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Hopf Bifurcation in a Delayed Epidemic Model with Vaccination

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***Abstract*—In this study, we construct a mathematical epidemic model with the vaccinated compartment as an infinite dimensional system with time delay. We assume that the rate at which susceptible individuals are vaccinated depends on the history of the infectious population. We obtain a sufficient condition for the Hopf bifurcation, which induces a nontrivial periodic solution. Such a periodic solution could explain the recurrent epidemic waves that have been observed in many infectious diseases.**

***Index Terms*—epidemic model, vaccination, time delay, Hopf bifurcation**

I. Introduction

The recurrent epidemic waves have been observed in many infectious diseases. For example, the coronavirus disease outbreak 2019 (COVID-19) has induced the recurrent epidemic waves in many countries [1]. The aim of this study is to discuss the mechanism of such recurrent epidemic waves from the viewpoint of mathematical modeling.

One of the most famous mathematical models of epidemics is the susceptible-infectious-removed (SIR) epidemic model, which was developed by Kermack and McKendrick [2]. The well-known factors of nontrivial periodic solutions in epidemic models are the time periodic coefficients, time delay, nonlinear incidence, variable population size and age structure [3]. In this study, we focus on the periodicity caused by the time delay. By incorporating the time delay, our model becomes an infinite dimensional system. We seek sufficient conditions for the Hopf bifurcation, which induces a nontrivial periodic solution.

To argue the effect of vaccination, mathematical models with the vaccinated compartment have been studied by many authors (see, e.g., [4]–[6]). In usual modeling, the rate at which susceptible individuals are vaccinated is assumed to be constant. In this study, we assume that this rate ($g(I)$ written bellow) depends on the history of the infectious population. This implies that the vaccination policy can be accelerated as the number of reported infectious individuals increases. On the other hand, when the epidemic is curbed and the number of reported infectious individuals decreases, the vaccination policy can be decelerated. Under this assumption, we aim to clarify the possible factors of the recurrent epidemic waves.

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The organization of this paper is as follows: In Section II, we construct our model as an infinite dimensional system. In Section III, we calculate the equilibria and the basic reproduction number [7] of our model. In Section IV, we study the stability of the equilibria and obtain a sufficient condition for the Hopf bifurcation. In Section V, we perform numerical simulations and obtain examples of the Hopf bifurcation. Finally, Section VI is devoted to the conclusion.

II. Model

Let $S(t)$, $V(t)$ and $I(t)$ be the susceptible, vaccinated and infectious populations at time t, respectively. We consider the following system of delay differential equations:

$$\begin{cases} S' = b - \beta SI - \mu S - g(I)S, \\ V' = g(I)S - \sigma\beta VI - \mu V, \\ I' = \beta SI + \sigma\beta VI - (\mu+\gamma)I, \end{cases} \tag{1}$$

where

$$g(I)(t) = v\int_0^\infty f(\tau)I(t-\tau)d\tau, \quad t \geq 0.$$

Here, b is the birth rate, β is the disease transmission rate, μ is the mortality, σ is the vaccine efficacy, γ is the removal rate, v is the vaccination rate and $f(\tau)$, $\tau \geq 0$ is the intensity of how the infectious population at τ time ago affects the vaccination rate at the current time.

We make the following assumptions:

(A1) b, β, μ, γ and v are strictly positive constants.
(A2) $0 < \sigma < 1$.
(A3) f is nonnegative on $[0,\infty)$ and $\int_0^\infty f(\tau)d\tau = 1$.
(A4) f has a compact support on $[0,\infty)$.

