The weak gravity conjecture and thermodynamics of spacetime

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Doctoral Dissertation

The weak gravity conjecture and thermodynamics of spacetime

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Abstract

The weak gravity conjecture states that in quantum gravity theories, gravity should be the weakest force. This conjecture presents various implications for cosmology and particle physics, and so it has been intensively inspected from the view points of string theory and low energy effective field theories. In this thesis we study higher derivative corrections to black brane thermodynamics and their implications for the weak gravity conjecture for higher form symmetries. In particular we show that higher derivative corrections increase charge-to-tension ratio of extremal black branes as implied by the weak gravity conjecture, if four-derivative couplings follow scattering positivity bounds. We also demonstrate that entropy corrections of extremal states are positive under the same assumptions. This extends earlier works in the Einstein-Maxwell theory to higher form symmetries in general spacetime dimensions, providing further evidences for the weak gravity conjecture.

Contents

1 Introduction

The Standard Model (SM) of particle physics is a self-consistent theory describing electromagnetic, weak and strong interactions in the universe and classifying all known elementary particles. The mathematical framework for the SM is provided by quantum field theory consistent with special relativity, and the SM can explain most particle physics phenomena which have been observed. However, the SM are widely considered to be incompatible with the most successful framework to describe gravity to date, general relativity, and it can not explain gravity which is one of the four fundamental interactions. Therefore, theoretical frameworks which can describe quantum gravity are sought and studied intensively.

One of the leading candidate for describing quantum gravity is string theory, which is a theoretical framework in which the point-like particles of particle physics are replaced by one-dimensional objects called strings. String theory describes how these strings propagate throught higher dimensional spacetime and interact with each other. String theory contains two type of strings, open and closed strings. It also contains branes which are objects that generalize the notion of point particles to higher dimensions. In particular, endpoints of open strings are on branes called D-branes. In string thoery, on distance scales larger than the string scale ($\sim 10^{-35}$ m typically), a string looks like an ordinary particle, with its mass, charge, and other properties determined by the vibrational state of the string. One can construct various models of effective field theories by determining the arrangement of D-branes and the shape of extra dimensions.

String thoery, which is consistent with general relativity, is expected as the theoretical framework for unifying the four fandamental interactions (gravity, electromagnetic, weak and strong interactions) and studied theoretically. However, its energy scale is too high to realize with our present technology, and so we can not test string theory directly by experiments. In this regard, the notion of what kind of low energy effective field theories can arise in string theory is a central question. The set of these effective field theories is called the string landscape, and the set of consistent looking effective field theories that are incompatible with quantum gravity is called the string swampland [1] (See figure.1). Studying criteria that distinguish the string landscape from the swampland, and this direction has been explored in the context of the swampland program, gives various implications for low energy physics.

The weak gravity conjecture (WGC) is a set of conjectures which says that any gauge force must be stronger than gravity [2]. The WGC has many interesting implications for cosmology and particle physics. For example,

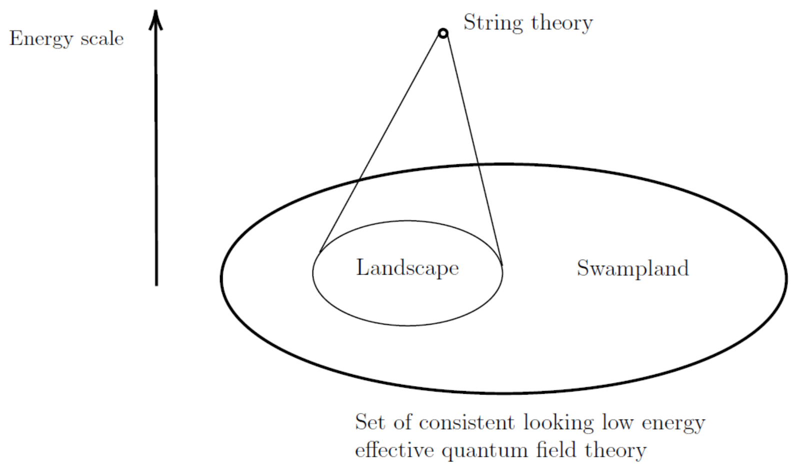

Figure 1: The set of the consistent looking low energy effective field theories is divided into a set of theories which arises from string theory and a set of those which cannot. The formar is named the string landscape, and the latter is named the string swampland.

the WGC provides nontrivial constraints on axion inflation models [3, 4]. These implications allow us to experimentally study the nature of quantum gravity. And so it has been studied theoretically and phenomenologically in the swampland program.

Here we note that the WGC is not a single conjecture, but rather a family of distinct conjectures, each of which is formulated based on the idea that any gauge force must be stronger than gravity in a different way. These various conjectures provide different implications for cosmology and particle physics. Some versions of the WGC have been rejected as counterexamples have been founded, while other versions have been supported by a body of evidences.

The mildest form of the WGC states that a consistent quantum gravity theory should contain a charged state whose charge-to-mass ratio is bigger than unity in an appropriate unit. An important question is at which scale there appears the charged state required by the WGC. String theory typically has charged states satisfying the WGC bound at various scales, both below and above the Planck scale. For example, for 0-form gauge symmetry, the WGC bound is given by the extremal condtion of macroscopic charged black holes that is determined in the Einstein-Maxwell theory. As the black hole mass decreases, the spacetime curvature increases and so higher derivative

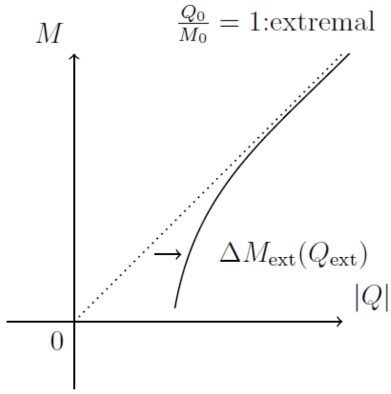

Figure 2: Since the curvature increases as the mass decreases, higher derivative corrections is relevant for small mass and push down the extremal curve monotonically.

corrections become relevant. As depicted in Fig. 2, the extremal condition (for non-supersymmetric black holes) is typically corrected monotonically. This feature has been tested in string theory examples [2,5–8] and studied in the effective field thoery framework, e.g., from the viewpoints of scattering amplitudes and black hole thermodynamics [9–22].

If we extrapolate the extremal curve down to the Planck scale, we expect a transition from black holes to microscopic objects. In string theory, excited strings would be responsible for this [23–25]. This perspective was illustrated in heteoric string examples [2] and then further studied, e.g., using modular invariance [26–30]. There are also some attempts to constrain the spectrum of light particles well below the Planck scale from consistency of gravitational scattering amplitudes [31–36]. The overall picture that there exist a tower of WGC states at various scales is summarized by the subLattic/Tower WGC [26, 27, 32, 37].

In this thesis we would like to generalize the story to higher form symmetries. More specifically, we study higher derivative corrections to extremal conditions of charged black branes from a thermodynamic perspective and discuss under which conditions the extremal curve is pushed down as implied by the WGC. We show that the WGC is satisfied if four-derivative couplings follow scattering positivity bounds [38]. We also show that entropy corrections for extremal black branes in the micro canonical ensemble are positeive under the same assumptions. This extends ealier works in the Einstein-Maxwell theory to higher form symmetries in general spacetime dimensions.

The organization of the thesis is as follows: In section 2 we introduce the weak gravity conjecture. In section 3 and 4, we review solutions of black holes and black branes and their thermodynamics. In section 5, we compute the higher derivative corrections to the black brane thermodynamics and show that higher derivative corrections increase charge-to-tension ratios of extremal black branes and entropy in the micro canonical ensemble, if four-derivative couplings follow the positivity bounds on scattering amplitudes. This section is based on the original paper [39] and contains the main results of the thesis. In the final section, we discuss our results. Technical details are collected in appendices.

2 Weak gravity conjecture

In this section we review the weak gravity conjecture [2]. This conjecture is one of the proposed swampland conditions [1], which distinguish low energy effective field theories which are compatible with quantum gravity from incompatible ones. Studying these criteria gives considerable implications for phenomena and allow us to experimentally test quantum gravity.

2.1 No global symmetry in quantum gravity

The WGC is a generalization of a widely accepted claim that theories of quantum gravity admit no global symmetries of any kind [40]. One motivation for this conjecture is the following. The global charge hidden behind the event horizon cannot be emitted outward through black hole evaporation with the Hawking radiation [41]. This differs from gauge charge, since a black hole discharge during evaporation by charged particle pair creation in the electric field outside of it [42–45]. This consideration of globally charged black hole evaporation suggests that black holes can violate global symmetries.

A more detailed explanation is the following. For example, we consider a quantum gravity theory with a $U(1)$ global symmetry. By throwing objects which are charged under this symmetry, we could produce large black holes with arbitrarily large global charge Q_{global}. These black holes can decay by emitting Hawking radiation [41] at least until they reach a radius r_f below which the effective field theory description is invalid. Black holes can lose masses and radii, but not the global charge, in the Hawking evaporation processes. Thus we can produce black holes of size r_f but arbitrarily large charge. The information which is stored in this charge is arbitrarily large, and in particular the entropy of this remnant exceeds the Bekenstein-Hawking entropy $\propto r_f^2$ [46]. This argument extends directly to any continuous global symmetry. Therefore the existence of remnants with any global charge violates the Bekenstein-Hawking area law [47], and so quantum gravity admit no global symmetry[1].

The above argument for global symmetry violation in quantum gravity is semiclasical. Other, more valid, arguments for no global symmetry conjecture have been given recently. For example, [51] showed that global symmetry violation in quantum gravity hold with the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [52]. It claims that any global symmetry in the bulk (AdS) leads a contradiction in the boundary (CFT). From these evidences, no global symmety conjecture seems considerably valid.

The strongest arguments for no global symmetries in quantum gravity are arguments against exact global symmetries. Then it is important to sharpen these arguments to ask to what extent approximate global symmetries are allowed or to what extend symmetires have to be gauged. As we will see, the WGC is one attempt to clear this question: the gauge coupling has a lower bound since the weak coupling limit of a gauge theory approximate a global symmetry, and should be forbidden in quantum gravity.

2.2 The weak gravity conjecture

As we see in the above discussion, the existence of black hole remnants with arbitraly charge violate the Bekenstein-Hawking entropy area law, and so is forbidden. That motivates existence of the lower bound on gauge couplings, as provided by the weak gravity conjecture[2].

2.2.1 The WGC for $U(1)$ gauge symmetry

We now consider quantum gravity theories coupled to a $U(1)$ gauge field, in d spacetime dimensions, with a low-energy action of the form,

$$I = \int d^d x \sqrt{-g} \left[\frac{1}{2\kappa^2} R - \frac{1}{2e^2} F^2 \right] , \tag{2.2}$$

[1]If a theory of quantum gravity does not have a black hole solution, global symmetries are admitted in it. For example, the string Polyakov action [48–50]

$$I[g, X^\mu] = \frac{T}{2} \int d^2\sigma \sqrt{-\sigma} \sigma^{ab} g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \tag{2.1}$$

has the global symmetry $X^\mu \to \Lambda^\mu_\nu X^\nu + a^\mu$. This is an action of the two-dimensional conformal field theory describing the worldsheet of a string in string theory, and there is no black hole solution.

[2]We will only focus on the WGC for p-form gauge symmetry for our purpose. Other variants of the WGC are reviewed, for example, in [53].

where $\kappa^2 = 8\pi G_d$ is the gravitational constant, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength, $F^2 = F_{\mu\nu}F^{\mu\nu}/2$, and e is the gauge coupling. As we will see in the next section, a spherically symmetric black hole solution is given by

$$ds^2 = -\left(1 - \frac{r_+^{d-3}}{r^{d-3}}\right)\left(1 - \frac{r_-^{d-3}}{r^{d-3}}\right)dt^2 + \left(1 - \frac{r_+^{d-3}}{r^{d-3}}\right)^{-1}\left(1 - \frac{r_-^{d-3}}{r^{d-3}}\right)^{-1}dr^2 + r^2 d\Omega_{d-2}^2, \tag{2.3}$$

$$F = \frac{e}{\kappa}\frac{\sqrt{2(d-2)(d-3)}(r_+ r_-)^{\frac{d-3}{2}}}{r^{d-2}} dt \wedge dr, \tag{2.4}$$

where $r_\pm$ is the outer(+) and inner(−) horizons radii respectively, and Ω_{d-2} collectively represents corrdinates of a $(d-2)$-dimensional unit sphere normal to the radial coordinates. The Arnowitt-Deser-Misner (ADM) mass M and the electric charge Q (subject to the Dirac quantization $Q \in \mathbb{Z}$) read[3]

$$M = \frac{V_{d-2}}{\kappa^2}\frac{d-2}{2}(r_+^{q-3} + r_-^{q-3}), \tag{2.5}$$

$$Q = \frac{V_{d-2}}{e\kappa}\sqrt{\frac{(d-2)(d-3)}{2}}(r_+ r_-)^{\frac{d-3}{2}}, \tag{2.6}$$

where V_{d-2} is the $(d-2)$-dimensioal unit sphere volume. To avoid the existence of the naked singularity, or from the cosmic censorship conjecture [54], mass M and electric charge Q of a black hole must satisfy the relation

$$\frac{Q}{M} \leq \sqrt{\frac{d-3}{d-2}}\frac{\kappa}{e} \equiv z. \tag{2.7}$$

In particular, a black hole with the charge-to-mass ratio $Q/M = z$ is called an extremal black hole.

A black hole with the mass M and the electric charge Q decays by emitting the Hawking radiation until the extremal condition $r_+ = r_-$ is satisfied. If we forbid the existence of black hole remnants with arbitrary masses and charges, an extremal black hole must have another way to decay to a smaller black hole (e.g. Schwinger effect [42–45]). Therefore it was conjectured that theories of quantum gravity must have a state with the charge-to-mass ratio q/m which is larger than z (See Fig. 3);

$$\frac{q}{m} \geq \sqrt{\frac{d-3}{d-2}}\frac{\kappa}{e}. \tag{2.8}$$

This is the (mild form of) WGC[4] [2].

[3]For the moment, we assume that electric charge is positive without loss of generality.

[4]The "mild" form means that it is enough if there exists a charged particle satisfying (2.8). One type of the "strong" form of WGC claims that a theory consistent with quantum gravity requires the existence of the lightest particle satisfying (2.8).

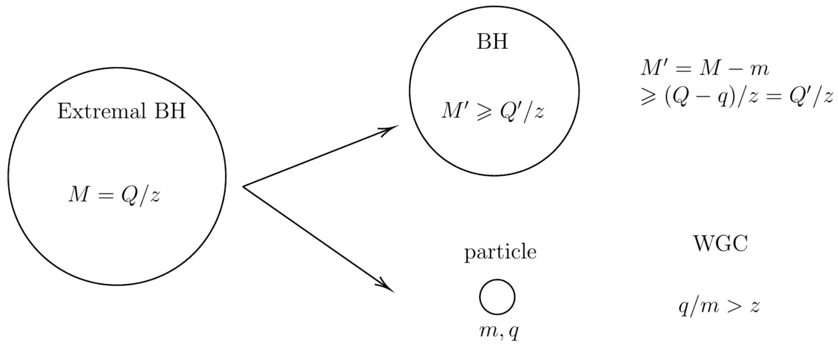

Figure 3: An extremal black hole can decay into another black hole by emitting the particle which satisfies the WGC bound.

2.2.2 The WGC for higher form gauge fields

We now consider quantum gravity theories coupled to a $(p+1)$-form gauge field, in d spacetime dimensions, with a low-energy action of the form,

$$I = \frac{1}{2\kappa^2}\int d^dx\sqrt{-g}R - \frac{1}{2e^2}\int d^dx\sqrt{-g}F^2\,, \tag{2.9}$$

where $F = dA$ is the $(p+2)$-form field strength and $F^2 \equiv \frac{1}{(p+2)!}F_{\mu_1\cdots\mu_{p+2}}F^{\mu_1\cdots\mu_{p+2}}$. As we will see in the next section, a spherically symmetric black brane solution is given by [55]:

$$ds^2 = f_-(r)^{\frac{2}{p+1}}\left[-\frac{f_+(r)}{f_-(r)}dt^2 + \delta_{ij}dy^idy^j\right] + \frac{dr^2}{f_+(r)f_-(r)} + r^2d\Omega_q^2, \tag{2.10}$$

$$\frac{F}{(p+2)!} = \frac{e}{\kappa}\frac{\sqrt{\frac{(p+q)(q-1)}{p+1}}(r_+r_-)^{\frac{q-1}{2}}}{r^q}dt\wedge dy^1\wedge\cdots\wedge dy^p\wedge dr, \tag{2.11}$$

where $q \equiv d-p-2$ and $f_\pm(r) \equiv 1 - \left(\frac{r_\pm}{r}\right)^{q-1}$. Also, t and y^i are coordinates along the p-brane and r is the radial coordinate. Then we have the ADM energy density and the electrical charge per unit brane volume as

$$M = \frac{V_q}{\kappa^2}\frac{p+q}{p+1}\left[r_-^{q-1} + \frac{q(p+1)}{2(p+q)}\left(r_+^{q-1} - r_-^{q-1}\right)\right], \tag{2.12}$$

$$Q = \frac{V_q}{e\kappa}\sqrt{\frac{(p+q)(q-1)}{p+1}}(r_+r_-)^{\frac{q-1}{2}}. \tag{2.13}$$

The extremal condition for black branes is given by

$$\frac{Q}{M} = \sqrt{\frac{(p+1)(q-1)}{p+q}}\frac{\kappa}{e}. \tag{2.14}$$

Therefore, an extremal charged black brane can decay into another black brane if and only if a theory has a state with the mass m and the charge q such that

$$\frac{q}{m} \geq \sqrt{\frac{(p+1)(q-1)}{p+q}}\frac{\kappa}{e}. \tag{2.15}$$

This is the (mild form of) WGC for $(p-1)$-form gauge field [37].

2.3 WGC and black holes

Ref. [2] also made a prediction regarding the charge-to-mass ratio of extremal black holes. Consider a particle with a mass M and a charge Q. If this particle is unstable, it must be able to decay into two or many particles whose total mass is smaller than M and total charge is equal to Q. To satisfy these conditions, at least one of the outgoing particles must have a larger charge-to-mass ratio than the original particle.

One can apply the above discussion to black holes, which are believed to be the low-energy description of a state whose masses are much above the Planck mass. Since the existence of an infinite number of stable particles is unnatural, the charge-to-mass ratio for extremal black holes $Q/M = \kappa/\sqrt{2}$ cannot be exact, and the extremal limit for Q/M for black holes is approached from above (See Fig. 2).

