ell^2\xi^2|\hat{q}|^2\\&\le C_{11}(1+\ell^2)(1+\ell^2+\xi^2)|\hat{s}|^2+(C_{12}+C_{14})(1+\ell^2)^2(1+\ell^2+\xi^2)|\hat{z}|^2\\&+(\rho_1\kappa-\rho_2b\kappa_0\ell^2)(1+\ell^2)\xi\mathrm{Re}(i\hat{s}\bar{\hat{y}})+\frac{\rho_1\kappa}{\rho_2b}(1+\ell^2)^2\xi\mathrm{Re}(i\hat{v}\bar{\hat{z}}),\end{aligned}$$

Thus this gives

$$
\begin{aligned}
&\partial_t\mathcal{E}_{11}+\frac{\rho_1\kappa}{b}(1+\ell^2)^2\xi\partial_t\mathrm{Re}(i\hat{z}\bar{\hat{y}})+\frac{1}{8}(1+\ell^2)\xi^2|\hat{v}|^2\\
&+\frac{\rho_1\kappa}{8}(1+\ell^2)^2\xi^2|\hat{y}|^2+\frac{1}{32}(1+\ell^2+\xi^2)\ell^2|\hat{p}|^2+\frac{\rho_1\kappa}{32}\ell^2\xi^2|\hat{q}|^2\\
&\le(C_{11}+C_{15})(1+\ell^2)(1+\ell^2+\xi^2)|\hat{s}|^2\\
&+\left(C_{12}+C_{14}+\frac{2\rho_1^2\kappa^2}{\rho_2^2b^2}\right)(1+\ell^2)^2(1+\ell^2+\xi^2)|\hat{z}|^2\\
&\le\tilde{C}_1(1+\ell^2)^2(1+\ell^2+\xi^2)(|\hat{s}|^2+|\hat{z}|^2),
\end{aligned}\tag{3.36}
$$

where

$$\tilde{C}_1:=\max\left\{C_{11}+C_{15},C_{12}+C_{14}+\frac{2\rho_1^2\kappa^2}{\rho_2^2b^2}\right\}.$$

Consequently, calculating $(3.5)+(3.36)\times\tilde{c}_1/(2\tilde{C}_1(1+\ell^2)^2(1+\ell^2+\xi^2))$, we arrive at

$$
\begin{aligned}
&\partial_t\tilde{\mathcal{E}}_1+\frac{\tilde{c}_1}{16\tilde{C}_1}\frac{\xi^2}{(1+\ell^2)(1+\ell^2+\xi^2)}|\hat{v}|^2+\frac{\rho_1\kappa\tilde{c}_1}{16\tilde{C}_1}\frac{\xi^2}{1+\ell^2+\xi^2}|\hat{y}|^2\\
&+\frac{\tilde{c}_1}{64\tilde{C}_1}\frac{\ell^2}{(1+\ell^2)^2}|\hat{p}|^2+\frac{\rho_1\kappa\tilde{c}_1}{64\tilde{C}_1}\frac{\ell^2\xi^2}{(1+\ell^2)^2(1+\ell^2+\xi^2)}|\hat{q}|^2+\frac{\tilde{c}_1}{2}(|\hat{s}|^2+|\hat{z}|^2)\le 0,
\end{aligned}\tag{3.37}
$$

where

$$\tilde{\mathcal{E}}_1:=\frac{1}{2}\mathcal{E}+\frac{\tilde{c}_1}{2\tilde{C}_1}\frac{1}{(1+\ell^2)^2(1+\ell^2+\xi^2)}\mathcal{E}_{11}+\frac{\rho_1\kappa\tilde{c}_1}{2b\tilde{C}_1}\frac{\xi}{1+\ell^2+\xi^2}\mathrm{Re}(i\hat{z}\bar{\hat{y}}).$$

Furthermore, there exist the positive constants $\tilde{c}_1^*$ and $\tilde{C}_1^*$ such that $\tilde{c}_1^*|\hat{u}|^2\le\tilde{\mathcal{E}}_1\le\tilde{C}_1^*|\hat{u}|^2$. Therefore, the estimate (3.37) gives

$$
\begin{aligned}
&|\hat{u}|^2+\frac{1}{\tilde{c}_1^*}\int_0^t\Big\{\frac{\tilde{c}_1}{16\tilde{C}_1}\frac{\xi^2}{(1+\ell^2)(1+\ell^2+\xi^2)}|\hat{v}|^2+\frac{\rho_1\kappa\tilde{c}_1}{16\tilde{C}_1}\frac{\xi^2}{1+\ell^2+\xi^2}|\hat{y}|^2\\
&+\frac{\tilde{c}_1}{64\tilde{C}_1}\frac{\ell^2}{(1+\ell^2)^2}|\hat{p}|^2+\frac{\rho_1\kappa\tilde{c}_1}{64\tilde{C}_1}\frac{\ell^2\xi^2}{(1+\ell^2)^2(1+\ell^2+\xi^2)}|\hat{q}|^2+\frac{\tilde{c}_1}{2}(|\hat{s}|^2+|\hat{z}|^2)\Big\}d\sigma\\
&\le\frac{\tilde{C}_1^*}{\tilde{c}_1^*}|\hat{u}_0|^2,
\end{aligned}
$$

and the pointwise estimate (3.3). Namely this completes the proof. □

4. Eigenvalue problem

To observe the optimality of the decay estimate obtained in Theorems 3.1 and 3.2, we analyze the profile of the eigenvalues for (2.6). The eigenvalues satisfy the corresponding characteristic equation described as $\det(\lambda I-\hat{\Phi}(\zeta))=0$. Precisely, the characteristic equation for (2.9) is described as

$$
\begin{aligned}
&(\rho_1\lambda+\alpha)\left(\frac{\lambda(\rho_1\lambda+\gamma)}{\kappa_0}-\zeta^2\right)\left\{\frac{\lambda+\delta}{b}+\frac{\lambda}{\kappa}\left(\frac{\rho_2\lambda(\lambda+\delta)}{b}-\zeta^2\right)\right\}\\
&-\zeta^2\left(\frac{\lambda(\rho_1\lambda+\gamma)}{\kappa_0}-\zeta^2\right)\left(\frac{\rho_2\lambda(\lambda+\delta)}{b}-\zeta^2\right)\\
&+\ell^2(\rho_1\lambda+\gamma)\left\{\frac{\lambda+\delta}{b}+\frac{\lambda}{\kappa}\left(\frac{\rho_2\lambda(\lambda+\delta)}{b}-\zeta^2\right)\right\}\\
&+\ell^2\left(\frac{\rho_2\lambda(\lambda+\delta)}{b}-\zeta^2\right)\left\{\left(\frac{\lambda(\rho_1\lambda+\alpha)}{\kappa_0}+\ell^2\right)+2\zeta^2\right\}=0.
\end{aligned}
\tag{4.1}
$$