Let E be the set of all bounded and continuous functions from $(-\infty,0]$ to $\mathbb{R}^3$, equipped with the norm

$$\|\varphi\|_E := \sup_{\rho\in(-\infty,0]}\max_{i\in\{1,2,3\}}|\varphi_i(\rho)|, \quad \varphi = (\varphi_1,\varphi_2,\varphi_3) \in E.$$

Let $E_+ := E \cap C((-\infty,0],\mathbb{R}^3_+)$ be the positive cone of E. We define the following set for system (1):

$$\Omega := \left\{ \varphi = (\varphi_1,\varphi_2,\varphi_3) \in E_+ : \sum_{i=1}^3 \varphi_i(\rho) \leq \frac{b}{\mu} \text{ for all } \rho \leq 0 \right\}.$$

Let $\psi = (\psi_1,\psi_2,\psi_3) \in \Omega$ be the initial condition for system (1), that is, $(S(t),V(t),I(t)) = (\psi_1(t),\psi_2(t),\psi_3(t))$ for all

$t \le 0$. We can then easily confirm that $u_t \in \Omega$ for all $t > 0$, where $u_t(\rho) = u(t+\rho) = (S(t+\rho), V(t+\rho), I(t+\rho))$ for all $t > 0$ and $\rho \le 0$. In other words, Ω is positively invariant for system (1).

III. Equilibria

Equilbria of system (1) can be obtained by solving the following algebraic equations:

$$\begin{cases} 0 = b - \beta SI - \mu S - vIS, \\ 0 = vIS - \sigma\beta VI - \mu V, \\ 0 = \beta SI + \sigma\beta VI - (\mu+\gamma)I. \end{cases}$$

It is easy to see that the disease-free equilibrium $(S,V,I) = (b/\mu, 0, 0) =: E_0$ always exists. On the other hand, at the endemic equilibrium $(S,V,I) = (S^*, V^*, I^*) =: E^*$, $I^* > 0$, it follows that

$$S^* = \frac{b}{(\beta+v)I^* + \mu}, \quad V^* = \frac{vI^*S^*}{\sigma\beta I^* + \mu},$$
$$\beta S^* + \sigma\beta V^* - (\mu+\gamma) = 0,$$

and thus, $\Phi(I^*) = 0$, where

$$\Phi(x) := \beta b \frac{\sigma(\beta+v)x + \mu}{(\beta+v)x + \mu} \frac{1}{\sigma\beta x + \mu} - (\mu+\gamma), \quad x \ge 0.$$

Since Φ is monotone decreasing on $\mathbb{R}_+$ and converges to $-(\mu+\gamma)$ as $x \to +\infty$, the positive root $I^* > 0$ of $\Phi(I^*) = 0$ exists if and only if $\Phi(0) = (\mu+\gamma)(\mathcal{R}_0 - 1) > 0$, where

$$\mathcal{R}_0 := \beta \frac{b}{\mu} \frac{1}{\mu+\gamma}$$

is the basic reproduction number for system (1). We can then conclude that the unique endemic equilibrium E^* exists if and only if the basic reproduction number is greater than 1.

IV. Stability

Let $(\bar{S}, \bar{V}, \bar{I})$ be an equilibrium of system (1). Setting $X = S - \bar{S}$, $W = V - \bar{V}$ and $Y = I - \bar{I}$, we obtain the following linearized system of (1) around $(\bar{S}, \bar{V}, \bar{I})$:

$$\begin{cases} X' = -[(\beta+v)\bar{I} + \mu]X - \beta\bar{S}Y - \bar{S}g(Y), \\ W' = v\bar{I}X + \bar{S}g(Y) - (\sigma\beta\bar{I} + \mu)W - \sigma\beta\bar{V}Y, \\ Y' = \beta\bar{I}(X + \sigma W) + [\beta(\bar{S} + \sigma\bar{V}) - (\mu+\gamma)]Y. \end{cases} \tag{2}$$

By (2), we obtain the characteristic equation for the disease-free equilibrium E_0 as follows:

$$\begin{vmatrix} \lambda+\mu & 0 & \beta\frac{b}{\mu} + \frac{b}{\mu}vF(\lambda) \\ 0 & \lambda+\mu & -\frac{b}{\mu}vF(\lambda) \\ 0 & 0 & \lambda - \beta\frac{b}{\mu} + \mu + \gamma \end{vmatrix} = 0,$$

where $F(\lambda) := \int_0^\infty f(\tau)e^{-\lambda\tau}d\tau$ is the Laplace transform of f. Note that $|F(\lambda)| < \infty$ for all $\lambda \in \mathbb{C}$ by virtue of (A3) and (A4). We then have that $\lambda = -\mu$ and $\lambda = (\mu+\gamma)(\mathcal{R}_0 - 1)$. Hence, the following proposition holds.