Now let us consider the spherically symmetric and electrically charged black holes derived from the equation of motion given by the corrected Einstein-Maxwell action

$$I = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + a(F_{\mu\nu}F^{\mu\nu})^2\right), \tag{2.16}$$

where a is a coefficient[5]. The charge-to-mass ratio for extremal black hole is given by [6]

$$\frac{\sqrt{2}}{\kappa}\frac{Q}{M} = 1 + \frac{16}{5\kappa^4}\frac{(4\pi)^2}{Q^2}a. \tag{2.17}$$

[5]We can consider another higher derivative operators like FFR and R^2 terms, but for simplicity we consider only F^4 term which is the leading correction for $M \gg M_{\rm Pl}$. Discussions including FFR and R^2 terms are given in [6–8, 10, 14]

As far as we consider effective field theories, the coefficient a is a completely arbitrary number. However, in any UV completion which respects unitarity of S-matrix, the a is forced to be positive

$$a > 0. \tag{2.18}$$

From (2.17) and (2.18), the charge-to-mass ratio for extremal black holes is corrected as $Q/M > 1$ by a higher derivative operator whose coefficient is restricted by unitarity of S-matrix.

The above result implies that black holes can satisfy the weak gravity bound (2.8) by wide classes of effective field theories in which higher-derivative corrections shift the charge-to-mass ratios of extremal black holes to larger values. We will show that the higher derivative corrections increase the charge-to-tension ratio for extremal black p-brane from unitarity constraints.

3 Black hole thermodynamics

With the existence of the event horizon, the four laws for stationary black holes are proposed by [56] as follows:

0. The horizon has constant surface gravity, angular velocity and electric potential for a stationary black hole.

1. For perturbations of stationary black holes, the change of mass is related to change of area, angular momentum and electric charge by

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega\delta J + \Phi\delta Q, \tag{3.1}$$

where M is the mass, κ is the surface gravity, A is the horizon area, Ω is the angular velocity, J is the angular momentum, Φ is the electric potential, and Q is the electric charge.

2. The horizon area does not decrease in classical processes:

$$\frac{dA}{dt} \geq 0. \tag{3.2}$$

3. It is not possible to form a black hole with vanishing surface gravity in finite processes.

The third law has not been proved yet, but no experimentally verified violations of the third law are known. These laws are often called the four laws of black hole thermodynamics for analogy to equilibrium thermodynamics. These laws give us hints or useful ways to analyze black hole mechanics. In particular, we will use the relation between the Euclidean action and thermodynamic free energy to study higher derivative corrections.

In this section we review black hole solutions and their first law of thermodynamics in Einstein gravity in four-dimensional spacetimes. For sake of simplicity, we ignore rotation and only consider spherically symmetric spacetimes.

3.1 Black hole solution in Einstein gravity

We consider a 1-form gauge field A coupled to gravity in 4-dimensional spacetime:

$$I = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}(R-2\Lambda) - \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}\right], \tag{3.3}$$

where Λ is the cosmological constant and $F = dA$ is the 2-form field strength. The equations of motion are

$$R_{\mu\nu} - \frac{1}{2}(R-2\Lambda)g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{3.4}$$

$$\nabla_\mu T^{\mu\nu} = 0, \tag{3.5}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, and

$$T_{\mu\nu} = \frac{1}{e^2}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}g_{\mu\nu}\right) \tag{3.6}$$

is the energy-momentum tensor. A spherically symmetric solution is given by the metric describing a Reissner-Nordstrøm black hole [57]. It takes the form:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + r^2 d\phi^2), \tag{3.7}$$

$$\boldsymbol{F} = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu = -\sqrt{\frac{G}{4\pi}}\frac{gq}{r}dt \wedge dr, \tag{3.8}$$

where

$$f(r) = 1 - \frac{2Gm}{r} + \frac{Gg^2q^2}{4\pi r^2} - \frac{\Lambda r^2}{3}. \tag{3.9}$$

Here m is the black hole mass and gq is its charge. It is convenient to rescale m, q and Λ to the quantites having dimension of length as

$$Gm = M, \qquad \frac{Gg^2q^2}{4\pi} = Q^2, \qquad \Lambda = \frac{3}{l^2}. \tag{3.10}$$

Then we rewrite f as

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{r^2}{l^2}. \tag{3.11}$$

If l is finite, f can be factorized as

$$f(r) = -\frac{1}{l^2 r^2}(r - r_{--})(r - r_-)(r - r_+)(r - r_C), \tag{3.12}$$

where r_{--} is negative, and the surfaces located at $r = r_-, r_+, r_C$ correspond to the inner horizon, the event horizon and the cosmological horizon. Comparing equations (3.11) and (3.12), we obtain the relation between the roots of $f(r)$ and the black hole parameters as

$$\begin{cases} r_{--} + r_- + r_+ + r_C = 0, \\ r_{--}(r_- + r_+ + r_C) + r_-(r_+ + r_C) + r_+ r_C = -l^2, \\ r_{--}r_-r_+ + r_{--}r_-r_C + r_{--}r_+r_C + r_-r_+r_C = -2Ml^2, \\ r_{--}r_-r_+r_C = -Q^2 l^2. \end{cases} \tag{3.13}$$

In the flat spacetime limit $l \to \infty$ ($\Lambda \to 0$), (3.13) are consistent by taking the limit $r_{--} \to -\infty$ and $r_C \to \infty$. Then $r_\pm$ are

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{3.14}$$

The structure of black hole spacetime (3.7) is determined by the number of real and independent roots of $f(r)$.

3.1.1 Asymptotically flat $\Lambda = 0$ limit.

Let us start with the simple case of $\Lambda = 0$, asymptotically flat black holes. In this case, we can ignore the roots r_{--} and r_C, and the property of black hole spacetime depends only on (3.14)

These $r_\pm$ are the zeros of (3.11), or the solutions of the equation,

$$P_{\Lambda=0}(r) \equiv r^2 f(r) = r^2 - 2Mr + Q^2 = 0. \tag{3.15}$$

The number of the real rooots of (3.11) are determined by the sign of the discriminant of $P_{\Lambda=0}(r)$:

$$D_{\Lambda=0}(M, Q) = 4(M^2 - Q^2). \tag{3.16}$$

We thus have three cases: (1) $Q^2 < M^2$, the two roots of (3.11) are real and positive with $0 < r_- < r_+$. The inner horizon located at $r = r_-$ is the

Cauchy horizon and hides the singularity at $r = 0$ from observers outside this horizon. The outer horizon located at $r = r_+$ is the event horizon, and observers at null future infinity can not recieve a signal from the inside of this horizon; (2) $Q^2 = M^2$, the two horizons are degenerate. This corresponds to the extremal black hole case. In this case, the temperature is zero, and the black holes is stable if black holes can decay only with Hawking radiation; (3) $Q^2 > M^2$, the two roots of (3.11) are complex, there is no horizon, the singularity at $r = 0$ is naked, and this case is forbidden by the weak cosmic censorship hypothesys [58].

3.1.2 de-Sitter $\Lambda \neq 0$ and nutral $Q = 0$ limit

In $\Lambda \neq 0$ and $Q = 0$ case, the inner horizon radius vanishes $r_- \to 0$, and the other horizon radii are the solutions of the equation,

$$P_{Q=0}(r) \equiv -l^2 r f(r) = r^3 - l^2 r + 2Ml^2 = 0. \tag{3.17}$$

This equation has one negative solution r_{--}, and the number of the other real solutions can be determined by the sign of the discriminant of $P_{Q=0}(r)$;

$$D_{Q=0}(M, l) = 4l^4(l^2 - 27M^2). \tag{3.18}$$

We thus have three cases: (1) $M^2 < \frac{l^2}{27}$, the two roots of (3.11) are real and positive with $0 < r_+ < r_C$. The surface located at $r = r_+$ is the event horizon, and the surface located at $r = r_C$ is the cosmological horizon; (2) $M^2 = \frac{l^2}{27}$, the two horizons are degenerate; $r_+ = r_C = 3M$. This corresponds to the extremal black hole case; (3) $M^2 > \frac{l^2}{27}$, the two roots of (3.11) are complex, $f(r)$ is always negative, there is no horizon, and the singularity at $r = 0$ is naked.

3.1.3 de-Sitter Reisnner-Nordstrøm black hole

The zeros of (3.11) r_{--}, r_-, r_+ and r_C are the solutions of the equation

$$P(r) \equiv -l^2 r^2 f(r) = r^4 - l^2 r^2 + 2Ml^2 r - Q^2 l^2 = 0. \tag{3.19}$$

Since $P(\pm\infty) > 0$ and $P(0) < 0$, $P(r)$ always has two real roots, one positive and one negative. The remaining two roots are either positive, or complex conjugate, depending on the sign of the discriminant of $P(r)$. The discriminant of $P(r)$ is

$$D(M, Q, l) = 16l^6 \left(-\frac{27}{l^2}(Ml)^4 + (l^2 + 36Q^2)(Ml)^2 - Q^2(l^2 + 4Q^2)^2 \right). \tag{3.20}$$

We thus have three cases: (1) $D > 0$, $r_{--} < 0$ and $0 < r_- < r_+ < r_C$. The surfaces located at r_-, r_+ and r_C correspond to the Cauchy horizon, the event horizon and the cosmological horizon respectively; (2) $D = 0$, $r_{--} < 0$ and either $0 < r_- = r_+ < r_C$ or $0 < r_- < r_+ = r_C$. Two horizons are degenerated; (3) $D < 0$, $r_{--} < 0$ and either $0 < r_C$ and $r_- = r_+^*$, or $0 < r_-$ and $r_C = r_+^*$. There is only one horizon.

The behavior of horizons in (2) and (3) depends on balance among the mass M, the electric charge Q and the cosmological constant l.

The discriminant (3.20) is a quadratic polynominal of $(Ml)^2$ with the discriminant:

$$\Delta(Q, l) = 256 l^{10}(l^2 - 12Q^2). \tag{3.21}$$

Thus, for $l^2 < 12Q^2$, the roots of D is given by

$$\frac{M_\pm^2(Q, l)}{l^2} = \frac{1}{54}\left(\left(1 + 36\frac{Q^2}{l^2}\right) \pm \left(1 - 12\frac{Q^2}{l^2}\right)^{\frac{3}{2}}\right), \tag{3.22}$$

and D is negative outside these roots. Since r_{--} is always negative, we focus on r_-, r_+ and r_C for $M_- < M < M_+$ and fixed $Q^2 < \frac{l^2}{12}$. These horizon radii define four regions: (1) $0 < r < r_-$. In this region, the repulsive electromagnetic force due to Q is dominant $f \sim Q^2/r^2$: (2) $r_- < r < r_+$. In this region, the attractive gravitational force is dominant $f \sim -2M/r$: (3) $r_+ < r < r_C$. In this region, the electromagnetic force, the gravitational force and the attractive force due to the cosmological constant are balanced, and the constant term is dominant $f \sim 1$: (4) $r_C < r$. In this region, the cosmological constant is dominant $f \sim -r^2/l^2$. As M increases, r_+ increases and r_-, r_C decrease. On the contrary, As M decreases, r_+ decreases and r_-, r_C increase. Therefore, when $D(M, Q, l) = 0$, the Cauchy horizon $r = r_-$ and the event horizon $r = r_+$ are degenerated if $M = M_-$, or the event horizon and the cosmological horizon $r = r_C$ are degenerated if $M = M_+$ (for fixed $Q^2 < l^2/12$).

3.2 Black hole thermodynamics in asymptotically flat spacetime

When a Schwartzscild black hole with the mass M is perturbed, the change of the mass satisfies the identity:

$$\delta M = \frac{\kappa}{8\pi G}\delta A, \tag{3.23}$$

where κ is the surface gravity and A is the area of the event horizon. (3.23) is called the first law of black hole mechanics.

The original derivation [59] employed the form of the Einstein equation. Wald generalized (3.23) to a theory with a deffeomorphism covariant Lagrangian,

$$L = L(g_{\mu\nu}; R_{\mu\nu\rho\sigma}, \nabla_\alpha R_{\mu\nu\rho\sigma}, ...; \phi_a, \nabla_\alpha \phi_a, ...), \tag{3.24}$$

where ∇_α is covariant derivative and ϕ_a is a matter field, by introducing Wald entropy [60]

$$S = -2\pi \int_\Sigma \frac{\delta L}{\delta R_{\mu\nu\rho\sigma}} n_{\mu\nu} n_{\rho\sigma}, \tag{3.25}$$

where Σ is the bifurcate Killing horizon and $n_{\mu\nu}$ is the binormal to Σ normalized so that $n_{\mu\nu}n^{\mu\nu} = -2$. For Einstein theory $L = \sqrt{-g}R$, the Wald entropy reproduces the Bekenstein-Hawking entropy [46]

$$S = S_{\mathrm{BH}} = \frac{A}{4G}. \tag{3.26}$$

(3.23) is derived from classical field theory. Let us consider relativistic quantum effect in the black hole spacetime. Since the whole spacetime does not have time translation symmetry, we can not define the state of the lowest energy. However, the spacetime has asymptotically time translation symmetry at null infinity defined as

$$t \to \pm\infty, \qquad r \to \infty, \qquad r \mp t = \text{constant}, \tag{3.27}$$

where the sign $-$ denotes the past null infinity and the sign $+$ denotes the future null infinity. Therefore we can define the vacua at the past and the future null inifinity $|0\rangle_-$ and $|0\rangle_+$. These vacua satisfy the relations

$$_\pm \langle 0|0\rangle_\pm = 1. \tag{3.28}$$

We can also define the creation and annihilation operator via

$$a_{\pm,i} |0\rangle_\pm = 0, \qquad [a_{\pm,i}, a_{\pm,j}] = [a^\dagger_{\pm,i}, a^\dagger_{\pm,j}] = 0, \qquad [a_{\pm,i}, a^\dagger_{\pm,j}] = \delta_{ij}. \tag{3.29}$$

These creation and annihilation operators are related to each other as the linear relation as

$$a_{+,i} = \sum_j \left(A^*_{ij} a_{-,j} + B^*_{ij} a^\dagger_{-,j} \right), \qquad a^\dagger_{+,i} = \sum_j \left(-B_{ij} a_{-,j} + A_{ij} a^\dagger_{-,j} \right), \tag{3.30}$$

where A_{ij}, B_{ij} are the Bogoliubov coefficient [61, 62] satisfying the relations[6]

$$AA^\dagger - BB^\dagger = \mathbf{1}, \qquad AB^{\mathrm{T}} - BA^{\mathrm{T}} = \mathbf{0}. \tag{3.31}$$

[6]These relations are derived from the above commutation relations (3.29).

Then the particle number operators of i-mode at the past and the future null infinity can be defined as

$$N_{\pm,i} = a^{\dagger}_{\pm,i} a_{\pm,i}. \tag{3.32}$$

Note that the number of particles of the state with no particle at the past null infinity is not necessarily zero at the future infinity;

$$\langle N_{+,i} \rangle_{-0} \equiv {}_{-}\langle 0| N_{+,i} |0\rangle_{-} = \mathrm{Tr}\left[B^{\dagger} B\right] \geq 0, \tag{3.33}$$

and vice versa;

$$\langle N_{-,i} \rangle_{+0} \equiv {}_{+}\langle 0| N_{-,i} |0\rangle_{+} = \mathrm{Tr}\left[B^{\dagger} B\right] \geq 0. \tag{3.34}$$

Considering a massless complex scalar field in the black hole spacetime, we have

$$\langle N_{+,i} \rangle_{-0} = \langle N_{-,i} \rangle_{+0} = \frac{1}{e^{2\pi\omega_i/\kappa} - 1}, \tag{3.35}$$

where ω_i is frequency of the i-mode. Then we can interpret a black hole as a black body with radiation with the temperature

$$T = \frac{\kappa}{2\pi}. \tag{3.36}$$

This temperature is called Hawking temperature [41, 63]. Therefore, we can rewrite (3.23) as

$$\delta M = T \delta S + \cdots , \tag{3.37}$$

where the "$\cdots$" denote possible additional contributions from long range matter fields. (3.37) is similar form to the first law of thermodynamics:

$$\delta E = T \delta S - P \delta V. \tag{3.38}$$

Comparing (3.37) and (3.38), the mass of the black hole has the same role as the energy of the thermodynamic system.

Let us check the first law for the Reisnner-Nordstrøm black hole with the mass M and the electric charge Q. M and Q are functions of the horizon radii $r_{\pm}$ as

$$M = M(r_+, r_-) = \frac{r_+ + r_-}{2}, \tag{3.39}$$

$$Q = Q(r_+, r_-) = \sqrt{r_+ r_-}. \tag{3.40}$$

We have the temperature and the entropy as

$$T = \frac{\kappa}{2\pi} = \frac{f'(r_+)}{4\pi} = \frac{r_+ - r_-}{4\pi r_+^2}, \tag{3.41}$$

$$S = \frac{A}{4} = \pi r_+^2. \tag{3.42}$$

We define the electric potential as

$$\Phi = -(\xi^\mu A_\mu|_{r=r_+} - \xi^\mu A_\mu|_{r=\infty}), \tag{3.43}$$

where ξ^μ is the timelike Killing vector. Then we have the electric potential as

$$\Phi = \sqrt{\frac{r_-}{r_+}}. \tag{3.44}$$

These expression gives us the Smarr formula [59, 64]

$$\frac{1}{2}M = TS + \frac{1}{2}\Phi Q. \tag{3.45}$$

Then we have the derivation of the mass, the entropy and the charge as

$$\delta M = \frac{\delta r_+ + \delta r_-}{2}, \tag{3.46}$$

$$\delta S = 2\pi r_+ \delta r_+, \tag{3.47}$$

$$\delta Q = \frac{1}{2}\sqrt{r_+ r_-}\left(\frac{\delta r_+}{r_+} + \frac{\delta r_-}{r_-}\right). \tag{3.48}$$

Hence we have the first law of black hole thermodynamics as

$$\delta M = T\delta S + \Phi \delta Q = \frac{\delta r_+ + \delta r_-}{2}. \tag{3.49}$$

3.2.1 The definition of energy in curved spacetime

We consider the gravitational action in the spacetime $\mathcal{M}$ with the boundary $\partial\mathcal{M}$. The variation of the Einstein-Hilbert action

$$I_{\rm EH} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}R \tag{3.50}$$

takes the form,

$$\begin{aligned}\delta I_{\rm EH} = &-\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right)\delta g_{\mu\nu} \\ &+ \int_{\partial\mathcal{M}} dS[n]\varepsilon(n)\left(A^\mu h^{\nu\rho}\nabla_\rho + B^{\mu\nu}n^\rho\nabla_\rho\right)\delta g_{\mu\nu},\end{aligned} \tag{3.51}$$

where n^μ is the outward unit normal of $\partial\mathcal{M}$, $\varepsilon(n) = n^2 = \pm 1$, $h^{\mu\nu} = g^{\mu\nu} - \varepsilon(n)n^\mu n^\nu$ is the induced metric on $\partial\mathcal{M}$, $dS[n]$ is the surface element of $\partial\mathcal{M}$, and A^μ and $B^{\mu\nu}$ is the appropriate tensor fields. Subject to the boundary condition

$$\delta g_{\mu\nu}|_{\partial\mathcal{M}} = 0, \tag{3.52}$$

the first surface term vanishes, and the second surface term is finite. Therefore, we can not derive the Einstein equations from varying the Einstein-Hilbert action.