To study the asymptotic expansion for λ, we suppose $\rho_1=\rho_2=1$ for simplicity. We apply the perturbation theory (see [6]) for one-parameter family of matrices for $|\zeta|\to 0$ to $\hat{\Phi}(\zeta)$. Then we deduce that $\lambda_j(\zeta)$ has a following asymptotic expansions for $|\zeta|\to 0$:

$$
\lambda_j(\zeta)=\sum_{k=0}^{\infty}\lambda_{j,k}\zeta^k \tag{4.2}
$$

for $j=1,\cdots,6$. Analogously, we study the asymptotic expansion for $|\zeta|\to\infty$. We rewrite $\hat{\Phi}(\zeta)$ that $\hat{\Phi}(\zeta)=-\zeta(A^0)^{-1}(A+\zeta^{-1}L)$ and apply the perturbation theory for $|\zeta|^{-1}\to 0$ to $-(A^0)^{-1}(A+\zeta^{-1}L)$. Then this gives the asymptotic expansion

$$
\lambda_j(\zeta)=\mu_{j,1}\zeta+\sum_{k=0}^{\infty}\mu_{j,-k}\zeta^{-k} \tag{4.3}
$$

for $j=1,\cdots,6$.

Firstly, we treat the asymptotic expansion in the case $\alpha>0$ and $\gamma>0$. In the case for $|\zeta|\to 0$, substituting the expansion for $\lambda_j(\zeta)$ in (4.2) into (4.1) with $\rho_1=\rho_2=1$, we obtain the eigenvalue expansion:

$$
\begin{aligned}
\lambda_j(\zeta)&=\lambda^{(0)}+O(|\zeta|),\\
\lambda_k(\zeta)&=\frac{1}{2}\left(-\alpha\pm\sqrt{\alpha^2-4\kappa_0\ell^2}\right)+O(|\zeta|),\\
\lambda_6(\zeta)&=-\delta+O(|\zeta|)
\end{aligned}
$$

for $j=1,2,3$ and $k=4,5$. Here $\lambda^{(0)}$ are roots of $f_1(\lambda)=0$ with $f_1(\lambda):=\lambda^3+\gamma\lambda^2+\kappa(1+\ell^2)\lambda+\gamma\kappa$. We note that $\mathrm{Re}\lambda^{(0)}<0$, which comes from $f_1(0)=\gamma\kappa>0$ and $f_1(-\gamma)=-\gamma\kappa\ell^2<0$. On the other hand, in the case for $|\zeta|\to\infty$, substituting the expansion for $\lambda_j(\zeta)$ in (4.3) into (4.1) with $\rho_1=\rho_2=1$, we also obtain the eigenvalue expansion:

$$
\begin{aligned}
\lambda_j(\zeta)&=\pm\sqrt{b}\zeta-\frac{\delta}{2}+O(|\zeta|^{-1}),\\
\lambda_k(\zeta)&=\pm\sqrt{\kappa}\zeta-\frac{\alpha}{2}+O(|\zeta|^{-1}),\\
\lambda_h(\zeta)&=\pm\sqrt{\kappa_0}\zeta-\frac{\gamma}{2}+O(|\zeta|^{-1})
\end{aligned}
$$

for $j=1,2$, $k=3,4$ and $h=5,6$ if $b\neq\kappa$, $\kappa\neq\kappa_0$ and $b\neq\kappa_0$,

$$\begin{aligned}
\lambda_j(\zeta) &= \sqrt{b}\zeta+\frac{1}{4}\left\{-(\alpha+\delta)\pm\sqrt{(\alpha+\delta)^2-4(\alpha\delta+b)}\right\}+O(|\zeta|^{-1}),\\
\lambda_k(\zeta) &= -\sqrt{b}\zeta+\frac{1}{4}\left\{-(\alpha+\delta)\pm\sqrt{(\alpha+\delta)^2-4(\alpha\delta+b)}\right\}+O(|\zeta|^{-1}),\\
\lambda_h(\zeta) &= \pm\sqrt{\kappa_0}\zeta-\frac{\gamma}{2}+O(|\zeta|^{-1})
\end{aligned}$$

for $j=1,2$, $k=3,4$ and $h=5,6$ if $b=\kappa$ and $\kappa\neq\kappa_0$,

$$\begin{aligned}
\lambda_j(\zeta) &= \sqrt{\kappa}\zeta+\frac{1}{4}\left\{-(\alpha+\gamma)\pm\sqrt{(\alpha+\gamma)^2-4(\alpha\gamma+4\kappa\ell^2)}\right\}+O(|\zeta|^{-1}),\\
\lambda_k(\zeta) &= -\sqrt{\kappa}\zeta+\frac{1}{4}\left\{-(\alpha+\gamma)\pm\sqrt{(\alpha+\gamma)^2-4(\alpha\gamma+4\kappa\ell^2)}\right\}+O(|\zeta|^{-1}),\\
\lambda_h(\zeta) &= \pm\sqrt{b}\zeta-\frac{\delta}{2}+O(|\zeta|^{-1})
\end{aligned}$$

for $j=1,2$, $k=3,4$ and $h=5,6$ if $b\neq\kappa$ and $\kappa=\kappa_0$, and

$$\lambda_j(\zeta)=\sqrt{b}\zeta+\mu_j^{(0)}+O(|\zeta|^{-1}),\qquad \lambda_k(\zeta)=-\sqrt{b}\zeta+\mu_{k-3}^{(0)}+O(|\zeta|^{-1})$$

for $j=1,2,3$ and $k=4,5,6$ if $b=\kappa=\kappa_0$. Here $\mu_j^{(0)}$ are roots of $f_2(\mu)=0$ with

$$f_2(\mu):=\mu^3+\frac{1}{2}(\alpha+\gamma+\delta)\mu^2+\frac{1}{4}(b+\alpha\gamma+\gamma\delta+\alpha\delta+4b\ell^2)\mu+\frac{1}{8}(b\gamma+\alpha\gamma\delta+4b\delta\ell^2).$$

We note that $\mathrm{Re}\mu_j^{(0)}<0$, which also comes from $f_2(0)=(b\gamma+\alpha\gamma\delta+4b\delta\ell^2)/8>0$ and

$$f_2\left(-\frac{1}{2}(\alpha+\gamma+\delta)\right)=-\frac{1}{8}(\alpha+\gamma)(\gamma+\delta)(\alpha+\delta)-\frac{1}{8}b(\alpha+\delta)-\frac{1}{2}b\ell^2(\alpha+\gamma)<0.$$