Proposition 1: If $\mathcal{R}_0 < 1$, then the disease-free equilibrium E_0 is asymptotically stable, whereas if $\mathcal{R}_0 > 1$, then E_0 is unstable.

On the other hand, the characteristic equation for the endemic equilibrium E^* is given as follows:

$$\begin{vmatrix} \lambda + (\beta+v)I^* + \mu & 0 & \beta S^* + S^*vF(\lambda) \\ -vI^* & \lambda + \sigma\beta I^* + \mu & \sigma\beta V^* - S^*vF(\lambda) \\ -\beta I^* & -\sigma\beta I^* & \lambda \end{vmatrix} = 0.$$

We then have that

$$\begin{aligned} 0 &= \begin{vmatrix} \lambda+\mu & \lambda+\mu & \lambda + \beta S^* + \sigma\beta V^* \\ -vI^* & \lambda + \sigma\beta I^* + \mu & \sigma\beta V^* - S^*vF(\lambda) \\ -\beta I^* & -\sigma\beta I^* & \lambda \end{vmatrix} \\ &= \begin{vmatrix} \lambda+\mu & 0 & \lambda+\mu+\gamma \\ -vI^* & \lambda + (\sigma\beta+v)I^* + \mu & \sigma\beta V^* - S^*vF(\lambda) \\ -\beta I^* & (1-\sigma)\beta I^* & \lambda \end{vmatrix} \\ &= \begin{vmatrix} \lambda+\mu & 0 & \lambda+\mu+\gamma \\ 0 & \lambda + \sigma(\beta+v)I^* + \mu & \sigma\beta V^* - S^*vF(\lambda) - \frac{v}{\beta}\lambda \\ -\beta I^* & (1-\sigma)\beta I^* & \lambda \end{vmatrix} \\ &= \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + (b_1\lambda + b_0)F(\lambda), \end{aligned} \tag{3}$$

where

$$\begin{aligned} a_2 =& [(1+\sigma)\beta + v]I^* + 2\mu, \\ a_1 =& \beta I^*[\sigma(\beta+v)I^* + \sigma(\mu+\gamma) + (1-\sigma)\beta S^*] \\ &+ \mu[(1+\sigma)\beta + v]I^* + \mu^2, \\ a_0 =& \sigma(\mu+\gamma)\beta I^*[(\beta+v)I^* + \mu] + (1-\sigma)\mu\beta^2 S^* I^*, \\ b_1 =& (1-\sigma)v\beta S^* I^*, \\ b_0 =& \mu b_1 = (1-\sigma)\mu v\beta S^* I^*. \end{aligned}$$

In the analysis of the stability of the endemic equilibrium E^*, we consider the special case where f is given by the following truncated uniform distribution:

$$f(\tau) = \begin{cases} \frac{1}{L}, & \tau \in (T, T+L), \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where $T > 0$ is the time delay until infectious individuals are reported and $L > 0$ is the length of the period in which the information of the past reported infectious individuals affects the current vaccination rate. The characteristic equation (3) can then be rewritten as follows:

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = -(b_1\lambda + b_0)\frac{1 - e^{-\lambda L}}{\lambda L}e^{-\lambda T}. \tag{5}$$

Substituting $\lambda = \pm i\omega$ and taking squares of the absolute values of both sides, we obtain the following equation:

$$\omega^8 + c_3\omega^6 + c_2\omega^4 + c_1\omega^2 + (d_1\omega^2 + d_0)(1 - \cos\omega L) = 0,$$

where

$$c_3 = a_2^2 - 2a_1, \quad c_2 = a_1^2 - 2a_2a_0, \quad c_1 = a_0^2,$$
$$d_1 = -\frac{2b_1^2}{L^2}, \quad d_0 = -\frac{2b_0^2}{L^2}.$$

We then see that a conjugate pair $\lambda = \pm i\omega$ of purely imaginary roots of (5) exists if $h(x) = 0$ has a positive root $x^* = \omega^2 > 0$, where

$$h(x) := x^4 + c_3x^3 + c_2x^2 + c_1x + (d_1x + d_0)(1 - \cos\sqrt{x}L).$$

In fact, we can establish the following proposition on the Hopf bifurcation regarding T as a bifurcation parameter (the detailed proof shall be given in [8]).