To solve this problem, we introduce the Gibbons-Hawking-York term [65, 66]:

$$I_{\text{GHY}} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{|h|}\varepsilon(n)K, \tag{3.53}$$

where $h = \det(h_{ij})$, and $K = \nabla_\mu n^\mu$ is the extrinsic curvature of $\partial\mathcal{M}$. Adding I_{GHY} does not change the equation of motion, since this is a surface term, and the variation of $I_{\text{EH+GHY}}$ is

$$\delta I_{\text{EH+GHY}} = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right)\delta g_{\mu\nu}, \tag{3.54}$$

subject to Dirichlet boundary condition. Therefore we can appropriately derive the Einstein equations from the variation principle. We define the gravitational action in Einstein gravity as

$$I_{\text{grav}} = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}R + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{|h|}\varepsilon(n)K + I_0[h_{ij}], \tag{3.55}$$

where $I_0[h_{ij}]$ is a surface term depending only on h_{ij}, which does not change the Einstein equations. $I_0[h_{ij}]$ is chosen such that the gravitational action is finite. In asymptotically flat spacetime, the full gravitational action is

$$I_{\text{grav}} = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}R + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{|h|}\varepsilon(n)(K - K_0), \tag{3.56}$$

where K_0 is the extrinsic curvature of boundary embedded in flat spacetime.

Let us construct the Hamiltonian for Einstein gravity. We assume that the spacetime is foliated into a family of spacelike hypersurface Σ_t, with coordinates on each slice given by x^i, labeled by the time coordinate t: $\mathcal{M} = \bigcup_{t_1\le t\le t_2}\Sigma_t$. Then $\partial\mathcal{M} = \Sigma_{t_1}\cup\Sigma_{t_2}\cup\mathcal{B}$, where $\mathcal{B} = \bigcup_t \partial\Sigma_t$ is a timelike hypersurface with coordinates x^a. Σ_t and $\mathcal{B}$ have the unit normals u^μ and r^μ, the induced metrices h_{ij} and γ_{ab}, and the extrinsic curvature $K_{\mu\nu}$ and $\mathcal{K}_{\mu\nu}$ respectively. Moreover, $\mathcal{B}_t = \partial\Sigma_t = \Sigma_t\cup\mathcal{B}$ is a two co-dimensions hypersurface with coordinate x^A, and the induced metric and the extrinsic curvature of $\mathcal{B}_t$ respect to r^μ are σ_{AB} and $k_{\mu\nu} = \sigma_\mu{}^\rho\sigma_\nu{}^\sigma\nabla_{(\rho}r_{\sigma)}$, where $\sigma_\mu{}^\nu \equiv \delta_\mu^\nu + u_\mu u^\nu - r_\mu r^\nu$, respectively. In the following, we assume that r^μ are orthogonal to n^μ (See Fig.4).

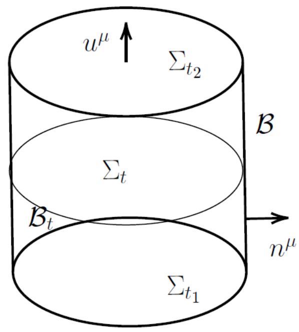

Figure 4: We consider the spacetime with the boundary.

We then have a $3+1$ decomposed metric as

$$ds^2 = \varepsilon(n)N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{3.57}$$

where $N = N(x)$ is the lapse function and $\vec{N} = \vec{N}(x)$ is the shift function. This is the ADM decompsition [67]. We then have the gravitational action as

$$\begin{aligned} I_{\rm grav} = &\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{h}N\left({}^3R + K_{ij}K^{ij} - K^2\right) \\ &+ \frac{1}{8\pi G}\int_{\mathcal{B}} dS[r](k - k_0) \equiv \int_{\mathcal{M}} d^4x\mathcal{L}_{\rm grav}, \end{aligned} \tag{3.58}$$

where ${}^3R$ is the Ricci scalar respect to the metric h_{ij}, and $K = h^{ij}K_{ij} = g^{\mu\nu}K_{\mu\nu}$. We then have the conjugate momenta to h_{ij}, N and N^i as

$$\pi^{ij}(t,\boldsymbol{x}) \equiv \frac{\partial\mathcal{L}_{\rm grav}}{\partial\dot{h}_{ij}} = \frac{\sqrt{h}}{16\pi G}\left(K^{ij} - h^{ij}K\right), \tag{3.59}$$

$$\Pi(t,\boldsymbol{x}) \equiv \frac{\partial\mathcal{L}_{\rm grav}}{\partial\dot{N}} = 0, \tag{3.60}$$

$$\Pi_i(t,\boldsymbol{x}) \equiv \frac{\partial\mathcal{L}_{\rm grav}}{\partial\dot{N}^i} = 0. \tag{3.61}$$

(3.60) and (3.61) give the primary constraints:

$$C(t,\boldsymbol{x}) \equiv \Pi(t,\boldsymbol{x}) = 0, \qquad C_i(t,\boldsymbol{x}) \equiv \Pi_i(t,\boldsymbol{x}) = 0. \tag{3.62}$$

We introduce the Lagrange multipliers λ, λ^i and subtract $(\lambda C + \lambda^i C_i)$ from the Lagrangian:

$$\bar{S} = \int_{\mathcal{M}} d^4x(\mathcal{L}_{\rm grav} - (\lambda C + \lambda^i C_i)). \tag{3.63}$$

Then we have the Hamiltonian as

$$\bar{H} = \int_{\Sigma_t} d^3x \left(\lambda C + \lambda^i C_i + N\mathcal{H} + N^i \mathcal{H}_i\right) - \int_{\mathcal{B}_t} d^2x \sqrt{\sigma} \left(\frac{N}{8\pi G}(k - k_0) - 2r_i N_j \frac{\pi^{ij}}{\sqrt{h}} \right), \tag{3.64}$$

where

$$\mathcal{H} \equiv \frac{16\pi G}{\sqrt{h}} \left(\pi_{ij}\pi^{ij} - \frac{1}{2}(h_{ij}\pi^{ij})^2 \right) - \frac{\sqrt{h}}{16\pi G}{}^3R, \tag{3.65}$$

$$\mathcal{H}_i \equiv -2h_{ij}\nabla_k \pi^{jk}. \tag{3.66}$$

Since the primary constraints (3.62) hold under the time evolution, we have the secondary constraints as

$$\dot{C}(t, \boldsymbol{x}) \equiv \{C(t, \boldsymbol{x}), \bar{H}\} = \mathcal{H}(t, \boldsymbol{x}) = 0, \tag{3.67}$$

$$\dot{C}_i(t, \boldsymbol{x}) \equiv \{C_i(t, \boldsymbol{x}), \bar{H}\} = \mathcal{H}_i(t, \boldsymbol{x}) = 0, \tag{3.68}$$

where $\{\cdot, \cdot\}$ is the Poisson braket defined as

$$\{h_{ij}(t, \boldsymbol{x}), \pi^{kl}(t, \boldsymbol{x}')\} = \delta^k_{(i}\delta^l_{j)}\delta^3(\boldsymbol{x} - \boldsymbol{x}'), \tag{3.69}$$

$$\{N(t, \boldsymbol{x}), \Pi(t, \boldsymbol{x}')\} = \delta^3(\boldsymbol{x} - \boldsymbol{x}'), \tag{3.70}$$

$$\{N^i(t, \boldsymbol{x}), \Pi_j(t, \boldsymbol{x}')\} = \delta^i_j\delta^3(\boldsymbol{x} - \boldsymbol{x}'). \tag{3.71}$$

We focus on the time evolution of N, N^i, h_{ij}, π^{ij} since C, C_i are constant under the time evolution. The Hamilton equation says that $\dot{N} = \lambda, \dot{N^i} = \lambda^i$. Since λ, λ^i are arbitrary functions, the time evolution of the lapse and shift are arbitrary. On the other hand, $(\lambda C + \lambda^i C_i)$ in $\bar{H}$ does not contribute to the time evolution of h_{ij} and π^{ij}. Then N and N^i can be regarded as the Lagrange multipliers to obtain the secondary constraints (3.67) and (3.68), and we have the gravitational Hamiltonian by subtracting terms proportional to C and C_i from $\bar{H}$ as

$$H_{\text{grav}} = \int_{\Sigma_t} d^3x \left(N\mathcal{H} + N^i\mathcal{H}_i\right) - \int_{\mathcal{B}_t} d^2x\sqrt{\sigma} \left(\frac{N}{8\pi G}(k - k_0) - 2r_i N_j \frac{\pi^{ij}}{\sqrt{h}} \right). \tag{3.72}$$

H_{grav} gives the same Hamilton equation for h_{ij}, π^{ij} as $\bar{H}$.

Evaluating the gravitational Hamiltonian in a classical solution gives the energy of spacetime. This is the integration on the boundary $\mathcal{B}_t$ since $\mathcal{H} = 0 = \mathcal{H}_i$ on the classical solution. For asymptotically flat spacetime, we define the energy by the gravitational Hamiltonian evaluated in a classical solution with $N = 1, N^i = 0$:

$$E \equiv H_{\text{grav}}|_{\text{classical}} = -\frac{1}{8\pi G}\int_{\mathcal{B}_t} d^2x\sqrt{\sigma}(k - k_0). \tag{3.73}$$

This definition is consistent with the ADM energy [67]

$$E_{\mathrm{ADM}} = \frac{1}{16\pi G} \oint_{S_2^\infty} dS r^i (\partial^j g_{ij} - \delta^{kl} \partial_i g_{kl}), \tag{3.74}$$

where S_2^∞ is the sphere with radius $r \to \infty$ and $x^i (i = 1, 2, 3)$ are the Cartesian coordinates at infinity.

We have the energy of the Schwarzschild black hole with the mass M as

$$E_{\mathrm{Sch}} = M. \tag{3.75}$$

3.2.2 Euclidean action and thermodynamics

In the path integral approach to the quantization of a field ϕ, the amplitude to go from a field configuration ϕ_1 at time t_1 to a field configuration ϕ_2 at time t_2 can be written as

$$\langle \phi_2, t_2 | \phi_1, t_1 \rangle = \int_{\phi_1, t_{\mathrm{E},1}}^{\phi_2, t_{\mathrm{E},2}} \mathcal{D}\phi \exp\left(-I_{\mathrm{E}}[\phi]\right), \tag{3.76}$$

where I_{E} is the Euclidean action with the Euclidean time

$$t \to t_{\mathrm{E}} = -it. \tag{3.77}$$

On the other hand, we can write this amplitude with the Hamiltonian H as

$$\langle \phi_2, t_2 | \phi_1, t_1 \rangle = \langle \phi_2 | \exp\left(-H(t_{\mathrm{E},2} - t_{\mathrm{E},1})\right) | \phi_1 \rangle. \tag{3.78}$$

If we set $t_{\mathrm{E},2} - t_{\mathrm{E},1} = \beta$ and $\phi_1 = \phi_2$, (3.76) and (3.78) give the relation

$$\mathrm{Tr}\left[\exp(-\beta H)\right] = \int \mathcal{D}\phi \exp\left[-I_{\mathrm{E}}[\phi]\right], \tag{3.79}$$

where the path integral is now taken over all fields with the periodicity of the period β in Euclidean time. The left-hand side of (3.79) is the partition function Z for the canonical ensemble consisting of the field ϕ at temperature $T = \beta^{-1}$. Thus we can express the partition function for the system in terms of a path integral over periodic fields:

$$Z = \int \mathcal{D}\phi \exp\left(-I_{\mathrm{E}}[\phi]\right). \tag{3.80}$$

If the path integral is approximated by the saddle point associated with a calssical solution, we have

$$Z = \exp\left(-I_{\mathrm{E}}[\phi]\right)\big|_{\phi=\varphi_{\mathrm{cl}}}. \tag{3.81}$$

We can also consider grand canonical ensembles with chemical potentials μ_i associated with conserved quantities C_i. In this case the partition function is

$$Z = \mathrm{Tr}\left[\exp\left(-\beta\left(H - \sum_i \mu_i C_i\right)\right)\right]. \tag{3.82}$$

Let us consider the partition function of an asymptotically flat Reissner-Nordstrøm black hole. The gravitational Euclidean action with GHY term is

$$I_{\mathrm{E}} = -\int_{\mathcal{M}} d^4x_{\mathrm{E}}\sqrt{g}\left[\frac{1}{16\pi G}(R-2\Lambda) - \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}\right] - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x_{\mathrm{E}}\sqrt{h}\varepsilon(n)(K-K_0). \tag{3.83}$$

As we see in the above discussion, this action imposed Dirichlet boundary condition $\delta g_{\mu\nu}|_{\partial\mathcal{M}}$ gives us the Einstein equations (3.4). This boundary condition is consistent with fixing the lapse function and the shift function on the boundary. Especially to fix the lapse function is equivalent to fixing the temperature of the system. Now we consider the variation of action (3.83) in the gauge field A_μ. We have

$$\delta_A I_{\mathrm{E}} = \frac{1}{e}\int_{\mathcal{M}} d^4x_{\mathrm{E}}\left(\nabla_\mu F^{\mu\nu}\right)\delta A_\nu - \frac{1}{e}\int_{\partial\mathcal{M}} dS_{\mathrm{E}}[n] n_\mu F^{\mu\nu}\delta A_\nu, \tag{3.84}$$

where n_μ is the unit normal to the boundary $\partial\mathcal{M}$ Then we have to impose the boudary condition

$$\delta A_\mu|_{\partial\mathcal{M}} = 0 \tag{3.85}$$

for derivation of the Maxwell equation (3.5). This boundary condition is consistent with fixing the electric potential Φ on the boundary $\partial\mathcal{M}$[7]. Then we have the partition function for the Reissner-Nordstrøm black hole as

$$Z = \exp\left(-I_E[g_{\mu\nu}, A_\mu]\right)|_{\mathrm{RN}}, \tag{3.88}$$

where RN denotes the Reissner-Nordstrøm solution, and the Gibbs free energy as

$$G = -\beta^{-1}\ln Z = T\ I_{\mathrm{E}}[g_{\mu\nu}, A_\mu]|_{\mathrm{RN}}. \tag{3.89}$$

[7] If we add the boundary term

$$\frac{1}{e}\int_{\partial\mathcal{M}} dS_{\mathrm{E}}[n] n_\mu A_\nu F^{\mu\nu} \tag{3.86}$$

to the action (3.83), we can derive the maxwell equation by imposing the boundary condition

$$h^{\mu\nu}\nabla_\mu A_\nu|_{\partial\mathcal{M}} = 0. \tag{3.87}$$

This is consistent with fixing the electric charge Q on the boundary $\partial\mathcal{M}$.

The Reissner-Nordstrøm solution is

$$ds^2 = f(r)dt_{\mathrm{E}}^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.90}$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{3.91}$$

To go to the near extremal limit, we take $r = r_+ + \epsilon$ and consider ϵ to be small. We also ignore the angular part. In that case $f(r) \approx f'(r_+)\epsilon$, then we have

$$\begin{aligned} ds^2 &\approx \epsilon f'(r_+)dt_{\mathrm{E}}^2 + \frac{1}{\epsilon f'(r_+)}d\epsilon^2 \\ &= dR^2 + R^2 d\Theta^2, \end{aligned} \tag{3.92}$$

where R and Θ are defined as

$$R = 2\sqrt{\frac{\epsilon}{f'(r_+)}}, \tag{3.93}$$

$$\Theta = \frac{f'(r_+)}{2}t_{\mathrm{E}}. \tag{3.94}$$

To avoid a conical singularity, Θ must be periodic with period 2π. Then the Euclidean time t_{E} is periodic with period

$$\beta = \frac{4\pi}{f'(r_+)}, \tag{3.95}$$

where β is the inverse of the Hawking temperature.

Substituting the Reissner-Nordstrøm solution (3.7), (3.8) to the gravitational Euclidean action (3.83), we have

$$\begin{aligned} I_{\mathrm{E}}|_{\mathrm{RN}} &= \int_0^\beta dt_{\mathrm{E}} \int d\theta d\phi \sin^2\theta \int_{r_+}^\infty dr r^2 \left(0 - \frac{Q^2}{4\pi}\frac{1}{r^4}\right) \\ &\quad - \frac{1}{8\pi}\int dt_{\mathrm{E}} d\theta d\phi \sin^2\theta \left(r\partial_r\sqrt{f(r)} + \frac{2}{r}\sqrt{f(r)}\right) \\ &= \frac{\beta}{2}(M - Q\Phi). \end{aligned} \tag{3.96}$$

Then we have the Gibbs free energy as

$$G = \frac{1}{2}(M - Q\Phi) = M - TS - Q\Phi. \tag{3.97}$$

Here the second equation is derived from the Smarr relation (3.45).

3.2.3 Free energy and other ensembles

In the above calculation, we have the Gibbs free energy (3.97) for a Reissner-Nordstrøm black hole. The natural variables for the Gibbs free energy are the temperature and the electric potential;

$$G = G(T, \Phi). \tag{3.98}$$

From the first law (3.49) and (3.97), we have the relation

$$dG = -SdT - Qd\Phi. \tag{3.99}$$

Then we have the entropy and the electric charge as functions of T and Φ as

$$S = S(T, \Phi) = -\left(\frac{\partial G}{\partial T}\right)_{\Phi}, \tag{3.100}$$

$$Q = Q(T, \Phi) = -\left(\frac{\partial G}{\partial \Phi}\right)_{T}. \tag{3.101}$$

Moreover we have the mass from the definition of the Gibbs free energy as

$$M = M(T, \Phi) = G(T, \Phi) + TS(T, \Phi) + \Phi Q(T, \Phi). \tag{3.102}$$

Starting from these relations, we turn to the other ensembles by Legendre transformation. For example, the Helmholtz free energy for a canonical ensemble is

$$H = G(T, \Phi(T, Q)) + Q\Phi(T, Q). \tag{3.103}$$

The natural variables for the Helmholtz free energy are the temperature and the electric charge. Then we have the first law as

$$dH = -TdS + \Phi dQ, \tag{3.104}$$

and the entropy, the electric potential, and the mass as functions of T and Q as

$$S = S(T, Q) = -\left(\frac{\partial H}{\partial T}\right)_{Q}, \tag{3.105}$$

$$\Phi = \Phi(T, Q) = \left(\frac{\partial H}{\partial Q}\right)_{T}, \tag{3.106}$$

$$M = M(T, Q) = H(T, Q) + TS(T, Q) - Q\Phi(T, Q). \tag{3.107}$$

Note that the above discussions are almost independent of the dimension[8]. So these discussions can be straightforwardly extended to higher dimension spacetimes.