Secondly, we consider the asymptotic expansion in the case for $\alpha>0$ and $\gamma=0$. Using the same argument as before, we obtain the following expansion.

$$\begin{aligned}
\lambda_1(\zeta) &= \lambda^{(2)}\zeta^2+O(|\zeta|^4),\\
\lambda_j(\zeta) &= \pm\sqrt{\kappa(1+\ell^2)}i+\lambda_\pm^{(2)}\zeta^2+O(|\zeta|^3),\\
\lambda_5(\zeta) &= -\delta+O(|\zeta|),\\
\lambda_k(\zeta) &= \frac{1}{2}\left(-\alpha\pm\sqrt{\alpha^2-4\kappa_0\ell^2}\right)+O(|\zeta|)
\end{aligned}\tag{4.4}$$

for $j=2,3$ and $k=5,6$, where

$$\lambda^{(2)}:=\frac{\alpha\delta+b\ell^4}{\delta\ell^2(1+\ell^2)},\qquad \lambda_\pm^{(2)}:=\lambda_1^{(2)}\pm\lambda_2^{(2)}\sqrt{\frac{\kappa}{1+\ell^2}}i\tag{4.5}$$

and

$$\lambda_1^{(2)} := \frac{1}{2(1+\ell^2)} \left\{ \frac{4\alpha\kappa\kappa_0\ell^2(1+\ell^2) + \alpha(\kappa_0\ell^2 - \kappa(1+\ell^2))^2}{\alpha^2\kappa(1+\ell^2) + (\kappa_0\ell^2 - \kappa(1+\ell^2))^2} + \frac{\delta b}{\delta^2 + \kappa(1+\ell^2)} \right\},$$

$$\lambda_2^{(2)} := \frac{1}{2} \left\{ \frac{(4\kappa_0\ell^2 - \alpha^2)(\kappa_0\ell^2 - \kappa(1+\ell^2))}{\alpha^2\kappa(1+\ell^2) + (\kappa_0\ell^2 - \kappa(1+\ell^2))^2} - \frac{b}{\delta^2 + \kappa(1+\ell^2)} - 1 \right\}.$$

Here we note that $\mathrm{Re}\lambda^{(2)} > 0$. On the other hand, we also obtain

$$\begin{aligned} &\lambda_j(\zeta) = \pm\sqrt{\kappa_0}\zeta \pm \frac{\sqrt{\kappa_0}(3\kappa+\kappa_0)\ell^2}{2(\kappa-\kappa_0)}\zeta^{-1} + \frac{\alpha(\kappa+\kappa_0)^2\ell^2}{2(\kappa-\kappa_0)^2}\zeta^{-2} + O(|\zeta|^{-3}), \\ &\lambda_k(\zeta) = \pm\sqrt{b}\zeta - \frac{\delta}{2} + O(|\zeta|^{-1}), \qquad \lambda_h(\zeta) = \pm\sqrt{\kappa}\zeta - \frac{\alpha}{2} + O(|\zeta|^{-1}) \end{aligned} \tag{4.6}$$

for $j = 1,2$, $k = 3,4$ and $h = 5,6$ if $b \neq \kappa$, $\kappa \neq \kappa_0$ and $b \neq \kappa_0$,

$$\begin{aligned} &\lambda_j(\zeta) = \pm\sqrt{\kappa_0}\zeta \pm \frac{\sqrt{\kappa_0}(3b+\kappa_0)\ell^2}{2(b-\kappa_0)}\zeta^{-1} + \frac{\alpha(b+\kappa_0)^2\ell^2}{2(b-\kappa_0)^2}\zeta^{-2} + O(|\zeta|^{-3}), \\ &\lambda_k(\zeta) = \sqrt{b}\zeta + \frac{1}{4}\left\{-(\alpha+\delta) \pm \sqrt{(\alpha+\delta)^2 - 4(\alpha\delta+b)}\right\} + O(|\zeta|^{-1}), \\ &\lambda_h(\zeta) = -\sqrt{b}\zeta + \frac{1}{4}\left\{-(\alpha+\delta) \pm \sqrt{(\alpha+\delta)^2 - 4(\alpha\delta+b)}\right\} + O(|\zeta|^{-1}) \end{aligned}$$

for $j = 1,2$, $k = 3,4$ and $h = 5,6$ if $b = \kappa$ and $\kappa \neq \kappa_0$,

$$\begin{aligned} &\lambda_j(\zeta) = \pm\sqrt{b}\zeta \pm \frac{\sqrt{b}(3\kappa+b)\ell^2}{2(\kappa-b)}\zeta^{-1} + \frac{(\alpha\delta+\kappa)(\kappa+b)^2}{2\delta(\kappa-b)^2}\zeta^{-2} + O(|\zeta|^{-3}), \\ &\lambda_k(\zeta) = \pm\sqrt{b}\zeta - \frac{\delta}{2} + O(|\zeta|^{-1}), \qquad \lambda_h(\zeta) = \pm\sqrt{\kappa}\zeta - \frac{\alpha}{2} + O(|\zeta|^{-1}) \end{aligned}$$

for $j = 1,2$, $k = 3,4$ and $h = 5,6$ if $b \neq \kappa$ and $b = \kappa_0$,

$$\begin{aligned} &\lambda_j(\zeta) = \sqrt{\kappa}\zeta + \frac{1}{4}\left(-\alpha \pm \sqrt{\alpha^2 - 16\kappa\ell^2}\right) + O(|\zeta|^{-1}), \\ &\lambda_k(\zeta) = -\sqrt{\kappa}\zeta + \frac{1}{4}\left(-\alpha \pm \sqrt{\alpha^2 - 16\kappa\ell^2}\right) + O(|\zeta|^{-1}), \\ &\lambda_h(\zeta) = \pm\sqrt{b}\zeta - \frac{\delta}{2} + O(|\zeta|^{-1}) \end{aligned}$$

for $j = 1,2$, $k = 3,4$ and $h = 5,6$ if $b \neq \kappa$ and $\kappa = \kappa_0$, and

$$\lambda_j(\zeta) = \sqrt{b}\zeta + \mu_j^{(0)} + O(|\zeta|^{-1}), \qquad \lambda_k(\zeta) = -\sqrt{b}\zeta + \mu_{k-3}^{(0)} + O(|\zeta|^{-1})$$

for $j = 1,2,3$ and $k = 4,5,6$ if $b = \kappa = \kappa_0$. Here $\mu_j^{(0)}$ are roots of $f_2(\mu) = 0$ with $\gamma = 0$. Especially, we find that $\mathrm{Re}\mu_j^{(0)} < 0$ for $j = 1,2,3$.