Proposition 2: Suppose that $\mathcal{R}_0 > 1$ and f is given by (4). If $h(x) = 0$ has a positive root $x^* > 0$, then the characteristic equation (5) has a conjugate pair $\lambda = \pm i\omega = \pm i\sqrt{x^*}$ of purely imaginary roots. Moreover, if $h'(x^*) > 0$ (resp. $h'(x^*) < 0$), then the pair $\lambda = \pm i\sqrt{x^*}$ crosses the imaginary axis from left to right (resp. right to left) as T increases.

Regarding $\lambda = \lambda(T)$ as the function of T, let $T^* > 0$ be a critical value such that $\lambda(T^*) = i\omega = i\sqrt{x^*}$. Substituting $\lambda = i\omega$ into (5) and using the Euler's formula, we obtain

$$\cos\omega T^* = \mathrm{Re}\left\{\frac{i\omega L[a_2\omega^2 - a_0 + i(\omega^3 - a_1\omega)]}{(ib_1\omega + b_0)(1 - e^{-i\omega L})}\right\} =: \chi_1(\omega),$$
$$\sin\omega T^* = -\mathrm{Im}\left\{\frac{i\omega L[a_2\omega^2 - a_0 + i(\omega^3 - a_1\omega)]}{(ib_1\omega + b_0)(1 - e^{-i\omega L})}\right\} =: \chi_2(\omega).$$

Hence, we can compute T^* as

$$T^* = T_0(\omega) + \frac{2n\pi}{\omega}, \quad n \in \{0, 1, 2, \ldots\},$$

where

$$T_0(\omega) := \begin{cases} \dfrac{\arccos\chi_1(\omega)}{\omega}, & \chi_2(\omega) \geq 0, \\ \dfrac{2\pi - \arccos\chi_1(\omega)}{\omega}, & \chi_2(\omega) < 0. \end{cases}$$

V. Numerical simulation

Let the unit time be one year and fix the following parameter values:

$$\begin{cases} b = \mu = \dfrac{1}{80}, \quad \gamma = 24, \quad \mathcal{R}_0 = 2.5, \\ \sigma = 0.3, \quad v = 10, \quad L = \dfrac{1}{4}. \end{cases} \tag{6}$$

Note that $\beta = \mathcal{R}_0\mu(\mu+\gamma)/b \approx 60$. One can then numerically compute $I^* \approx 2.9 \times 10^{-4}$ and

$$\begin{aligned} &a_2 \approx 0.0505, \quad a_1 \approx 0.4047, \quad a_0 \approx 0.0076, \\ &b_1 \approx 0.0464, \quad b_0 \approx 5.8036 \times 10^{-4}, \\ &c_3 \approx -0.8069, \quad c_2 \approx 0.1630, \quad c_1 \approx 5.7739 \times 10^{-5}, \\ &d_1 \approx -0.069, \quad d_0 \approx -1.0778 \times 10^{-5}. \end{aligned}$$

Moreover, one can numerically see that $h(x) = 0$ has two positive roots $x_+^* \approx 0.4454$ and $x_-^* \approx 0.3619$ such that $h'(x_+^*) > 0$ and $h'(x_-^*) < 0$ (see Fig. 1). Let $\omega_+ := \sqrt{x_+^*}$, $\omega_- := \sqrt{x_-^*}$ and

$$T_n^+ := T_0(\omega_+) + \frac{2n\pi}{\omega_+}, \quad T_n^- := T_0(\omega_-) + \frac{2n\pi}{\omega_-}, \quad n \in \mathbb{N} \cup \{0\}.$$

By Proposition 2, one can conclude that the endemic equilibrium E^* can be destabilized (resp. stabilized) and a nontrivial periodic solution can appear (resp. disappear) by a Hopf bifurcation at $T = T_n^+$ (resp. $T = T_n^-$). For $n = 0, 1$, we now have

$$T_0^+ \approx 0.5966, \ T_0^- \approx 4.4058, \ T_1^+ \approx 10.0113, \ T_1^- \approx 14.8503.$$