[8]For example, the form of the Smarr formula (3.45) depends on dimensions of the spacetime as we see in the next section.

4 Black brane thermodynamics

A black brane is a solution of the equation of motion for d-dimensional Einstein gravity coupled to a $(p+1)$-form gauge field. The mechanics of black branes has a relation [68]

$$dE = TdS + \Phi dQ, \tag{4.1}$$

similar to the first law of black hole thermodynamics. In the following, we review black brane solutions and their thermodynamics.

4.1 Black brane solutions

We consider a $(p+1)$-form gauge field A coupled to gravity in d-dimensional spacetime:

$$I = \frac{1}{2\kappa^2}\int d^dx\sqrt{-g}R - \frac{1}{2e^2}\int d^dx\sqrt{-g}F^2 + \frac{1}{\kappa^2}\int d^{d-1}x\sqrt{|h|}dS[n]\varepsilon(n)(K-K_0)\,, \tag{4.2}$$

where $F = dA$ is the $(p+2)$-form field strength and $F^2 \equiv \frac{1}{(p+2)!}F_{\mu_1\cdots\mu_{p+2}}F^{\mu_1\cdots\mu_{p+2}}$. The equation of motion for the action (4.2) is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{4.3}$$

$$\nabla_\mu T^{\mu\nu} = 0, \tag{4.4}$$

where $T_{\mu\nu}$ is the energy-momentum tensor:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\rm matter})}{\delta g^{\mu\nu}} = \frac{1}{e^2}\left(\frac{1}{(p+1)!}F_{\mu\mu_1\cdots\mu_{p+1}}F_\nu{}^{\mu_1\cdots\mu_{p+1}} - \frac{1}{2}F^2 g_{\mu\nu}\right). \tag{4.5}$$

We seek the electrically charged black brane solution for the action (4.2). The electric brane worldvolume has $p+1$ dimensions: We take an anzats that is translationally and rotationally invariant in the p spatial world volume coordinates $y^i (i = 1, ..., p)$. Then we have a black brane solution [55]:

$$ds^2 = f_-(r)^{\frac{2}{p+1}}\left[-\frac{f_+(r)}{f_-(r)}dt^2 + \delta_{ij}dy^i dy^j\right] + \frac{dr^2}{f_+(r)f_-(r)} + r^2 d\Omega_q^2, \tag{4.6}$$

$$\frac{F}{(p+2)!} = \frac{e}{\kappa}\frac{\sqrt{\frac{(p+q)(q-1)}{p+1}}(r_+r_-)^{\frac{q-1}{2}}}{r^q}dt\wedge dy^1\wedge\cdots\wedge dy^p\wedge dr, \tag{4.7}$$

where $q \equiv d-p-2$ and $f_\pm(r) \equiv 1 - \left(\frac{r_\pm}{r}\right)^{q-1}$. Also, t and y^i are corrdinates along the p-brane and r is the radial coordinate. Ω_q collectively represents coordinates of a q-dimensional unit sphere normal to the radial coordinates. An explicit form of $d\Omega_q^2$ is

$$d\Omega_q^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \sin^2\theta_1 \sin^2\theta_2 d\theta_3^2 + \cdots + \left(\prod_{i=1}^{q-1} \sin^2\theta_i\right) d\theta_q^2. \quad (4.8)$$

To identify the electric charge of the p-brane, it is convenient to write down the Hodge dual of the field strength:

$$\star F = \frac{e}{\kappa}\sqrt{\tfrac{(p+q)(q-1)}{p+1}}(r_+ r_-)^{\frac{q-1}{2}}\omega\,, \quad (4.9)$$

where ω is the volume form of a q-dimensional unit sphere given by

$$\omega = \left(\prod_{i=1}^{q-1} \sin^i\theta_{q-i}\right) d\theta_1 \wedge \cdots \wedge d\theta_q\,. \quad (4.10)$$

Then, the electric charge (subject to the Dirac quantization $Q \in \mathbb{Z}$) reads

$$Q = \frac{1}{e^2}\int_{S^q} \star F = \frac{V_q}{e\kappa}\sqrt{\tfrac{(p+q)(q-1)}{p+1}}(r_+ r_-)^{\frac{q-1}{2}}\,, \quad (4.11)$$

where V_q is the volume of a q-dimensional unit sphere, i.e.,

$$V_q = \int_{S^q}\omega = \frac{2\pi^{\frac{q+1}{2}}}{\Gamma(\frac{q+1}{2})}\,. \quad (4.12)$$

On the other hand, the ADM tension[9] reads [69]

$$M = \frac{V_q}{\kappa^2}\frac{p+q}{p+1}\left[r_-^{q-1} + \frac{q(p+1)}{2(p+q)}\left(r_+^{q-1} - r_-^{q-1}\right)\right]. \quad (4.13)$$

Now, the extremal bound follows from $r_+ \geq r_-$ as

$$\left|\frac{Q}{M}\right| \leq \sqrt{\frac{(p+1)(q-1)}{p+q}}\frac{\kappa}{e}\,. \quad (4.14)$$

The WGC requires a charged state whose charge-to-tension ratio is bigger than that of the macroscopic extremal black brane [2], i.e.,

$$\left|\frac{Q}{M}\right| \geq \sqrt{\frac{(p+1)(q-1)}{p+q}}\frac{\kappa}{e}\,. \quad (4.15)$$

[9]The ADM tension conventiaonally means the ADM energy density, and not means the ADM average tension defined as (4.49) in the following subsection. We also use the ADM tension as the ADM energy density in this paper.

4.2 Black brane thermodynamics

4.2.1 ADM energy of black p-branes

We start to define the ADM energy of black p-branes.

We consider the following action;

$$I = \frac{1}{2} \int d^d x \sqrt{-g} R + I^{(\mathrm{mat})}, \tag{4.16}$$

where $g = \det(g_{\mu\nu})$, $I^{(\mathrm{mat})} = \int d^4 x \sqrt{-g} \mathcal{L}^{(\mathrm{mat})}$ we set $\kappa_d^2 = (8\pi G_d)^{-1} = 1$. We assume that the spacetime is asymptotically flat, and the metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{4.17}$$

where $\eta_{\mu\nu}$ is the metric for the d-dimension Minkowski spacetime, and $h_{\mu\nu}$ is a perturbation to $\eta_{\mu\nu}$. Then we have the Einstein equation as

$$G^{(\mathrm{lin})}_{\mu\nu} = T^{(\mathrm{mat})}_{\mu\nu} + \tau_{\mu\nu}, \tag{4.18}$$

where

$$G^{(\mathrm{lin})}_{\mu\nu} \equiv R^{(\mathrm{lin})}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R^{(\mathrm{lin})}, \tag{4.19}$$

$$= \frac{1}{2} \left(\partial_\rho \partial_\mu h^\rho{}_\nu + \partial_\rho \partial_\nu h^\rho{}_\mu - \partial_\mu \partial_\nu h - \partial_\rho \partial^\rho h_{\mu\nu} - \partial_\rho \partial_\sigma h^{\rho\sigma} \eta_{\mu\nu} + \partial_\rho \partial^\rho h \eta_{\mu\nu} \right), \tag{4.20}$$

where $h = h^\mu{}_\mu$, is the linearized Einstein tensor, and

$$T^{(\mathrm{mat})}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta I^{(\mathrm{mat})}}{\delta g^{\mu\nu}}, \tag{4.21}$$

$$\tau_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\gamma)}{\delta g^{\mu\nu}}, \tag{4.22}$$

where

$$\gamma \equiv \eta^{\mu\nu} \left(\Gamma^\rho_{\sigma\mu} \Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\mu\nu} \Gamma^\sigma_{\rho\sigma} \right). \tag{4.23}$$

We introduce the tensor $K^{\mu\rho\nu\sigma}$ defined as

$$K^{\mu\rho\nu\sigma} \equiv \frac{1}{2} \left(\eta^{\mu\sigma} H^{\nu\rho} + \eta^{\nu\rho} H^{\mu\sigma} - \eta^{\mu\nu} H^{\rho\sigma} - \eta^{\rho\sigma} H^{\mu\nu} \right), \tag{4.24}$$

where

$$H_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h. \tag{4.25}$$

$K^{\mu\rho\nu\sigma}$ has the same symmetry as the Riemann tensor:

$$K^{\mu\rho\nu\sigma} = -K^{\rho\mu\nu\sigma} = -K^{\mu\nu\sigma\rho} = K^{\nu\sigma\mu\rho}. \tag{4.26}$$

Then the Einstein equations (4.18) can be rewritten as

$$\partial_\rho \partial_\sigma K^{\mu\rho\nu\sigma} = T^{\mu\nu}_{(\mathrm{mat})} + \tau^{\mu\nu} = T^{\mu\nu}. \tag{4.27}$$

From (4.26) and (4.27), we have the divergence of the energy-momentum tensor to be zero;

$$\partial_\mu T^{\mu\nu} = 0. \tag{4.28}$$

We define $X^\mu = (t, x^i, y^m)(i = 1, ..., d-p-1, m = 1, ..., p)$ as the Cartesian coordinates about the background Minkowski spacetime:

$$ds^2(\eta) = -dt^2 + \sum_{i=1}^{d-p-1} dx^i dx^i + \sum_{m=1}^{p} dy^m dy^m. \tag{4.29}$$

Then

$$\bar{k} = \frac{\partial}{\partial t}, \qquad \bar{l}_m = \frac{\partial}{\partial y^m} \tag{4.30}$$

are commuting Killing vector fields of this background metric, which we write collectively as $\bar{k}_{(M)}(M = 0, ..., p)$. We will assume that the metric g approaches η as $|x| \to \infty$. That is consistent to the energy-momentum tensor $T^{\mu\nu}$ vanishes at infinity. From (4.28), the currents

$$J^\mu_{(M)} \equiv T^\mu{}_\nu \bar{k}^\nu_{(M)} \tag{4.31}$$

are divergence-free. We also construct a set of p-form corrent as

$$A^{\mu_1 \dots \mu_p}_{(0)} \equiv \bar{l}^{[\mu_1}_{(1)} \cdots \bar{l}^{\mu_p]}_{(p)}, \tag{4.32}$$

$$A^{\mu_1 \dots \mu_p}_{(m)} \equiv \bar{l}^{[\mu_1}_{(1)} \cdots \bar{k}^{\mu_m}_{(m)} \cdots \bar{l}^{\mu_p]}_{(p)}. \tag{4.33}$$

Since the Killing vector fields are commutable, these p-form currents are also divergence-free;

$$\partial_\mu A^{\mu\mu_1 \dots \mu_{p-1}}_{(M)} = 0 \qquad (M = 0, ..., p) \tag{4.34}$$

We now define the brane stress density tensor as

$$\theta_{MN} \equiv \int_\Sigma dS_{\mu\mu_1 \dots \mu_p} J^\mu_{(M} A^{\mu_1 \dots \mu_p}_{N)}, \tag{4.35}$$

where $dS_{\mu\mu_1 \dots \mu_p}$ is the area element of the hypersurface orthogonal to the Killing vector fields $\{\bar{k}_{(M)}\}$. In cartesian coordinate to the background metric, we have

$$\theta_{MN} = \int d^{d-p-1}x T_{MN}. \tag{4.36}$$

Therefore, θ_{MN} can be written as a surface integral at transverse spatial infinity if and only if it is independent of the worldvolume coordinates $y^M = (t, y^m)$. But this condition is consistent with

$$(\mathscr{L}_{\bar{k}} T)_{MN} = 0, \tag{4.37}$$

$$\left(\mathscr{L}_{\bar{l}_{(j)}} T\right)_{MN} = 0 \qquad (j = 0, ..., p), \tag{4.38}$$

where $\mathscr{L}$ represents Lie derivative. These conditions are quite severe since they are effectively true in spacetimes that are everywhere both stationally and translationally invariant, rather than just asymptotically so. If we consider only the component θ_{00} then we need only require the condition (4.38), but this still restricts us to translationally invariant spacetimes. To circumvent this restriction, we impose periodicity in the brane directions, with total p-volume V_p, and define

$$\langle A \rangle \equiv \frac{1}{V_p} \int d^p y A, \tag{4.39}$$

for any function A on the background spacetime. Then we have the average energy density as

$$\mathscr{E} \equiv \langle \theta_{00} \rangle = \int d^{d-p-1}x \, \langle T^{00} \rangle \, . \tag{4.40}$$

For stationally spacetime, we similary define the average tension as

$$\mathscr{T} \equiv -\frac{1}{p} \sum_m \langle \theta_{mm} \rangle = -\frac{1}{p} \sum_m \int d^{d-p-1}x \, \langle T^{mm} \rangle \tag{4.41}$$

From the Einstein equation (4.27), we rewrite the average energy density $\mathscr{E}$ as a surface integral at spatial infinity:

$$\mathscr{E} = \frac{1}{2} \oint_\infty dS_i (\partial_j h_{ij} - \partial_i h_{jj} - \partial_i h_{mm}) \qquad (i, j = 1, ..., d-p-1, m = 1, ..., p), \tag{4.42}$$

where we take sum for the same indices. This is ADM energy density.

4.2.2 Komar integral

We can rewrite the ADM energy density (4.42) as a integral on the bulk;

$$\mathscr{E} = \frac{1}{2} \int d^{d-p-1}x \left(h_{ij,ij} - h_{jj,ii} - h_{mm,ii} \right), \tag{4.43}$$

where we take sum for the same indices. Let us rewrite the ADM energy density (4.43) as the Komar integral [70].

From the Einstein equations (4.18), we have the Pauli-Fierz equations

$$\Box h_{\mu\nu} + h_{,\mu\nu} - 2h_{(\mu,\nu)} = -2\left(T_{\mu\nu} - \frac{1}{d-2}\eta_{\mu\nu}T\right), \tag{4.44}$$

where $\Box = \partial_\mu\partial^\mu$ and $T = \eta^{\mu\nu}T_{\mu\nu}$. If we assume that $h_{\mu\nu}$ is independent of time, we have the relation

$$(d-3)\left[\langle h_{ij}\rangle_{,ij} - \langle h_{jj}\rangle_{,ii} - \langle h_{mm}\rangle_{,ii}\right] = -\left[(d-2)\nabla^2\langle h_{00}\rangle + 2\left(\langle T_{ii}\rangle + \langle T_{mm}\rangle\right)\right], \tag{4.45}$$

by taking p-average and contracting the Pauli-Fierz equations (4.44). Since the integration of the left-hand side of (4.45) on $(d-p-1)$-dimension Euclidean spacetime is proportional to the ADM energy density (4.42), then we have

$$\mathscr{E} = -\left(\frac{d-2}{2(d-3)}\right)\oint_\infty dS_i\partial_i h_{00} + \frac{p}{d-3}\mathscr{T} - \frac{1}{d-3}\int d^jx\,\langle T_{ii}\rangle. \tag{4.46}$$

If we assume that the energy momentum tensor vanishes at spatial infinity, the energy conservation law

$$\partial_i\langle T_{ij}\rangle = 0 \tag{4.47}$$

gives

$$\int d^{d-p-1}x\,\langle T_{ii}\rangle = \oint_\infty dS_i\,\langle x^jT_{ij}\rangle = 0, \tag{4.48}$$

that is the third term of the right-hand side of (4.46) does not contribute to $\mathscr{E}$. Moreover the tension $\mathscr{T}$ does not contribute to $\mathscr{E}$ in case $p=0$. If $p>0$, the Einstein equation (4.27) and the Pauli-Fierz equations (4.44) give us the $\mathscr{T}$ as a surface integration as

$$\mathscr{T} = -\frac{1}{2p(d-p-3)}\oint_\infty dS_i\partial_i\left[ph_{00} - (d-3)h_{mm}\right]. \tag{4.49}$$

Then we can rewrite the energy density as

$$\mathscr{E} = -\frac{1}{2(d-p-3)}\oint_\infty dS_i\partial_i\left[(d-p-2)h_{00} - h_{mm}\right]. \tag{4.50}$$

Let us rewrite the energy density (4.50) and the tension (4.49) as covariant forms. We now set that the Killing vector fields $\{k, l_{(m)}\}$ are approximate to the Killing vector fields of the background Minkowski spacetime $\{\bar{k}, \bar{l}_{(m)}\}$ at spatial infinity $|x| \to \infty$. Since $k \to \partial/\partial t$ at spatial infinity, we have

$\nabla_i k^0 \sim \Gamma^0{}_{i0}$. If the metric is independent of time, $\Gamma^0{}_{i0} \sim h_{00,i}/2$. From anti-symmetry of the indices of the area element and $D^\mu k^\mu$, we have

$$dS_{\mu\nu\mu_1\ldots\mu_p}\left(D^\mu k^\nu\right) l^{\mu_1}_{(1)}\cdots l^{\mu_p}_{(p)} \sim dS_i \partial_i h_{00}. \tag{4.51}$$

The same applies to h_{mm}. Therefore, we have the energy density and the tension as Komar integral as

$$\mathscr{E} = -\frac{1}{2(d-p-3)}\oint_\infty dS_{\mu\nu\mu_1\ldots\mu_p}\left[(d-p-2)(D^\mu k^\nu) l^{\mu_1}_{(1)}\cdots l^{\mu_p}_{(p)} + k^\nu D^\mu (l^{\mu_1}_{(1)}\cdots l^{\mu_p}_{(p)})\right], \tag{4.52}$$

$$\mathscr{T} = -\frac{1}{2p(d-p-3)}\oint_\infty dS_{\mu\nu\mu_1\ldots\mu_p}\left[p(D^\mu k^\nu) l^{\mu_1}_{(1)}\cdots l^{\mu_p}_{(p)} + (d-p-3)k^\nu D^\mu (l^{\mu_1}_{(1)}\cdots l^{\mu_p}_{(p)})\right]. \tag{4.53}$$

4.2.3 Smarr formula

Let us derive an analog for branes of the Smarr formula for the mass of a electrically charged black hole (3.45). We will suppose that the timelike Killing vector field k is normalized such that $k^2 \to -1$ at transverse spatial infinity. Also we assume that the spacelike vector fields $k_{(m)}$ are everywhere Killing, and not just asymptotically Killing.