Consequently, these asymptotic expansions for the eigenvalues guarantee the optimality of the pointwise estimates in Theorems 3.1 and 3.2.

5. Spectral representation

In this section, we consider the case that $\alpha > 0$, $\gamma = 0$ and $b \neq \kappa$, $\kappa \neq \kappa_0$, $b \neq \kappa_0$. Let $P_j(\zeta)$ be the corresponding eigenprojections of the eigenvalue $\lambda_j(\zeta)$. Here we note that the eigenprojection $P_j(\zeta)$ is described as

$$P_j(\zeta) = \prod_{k \neq j} \frac{\hat{\Phi}(\zeta) - \lambda_k(\zeta) I}{\lambda_j(\zeta) - \lambda_k(\zeta)}$$

as long as $\lambda_j(\zeta)$ are distinct to each other. To get more detailed properties of the eigenprojections, we apply the perturbation theory to (2.4), which is mentioned in [6] for one-parameter family of matrices for $|\zeta| \to 0$ and $|\zeta| \to \infty$. Employing the perturbation theory for $|\zeta| \to 0$, we can deduce that $\lambda_j(\zeta)$ have the asymptotic expansions (4.4), and $P_j(\zeta)$ also have the asymptotic expansions

$$P_j(\zeta) = \sum_{k=0}^{\infty} P_j^{(k)} \zeta^k \tag{5.1}$$

for $j = 1, 2, 3$ with

$$P_1^{(0)} = \frac{1}{1+\ell^2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ell^2 & 0 & -\ell \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\ell & 0 & 1 \end{pmatrix},$$

$$P_2^{(0)} = \frac{1}{2(1+\ell^2)} \begin{pmatrix} 1+\ell^2 & 0 & 0 & -\sqrt{\kappa(1+\ell^2)}i & 0 & -\ell\sqrt{\kappa(1+\ell^2)}i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{(1+\ell^2)/\kappa}i & 0 & 0 & 1 & 0 & \ell \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \ell\sqrt{(1+\ell^2)/\kappa}i & 0 & 0 & \ell & 0 & \ell^2 \end{pmatrix}, \tag{5.2}$$

$$P_3^{(0)} = \frac{1}{2(1+\ell^2)} \begin{pmatrix} 1+\ell^2 & 0 & 0 & \sqrt{\kappa(1+\ell^2)}i & 0 & \ell\sqrt{\kappa(1+\ell^2)}i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{(1+\ell^2)/\kappa}i & 0 & 0 & 1 & 0 & \ell \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\ell\sqrt{(1+\ell^2)/\kappa}i & 0 & 0 & \ell & 0 & \ell^2 \end{pmatrix}$$

On the other hand, we next employ the perturbation theory for $|\zeta| \to \infty$. Then we obtain (4.6) and the expansion of the eigenprojection $P_j(\zeta)$ for $\lambda_j(\zeta)$ that

$$P_j(i\xi) = \sum_{k=0}^{\infty} Q_j^{(k)} \zeta^{-k} \tag{5.3}$$

for $j=1,2$ with

$$(5.4)\qquad Q_1^{(0)}=\frac{1}{2}\begin{pmatrix}0\,0\,0\,0 & 0 & 0\\ 0\,0\,0\,0 & 0 & 0\\ 0\,0\,0\,0 & 0 & 0\\ 0\,0\,0\,0 & 0 & 0\\ 0\,0\,0\,0 & 1 & \sqrt{\kappa_0}\\ 0\,0\,0\,0 & 1/\sqrt{\kappa_0} & 1\end{pmatrix},\qquad Q_2^{(0)}=\frac{1}{2}\begin{pmatrix}0\,0\,0\,0 & 0 & 0\\ 0\,0\,0\,0 & 0 & 0\\ 0\,0\,0\,0 & 0 & 0\\ 0\,0\,0\,0 & 0 & 0\\ 0\,0\,0\,0 & 1 & -\sqrt{\kappa_0}\\ 0\,0\,0\,0 & -1/\sqrt{\kappa_0} & 1\end{pmatrix}.$$

Our next purpose is to get the asymptotic representations of $e^{t\hat{\Phi}(i\xi)}$ as $|\xi|\to 0$ and $|\xi|\to\infty$ by using the above expressions. We have a spectral decomposition of the matrix $e^{t\hat{\Phi}(i\xi)}$ that

$$(5.5)\qquad e^{t\hat{\Phi}(i\xi)}=\sum_{j=1}^{3}e^{\lambda_j(i\xi)t}P_j(i\xi).$$

We first consider the case $|\xi|\to 0$. Inspired by (4.4) and (5.1) with (5.2), we define

$$(5.6)\qquad \hat{G}(t,i\xi):=\sum_{j=1}^{3}e^{\lambda_j^0(i\xi)t}P_j^{(0)},\qquad \hat{R}_0(t,i\xi):=e^{t\hat{\Phi}(i\xi)}-\hat{G}(t,i\xi),$$

where we define

$$\lambda_1^0(i\xi):=-\lambda^{(2)}\xi^2,\qquad \lambda_j^0(i\xi):=\pm\sqrt{\kappa(1+\ell^2)}i-\lambda_\pm^{(2)}\xi^2$$

for $j=2,3$, and $\lambda^{(2)}$ and $\lambda_\pm^{(2)}$ are defined in (4.5). Then we have the following pointwise estimate for the low-frequency part.

Lemma 5.1. *There is a positive constant r_0 such that if $|\xi|\le r_0$, then the remainder term $\hat{R}_0(t,i\xi)$ satisfies*

$$(5.7)\qquad |\hat{R}_0(t,i\xi)|\le C|\xi|e^{-c\xi^2t}+Ce^{-ct},$$

where C and c are certain positive constants.