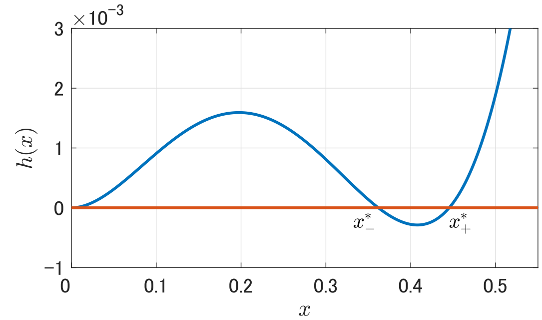


Fig. 1. $h(x)$ $(0 \leq x \leq 0.55)$ for parameter values (6).

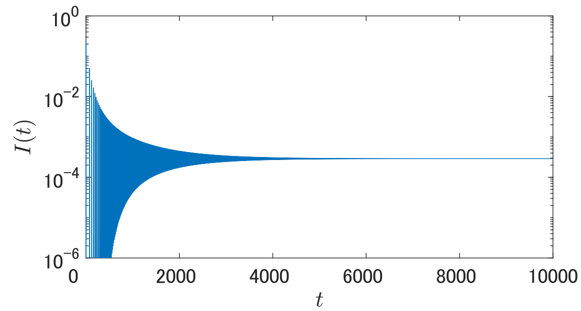


Fig. 2. $I(t)$ $(0 \leq t \leq 10000)$ for parameter values (6) and $T = 0.55 \in \mathcal{T}_s$.

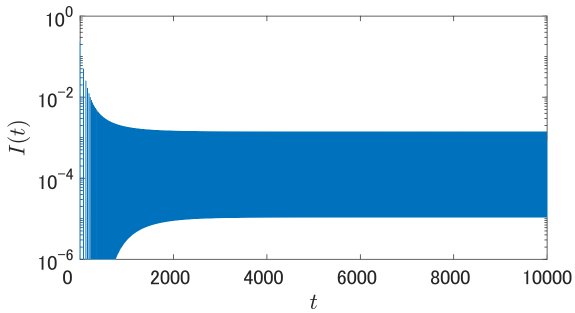
One can then guess that if $T \in (T_0^+, T_0^-) \cup (T_1^+, T_1^-) =: \mathcal{T}_u$, then a nontrivial periodic solution exists, whereas if $T \in (0, T_0^+) \cup (T_0^-, T_1^+) =: \mathcal{T}_s$, then there is no such solution.

Let the initial condition be $\psi = (\psi_1, \psi_2, \psi_3) = (1 - 10^{-7}, 0, 10^{-7}) \in \Omega$. For $T = 0.55 \in \mathcal{T}_s$, we numerically see that the infectious population $I(t)$ converges to I^* as t increases in Figure 2. On the other hand, for $T = 0.65 \in \mathcal{T}_u$, $I(t)$ converges to a nontrivial periodic solution in Figure 3. Similarly, for $T = 4.3 \in \mathcal{T}_u$ (resp. $T = 4.5 \in \mathcal{T}_s$), $I(t)$ converges to a nontrivial periodic solution (resp. I^*) in Fig. 4 (resp. Fig. 5).

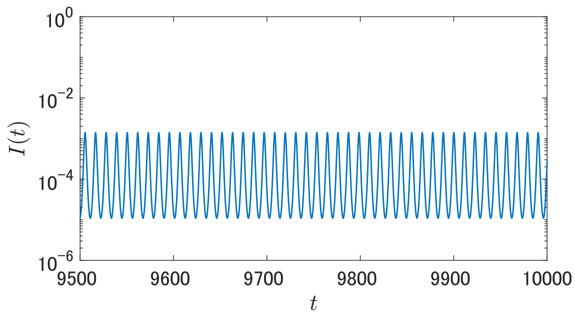
Finally, we plot the curves of T_n^+ and T_n^- $(n = 0, 1, 2)$ for the vaccination rate v in Fig. 6. By the above argument, we can guess that a nontrivial periodic solution arises in the regions bounded by the curves of T_n^+ and T_n^- $(n = 0, 1, 2)$. Fig. 6 suggests that there is no periodic solution if v is sufficiently small. In other words, the vaccination rate and the time delay until infectious individuals are reported could play an important role in causing the recurrent epidemic waves.