An application of the Gauss' law to the other surface integral on the right hand side of (4.52) gives both an surface integral over the event horizon $\mathcal{H}$ and a bulk integral over the hypersuface Σ orthogonal to the horizon:

$$\begin{aligned}\mathscr{E} = &-\frac{1}{2(d-p-3)}\oint_{\mathcal{H}} dS_{\mu\nu\mu_1\ldots\mu_p}\left[(d-p-2)(D^\mu k^\nu) l^{\mu_1}_{(1)}\cdots l^{\mu_p}_{(p)} + k^\nu D^\mu (l^{\mu_1}_{(1)}\cdots l^{\mu_p}_{(p)})\right]\\ &+\mathscr{E}^{(\mathrm{mat})},\end{aligned} \tag{4.54}$$

where

$$\begin{aligned}\mathscr{E}^{(\mathrm{mat})} = &\frac{1}{d-p-3}\int_\Sigma dS_{\mu\nu\mu_1\ldots\mu_{p-1}}\left[l^{\mu_1}_{(1)}\cdots l^{\mu_{p-1}}_{(p-1)} l^\nu_{(p)}\left[(d-p-2)\left(T^{(\mathrm{mat})}\right)^\mu{}_\rho k^\rho - T^{(\mathrm{mat})}k^\mu\right]\right.\\ &\left.-p l^{\mu_1}_{[(1)}\cdots l^{\mu_{p-1}}_{(p-1)} l^\nu_{(p)]}\left(T^{(\mathrm{mat})}\right)^\mu{}_\rho k^\rho\right].\end{aligned} \tag{4.55}$$

Also $T^{(\mathrm{mat})}$ is the energy-momentum tensor defined via the Einstein equation

$$R_{\mu\nu} = T^{(\mathrm{mat})}_{\mu\nu} - \frac{1}{d-2}g_{\mu\nu}T^{(\mathrm{mat})}, \qquad (T^{(\mathrm{mat})} \equiv g^{\mu\nu}T^{(\mathrm{mat})}_{\mu\nu}). \tag{4.56}$$

For the moment, we will assume that the energy stress tensor, or matter energy density, vanishes. This means that we are now considering uncharged black branes.

Given $\mathscr{E}^{(\mathrm{mat})} = 0$, we have to consider only the surface integral on the horizon in (4.54). The area element on the horizon $dS_{\mu\nu\mu_1\dots\mu_p}$ is determined by the normalized Killing vector fields $\{k, l_{(m)}\}$ and a null vector n such that

$$k \cdot n = -1. \tag{4.57}$$

Since k is tangent and normal to the horizon, it must be orthogonal to the other p normals;

$$k \cdot \zeta_{(m)} = 0. \tag{4.58}$$

Then the area element are given by

$$dS_{\mu\nu\mu_1\dots\mu_p} = [(p+2)!]v^{-1}(\zeta)d\mathcal{A}k_{[\mu}n_\nu\zeta^{(1)}_{\mu_1}\cdots\zeta^{(p)}_{\mu_p]}, \tag{4.59}$$

where $d\mathcal{A}$ is the measure of the area element of the unit p-volume, and $v(\zeta) = |\zeta_{(1)} \wedge \cdots \wedge \zeta_{(p)}|$. Here we choose $\zeta_{(m)} = l_{(m)}$.

We define the surface gravity on the horizon κ via

$$k^\nu \nabla_\nu k^\mu|_{\mathcal{H}} = \kappa k^\mu|_{\mathcal{H}}\,. \tag{4.60}$$

Then we have

$$\begin{aligned} dS_{\mu\nu\mu_1\dots\mu_p}\left(\nabla^\mu k^\nu\right) l^{\mu_1}_{(1)}\cdots l^{\mu_p}_{(p)} &= 2v^{-1}(l)d\mathcal{A}k_{[\mu}n_{\nu]}l^{(1)}_{\mu_1}\cdots l^{(p)}_{\mu_p}\left(\nabla^{[\mu}k^{\nu]}\right) l^{\mu_1}_{(1)}\cdots l^{\mu_p}_{(p)} \\ &= 2v^{-1}(l)d\mathcal{A}k_\mu n_\nu\left(\nabla^\mu k^\nu\right) l^{\mu_1}_{(1)}\cdots l^{\mu_p}_{(p)}\left(\nabla^\mu k^\nu\right) v^2(l) \\ &= -2\kappa v(l)d\mathcal{A}, \end{aligned} \tag{4.61}$$

and

$$dS_{\mu\nu\mu_1\cdots\mu_p}k^\nu\nabla^\mu\left(l^{\mu_1}_{(1)}\cdots l^{\mu_p}_{(p)}\right) = 0. \tag{4.62}$$

Substituting the above relations to (4.54), we have the Smarr formula for uncharged black brane as

$$\mathscr{E} = \frac{d-p-2}{d-p-3}\kappa\mathcal{A}_{\mathrm{eff}}, \tag{4.63}$$

where $\mathcal{A}_{\mathrm{eff}}$ is the effective horizon area defined as

$$\mathcal{A}_{\mathrm{eff}} \equiv \oint_{\mathcal{H}} v(l)d\mathcal{A}. \tag{4.64}$$

We now introduce the electric charge. A charged p-brane is a source of a $(p+1)$-gauge field A. we have the energy momentum tensor

$$T^{(\mathrm{mat})}_{\mu\nu} = \frac{1}{(p+1)!}F_{\mu\mu_1\dots\mu_{p+1}}F_\nu{}^{\mu_1\dots\mu_{p+1}} - \frac{1}{2}g_{\mu\nu}F^2, \tag{4.65}$$

where $F^2 = F_{\mu_1\dots\mu_{p+2}} F^{\mu_1\dots\mu_{p+2}}/(p+2)!$. Substituting (4.65) to (4.55), we then have

$$\begin{aligned}\mathscr{E}^{(\mathrm{mat})} &= \frac{1}{(d-p-3)(p+1)!}\int_\Sigma dS_{\mu\nu\mu_1\dots\mu_p}\Big[l^{\mu_1}_{(1)}\cdots l^{\mu_p}_{(p)} \\ &\quad\times\left[(n-1)F^{\mu\nu_1\dots\nu_{p+1}}k^\rho F_{\rho\nu_1\dots\nu_{p+1}} - (p+1)(p+1)!F^2k^\mu\right] \\ &\quad -pl^{\mu_1}_{[(1)}\cdots l^{\mu_{p-1}}_{(p-1)}l^\rho_{(p)]}F_{\rho\nu_1\dots\nu_{p+1}}F^{\mu\nu_1\dots\nu_{p+1}}k^\nu\Big]. \end{aligned} \quad (4.66)$$

We assume that the field strength F is invariant under the transformation generated by the Killing vector k and $l_{(m)}$, and we choose a gauge for A such that

$$\mathscr{L}_k A = 0, \qquad \mathscr{L}_{l_{(m)}} A = 0, \qquad A \xrightarrow{|x|\to\infty} 0. \quad (4.67)$$

Then the bulk integration reduces to the surface integration on the horizon, and we have

$$\begin{aligned}\mathscr{E}^{(\mathrm{mat})} &= \frac{1}{2(d-p-3)p!}\oint_{\mathcal{H}} dS_{\mu\nu\mu_1\dots\mu_{p-1}\rho}\Big[l^{\mu_1}_{(1)}\cdots l^{\mu_{p-1}}_{(p-1)}l^\nu_{(p)} \\ &\quad\times\left[(d-p-2)F^{\mu\rho\nu_1\dots\nu_p}k^\sigma + 2F^{\rho\sigma\nu_1\dots\nu_p}k^\mu\right]A_{\sigma\nu_1\dots\nu_p} \\ &\quad -pl^{\mu_1}_{[(1)}\cdots l^{\mu_{p-1}}_{(p-1)}l^\sigma_{(p)]}k^\nu F^{\mu\rho\nu_1\dots\nu_p}A_{\sigma\nu_1\dots\nu_p}\Big]. \end{aligned} \quad (4.68)$$

From the Einstein equations (4.27), we have

$$k_\mu\left(k^{[\mu}F^{\nu\rho\nu_1\dots\nu_p]}\right)\Big|_{\mathcal{H}} = 0. \quad (4.69)$$

Since the field strength is translationally invariant, we also have

$$l^{[\mu}_{(m)}F^{\nu_1\dots\nu_{p+2}]} = 0. \quad (4.70)$$

(4.69) and (4.70) simplify the surface integral (4.68) as

$$\mathscr{E}^{(\mathrm{mat})} = \frac{1}{(p+2)!}\oint_{\mathcal{H}} dS_{\mu\nu\mu_1\dots\mu_p}F^{\mu\nu\mu_1\dots\mu_p}\Phi, \quad (4.71)$$

where Φ is the electric potential on the horizon defined as

$$\Phi \equiv -k^\mu l^{\mu_1}_{(1)}\cdots l^{\mu_p}_{(p)}A_{\mu\mu_1\dots\mu_p}\Big|_{\mathcal{H}}. \quad (4.72)$$

Since the gauge field A is invariant under the transformation generated by the Killing vectors, the electric potential is constant everywhere on the horizon. Hence we can unplug Φ from the surface integral as

$$\mathscr{E}^{(\mathrm{mat})} = \frac{1}{(p+2)!}\Phi\oint_{\mathcal{H}} dS_{\mu\nu\mu_1\dots\mu_p}F^{\mu\nu\mu_1\dots\mu_p}. \quad (4.73)$$

Applying the Gauss' law again, we have

$$\mathscr{E}^{(\mathrm{mat})} = \Phi Q, \tag{4.74}$$

where Q is the charge per unit p-volume defined as

$$\frac{1}{(p+2)!}\oint_\infty dS_{\mu\nu\mu_1\ldots\mu_p}F^{\mu\nu\mu_1\ldots\mu_p}. \tag{4.75}$$

From (4.74) and (4.63), we have the Smarr formula for a charged blakc brane as

$$\mathscr{E} = \frac{d-p-2}{d-p-3}\kappa\mathcal{A}_{\mathrm{eff}} + \Phi Q. \tag{4.76}$$

4.2.4 The first law of black brane thermodynamics

We derived the Smarr formula for a charged black brane, which is a solution of the Euler-Lagrange equations for the action

$$I = \frac{1}{2}\int d^dx\sqrt{-g}\left[R - F^2\right] + (\mathrm{surfaceterm}). \tag{4.77}$$

If the metric and the gauge field are dimensionless, the dimension of each term of the action is equal to the number of derivations. Since R and F^2 have two derivatives, they have the same dimensions, and the Euler-Lagrange equations are scale invariant. Therefore, any dimensionful quantitiy must be a weighted homogeneous function. Each weight are determined dimensions of each parameters.

The energy density is a function of the electric charge Q and the effective horizon area $\mathcal{A}_{\mathrm{eff}}$;

$$\mathscr{E} = \mathscr{E}(Q, \mathcal{A}_{\mathrm{eff}}). \tag{4.78}$$

From (4.52) and (4.75), we have

$$[\mathscr{E}] = [Q] = L^{d-p-3}, \tag{4.79}$$

where $[A]$ represents dimensions of length of A. We also have

$$[\mathcal{A}_{\mathrm{eff}}] = L^{d-p-2}. \tag{4.80}$$

Applying the Euler's homogeneous function theorem[10], we have

$$\mathscr{E} = Q\frac{\partial \mathscr{E}}{\partial Q} + \frac{d-p-2}{d-p-3}\mathcal{A}_{\text{eff}}\frac{\partial \mathscr{E}}{\partial \mathcal{A}_{\text{eff}}}. \tag{4.83}$$

Comparing (4.83) and the Smarr formula (4.76), we have

$$\frac{\partial \mathscr{E}}{\partial Q} = \Phi, \qquad \frac{\partial \mathscr{E}}{\partial \mathcal{A}_{\text{eff}}} = \kappa. \tag{4.84}$$

Moreover, we have the first law of black brane thermodynamics [68];

$$d\mathscr{E} = \kappa d\mathcal{A}_{\text{eff}} + \Phi dQ. \tag{4.85}$$

We can rewrite the first term as TdS, where T is the Hawking temperature and S is the Bekenstein-Hawking temperature. Thus we have another form of the first law of thermodynamics as

$$d\mathscr{E} = TdS + \Phi dQ. \tag{4.86}$$

This is the same form of the first law of black hole thermodynamics (3.49).

4.3 Thermodynamic quantities of black brane solutions in Einstein gravity

The Hawking temperature is determined by requiring that the metric (4.6) has no conical singularity at the horizon $r = r_+$ as

$$T = \frac{q-1}{4\pi}\frac{1}{r_+}\left[1 - \left(\frac{r_-}{r_+}\right)^{q-1}\right]^{\frac{1}{p+1}}. \tag{4.87}$$

Similarly, the Bekenstein-Hawking entropy per unit brane volume[11] is given by the horizon area as

$$S = \frac{2\pi V_q}{\kappa^2} r_+^q \left[1 - \left(\frac{r_-}{r_+}\right)^{q-1}\right]^{\frac{p}{p+1}}. \tag{4.88}$$

[10] If $f(x_1, ..., x_m)$ is a funtion of homogeneous of order n, i.e.

$$f(sx_1, ..., sx_m) = s^n f(x_1, ..., x_m), \tag{4.81}$$

we have

$$nf(x_1, ..., x_m) = \sum_{i=1}^{m} x_i \frac{\partial f}{\partial x_i}. \tag{4.82}$$

[11] The brane volume $\mathcal{V} = \int d^p y$ is evaluated at the infinity $r = \infty$.

The electric potential is the value of the gauge potential at the event horizon,

$$\Phi = \frac{e}{\kappa}\sqrt{\frac{p+q}{(p+1)(q-1)}}\left(\frac{r_-}{r_+}\right)^{\frac{q-1}{2}} . \tag{4.89}$$

Note that they satisfy the Smarr-like formula $M = \frac{q}{q-1}TS + Q\Phi$. In the microcanonical ensemble, where the tension M and the charge Q are identified as independent variables, we confirm the following first law of black brane thermodynamics (4.86):

$$dM = TdS + \Phi dQ\,. \tag{4.90}$$

4.3.1 Other ensembles

In the WGC context, we are interested in the charge-to-tension ratio of black branes of zero temperature, so that it is convenient to work in the canonical ensemble. Also, the grand canonical ensemble naturally appears in the Euclidean path integral analysis employed in the next section to study higher derivative corrections. Having this in mind, we introduce several thermodynamic ensembles and the corresponding potentials.

Generality. The Helmholtz free energy $H = H(T, Q)$ in the canonical ensemble and the Gibbs free energy $G = G(T, \Phi)$ in the grand canonical ensemble are defined by

$$H = M - TS\,, \qquad G = H - Q\Phi\,. \tag{4.91}$$

The first law (4.90) in the microcanonical ensemble then implies

$$dH = -SdT + \Phi dQ\,, \qquad dG = -SdT - Qd\Phi\,. \tag{4.92}$$

We emphasize that these relations hold in general (beyond the Einstein gravity), as long as the first law (4.86) in the micro canonical ensemble is satisfied. Indeed, the Wald entropy [60] is defined such that the first law is satisfied, which provides a natural generalization of the Bekenstein-Hawking entropy formula in higher derivative theories.

Rescaled variables. For notational simplicity, it is convenient to introduce rescaled thermodynamic quantities as follows:

$$\mathcal{M} = \frac{p+1}{p+q}\frac{\kappa^2}{V_q}M\,, \quad \mathcal{T} = \frac{4\pi}{q-1}T\,, \quad \mathcal{S} = \frac{(p+1)(q-1)}{4\pi(p+q)}\frac{\kappa^2}{V_q}S\,, \tag{4.93}$$

$$\mathcal{Q} = \sqrt{\frac{p+1}{(p+q)(q-1)}}\,\frac{e\kappa}{V_q}Q\,, \quad \mathit{\Phi} = \sqrt{\frac{(p+1)(q-1)}{p+q}}\,\frac{\kappa}{e}\Phi\,. \tag{4.94}$$

Correspondingly, we define the rescaled free energies as

$$\mathcal{H} = \mathcal{M} - \mathcal{T}\mathcal{S} = \frac{p+1}{p+q}\frac{\kappa^2}{V_q}H\,, \quad \mathcal{G} = \mathcal{H} - \mathcal{Q}\mathit{\Phi} = \frac{p+1}{p+q}\frac{\kappa^2}{V_q}G\,. \tag{4.95}$$

Note that the first laws are in the standard form even after the rescaling:

$$d\mathcal{M} = \mathcal{T}d\mathcal{S} + \mathit{\Phi}d\mathcal{Q}\,, \quad d\mathcal{H} = -\mathcal{S}d\mathcal{T} + \mathit{\Phi}d\mathcal{Q}\,, \quad d\mathcal{G} = -\mathcal{S}d\mathcal{T} - \mathcal{Q}d\mathit{\Phi}\,. \tag{4.96}$$

Grand canonical ensemble in Einstein gravity. For later use, we provide an explicit form of thermodynamic quantities in the Einstein gravity for the grand canonical ensemble, where the temperature T and the electric potential Φ are independent thermodynamic variables. First, the electric charge $\mathcal{Q}$, the ADM tension $\mathcal{M}$, and the entropy density $\mathcal{S}$ in the rescaled language are given by

$$\mathcal{Q}_0(\mathcal{T},\mathit{\Phi}) = \mathit{\Phi}\frac{(1-\mathit{\Phi}^2)^{\frac{q-1}{p+1}}}{\mathcal{T}^{q-1}}\,, \tag{4.97}$$

$$\mathcal{M}_0(\mathcal{T},\mathit{\Phi}) = \left[\mathit{\Phi}^2 + \frac{q(p+1)}{2(p+q)}(1-\mathit{\Phi}^2)\right]\frac{(1-\mathit{\Phi}^2)^{\frac{q-1}{p+1}}}{\mathcal{T}^{q-1}}\,, \tag{4.98}$$

$$\mathcal{S}_0(\mathcal{T},\mathit{\Phi}) = \frac{(q-1)(p+1)}{2(p+q)}\frac{(1-\mathit{\Phi}^2)^{\frac{p+q}{p+1}}}{\mathcal{T}^{q}}\,. \tag{4.99}$$