Proof. We have from (5.5) and (5.6)

$$\begin{aligned}\hat{R}_0(t,i\xi)&=\sum_{j=1}^{3}e^{\lambda_j(i\xi)t}(P_j(i\xi)-P_j^{(0)})\\&\qquad+\sum_{j=1}^{3}e^{\lambda_j^0(i\xi)t}\{e^{(\lambda_j(i\xi)-\lambda_j^0(i\xi))t}-1\}P_j^{(0)}+\sum_{j=4}^{6}e^{\lambda_j(i\xi)t}P_j(i\xi)\\&=:R_{01}+R_{02}+R_{03}.\end{aligned}$$

It follows from (4.4) and (5.1) with (5.2) that $\mathrm{Re}\lambda_j(i\xi)\le -c|\xi|^2$ and $|P_j(i\xi)-P_j^{(0)}|\le C|\xi|$ for $|\xi|\le r_0$, where $j=1,2,3$. Hence we obtain $|R_{01}|\le C|\xi|e^{-c\xi^2t}$ for $|\xi|\le r_0$. On the other hand, from the fact that $|e^{(\lambda_j(i\xi)-\lambda_j^0(i\xi))t}-1|\le C\xi^4te^{C\xi^4t}$ for $|\xi|\le r_0$, where $j=1,2,3$, we have

$$|R_{02}|\le Ce^{-c\xi^2t}\xi^4te^{C\xi^4t}\le C\xi^4te^{-c\xi^2t}\le C\xi^2e^{-c\xi^2t}$$

for $|\xi| \le r_0$. Furthermore, it is easy to derive that $|R_{03}| \le Ce^{-ct}$ for $|\xi| \le r_0$. Consequently, these observations give us the desired estimate (5.7) and hence the proof is completed. □

Similarly, we next consider the case $|\xi| \to \infty$. Inspired by (4.6) and (5.3) with (5.4), we define

$$\hat{H}(t, i\xi) := e^{\lambda_1^\infty(i\xi)t} Q_1^{(0)} + e^{\lambda_2^\infty(i\xi)t} Q_2^{(0)}, \qquad \hat{R}_1(t, i\xi) := e^{t\hat{\Phi}(i\xi)} - \hat{H}(t, i\xi), \tag{5.8}$$

where

$$\lambda_j^\infty(i\xi) := \pm\sqrt{\kappa_0} i\xi \mp \frac{\sqrt{\kappa_0}(3\kappa + \kappa_0)\ell^2}{2(\kappa - \kappa_0)} i\xi^{-1} - \frac{\alpha(\kappa + \kappa_0)^2\ell^2}{2(\kappa - \kappa_0)^2}\xi^{-2}$$

for $j = 1, 2$. Then we also get the following pointwise estimate for the high-frequency part.

Lemma 5.2. *There is a positive constant r_1 such that if $|\xi| \ge r_1$, then the remainder term $\hat{R}_1(t, i\xi)$ satisfies*

$$|\hat{R}_1(t, i\xi)| \le C|\xi|^{-1} e^{-c\xi^{-2}t} + Ce^{-ct}, \tag{5.9}$$

where C and c are certain positive constants.

Proof. Employing (5.5) and (5.8), we divide $\hat{R}_1(t, \xi)$ as follows.

$$\begin{aligned}
\hat{R}_1(t, i\xi) &= \sum_{j=1}^{2} e^{\lambda_j(i\xi)t}(P_j(i\xi) - Q_j^{(0)}) \\
&\quad + \sum_{j=1}^{2} e^{\lambda_j^\infty(i\xi)t}\{e^{(\lambda_j(i\xi) - \lambda_j^\infty(i\xi))t} - 1\}Q_j^{(0)} + \sum_{j=3}^{6} e^{\lambda_j(i\xi)t}P_j(i\xi) \\
&=: R_{11} + R_{12} + R_{13}.
\end{aligned}$$

Using (4.6) and (5.3) with (5.4), we have for $j = 1, 2$ that $\mathrm{Re}\lambda_j(i\xi) \le -c|\xi|^{-2}$ and $|P_j(i\xi) - Q_j^{(0)}| \le C|\xi|^{-1}$ for $|\xi| \ge r_1$. Hence we obtain $|R_{11}| \le C|\xi|^{-1}e^{-c\xi^{-2}t}$ for $|\xi| \ge r_1$. On the other hand, we also obtain for $j = 1, 2$ the fact $|e^{(\lambda_j(i\xi) - \lambda_j^\infty(\xi))t} - 1| \le C|\xi|^{-3}te^{C|\xi|^{-3}t}$ for $|\xi| \ge r_1$. Thus we can estimate

$$|R_{12}| \le Ce^{-c\xi^{-2}t}|\xi|^{-3}te^{C|\xi|^{-3}t} \le C|\xi|^{-3}te^{-c\xi^{-2}t} \le C|\xi|^{-1}e^{-c\xi^{-2}t}$$

for $|\xi| \ge r_1$. Furthermore, it is clear that $|R_{13}| \le Ce^{-ct}$ for $|\xi| \ge r_1$. Consequently, we combine these estimates and arrive at the desired estimate (5.9). This completes the proof. □

Finally we introduce $\hat{R}(t, i\xi)$ that

$$e^{t\hat{\Phi}(i\xi)} = \hat{G}(t, i\xi) + \hat{H}(t, i\xi) + \hat{R}(t, i\xi).$$

Then, from Lemma 5.1 and 5.2, we conclude the following pointwise estimate for the remainder term $\hat{R}(t, i\xi)$.

Proposition 5.3. *There are positive constants r_0 and r_1 such that the remainder term $\hat{R}(t,i\xi)$ satisfies*

$$|\hat{R}(t,i\xi)| \le \begin{cases} C|\xi|e^{-c\xi^2 t} + Ce^{-ct} & for \quad |\xi| \le r_0, \\ Ce^{-ct} & for \quad r_0 \le |\xi| \le r_1, \\ C|\xi|^{-1}e^{-c\xi^{-2}t} + Ce^{-ct} & for \quad |\xi| \ge r_1, \end{cases} \tag{5.10}$$

where C and c are certain positive constants.

Proof. We first prove in the case $|\xi| \le r_0$. From the relation $\hat{R}(t,i\xi) = \hat{R}_0(t,i\xi) - \hat{H}(t,i\xi)$ and Lemma 5.1, we compute

$$|\hat{R}(t,i\xi)| \le |\hat{R}_0(t,i\xi)| + |\hat{H}(t,i\xi)| \le C|\xi|e^{-c\xi^2 t} + Ce^{-ct}$$

for $|\xi| \le r_0$. Similarly, we can show in the case $|\xi| \ge r_1$. Using $\hat{R}(t,i\xi) = \hat{R}_1(t,i\xi) - \hat{G}(t,i\xi)$ and Lemma 5.2, we compute

$$|\hat{R}(t,i\xi)| \le |\hat{R}_1(t,i\xi)| + |\hat{G}(t,i\xi)| \le C|\xi|^{-1}e^{-c\xi^{-2}t} + Ce^{-ct}$$

for $|\xi| \ge r_1$. Finally we derive in the case $r_0 \le |\xi| \le r_1$. We have from (5.6) and (5.8) that $|\hat{G}(t,i\xi)| + |\hat{H}(t,i\xi)| \le Ce^{-ct}$ for $r_0 \le |\xi| \le r_1$. Furthermore, we can obtain

$$|e^{t\hat{\Phi}(i\xi)}| \le Ce^{-c\frac{\xi^2}{(1+\xi^2)^2}t}$$

for $\xi \in \mathbb{R}$ by Theorem 3.2. This gives $|e^{t\hat{\Phi}(i\xi)}| \le Ce^{-ct}$ for $r_0 \le |\xi| \le r_1$, and hence

$$|\hat{R}(t,i\xi)| \le |e^{t\hat{\Phi}(i\xi)}| + |\hat{G}(t,i\xi)| + |\hat{H}(t,i\xi)| \le Ce^{-ct}$$

for $r_0 \le |\xi| \le r_1$. Consequently, these observations give us the desired estimate (5.10) and hence the proof is completed. □