VI. Conclusion

In this paper, we have constructed a delayed epidemic model with vaccination and obtained sufficient conditions for the Hopf bifurcation. Our numerical simulation suggested that the vaccination rate and the time delay could play important roles in causing the recurrent epidemic waves. In future work, we may have to take into account other important factors such as age structure, seasonality and virus mutation to obtain a more plausible result by using a more realistic model.

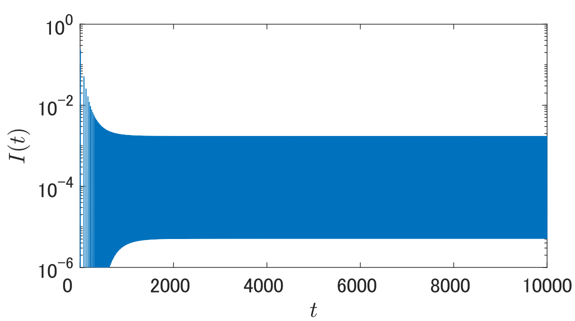


(a) $0 \leq t \leq 10000$.

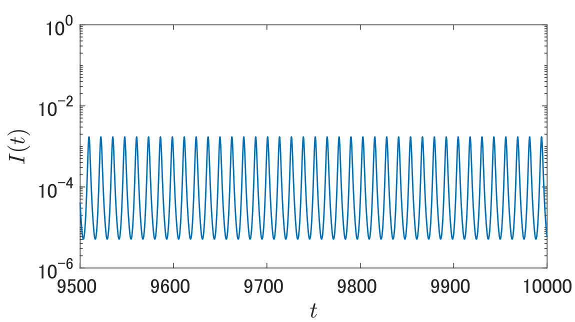


(b) $9500 \leq t \leq 10000$.

Fig. 3. $I(t)$ for parameter values (6) and $T = 0.65 \in \mathcal{T}_u$.



(a) $0 \leq t \leq 10000$.



(b) $9500 \leq t \leq 10000$.

Fig. 4. $I(t)$ for parameter values (6) and $T = 4.3 \in \mathcal{T}_u$.

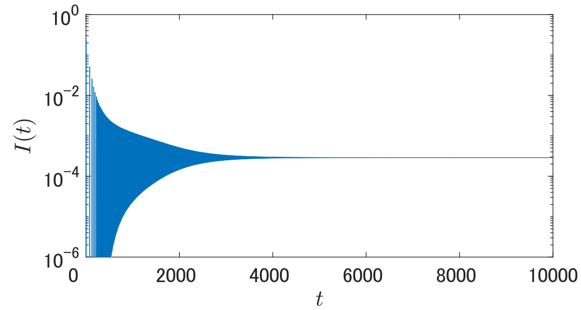


Fig. 5. $I(t)$ for parameter values (6) and $T = 4.5 \in \mathcal{T}_s$.

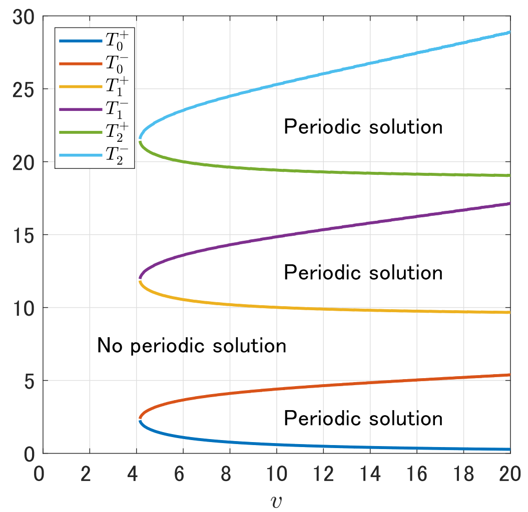


Fig. 6. T_n^+ and T_n^- $(n = 0, 1, 2)$ versus $v \in (0, 20)$.

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