Here and in what follows we use the subscript 0 to indicate that the expression is for the Einstein gravity without higher derivative corrections. Accordingly, the rescaled Gibbs free energy reads

$$\mathcal{G}_0(\mathcal{T},\mathit{\Phi}) = \mathcal{M}_0(\mathcal{T},\mathit{\Phi}) - \mathcal{T}\mathcal{S}_0(\mathcal{T},\mathit{\Phi}) - \mathit{\Phi}\mathcal{Q}_0(\mathcal{T},\mathit{\Phi}) = \frac{p+1}{2(p+q)}\frac{(1-\mathit{\Phi}^2)^{\frac{p+q}{p+1}}}{\mathcal{T}^{q-1}}\,. \tag{4.100}$$

As a consistency check, one may explicitly show that the first law (4.96) is satisfied, i.e.,

$$\mathcal{Q}_0(\mathcal{T},\mathit{\Phi}) = -\frac{\partial\mathcal{G}_0(\mathcal{T},\mathit{\Phi})}{\partial\mathit{\Phi}}\,, \quad \mathcal{S}_0(\mathcal{T},\mathit{\Phi}) = -\frac{\partial\mathcal{G}_0(\mathcal{T},\mathit{\Phi})}{\partial\mathcal{T}}\,. \tag{4.101}$$

5 Higher derivative corrections

In this section we study higher derivative corrections to the black brane thermodynamics and their implications for the WGC. Assuming parity invariance, we consider the following Lorentzian effective action with the leading order corrections relevant to our problem[12]:

$$\mathcal{I} = \mathcal{I}_0 + \Delta\mathcal{I}\,, \tag{5.1}$$

$$\Delta\mathcal{I} = \frac{\kappa^2}{e^4}\sum_j \alpha_j \int d^dx\sqrt{g}\left(F^4\right)_j + \frac{\beta}{e^2}\int d^dx\sqrt{g}FFW + \frac{\gamma}{\kappa^2}\int d^dx\sqrt{g}W^2\,, \tag{5.2}$$

where $\mathcal{I}_0$ is the Einstein gravity action (3.3) and $W_{\mu\nu\rho\sigma}$ is the Weyl tensor. The first term in Eq. (5.2) denotes operators with four field-strengths and j is a label for tensor structures. For example, for $p = 0$, we have two independent tensor structures ($j = 1, 2$):

$$(F_{\mu\nu}F^{\mu\nu})^2\,, \quad F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}\,. \tag{5.3}$$

Similarly, for $p = 1$, we have three independent tensor structures ($j = 1, 2, 3$):

$$(F_{\mu\nu\rho}F^{\mu\nu\rho})^2\,, \quad F^{\mu\alpha\beta}F_\mu{}^{\gamma\delta}F^\nu{}_{\alpha\beta}F_{\nu\gamma\delta}\,, \quad F^{\mu\alpha\beta}F_\mu{}^{\gamma\delta}F^\nu{}_{\alpha\gamma}F_{\nu\beta\delta}\,. \tag{5.4}$$

The second term FFW denotes an operator with two field-strengths and one Weyl tensor, for which there exists only one independent tensor structure:

$$FFW = F_{\mu_1\mu_2\nu_1\nu_2\ldots\nu_p}F_{\mu_3\mu_4}{}^{\nu_1\nu_2\ldots\nu_p}W^{\mu_1\mu_2\mu_3\mu_4}\,. \tag{5.5}$$

The third term W^2 is $W^2 = W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}$. Besides, higher derivative corrections to the boundary action are irrelevant in our analysis as long as $q \geq 2$, essentially because higher derivative operators decay quickly far away from the brane. In the following, we study the effects of the four-derivative operators in Eq. (5.2) on the black brane thermodynamics.

5.1 Outline

5.1.1 Gibbs free energy from Euclidean path integral.

In the thermodynamic analysis, we employ the Euclidean path integral formalism based on the following Euclidean action for the setup (5.1):

$$I[\phi] = I_0[\phi] + \Delta I[\phi]\,, \tag{5.6}$$

[12]When $p+1 = 4n$ ($n = 1, 2, \ldots$), the effective action contains three-derivative operators schematically of the form, F^3, but we ignore these terms because they do not modify the profile of electric brane solutions and their thermodynamics.

$$I_0[\phi] = -\frac{1}{2\kappa^2}\int_{\mathcal{M}} d^dx\sqrt{g}R + \frac{1}{2e^2}\int_{\mathcal{M}} d^dx\sqrt{g}F^2 - I_{\rm GHY}\,, \tag{5.7}$$

$$\Delta I = -\frac{\kappa^2}{e^4}\sum_j \alpha_j \int d^dx\sqrt{g}\left(F^4\right)_j - \frac{\beta}{e^2}\int d^dx\sqrt{g}FFW - \frac{\gamma}{\kappa^2}\int d^dx\sqrt{g}W^2\,, \tag{5.8}$$

where $\phi = g_{\mu\nu}, A$ collectively denotes the metric $g_{\mu\nu}$ and the p-form gauge field A. $I_{\rm GHY}$ is the Gibbons-Hawking-York term [65, 66],

$$I_{\rm GHY} = \frac{1}{\kappa^2}\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{h}\left(K - K_0\right)\,, \tag{5.9}$$

where $\partial\mathcal{M}$ is the boundary of the spacetime manifold $\mathcal{M}$, h is the induced metric on $\partial\mathcal{M}$, K is the trace of the extrinsic curvature of $\partial\mathcal{M}$, and K_0 is that for flat space necessary to make the on-shell action finite.

For the action (5.6), it is appropriate to impose Dirichlet boundary conditions on the electric potential. Also, we impose periodic boundary conditions along the Euclidean time specified by the temperature T. Then, the Euclidean path integral gives the grand canonical ensemble, where the temperature T and the electric potential Φ are thermodynamic variables. At the semiclassical level, the Gibbs free energy per unit brane volume reads

$$G = \frac{TI[\bar{\phi}]}{\mathcal{V}}\,, \tag{5.10}$$

where $\mathcal{V} = \int d^py$ is the brane volume evaluated at the infinity $r = \infty$ and $\bar{\phi}$ stands for the classical solution in the setup (5.6). Other thermodynamic quantities follow from the Gibbs free energy based on the standard thermodynamic argument, so that our first task is to evaluate higher derivative corrections to the on-shell Euclidean action.

5.1.2 Higher derivative analysis.

For this purpose, let us expand the solution $\bar{\phi}$ in the full theory (5.6) as

$$\bar{\phi} = \bar{\phi}_0 + \Delta\bar{\phi}\,, \tag{5.11}$$

where $\bar{\phi}_0$ is the solution in the Einstein gravity and $\Delta\bar{\phi}$ is the higher derivative correction. Note that we are working in the grand canonical ensemble and so the solution $\bar{\phi}$ should be regarded as a function of the temperature T, the electric potential Φ, and the Wilson coefficients α_i, β, and γ. More explicitly,

$$\Delta\bar{\phi}(T,\Phi) := \bar{\phi}(T,\Phi) - \bar{\phi}_0(T,\Phi)\,. \tag{5.12}$$

Accordingly, we can expand the on-shell action as

$$
\begin{aligned}
I[\bar{\phi}] &= I_0[\bar{\phi}_0 + \Delta\bar{\phi}] + \Delta I[\bar{\phi}_0 + \Delta\bar{\phi}] \\
&\simeq I_0[\bar{\phi}_0] + \int d^d x \, \frac{\delta I_0}{\delta \phi}\bigg|_{\phi=\bar{\phi}_0} \Delta\bar{\phi} + \Delta I[\bar{\phi}_0] \,,
\end{aligned}
\tag{5.13}
$$

where we kept the leading order correction only. A simple, but useful observation is that the equations of motion guarantee that the second term vanishes up to a boundary term. Also, the boundary term vanishes as long as $q \geq 2$, essentially because the spacetime curvature and the electric flux decay quickly away from the brane. As a result, we find

$$
I[\bar{\phi}] \simeq I_0[\bar{\phi}_0] + \Delta I[\bar{\phi}_0] \tag{5.14}
$$

at the leading order. In particular, we do not need to evaluate the correction $\Delta\bar{\phi}$ to the solution explicitly, which simplifies the analysis significantly[13] [71]. Then, the Gibbs free energy reads

$$
G(T,\Phi) = G_0(T,\Phi) + \Delta G(T,\Phi) \tag{5.15}
$$

with

$$
G_0(T,\Phi) = \frac{T I_0[\bar{\phi}_0]}{\mathcal{V}} \,, \quad \Delta G(T,\Phi) = \frac{T \Delta I[\bar{\phi}_0]}{\mathcal{V}} \,. \tag{5.16}
$$

Note that one can explicitly show that G_0 evaluated in this manner agrees with Eq. (4.100). Below, we first evaluate ΔG and then perform thermodynamic arguments.

Now we compute the higher derivative corrections to the Gibbs free energy.

F^4 term. First, it is convenient to notice that for the Einstein gravity solution (4.6)-(4.7) the α_j operators schematically of the form F^4 are simply

$$
\left(F^4\right)_j = A_j \, (g^{t_E t_E} g^{rr} g^{y_1 y_1} \cdots g^{y_p y_p})^2 (F_{t_E r y^1 \ldots y^p})^4 = A_j \, (F_{t r y^1 \ldots y^p})^4 \tag{5.17}
$$

with a combinatorial factor A_j. For example, for $p = 0$, there are two operators,

$$
\left(F^4\right)_1 = (F_{\mu\nu}F^{\mu\nu})^2 \,, \quad \left(F^4\right)_2 = F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} \,, \tag{5.18}
$$

[13] Since solutions of the Einstein equations are stationary points of the Euclidean action, the variation of $I_0[\phi]$ vanishes at $\phi = \phi_0$ in first order.

and the corresponding combinatorial factors read

$$A_1 = 4\,, \quad A_2 = 2\,. \tag{5.19}$$

Similarly, for $p = 1$, we have

$$\left(F^4\right)_1 = (F_{\mu\nu\rho}F^{\mu\nu\rho})^2\,,\ \left(F^4\right)_2 = F^{\mu\alpha\beta}F_\mu{}^{\gamma\delta}F^\nu{}_{\alpha\beta}F_{\nu\gamma\delta}\,,\ \left(F^4\right)_3 = F^{\mu\alpha\beta}F_\mu{}^{\gamma\delta}F^\nu{}_{\alpha\gamma}F_{\nu\beta\delta}\,, \tag{5.20}$$

and

$$A_1 = 36\,, \quad A_2 = 12\,, \quad A_3 = 6\,. \tag{5.21}$$

It is straightforward to compute the correction to the Gibbs free energy as

$$\begin{aligned}\Delta G|_{F^4} &= -\Big(\sum_i A_i\alpha_i\Big)\frac{(p+q)^2(q-1)^2}{(p+1)^2}\frac{V_q}{\kappa^2}\int_{r_+}^\infty dr\frac{(r_+r_-)^{2(q-1)}}{r^{3q}} \\ &= -\Big(\sum_i A_i\alpha_i\Big)\frac{(p+q)^2(q-1)^2}{(p+1)^2(3q-1)}\frac{V_q}{\kappa^2}r_+^{q-3}\left(\frac{r_-}{r_+}\right)^{2(q-1)} \\ &= -\Big(\sum_i A_i\alpha_i\Big)\frac{(p+q)^2(q-1)^2}{(p+1)^2(3q-1)}\frac{V_q}{\kappa^2}\mathcal{T}^{3-q}\Phi^4(1-\Phi^2)^{\frac{q-3}{p+1}}\,.\end{aligned} \tag{5.22}$$

Here and in what follows, we use $O|_{F^4}$ to denote the contribution from the α_j couplings to the quantity O, and similarly for other higher derivative operators. In terms of the rescaled Gibbs free energy, we have

$$\Delta\mathcal{G}|_{F^4} = -\Big(\sum_i A_i\alpha_i\Big)\frac{(p+q)(q-1)^2}{(p+1)(3q-1)}\mathcal{T}^{3-q}\Phi^4(1-\Phi^2)^{\frac{q-3}{p+1}}\,. \tag{5.23}$$

FFW **term.** Other contributions can also be evaluated in a straightforward manner. First, the correction from the FFW term to the rescaled Gibbs free energy reads[14]

$$\Delta\mathcal{G}_{FFW} = \beta(q-1)^2p!\left[\frac{2\mathcal{R}_{FFW}}{(p+1)(p+q+1)(3q-1)} + (\Phi^{-2}-1)\right]\mathcal{T}^{3-q}\Phi^4\left(1-\Phi^2\right)^{\frac{q-3}{p+1}}\,. \tag{5.24}$$

Here $\mathcal{R}_{FFW}$ is a (p,q)-dependent factor defined by

$$\mathcal{R}_{FFW} = p^2(2q-1) + p(5q-2) - (q^2-4q+1)\,. \tag{5.25}$$

Note that the second term in the brackets of Eq. (5.24) vanishes in the limit $\Phi^2 \to 1$, which corresponds to the extremal limit in the Einstein gravity. Essentially because of this, it does not affect the extremal condition and the entropy of the $\mathcal{M} = \mathcal{Q}$ black branes as we explicitly see later.

[14]In Appendix A for, we collect useful formulae including an explicit form of curvature tensors for the black brane solution (4.6) in the Einstein gravity.

W^2 term. Similarly, the correction from the W^2 term reads

$$\Delta\mathcal{G}_{W^2} = -\gamma\left[\frac{2(q-1)\mathcal{R}_{W^2}}{(p+1)^2(p+q+1)(3q-1)} + \mathcal{O}(1-\Phi^2)\right]\mathcal{T}^{3-q}\left(1-\Phi^2\right)^{\frac{q-3}{p+1}} . \tag{5.26}$$

Here the second term in the brackets denotes subleading terms in the limit $\Phi^2 \to 1$, whose explicit form is given in Appendix B. Similarly to the second term in the brackets of Eq. (5.24), they do not affect the extremal condition and the entropy of the $\mathcal{M} = \mathcal{Q}$ black branes. Also, $\mathcal{R}_{W^2}$ is a (p, q)-dependent factor given by

$$\mathcal{R}_{W^2} = p^3q(2q-1) + 2p^2(q^3+2q-1) + p(3q^3-6q^2+12q-4) + 2(q^3-3q^2+4q-1) . \tag{5.27}$$

Summary of the subsection. Collecting all the above results and picking up the leading order term in the limit $\Phi^2 \to 1$, we conclude that

$$\Delta\mathcal{G} = -\Big[\mathcal{C} + \mathcal{O}(1-\Phi^2)\Big]\mathcal{T}^{3-q}\left(1-\Phi^2\right)^{\frac{q-3}{p+1}} \tag{5.28}$$

with a constant factor $\mathcal{C}$ given by

$$\begin{aligned}\mathcal{C} = \Big(\sum_i A_i\alpha_i\Big)\frac{(p+q)(q-1)^2}{(p+1)(3q-1)} - \beta\frac{2(q-1)^2p!\mathcal{R}_{FFW}}{(p+1)(p+q+1)(3q-1)}\\ + \gamma\frac{2(q-1)\mathcal{R}_{W^2}}{(p+1)^2(p+q+1)(3q-1)} .\end{aligned} \tag{5.29}$$

We will find that the sign of $\mathcal{C}$ is crucial in the following analysis.

An explicit form for $p = 0$. To compare our results with the black hole analysis in the literature (see, e.g., Refs. [6,9,10]), it is convenient to explicitly write down $\mathcal{C}$ for $p = 0$:

$$\begin{aligned}\mathcal{C} &= (2\alpha_1+\alpha_2)\frac{2q(q-1)^2}{3q-1} + \beta\frac{2(q-1)^2(q^2-4q+1)}{(q+1)(3q-1)} + \gamma\frac{4(q-1)(q^3-3q^2+4q-1)}{(q+1)(3q-1)}\\ &= (2\alpha_1+\alpha_2)\frac{2(d-2)(d-3)^2}{3d-7} + \beta\frac{2(d-3)^2(d^2-8q+13)}{(d-1)(3d-7)}\\ &\quad + \gamma\frac{4(d-3)(d^3-9d^2+28d-29)}{(d-1)(3d-7)} ,\end{aligned} \tag{5.30}$$

where we used $q = d-2$ in the second equality. This corresponds to $\mathcal{F}(a_i)$ in Eq. (S26) of Ref. [10] after identification $\alpha_{1,2} \to a_{1,2}$, $\beta \to a_3$ and $\gamma \to a_4$, and setting $e = 1$.

5.2 Extremal condition

Next we compute higher derivative corrections to the extremal condition.

General consideration. In the WGC context, we are interested in the tension-to-charge ration of the zero temperature black branes, so that the canonical ensemble is useful. Let us define the tension-to-charge ratio in the canonical ensemble in the rescaled language as

$$\mu(\mathcal{T},\mathcal{Q})=\frac{\mathcal{M}(\mathcal{T},\mathcal{Q})}{\mathcal{Q}}=\frac{\mathcal{H}(\mathcal{T},\mathcal{Q})+\mathcal{T}\mathcal{S}(\mathcal{T},\mathcal{Q})}{\mathcal{Q}}\,, \tag{5.31}$$

where $\mathcal{H}(\mathcal{T},\mathcal{Q})$ is the rescaled Helmholtz free energy in particular. A standard thermodynamic analysis shows that its derivative in $\mathcal{Q}$ is of the form,

$$\begin{aligned}\frac{\partial\mu(\mathcal{T},\mathcal{Q})}{\partial\mathcal{Q}}&=\frac{1}{\mathcal{Q}^2}\left[\mathcal{Q}\Phi(\mathcal{T},\mathcal{Q})+\mathcal{T}\mathcal{Q}\frac{\partial\mathcal{S}(\mathcal{T},\mathcal{Q})}{\partial\mathcal{Q}}-\Big(\mathcal{H}(\mathcal{T},\mathcal{Q})+\mathcal{T}\mathcal{S}(\mathcal{T},\mathcal{Q})\Big)\right]\\&=\frac{1}{\mathcal{Q}^2}\left[-\mathcal{G}(\mathcal{T},\Phi(\mathcal{T},\mathcal{Q}))+\mathcal{T}\left(\mathcal{Q}\frac{\partial\mathcal{S}(\mathcal{T},\mathcal{Q})}{\partial\mathcal{Q}}-\mathcal{S}(\mathcal{T},\mathcal{Q})\right)\right]\,,\end{aligned} \tag{5.32}$$

where $\mathcal{G}(\mathcal{T},\Phi)$ is the Gibbs free energy in the grand canonical ensemble and $\Phi(\mathcal{T},\mathcal{Q})$ is the electric potential in the canonical ensemble. We then define the tension-to-charge ratio of the extremal back brane by

$$\mu_{\rm ext}(\mathcal{Q})=\mu(0,\mathcal{Q})\,. \tag{5.33}$$

If the entropy $\mathcal{S}(\mathcal{T},\mathcal{Q})$ and its derivative in the electric charge $\mathcal{Q}$ are finite in the zero temperature limit $\mathcal{T}\to 0$ (it is true in our setup including higher derivative corrections), the second term in Eq. (5.32) vanishes in the extremal limit and therefore

$$\frac{d\mu_{\rm ext}(\mathcal{Q})}{dQ}=-\frac{\mathcal{G}(0,\Phi(0,\mathcal{Q}))}{\mathcal{Q}^2}\,. \tag{5.34}$$

Our task is now to evaluate the Gibbs free energy in the canonical ensemble and take the zero temperature limit.