6. Large time approximation

We study the representations for the inverse Fourier transform of $\hat{G}(t,i\xi)$ and $\hat{H}(t,i\xi)$ in this section. Let $\{e_1,e_2,e_3,e_4,e_5,e_6\}$ be a standard orthogonal basis of $\mathbb{R}^6$, and define the three-dimensional subspace X that $X = \mathrm{Span}\{e_1,e_4,e_6\}$. Then we introduce a matrix S_0 which corresponds to the restriction operator from $\mathbb{R}^6$ to X. Namely we define

$$S_0 := \begin{pmatrix} 1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,1 \end{pmatrix}.$$

Here we note that the matrix $S_0^\top$ denotes the extension operator from X to $\mathbb{R}^6$. Using this operator, we define $\mathcal{P}_j^{(0)} := S_0 P_j^{(0)} S_0^\top$ for $j = 1, 2, 3$, which means

$$\mathcal{P}_1^{(0)} = \frac{1}{1+\ell^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \ell^2 & -\ell \\ 0 & -\ell & 1 \end{pmatrix},$$

$$\mathcal{P}_2^{(0)} = \frac{1}{2(1+\ell^2)}\begin{pmatrix} 1+\ell^2 & -\sqrt{\kappa(1+\ell^2)}i & -\ell\sqrt{\kappa(1+\ell^2)}i \\ \sqrt{(1+\ell^2)/\kappa}i & 1 & \ell \\ \ell\sqrt{(1+\ell^2)/\kappa}i & \ell & \ell^2 \end{pmatrix},$$

$$\mathcal{P}_3^{(0)} = \frac{1}{2(1+\ell^2)}\begin{pmatrix} 1+\ell^2 & \sqrt{\kappa(1+\ell^2)}i & \ell\sqrt{\kappa(1+\ell^2)}i \\ -\sqrt{(1+\ell^2)/\kappa}i & 1 & \ell \\ -\ell\sqrt{(1+\ell^2)/\kappa}i & \ell & \ell^2 \end{pmatrix}.$$

Because the pair $\{\mathcal{P}_1^{(0)}, \mathcal{P}_2^{(0)}, \mathcal{P}_3^{(0)}\}$ is an orthogonal projection, we define

$$\hat{\Phi}_0(i\xi) := \sum_{j=1}^{3} \lambda_j^0(i\xi)\mathcal{P}_j^{(0)} = L_0 - \xi^2 B_0, \tag{6.1}$$

where

$$L_0 := \begin{pmatrix} 0 & \kappa & \kappa\ell \\ -1 & 0 & 0 \\ -\ell & 0 & 0 \end{pmatrix},$$

$$B_0 := \frac{1}{1+\ell^2}\begin{pmatrix} (1+\ell^2)\lambda_1^{(2)} & \kappa\lambda_2^{(2)} & \kappa\ell\lambda_2^{(2)} \\ -\lambda_2^{(2)} & \ell^2\lambda^{(2)} + \lambda_1^{(2)} & -\ell(\lambda^{(2)} - \lambda_1^{(2)}) \\ -\ell\lambda_2^{(2)} & -\ell(\lambda^{(2)} - \lambda_1^{(2)}) & \lambda^{(2)} + \ell^2\lambda_1^{(2)} \end{pmatrix}, \tag{6.2}$$

and the corresponding matrix exponential can be obtained by

$$e^{t\hat{\Phi}_0(i\xi)} = \sum_{j=1}^{3} e^{\lambda_j^0(i\xi)t}\mathcal{P}_j^{(0)}. \tag{6.3}$$

From (5.6) and (6.3), this yields

$$\hat{G}(t, i\xi) = S_0^\top e^{t\hat{\Phi}_0(i\xi)} S_0.$$

We define $(e^{t\Phi_0}\varphi)(x) := \mathcal{F}^{-1}[e^{t\hat{\Phi}_0(i\xi)}\hat{\varphi}(\xi)](x)$, and get the representation of the operator G that

$$(G(t)\varphi)(x) = \mathcal{F}^{-1}[S_0^\top e^{t\hat{\Phi}_0(i\xi)} S_0\hat{\varphi}(\xi)](x) = (S_0^\top e^{t\Phi_0} S_0\varphi)(x).$$

Similarly, we define the two-dimensional subspace Y that $Y = \mathrm{Span}\{e_5, e_6\}$, and introduce the 2×6 matrix S_∞ which corresponds to the restriction operator from $\mathbb{R}^6$ to Y:

$$S_\infty := \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here we remark that the 6×2 matrix $S_\infty^\top$ denotes the extension operator from Y to $\mathbb{R}^6$. We also define $\mathcal{Q}_1^{(0)} := S_\infty Q_1^{(0)} S_\infty^\top$ and $\mathcal{Q}_2^{(0)} := S_\infty Q_2^{(0)} S_\infty^\top$, and hence

$$\mathcal{Q}_1^{(0)} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{\kappa_0} \\ 1/\sqrt{\kappa_0} & 1 \end{pmatrix}, \qquad \mathcal{Q}_2^{(0)} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{\kappa_0} \\ -1/\sqrt{\kappa_0} & 1 \end{pmatrix}.$$

Because the pair $\{\mathcal{Q}_1^{(0)}, \mathcal{Q}_2^{(0)}\}$ is an orthogonal projection, we define

$$\begin{aligned}\hat{\Phi}_\infty(i\xi) &:= \lambda_1^\infty(i\xi)\mathcal{Q}_1^{(0)} + \lambda_2^\infty(i\xi)\mathcal{Q}_2^{(0)} \\ &= i\xi\begin{pmatrix} 0 & \kappa_0 \\ 1 & 0 \end{pmatrix} - \frac{(3\kappa+\kappa_0)\ell^2}{2(\kappa-\kappa_0)} i\xi^{-1}\begin{pmatrix} 0 & \kappa_0 \\ 1 & 0 \end{pmatrix} - \frac{\alpha(\kappa+\kappa_0)^2\ell^2}{2(\kappa-\kappa_0)^2}\xi^{-2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\end{aligned}$$

and the corresponding matrix exponential can be obtained by

$$e^{t\hat{\Phi}_\infty(i\xi)} = e^{\lambda_1^\infty(i\xi)t}\mathcal{Q}_1^{(0)} + e^{\lambda_2^\infty(i\xi)t}\mathcal{Q}_2^{(0)}. \tag{6.4}$$

Using (5.8) and (6.4), we have

$$\hat{H}(t, i\xi) = S_\infty^\top e^{t\hat{\Phi}_\infty(i\xi)} S_\infty.$$

We also define $(e^{t\Phi_\infty}\varphi)(x) := \mathcal{F}^{-1}[e^{t\hat{\Phi}_\infty(i\xi)}\hat{\varphi}(\xi)](x)$, and get the representation of the operator H that

$$(H(t)\varphi)(x) = \mathcal{F}^{-1}[S_\infty^\top e^{t\hat{\Phi}_\infty(i\xi)} S_\infty \hat{\varphi}(\xi)](x) = (S_\infty^\top e^{t\Phi_\infty} S_\infty \varphi)(x).$$

Now we can show the asymptotic decay estimates by virtue of Proposition 5.3.

Theorem 6.1. *Let $\sigma \geq 0$ and $1 \leq \theta \leq 2$. Suppose that $\varphi \in H^\sigma \cap L^\theta$. Then the semigroup operator $e^{t\Phi}$ defined in* (2.5) *satisfies*

$$\begin{aligned}&\|\partial_x^j(e^{t\Phi} - G(t))\varphi\|_{L^2} \\ &\qquad \leq C(1+t)^{-\frac{1}{2}(\frac{1}{\theta}-\frac{1}{2})-\frac{j}{2}-\frac{1}{2}}\|\varphi\|_{L^\theta} + C(1+t)^{-\frac{k}{2}}\|\partial_x^{j+k}\varphi\|_{L^2},\end{aligned} \tag{6.5}$$

$$\begin{aligned}&\|\partial_x^j(e^{t\Phi} - H(t))\varphi\|_{L^2} \\ &\qquad \leq C(1+t)^{-\frac{1}{2}(\frac{1}{\theta}-\frac{1}{2})-\frac{j}{2}}\|\varphi\|_{L^\theta} + C(1+t)^{-\frac{k}{2}}\|\partial_x^{j+k-1}\varphi\|_{L^2},\end{aligned} \tag{6.6}$$

$$\begin{aligned}&\|\partial_x^j(e^{t\Phi} - G(t) - H(t))\varphi\|_{L^2} \\ &\qquad \leq C(1+t)^{-\frac{1}{2}(\frac{1}{\theta}-\frac{1}{2})-\frac{j}{2}-\frac{1}{2}}\|\varphi\|_{L^\theta} + C(1+t)^{-\frac{k}{2}}\|\partial_x^{j+k-1}\varphi\|_{L^2}\end{aligned} \tag{6.7}$$

for $j, k \geq 0$, where $0 \leq j+k \leq \sigma$ in (6.5), *and $0 \leq j+k-1 \leq \sigma$ in* (6.6) *and* (6.7). *Here C is a certain positive constant.*

Remark 3. *We can obtain the estimates for G and H from* (5.6) *and* (5.8) *that*

$$\|\partial_x^j G(t)\varphi\|_{L^2} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{\theta}-\frac{1}{2})-\frac{j}{2}}\|\varphi\|_{L^\theta} + Ce^{-ct}\|\partial_x^j\varphi\|_{L^2}, \tag{6.8}$$

$$\|\partial_x^j H(t)\varphi\|_{L^2} \leq Ce^{-ct}\|\varphi\|_{L^\theta} + C(1+t)^{-\frac{k}{2}}\|\partial_x^{j+k}\varphi\|_{L^2} \tag{6.9}$$

for $1 \leq \theta \leq 2$ and $j, k \geq 0$. These estimates and (6.5), (6.6), (6.7) *tell us that G and H are dominant terms of the solutions for* (2.9), (3.1). *More precisely, the estimates* (6.5), (6.6) *and* (6.7) *imply that the decay rate $t^{-1/2}$ can be obtained by subtracting the operator $G(t)$. On the other hand, we can weaken the order 1-regularity is required* *on the initial function by subtracting the operator $H(t)$. The derivations of* (6.8) *and* (6.9) *are same as the proof of Theorem 6.1, and omitted in detail.*

If we consider the special case that $\varphi = (0, \varphi_2, \varphi_3, 0, 0, 0)^\top$, this yields $G(t)\varphi = H(t)\varphi = 0$. Therefore, as a corollary of Theorem 6.1, we derive the following.

Corollary 6.2. *Suppose the same assumption as in Theorem 6.1. Then the semigroup operator $e^{t\Phi}$ satisfies for $\varphi = (0, \varphi_2, \varphi_3, 0, 0, 0)^\top$ that*

$$\|\partial_x^j e^{t\Phi}\varphi\|_{L^2} \le C(1+t)^{-\frac{1}{2}(\frac{1}{\theta}-\frac{1}{2})-\frac{j}{2}-\frac{1}{2}}\|\varphi\|_{L^\theta} + C(1+t)^{-\frac{k}{2}}\|\partial_x^{j+k-1}\varphi\|_{L^2}$$

for $j, k \ge 0$ with $0 \le j+k-1 \le \sigma$, where C is a certain positive constant.