Einstein gravity. Before applying this consideration to our higher derivative analysis, it is instructive to summarize its implication in the Einstein gravity case. First, let us introduce a dictionary between the grand canonical and canonical ensembles. Recall that the rescaled charge in the grand canonical ensemble is given in Eq. (4.97) as

$$\mathcal{Q}_0(\mathcal{T},\Phi)=\Phi\frac{(1-\Phi^2)^{\frac{q-1}{p+1}}}{\mathcal{T}^{q-1}}\,. \tag{5.35}$$

This gives the rescaled electric potential Φ in the canonical ensemble as

$$\Phi_0(\mathcal{T},\mathcal{Q}) = 1 - \frac{1}{2}\mathcal{T}^{p+1}\mathcal{Q}^{\frac{p+1}{q-1}} + \cdots\,, \tag{5.36}$$

where the dots stand for higher order terms in $\mathcal{T}$ that are negligible in the low temperature regime. Also, without loss of generality we assumed that the charge is positive $\mathcal{Q} > 0$. Then, in the extremal limit $\mathcal{T} \to 0$, the Gibbs free energy in the canonical ensemble reads

$$\mathcal{G}_0(\mathcal{T},\Phi(\mathcal{T},\mathcal{Q})) = \frac{p+1}{2(p+q)}\frac{(1-\Phi(\mathcal{T},\mathcal{Q})^2)^{\frac{q-1}{p+1}}}{\mathcal{T}^{q-1}}(1-\Phi^2) \to 0\,. \tag{5.37}$$

We then conclude that the extremal curve is linear in the Einstein gravity:

$$\frac{d\mu_{\mathrm{ext},0}(\mathcal{Q})}{d\mathcal{Q}} = -\frac{\mathcal{G}_0(0,\Phi(0,\mathcal{Q}))}{\mathcal{Q}^2} = 0\,, \tag{5.38}$$

which of course agrees with the extremal condition $\mu_{\mathrm{ext},0}(\mathcal{Q}) = 1$ in the Einstein gravity.

Higher derivative analysis. Now we move on to our higher derivative analysis. Again, we first introduce a dictionary between the two ensembles. In terms of the higher derivative correction $\Delta\mathcal{G}$ to the Gibbs free energy evaluated in the previous subsection, we have

$$\mathcal{Q}(\mathcal{T},\Phi) = \mathcal{Q}_0(\mathcal{T},\Phi) + \Delta\mathcal{Q}(\mathcal{T},\Phi) \quad \text{with} \quad \Delta\mathcal{Q}(\mathcal{T},\Phi) = -\frac{\partial\Delta\mathcal{G}(\mathcal{T},\Phi)}{\partial\Phi}\,. \tag{5.39}$$

Then, the electric potential in the canonical ensemble reads

$$\Phi(\mathcal{T},\mathcal{Q}) = \Phi_0(\mathcal{T},\mathcal{Q}) + \Delta\Phi(\mathcal{T},\mathcal{Q}) \tag{5.40}$$

with the correction term

$$\Delta\Phi(\mathcal{T},\mathcal{Q}) = -\left.\frac{\Delta\mathcal{Q}(\mathcal{T},\Phi)}{\frac{\partial\mathcal{Q}_0(\mathcal{T},\Phi)}{\partial\Phi}}\right|_{\Phi=\Phi_0(\mathcal{T},\mathcal{Q})} \tag{5.41}$$

at the leading order. In the extremal limit, higher derivative corrections to the Gibbs free energy in the canonical ensemble are

$$\mathcal{G}(0,\Phi(0,\mathcal{Q})) \simeq \mathcal{G}_0(0,\Phi_0(0,\mathcal{Q})) + \left.\frac{\partial\mathcal{G}_0(0,\Phi)}{\partial\Phi}\right|_{\Phi=\Phi_0(0,\mathcal{Q})}\Delta\Phi(0,\mathcal{Q}) + \Delta\mathcal{G}(0,\Phi_0(0,\mathcal{Q}))$$

$$= -\mathcal{Q}\Delta\Phi(0,\mathcal{Q}) + \Delta\mathcal{G}(0,\Phi_0(0,\mathcal{Q}))\,, \tag{5.42}$$

where at the second equality we used

$$\mathcal{G}_0(0,\Phi_0(0,\mathcal{Q})) = 0\,, \qquad \frac{\partial\mathcal{G}_0(0,\Phi)}{\partial\Phi}\bigg|_{\Phi=\Phi_0(0,\mathcal{Q})} = -\mathcal{Q}_0(0,\Phi_0(0,\mathcal{Q})) = -\mathcal{Q}\,. \tag{5.43}$$

Using the explicit form (5.28) of $\Delta\mathcal{G}(\mathcal{T},\Phi)$ together with Eq. (5.39) and Eq. (5.41), we find

$$\mathcal{G}(0,\Phi(0,\mathcal{Q})) \simeq -\frac{2\mathcal{C}}{q-1}\mathcal{Q}^{1-\frac{2}{q-1}}\,. \tag{5.44}$$

Using the relation (5.34), we conclude that the correction to the tension-to-charge is

$$\mu_{\rm ext}(\mathcal{Q}) = 1 + \Delta\mu_{\rm ext}(\mathcal{Q}) \quad \text{with} \quad \Delta\mu_{\rm ext}(\mathcal{Q}) \simeq -\mathcal{C}\mathcal{Q}^{-\frac{2}{q-1}}\,, \tag{5.45}$$

where the integration constant is determined by the extremal condition in the Einstein gravity: $\mu_{\rm ext}(\mathcal{Q}) \to 1$ ($\mathcal{Q}\to\infty$). The WGC implies that higher derivative corrections decrease the tension-to-charge ratio of extremal black branes, i.e., $\mathcal{C} > 0$.

Note that Eq. (5.45) for $p = 0$ reproduces the black hole result in the literature. More explicitly, the charge-to-mass ratio $z_{\rm ext} = \mu_{\rm ext}^{-1}$ of extremal black holes reads

$$z_{\rm ext} = 1 - \Delta\mu_{\rm ext} = 1 + \mathcal{C}\mathcal{Q}^{-\frac{2}{q-1}} = 1 + \mathcal{C}\left(\frac{(d-2)(d-3)V_{d-2}^2}{e^2\kappa^2Q^2}\right)^{\frac{1}{d-3}}\,, \tag{5.46}$$

which agrees with Eq. (S24) of Ref. [10] (see below Eq.(5.30) for the dictionary between our notation and that of Ref. [10]).

5.3 Entropy of $\mathcal{Q} = \mathcal{M}$ black branes

It is known that higher derivative corrections to entropy of $\mathcal{Q} = \mathcal{M}$ black holes are positive, if and only if corrections to the mass-to-charge ratio of extremal black holes are negative as implied by the WGC (see, e.g., Refs. [9, 10, 14, 15]). In this subsection we explicitly show that the same story holds for black p-branes in general spacetime dimensions.

Temperature. To discuss higher derivative corrections to thermodynamics of $\mathcal{Q} = \mathcal{M}$ black branes, we first identify the temperature $\mathcal{T}_*(\mathcal{Q})$ of the $\mathcal{Q} = \mathcal{M}$ black branes as a function of the charge $\mathcal{Q}$. For this purpose, let us first write down the tension $\mathcal{M}$ in the canonical ensemble as

$$\mathcal{M}(\mathcal{T}, \mathcal{Q}) = \mathcal{M}_0(\mathcal{T}, \mathcal{Q}) + \Delta\mathcal{M}(\mathcal{T}, \mathcal{Q})\,. \tag{5.47}$$

Here the first term is the tension in the Einstein gravity, whose explicit form in the low-temperature regime is given by

$$\mathcal{M}_0(\mathcal{T}, \mathcal{Q}) = \mathcal{Q}\left[1 + \frac{p(q-1)}{2(p+q)}\epsilon + \frac{2p+1}{8}\epsilon^2 + \mathcal{O}(\epsilon^3)\right] \quad \text{with} \quad \epsilon = \mathcal{Q}^{\frac{p+1}{q-1}}\mathcal{T}^{p+1}\,. \tag{5.48}$$

This shows that $\mathcal{Q} = \mathcal{M}$ black branes have zero temperature $\epsilon = 0$ in the Einstein gravity. Note that for $p = 0$ the second term in the brackets vanishes and so the third term is the next-to-leading order term in the ϵ expansion.

On the other hand, the second term in Eq. (5.47) is the higher derivative correction to the tension in the canonical ensemble. In the low-temperature limit, it reads

$$\Delta\mathcal{M}(\mathcal{T}, \mathcal{Q}) \simeq \Delta\mathcal{M}(0, \mathcal{Q}) = \mathcal{Q}\Delta\mu_{\text{ext}}(\mathcal{Q})\,. \tag{5.49}$$

This additional contribution to the tension shifts the temperature of $\mathcal{Q} = \mathcal{M}$ black branes. More explicitly, at the leading order, $\mathcal{M}(\mathcal{T}, \mathcal{Q}) = \mathcal{Q}$ implies

$$-\Delta\mu_{\text{ext}}(\mathcal{Q}) = \begin{cases} \dfrac{1}{8}\epsilon^2 & (p = 0)\,, \\ \dfrac{p(q-1)}{2(p+q)}\epsilon & (p \geq 1)\,. \end{cases} \tag{5.50}$$

Note that $\Delta\mu_{\text{ext}}(\mathcal{Q}) < 0$ (which is identical to the WGC bound) is required for the shifted temperature to be real. This is simply because when $\Delta\mu_{\text{ext}}(\mathcal{Q}) > 0$, the corrected extremal bound is not satisfied by $\mathcal{Q} = \mathcal{M}$, so that $\mathcal{Q} = \mathcal{M}$ solutions have no horizon and therefore they are not black branes anymore. Solving Eq. (5.50) gives the temperature $\mathcal{T}_*(\mathcal{Q})$ of $\mathcal{Q} = \mathcal{M}$ black branes as a function of the charge $\mathcal{Q}$:

$$\mathcal{T}_*(\mathcal{Q}) = \epsilon_*^{\frac{1}{p+1}}\mathcal{Q}^{-\frac{1}{q-1}} \quad \text{with} \quad \epsilon_*(\mathcal{Q}) = \begin{cases} \sqrt{-8\Delta\mu_{\text{ext}}(\mathcal{Q})} & (p = 0)\,, \\ -\dfrac{2(p+q)}{p(q-1)}\Delta\mu_{\text{ext}}(\mathcal{Q}) & (p \geq 1)\,. \end{cases} \tag{5.51}$$

Since Eqs. (5.50)-(5.51) are qualitatively different for $p \geq 1$ and $p = 0$, below we study the two cases separately.

5.3.1 Black branes with $p \geq 1$

We start from the $p \geq 1$ case and evaluate higher derivative corrections to the electric potential, Gibbs free energy, and then entropy, respectively.

Electric potential. Let us write the electric potential of the $\mathcal{Q} = \mathcal{M}$ black branes as

$$\Phi_*(\mathcal{Q}) = \Phi_0(\mathcal{T}_*(\mathcal{Q}), \mathcal{Q}) + \Delta\Phi(\mathcal{T}_*(\mathcal{Q}), \mathcal{Q})\,. \tag{5.52}$$

Here $\Phi_0(\mathcal{T}, \mathcal{Q})$ is the electric potential in the canonical ensemble for the Einstein gravity, which is given by

$$\Phi_0(\mathcal{T}_*(\mathcal{Q}), \mathcal{Q}) = 1 - \frac{1}{2}\epsilon_*(\mathcal{Q}) + \mathcal{O}(\epsilon_*^2)\,. \tag{5.53}$$

On the other hand, $\Delta\Phi(\mathcal{T}, \mathcal{Q})$ is the higher derivative correction, whose low-temperature behavior reads

$$\Delta\Phi(\mathcal{T}_*(\mathcal{Q}), \mathcal{Q}) \simeq \frac{q-3}{q-1}\mathcal{C}\mathcal{Q}^{-\frac{2}{q-1}} = -\frac{q-3}{q-1}\Delta\mu_{\rm ext}(\mathcal{Q}) = \frac{p(q-3)}{2(p+q)}\epsilon_*(\mathcal{Q})\,. \tag{5.54}$$

We then find

$$\Phi_*(\mathcal{Q}) = 1 - \frac{1}{2}\left(1 - \frac{p(q-3)}{p+q}\right)\epsilon_*(\mathcal{Q}) + \mathcal{O}({\epsilon_*}^2)\,. \tag{5.55}$$

Gibbs free energy. Similarly, we decompose the Gibbs free energy of $\mathcal{Q} = \mathcal{M}$ black branes as

$$\mathcal{G}_*(\mathcal{Q}) = \mathcal{G}_0(\mathcal{T}_*(\mathcal{Q}), \Phi_*(\mathcal{Q})) + \Delta\mathcal{G}(\mathcal{T}_*(\mathcal{Q}), \Phi_*(\mathcal{Q}))\,. \tag{5.56}$$

Here the first term is the Einstein gravity result:

$$\mathcal{G}_0(\mathcal{T}_*(\mathcal{Q}), \Phi_*(\mathcal{Q})) = \mathcal{Q}\left[\frac{(1+p)}{2(p+q)}\epsilon_*(\mathcal{Q}) + \mathcal{O}(\epsilon^2)\right]\,. \tag{5.57}$$

The second term is the higher derivative correction, whose low-temperature behavior is

$$\Delta\mathcal{G}(\mathcal{T}_*(\mathcal{Q}), \Phi_*(\mathcal{Q})) \simeq -\mathcal{C}\mathcal{Q}^{1-\frac{2}{q-1}} = \mathcal{Q}\Delta\mu_{\rm ext}(\mathcal{Q}) = -\frac{p(q-1)}{2(p+q)}\mathcal{Q}\epsilon_*(\mathcal{Q})\,. \tag{5.58}$$

Therefore, we have

$$\mathcal{G}_*(\mathcal{Q}) = \mathcal{Q}\left[\frac{1+2p-pq}{2(p+q)}\epsilon_*(\mathcal{Q}) + \mathcal{O}(\epsilon_*^2)\right]\,. \tag{5.59}$$

Entropy. Finally, we compute the entropy of $\mathcal{Q} = \mathcal{M}$ black branes. For this purpose, it is convenient to note that the thermodynamic relation $\mathcal{M} = \mathcal{G} + \mathcal{T}\mathcal{S} + \mathcal{Q}\Phi$ implies that the entropy of $\mathcal{Q} = \mathcal{M}$ black brane is

$$\mathcal{S}_*(\mathcal{Q}) = \frac{(1 - \Phi_*(\mathcal{Q}))\mathcal{Q} - \mathcal{G}_*(\mathcal{Q})}{\mathcal{T}_*(\mathcal{Q})} \,. \tag{5.60}$$

Using Eqs. (5.51), (5.55), and (5.59), we find

$$\mathcal{S}_*(\mathcal{Q}) = \frac{2p + q - 1}{2(p+q)} \mathcal{Q}^{\frac{q}{q-1}} \epsilon_*^{\frac{p}{p+1}} \left[1 + \mathcal{O}(\epsilon_*)\right] \,, \tag{5.61}$$

which is consistent with the fact that extremal black branes with $p \geq 1$ have vanishing entropy $\mathcal{S}_{*0}(\mathcal{Q}) = 0$. We also find that entropy correction is positive as long as $\Delta\mu_{\rm ext}(\mathcal{Q}) < 0$ and therefore $T_*(\mathcal{Q})$ is real and $\epsilon_*(\mathcal{Q}) > 0$. A direct relation between entropy correction $\Delta\mathcal{S}_*(\mathcal{Q}) = \mathcal{S}_*(\mathcal{Q}) - \mathcal{S}_{0*}(\mathcal{Q})$ and the correction to the extremal bound $\Delta\mu_{\rm ext(\mathcal{Q})}$ reads

$$\Delta\mathcal{S}_*(\mathcal{Q}) \simeq \frac{2p + q - 1}{2(p+q)} \left(\frac{2(p+q)}{p(q-1)}\right)^{\frac{p}{p+1}} \mathcal{Q}^{\frac{q}{q-1}} \Big(- \Delta\mu_{\rm ext}(\mathcal{Q})\Big)^{\frac{p}{p+1}} \tag{5.62}$$

at the leading order.

5.3.2 Black holes ($p = 0$)

The analysis for $p = 0$ can also be performed in the same manner. The only thing to take care here is that $\Delta\mu_{\rm ext}(\mathcal{Q}) = \mathcal{O}(\epsilon_*^2)$ in contrast to the $p \geq 1$ case, where $\Delta\mu_{\rm ext}(\mathcal{Q}) = \mathcal{O}(\epsilon_*)$. See Eq. (5.51). Accordingly, we include the next order of the ϵ_* expansion in the analysis.

Electric potential. First, the electric potential of $\mathcal{Q} = \mathcal{M}$ black holes reads

$$\Phi_*(\mathcal{Q}) = \Phi_0(\mathcal{T}_*(\mathcal{Q}), \mathcal{Q}) + \Delta\Phi(\mathcal{T}_*(\mathcal{Q}), \mathcal{Q}) \tag{5.63}$$

with

$$\Phi_0(\mathcal{T}_*(\mathcal{Q}), \mathcal{Q}) = 1 - \frac{1}{2}\epsilon_*(\mathcal{Q}) - \frac{q+1}{8(q-1)}\epsilon_*(\mathcal{Q})^2 + \mathcal{O}(\epsilon_*^3) \,, \tag{5.64}$$

$$\Delta\Phi(\mathcal{T}_*(\mathcal{Q}), \mathcal{Q}) \simeq \frac{q-3}{q-1}\mathcal{C}\mathcal{Q}^{-\frac{2}{q-1}} = -\frac{q-3}{q-1}\Delta\mu_{\rm ext}(\mathcal{Q}) = \frac{q-3}{8(q-1)}\epsilon_*(\mathcal{Q})^2 \,. \tag{5.65}$$

We then have

$$\Phi_*(\mathcal{Q}) = 1 - \frac{1}{2}\epsilon_*(\mathcal{Q}) - \frac{1}{2(q-1)}\epsilon_*(\mathcal{Q})^2 + \mathcal{O}(\epsilon_*^3) \,. \tag{5.66}$$

Gibbs free energy. Similarly, the Gibbs free energy of $\mathcal{Q} = \mathcal{M}$ black holes is

$$\mathcal{G}_*(\mathcal{Q}) = \mathcal{G}_0(\mathcal{T}_*(\mathcal{Q}), \Phi_*(\mathcal{Q})) + \Delta\mathcal{G}(\mathcal{T}_*(\mathcal{Q}), \Phi_*(\mathcal{Q})) \tag{5.67}$$

with

$$\mathcal{G}_0(\mathcal{T}_*(\mathcal{Q}), \Phi_*(\mathcal{Q})) = \mathcal{Q}\left[\frac{1}{2q}\epsilon_*(\mathcal{Q}) - \frac{q-5}{8(q-1)}\epsilon_*(\mathcal{Q})^2 + \mathcal{O}(\epsilon^3)\right], \tag{5.68}$$

$$\Delta\mathcal{G}(\mathcal{T}_*(\mathcal{Q}), \Phi_*(\mathcal{Q})) \simeq -\mathcal{C}\mathcal{Q}^{1-\frac{2}{q-1}} = \mathcal{Q}\Delta\mu_{\rm ext}(\mathcal{Q}) = -\frac{1}{8}\epsilon_*(\mathcal{Q})^2 . \tag{5.69}$$

Therefore, we have

$$\mathcal{G}_*(\mathcal{Q}) = \mathcal{Q}\left[\frac{1}{2q}\epsilon_*(\mathcal{Q}) - \frac{q-3}{4(q-1)}\epsilon_*(\mathcal{Q})^2 + \mathcal{O}(\epsilon_*^3)\right] . \tag{5.70}$$

Entropy. Finally, the entropy of $\mathcal{Q} = \mathcal{M}$ black holes reads

$$\begin{aligned}\mathcal{S}_*(\mathcal{Q}) &= \frac{(1-\Phi_*(\mathcal{Q}))\mathcal{Q} - \mathcal{G}_*(\mathcal{Q})}{\mathcal{T}_*(\mathcal{Q})} \\ &= \frac{\mathcal{Q}}{\mathcal{T}_*(\mathcal{Q})}\left[\frac{q-1}{2q}\epsilon_*(\mathcal{Q}) + \frac{1}{4}\epsilon_*(\mathcal{Q})^2 + \mathcal{O}(\epsilon_*^3)\right] \\ &= \mathcal{Q}^{\frac{q}{q-1}}\left[\frac{q-1}{2q} + \frac{1}{4}\epsilon_*(\mathcal{Q}) + \mathcal{O}(\epsilon_*^2)\right] ,\end{aligned} \tag{5.71}$$

where the first term reproduces the entropy of $\mathcal{Q} = \mathcal{M}$ black holes $\mathcal{S}_{*0}(\mathcal{Q}) = \dfrac{q-1}{2q}\mathcal{Q}^{\frac{q}{q-1}}$. Then, the entropy correction reads

$$\Delta\mathcal{S}_*(\mathcal{Q}) = \mathcal{S}_*(\mathcal{Q}) - \mathcal{S}_{*0}(\mathcal{Q}) \simeq \frac{1}{4}\epsilon_*(\mathcal{Q})\mathcal{Q}^{\frac{q}{q-1}} , \tag{5.72}$$

which is positive as long as $\Delta\mu_{\rm ext}(\mathcal{Q}) < 0$ and therefore $T_*(\mathcal{Q})$ is real and $\epsilon_*(\mathcal{Q}) > 0$. A more direct relation between $\Delta\mathcal{S}_*$ and $\Delta\mu_{\rm ext}$ is

$$\Delta\mathcal{S}_*(\mathcal{Q}) \simeq \frac{1}{\sqrt{2}}\sqrt{-\Delta\mu_{\rm ext}(\mathcal{Q})}\,\mathcal{Q}^{\frac{q}{q-1}} , \tag{5.73}$$

which agrees with the known result in the literature [10][15].

[15] The agreement is after correcting typos in Ref. [10]. First, Eq. (S53) has to be corrected as

$$\frac{\Delta r_H^2}{r_H^2} = -\frac{2\Delta g(r_H)}{r_H^2 g''_{EM}(r_H)} = \frac{2\mathcal{F}(a_i)}{(D-3)^2 m^{\frac{2}{D-3}}} . \tag{5.74}$$

5.4 Implications of positivity bounds

So far, we have shown that positive sign of $\mathcal{C}$ given in Eq. (5.29) is favored by the WGC and also the same sign follows from positivity of the entropy correction of $\mathcal{M} = \mathcal{Q}$ black branes. In the following, we conclude our higher derivative analysis by demonstrating that the same sign is favored by positivity bounds on scattering amplitudes [38].

Positivity bounds and their limitation. Positivity bounds provide a necessary condition for an IR scattering amplitude to have a consistent UV completion that respects unitarity, analyticity and locality [38]. They are formulated in terms of the IR expansion of an s-u symmetric scattering amplitude $M(s,t)$. Here for notational simplicity we assume that external particles are massless. If the amplitude in the forward limit enjoys the IR expansion of the form,

$$\mathcal{M}(s, t = 0) = \sum_{n=0}^{\infty} a_{2n} s^{2n}\,, \tag{5.76}$$

the UV assumptions mentioned earlier imply that the coefficients a_{2n} of s^{2n} $(2n = 2, 4, \ldots)$ are all positive, which are called the positivity bounds.

However, the story changes in the presence of gravity because the IR expansion (5.76) does not work anymore. For example, tree-level graviton exchange gives a t-channel pole $\sim \kappa^2 \dfrac{s^2}{t}$, which dominates over the s^2 term in the forward limit. As a consequence, positivity of the s^2 coefficient does not follow straightforwardly. The best we can do in the present technology is to show that positivity of the s^2 coefficient holds approximately as long as gravitational effects are subdominant. See, e.g., Refs. [10, 11, 72–76] for more details.

Positivity vs $\mathcal{C} > 0$. Having said this caveat, we discuss implications of the approximate positivity of the s^2 coefficient for our higher derivative analysis. Since the approximate positivity works when gravity is subdominant, here we focus on the situation where contributions of FFW and W^2 are subdominant

---

Accordingly, Eq. (S54) should be modified as

$$\frac{\Delta S}{S_{EM}} \simeq \frac{\Delta S_{\text{horizon}}}{S_{EM}} = \frac{\sqrt{2}(D-2)}{(D-3)m^{\frac{1}{D-3}}} \sqrt{\mathcal{F}(a_i)}\,. \tag{5.75}$$

compared to the F^4 term, i.e.,

$$\mathcal{C} \simeq \Big(\sum_i A_i \alpha_i\Big) \frac{(p+q)(q-1)^2}{(p+1)(3q-1)} \,. \tag{5.77}$$

Below we show that the approximate positivity implies $\sum_i A_i \alpha_i \gtrsim 0$ and therefore $\mathcal{C} \gtrsim 0$.[16]

For this purpose, we consider four-point scattering of the $(p+1)$-form A and decompose the spacetime coordinates into 3 dimensions x^a $(a = 0, 1, 2)$ on the scattering plane and $(d-3)$ dimensions x^I $(I = 3, \ldots, d-1)$ orthogonal to it. Also let us suppose that all the external particles are polarized on the same $(p+1)$-dimensional plane, say on the plane $(x^3, x^4, \ldots, x^{p+3})$. By analogy with the Kaluza-Klein decomposition, we identify those external modes with a scalar field $\phi(x^a)$ on the 3-dimensional scattering plane:

$$\phi(x^a) = A_{3\,4\,\ldots\,p+3}(x^a) \,. \tag{5.78}$$

Then, the scattering amplitude can be computed by rewriting the effective action (5.1) in terms of ϕ (and other modes relevant to our scattering problem, if any). If we ignore gravity by our working assumption, the only interaction relevant to four-point scattering of ϕ is $(\partial_a \phi \partial^a \phi)^2$ originating from the F^4 operator.[17] Noticing that $F_{a\,3\,4\,\ldots\,p+3} = \partial_a \phi$, the effective Lagrangian relevant to our scattering problem is

$$\mathcal{L} = -\frac{1}{2}(\partial_a \phi)^2 + \frac{\kappa^2}{e^4}\Big(\sum_i A_i \alpha_i\Big)(\partial_a \phi \partial^a \phi)^2 \,. \tag{5.79}$$

We emphasize that the same combinatorial factor A_i defined in Eq. (5.17) appears in the effective Lagrangian. Then, the approximate positivity bound on the s^2 coefficient reads

$$\sum_i A_i \alpha_i \gtrsim 0 \,. \tag{5.80}$$

To summarize, we have shown that when gravity is subdominant, the approximate positivity on the s^2 coefficient implies $\sum_i A_i \alpha_i \gtrsim 0$ and therefore $\mathcal{C} \gtrsim 0$ as expected by the WGC and positivity of the entropy correction of $\mathcal{Q} = \mathcal{M}$ black branes.

[16] We use $\gtrsim$ rather than $>$ to emphasize that known positivity bounds on the s^2 coefficient are only approximate in the presence of gravity.

[17] As we mentioned earlier, when $p+1 = 4n$ $(n = 1, 2, \ldots)$, the effective action contains three-derivative operators of the form F^3, but they do not play any role in four-point scattering of ϕ at the tree-level.

6 Discussion

In this thesis we studied the WGC for higher form gauge fields as a generalization of the WGC for ordinary $U(1)$ 1-form gauge symmeties. As we reviewed in section 2, the WGC was originally motivated by the idea that non-supersymmetric black holes should be able to decay. In the Einstein-Maxwell theory, extremal black holes are thermodynamically stable and any black holes cannot satisfy the weak gravity bound. However, higher derivative corrections push down the extremal curve and black holes can satisfy the weak gravity bound. We generalized this story to black branes charged under a higher form gauge field.

To apply to computing higher derivative corrections, in section 3 and 4, we reviewed black hole and black brane thermodynamics. Black holes and black branes can be regarded as black bodies with temperature proportional to their surface gravity and entropy proportional to their surface area. We can interpret black hole mass and black brane tension as energy of spacetime defined as the Hamiltonian for the gravitational action, and the first law relates these thermodynamic quantites. Moreover, we derived the relation between the Euclidean action and thermodynamic free energy, and it allows us to compute higher derivative corrections to black brane thermodynamics.

In section 5, we studied higher derivative corrections to black brane thermodynamics and their implications for the WGC. We found that higher derivative corrections decrease the tension-to-charge ratio of extremal black branes as expected by the WGC, if and only if the constant factor $\mathcal{C}$ defined in Eq. (5.29) is positive. In particular, we demonstrated that the positive sign is favored by the approximate positivity bounds on scattering amplitudes. We also found that for the same sign of $\mathcal{C}$, the entropy correction of $\mathcal{Q} = \mathcal{M}$ black branes is positive. This extends earlier works in the Einstein-Maxwell theory to p-form gauge fields in general spacetime dimensions.

There are various future directions along the line of our present work. First, it is natural to extend our analysis to p-form gauge fields coupled to dilaton and axion. There will be no obstruction to the thermodynamic analysis besides technical complication. For example, it would be interesting to study the role of duality constraints in this context. Another interesting direction is to go beyond the leading order correction. In the context of the WGC for 1-form gauge fields, it has been studied how to interpolate the extremal curve of large black holes and the charged state spectrum below the Planck scale, e.g., based on modular invariance of the world-sheet and/or the dual CFT [26,27,30]. It would be interesting to generalize the discussion to p-form gauge fields. Moreover, it would be great if we can reinterpret the earlier implication of modular invariance from the thermodynamic perspective. Our question is what the thermodynamics would tell us about microscopic back holes/branes and the subPlanckian spectrum.

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A Geometry of black brane solutions

Riemann tensor. For the black brane solution (4.6) in the Einstein gravity, nontrivial components of the Riemann tensor are given in terms of the functions $f_\pm(r)$ as

$$
\begin{aligned}
R^{t}{}_{rtr} &= -\frac{1}{2}\left(\frac{f_+''}{f_+} - \frac{p-2}{p+1}\frac{f_+'}{f_+}\frac{f_-'}{f_-} - \frac{p-1}{p+1}\frac{f_-''}{f_-} + \frac{p-1}{p+1}\frac{p}{p+1}\frac{f_-'^2}{f_-^2}\right) ,\\
R^{t}{}_{\theta_i t\theta_i} &= -\frac{r}{2}\left(f_+' f_- - \frac{p-1}{p+1} f_+ f_-'\right)\prod_{k=1}^{i-1}\sin^2\theta_k ,\\
R^{t}{}_{y^i t y^i} &= -\frac{1}{2(p+1)}\left(f_+' f_- - \frac{p-1}{p+1} f_+ f_-'\right) f_-' f_-^{-\frac{p-1}{p+1}} ,\\
R^{r}{}_{\theta_i r\theta_i} &= -\frac{r}{2}\left(f_+' f_- + f_+ f_-'\right)\prod_{k=1}^{i-1}\sin^2\theta_k ,\\
R^{r}{}_{y^i r y^i} &= -\frac{1}{2(p+1)}\left(f_+' f_-' + 2 f_+ f_-'' - \frac{p-1}{p+1} f_+ \frac{f_-'^2}{f_-}\right) f_-^{\frac{2}{p+1}} ,\\
R^{\theta_i}{}_{\theta_j\theta_i\theta_j} &= (1 - f_+ f_-)\prod_{k=1}^{j-1}\sin^2\theta_k \qquad (i>j) ,\\
R^{\theta_i}{}_{y^j\theta_i y^j} &= -\frac{1}{p+1}\frac{1}{r} f_+ f_-' f_-^{\frac{2}{p+1}} ,\\
R^{y^i}{}_{y^j y^i y^j} &= -\frac{1}{(p+1)^2} f_+ f_-'^2 f_-^{-\frac{p-1}{p+1}} \qquad (i\neq j) .
\end{aligned}
$$

Ricci curvatures. Nonzero components of the Ricci tensor and the Ricci scalar are

$$
\begin{aligned}
R_{tt} &= \frac{1}{2} f_+ f_-^{-\frac{p-1}{p+1}}\left[f_+'' f_- + \frac{2}{p+1} f_+' f_-' - \frac{p-1}{p+1} f_+ f_-'' + \frac{q}{r}\left(f_+' f_- - \frac{p-1}{p+1} f_+ f_-'\right)\right] ,\\
R_{rr} &= -\frac{1}{2} f_+^{-1} f_-^{-1}\left[f_+'' f_- + \frac{2}{p+1} f_+' f_-' + f_+ f_-'' + \frac{q}{r}\left(f_+' f_- + f_+ f_-'\right)\right] ,\\
R_{\theta_i\theta_i} &= -\left[r\left(f_+' f_- + f_+ f_-'\right) - (q-1)(1 - f_+ f_-)\right]\prod_{k=1}^{i-1}\sin^2\theta_k ,\\
R_{y^i y^i} &= -\frac{1}{p+1} f_-^{\frac{2}{p+1}}\left[f_+' f_-' + f_+ f_-'' + \frac{q}{r} f_+ f_-'\right] ,\\
R &= \frac{q(q-1)}{r^2}(1 - f_+ f_-) - 2\frac{q}{r}\left(f_+' f_- + f_+ f_-'\right) - \left(f_+'' f_- + \frac{p+2}{p+1} f_+' f_-' + f_+ f_-''\right) .
\end{aligned}
$$

Weyl tensor. The Weyl tensor is defined in spacetime d dimensions as

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{d-2}\left(R_{\mu\sigma}g_{\nu\rho} - R_{\mu\rho}g_{\nu\sigma} + R_{\nu\rho}g_{\mu\sigma} - R_{\nu\sigma}g_{\mu\rho}\right) + \frac{1}{(d-2)(d-1)}R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})\,. \tag{A.1}$$

Its square can be reformulated in terms of the Riemann tensor as

$$W^2 \equiv W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - \frac{4}{d-2}R^2_{\mu\nu} + \frac{2}{(d-1)(d-2)}R^2\,, \tag{A.2}$$

which can be evaluated for the solution (4.6) in a straightforward manner using the formulae provided in this appendix.

B An explicit form of $\Delta\mathcal{G}_{W^2}$

An explicit form of $\Delta\mathcal{G}_{W^2}$ including subleading terms omitted in Eq. (5.26) is

$$\Delta\mathcal{G}_{W^2} = -\gamma\,\widetilde{\mathcal{C}}_{W^2}\mathcal{T}^{3-q}\left(1-\Phi^2\right)^{\frac{q-3}{p+1}} \tag{B.1}$$

with $\widetilde{\mathcal{C}}_{W^2}$ defined by

$$\widetilde{\mathcal{C}}_{W^2} = -\frac{p(p-1)q(q-1)^3}{(p+1)^2(p+q)}\Phi^{-2} + \mathcal{P}_0 + \mathcal{P}_2\Phi^2 + \mathcal{P}_4\Phi^4 + \frac{p(p-1)q(q-1)^3}{(p+1)^2(p+q)}\frac{(1-\Phi^2)^2}{\Phi^2}\,{}_2F_1\left(1, \frac{2}{q-1}; 1+\frac{2}{q-1}; \Phi^2\right)\,. \tag{B.2}$$

Here $\mathcal{P}_{0,2,4}$ are (p,q)-dependent factors given by

$$\begin{aligned}
\mathcal{P}_0 &= \frac{q^2(q-1)}{(p+1)^2(p+q)(q+1)}\left[p^3(q+1) + p^2(2q^2-q+5) + p(-2q^2+7q+1) + (q+1)\right]\,,\\
\mathcal{P}_2 &= -\frac{(q-1)^2}{(p+1)^2(q+1)}\left[p^2(q+1)^2 - p(q^2-8q-1) + 2(q+1)\right]\,,\\
\mathcal{P}_4 &= \frac{(q-1)^2}{(p+1)^2(p+q+1)(q+1)(3q-1)}\left[p^3(4q-1)(q+1) + 2p^2(2q^3+7q^2-2q+1)\right.\\
&\qquad \left. +5p(q+1)(3q^2-3q+1) + (q-1)(q+1)(7q-2)\right]\,.
\end{aligned}$$

Notice in particular that the second line of Eq. (B.2) vanishes in the limit $\Phi^2 \to 1$.

References

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