Proof of Theorem 6.1. We first derive (6.7). Using the Plancherel theorem, we have

$$\begin{aligned} &\|\partial_x^k(e^{t\Phi} - G(t) - H(t))\varphi\|_{L^2}^2 \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} |\xi|^{2k}|e^{t\hat{\Phi}(i\xi)} - \hat{G}(t, i\xi) - \hat{H}(t, i\xi)|^2|\hat{\varphi}(\xi)|^2 d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} |\xi|^{2k}|\hat{R}(t, i\xi)|^2|\hat{\varphi}(\xi)|^2 d\xi \\ &= \int_{|\xi|\le r_0} + \int_{r_0\le|\xi|\le r_1} + \int_{|\xi|\ge r_1} =: J_1 + J_2 + J_3, \end{aligned} \tag{6.10}$$

where r_0 and r_1 are given in Proposition 5.3. We apply the Hölder inequality and Proposition 5.3 to the low-frequency part J_1, obtaining

$$\begin{aligned} J_1 &\le \frac{1}{2\pi}\||\xi|^{2j}|\hat{R}(t, i\xi)|^2\|_{L^{\theta_1}(|\xi|\le r_0)}\|\hat{\varphi}^2\|_{L^{\theta_2}(|\xi|\le r_0)} \\ &\le C(\||\xi|^{2(j+1)}e^{-c|\xi|^2 t}\|_{L^{\theta_1}(|\xi|\le r_0)} + e^{-ct})\|\hat{\varphi}^2\|_{L^{\theta_2}(|\xi|\le r_0)} \\ &\le C(1+t)^{-\frac{1}{2\theta_1}-j-1}\|\hat{\varphi}\|_{L^{2\theta_2}}^2 \end{aligned}$$

for $1 \le \theta_1, \theta_2 \le \infty$ with $1/\theta_1 + 1/\theta_2 = 1$. Moreover, using Hausdorff-Young inequality, we get

$$J_1 \le C(1+t)^{-(\frac{1}{\theta}-\frac{1}{2})-j-1}\|\varphi\|_{L^\theta}^2$$

for $1 \le \theta \le 2$. On the other hand, for the high-frequency part J_3, we compute

$$\begin{aligned} J_3 &\le C(\sup_{|\xi|\ge r_1}\{|\xi|^{-2k}e^{-c|\xi|^{-2}t}\} + e^{-ct})\int_{|\xi|\ge r_1} |\xi|^{2(j+k-1)}|\hat{\varphi}(\xi)|^2 d\xi \\ &\le C(1+t)^{-k}\|\partial_x^{j+k-1}\varphi\|_{L^2}^2. \end{aligned}$$

for $j+k-1 \ge 0$. For the middle-frequency part J_2, it is easy to calculate

$$J_2 \le Ce^{-ct}\int_{r_0\le|\xi|\le r_1} |\xi|^{2j}|\hat{\varphi}(\xi)|^2 d\xi \le Ce^{-ct}\|\partial_x^{j+k-1}\varphi\|_{L^2}^2.$$

Eventually, substituting the estimates of J_1, J_2 and J_3 into (6.10), we arrive at the decay estimate (6.7).

The proof of (6.5) and (6.6) is simple. Indeed, (6.5) follows from (6.7) and (6.9). On the other hand, combining (6.7) and (6.8), we arrive at (6.6). Thus this completes the proof. □

In the rest of this section, we study the asymptotic profile of the solution to (2.9). We define new functions that $h(t,x) := e^{-tL_0}(v(t,x), y(t,x), q(t,x))^\top$ and $h_0(x) := (v_0(x), y_0(x), q_0(x))^\top$. Then these functions satisfy the Cauchy problem of the parabolic system, that is,

$$h_t - B_0 h_{xx} = 0, \qquad h(0,x) = h_0(x). \tag{6.11}$$

Here L_0 and B_0 are defined in (6.2). We introduce the fundamental solution to (6.11) that $g(t,x) := \mathcal{F}^{-1}[e^{-t\xi^2 B_0}](x)$. Remark that g has a self-similar property and $\int_{\mathbb{R}^n} g(t,x)dx = I$ for $t > 0$, where I is an identity matrix. Furthermore, it is not difficult for the solution to (6.11) to obtain

$$\|\partial_t^j \partial_x^k (h(t) - g(t+1)M_0)\|_{L^2} \le C(1+t)^{-\frac{3}{4}-j-\frac{k}{2}}(\|h_0\|_{L^1_1} + \|\partial_x^{2j+k} h_0\|_{L^2}) \tag{6.12}$$

for $j,k \ge 0$ with $2j+k \le \sigma$, provided by $h_0 \in H^\sigma \cap L^1_1$ for $\sigma \ge 0$, where $M_0 := \int_{\mathbb{R}} h_0(x)dx$. For the derivation of (6.12), we refer [10] to readers.

To state a result of the asymptotic profile, we introduce a diffusion waves for our problem. The diffusion wave corresponding to $G(t)$ is defined by

$$\bar{u}(t,x) := S_0^\top e^{tL_0} g(t+1,x) S_0 M, \qquad M := \int_{\mathbb{R}} u_0(x)dx. \tag{6.13}$$

We conclude this paper with the following theorem.

Theorem 6.3. *Let $\sigma \ge 2$ be an integer. Suppose that the initial data u_0 belongs to $H^\sigma \cap L^1_1$. Let $\bar{u}$ be the diffusion waves defined by (6.13). Then the solution u to the problem (2.9), (3.1) satisfies the asymptotic profile:*

$$\|\partial_x^j (u - \bar{u})(t)\|_{L^2} \le C(1+t)^{-\frac{3}{4}-\frac{j}{2}}(\|u_0\|_{L^1_1} + \|u_0\|_{H^\sigma})$$

for $0 \le j \le \sigma/2 - 3/4$, where C is a certain positive constant.

Proof. We have

$$\begin{aligned} \|\partial_x^j (G(t)u_0 - \bar{u}(t))\|_{L^2} &= \|\partial_x^j S_0^\top e^{tL_0}(e^{-t\xi^2 B_0} S_0 u_0 - g(t+1) S_0 M)\|_{L^2} \\ &\le C(1+t)^{-\frac{3}{4}-\frac{j}{2}}(\|S_0 u_0\|_{L^1_1} + \|\partial_x^j S_0 u_0\|_{L^2}) \end{aligned}$$

given by (6.1) and (6.12). Thus this estimate and (6.5) give

$$\begin{aligned} \|\partial_x^j (u - \bar{u})(t)\|_{L^2} &\le \|\partial_x^j (e^{t\Phi} - G(t))u_0\|_{L^2} + \|\partial_x^j (G(t)u_0 - \bar{u}(t))\|_{L^2} \\ &\le C(1+t)^{-\frac{3}{4}-\frac{j}{2}}(\|u_0\|_{L^1_1} + \|\partial_x^j u_0\|_{L^2}) + C(1+t)^{-\frac{k}{2}}\|\partial_x^{j+k} u_0\|_{L^2} \end{aligned}$$

for $j,k \ge 0$. Therefore we have to assume $2j + 3/2 \le \sigma$ to get the desired decay rate, and hence this completes the proof. □

Acknowledgments: The second author is partially supported by Grant-in-Aid for Scientific Research (C) No. 21K03327 from Japan Society for the Promotion of Science. The authors would like to thank the anonymous referee for pointing out suitable remarks on a previous version, whose corrections culminated in the current one.

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