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Dissipative structures for the system of Moore–Gibson–Thompson thermoelasticity in the whole space

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Abstract:

We investigate the dissipative structure for the system of Moore–Gibson–Thompson thermoelasticity in the whole space. To analyze the dissipative structure, it is very useful to rewrite the equations into a symmetric hyperbolic system and apply the so-called stability condition. When we rewrite our system into the symmetric hyperbolic form in the multi-dimensional case, it is important to take the constraint conditions into account. Indeed, the stability condition with the constraint conditions guarantees the dissipative property for our system in some cases. In this paper, we introduce the stability condition with constraints for the general problem and apply this argument to the system of Moore–Gibson–Thompson thermoelasticity. Furthermore, we discuss the optimality of the decay estimates we obtain, together with the no regularity-loss phenomenon.

1. Introduction

It is well known that the Moore-Gibson-Thompson equation was proposed in the context of fluid mechanics (see [28]). This equation has been the subject of intense studies in the last fifteen years (see, for instance, [5, 13, 14, 15, 20]). Recently, it has been proposed to describe heat conduction ([21]). Indeed, since Green and Naghdi's type III heat equation ([9, 10]) allows instantaneous propagation of waves (see [25]), it is natural to introduce a relaxation time that would lead us to a hyperbolic type heat equation, and that, therefore, it is compatible with the principle of causality[1]. This equation has given rise to the possibility of obtaining an integro-differential heat equation different from those known as Gurtin or Gurtin and Pipkin type (see [4]).

[1]This idea is similar to the one introduced by Maxwell and Cattaneo with respect to the classical Fourier theory which, as it is well known, also allows the instantaneous propagation of waves.

We can also remember that the linear theories of viscoelasticity lead us back to incompatibility with the principle of causality. A procedure similar to the one previously mentioned can be developed, but now in a mechanical context (see [7]). This allows us to recover the theory of viscoelasticity proposed by Zener in [8, 24, 33] and, therefore, we obtain again the Moore-Gibson-Thompson equation in a mechanical context.

Once the heat equation has been obtained, we can consider the corresponding thermoelastic theory (see, for instance, [3, 21]). This problem has been intensively studied in the last four years in works such as [27]. Several different results have been obtained in the case that we consider bounded domains. It is appropriate to highlight the existence and uniqueness results in the general case when the elasticity tensor is positive definite, as well as the exponential decay of the solutions in the one-dimensional casem, or the existence of radial solutions (see [2]). Uniqueness and instability of solutions have also been tested when the elasticity tensor is not positive definite using the logarithmic convexity [16] method.

We can also recall some recent contributions in which the authors study different systems that are similar (from a mathematical point of view) to the system of thermoelasticity of the Moore-Gibson-Thompson type (such as [1, 6, 18, 19, 32]). These systems have been considered either in the case of bounded or unbounded domains.

However, we are not aware of any contribution that studies the Moore-Gibson-Thompson thermoelasticity system in the case of unbounded domains. The objective of this article is to make a first contribution in this line, so we aim at studying this thermoelastic theory in the case that the material is defined in the whole space. Therefore, our objective is the study the so-called Cauchy problem for the thermoelasticity of the MGT type.

We consider the system of the Moore–Gibson–Thompson thermoelasticity.

$$\begin{aligned} \rho u_{tt} - \alpha\Delta u - (\alpha+\alpha^*)\nabla \mathrm{div} u + \beta(\nabla\theta_t + \tau\nabla\theta_{tt}) = 0, \\ \kappa\theta_{tt} + \kappa\tau\theta_{ttt} + \beta \mathrm{div} u_t - \gamma\Delta\theta_t - \gamma^*\Delta\theta = 0 \end{aligned} \tag{1.1}$$

for $t > 0$ and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. That is, we consider the displacement of an elastic material coupled with heat conduction of MGT type (see [21]). Here, $u = u(t,x) = (u^1, \cdots, u^n)(t,x)$ and $\theta = \theta(t,x)$ are unknown functions representing the velocity and temperature of the material. The parameters ρ, α, α^*, β, κ, γ, γ^* and τ are constitutive coefficients which satisfy

$$\rho, \alpha, \alpha^*, \kappa, \gamma, \gamma^* > 0, \quad \beta \neq 0,$$

and

$$\gamma - \tau\gamma^* > 0. \tag{1.2}$$

The study of problems about regions that occupy the whole space is very common from a mathematical point of view, but at the same time it is useful from a realistic point of view since it describes the case of domains whose extension is so large that it is difficult to delimit.

The condition (1.2) is called the dissipative condition in this article. The parameter τ denotes the relaxation parameter in the heat coupling of MGT type. That is, if $\tau = 0$, we would obtain a system modeling an elastic material coupled with heat conduction of the Green-Naghdi type III. Formally, letting $\tau = 0$ in (1.1), yields the system

$$\begin{aligned} \rho u_{tt} - \alpha\Delta u - (\alpha + \alpha^*)\nabla \mathrm{div} u + \beta\nabla\theta_t = 0, \\ \kappa\theta_{tt} + \beta \mathrm{div} u_t - \gamma\Delta\theta_t - \gamma^*\Delta\theta = 0. \end{aligned} \tag{1.3}$$

In this article, we focus on the Cauchy problems defined by the systems (1.1) and (1.3) with the initial data

$$\begin{aligned} &u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \\ &\theta(0, x) = \theta_0(x), \quad \theta_t(0, x) = \theta_1(x), \quad \theta_{tt}(0, x) = \theta_2(x) \end{aligned}$$

and

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad \theta(0, x) = \theta_0(x), \quad \theta_t(0, x) = \theta_1(x)$$

for $x \in \mathbb{R}^n$, respectively, where $u_0 = u_0(x)$, $u_1 = u_1(x)$, $\theta_0 = \theta_0(x)$, $\theta_1 = \theta_1(x)$ and $\theta_2 = \theta_2(x)$ are given scalar functions. Then our purpose is to analyze the dissipative structure and the dependence on τ and the limit $\tau \to 0$ in (1.1) both in the one-dimensional and the multi-dimensional cases.

This paper is organized as follows. In Section 2 we consider a general linear system with constraints and review the stability condition to derive the decay estimate of the corresponding solutions. As an application of this stability condition, in Section 3 we study the decay rate of the one-dimensional problem of the MGT-thermoelasticity and the type III Green–Naghdi thermoelasticity equations (1.1) and (1.3), respectively. Furthermore, in Section 4, we focus on the multi-dimensional problem of the previous two equations and obtain their decay estimates. In both cases, we can observe no regularity-loss of the initial conditions in obtaining the corresponding decay rate estimates. Finally, we analyze the asymptotic expansion of the corresponding eigenvalue of the MGT-thermoelasticity equation, and discuss the optimality of the decay estimate in Section 5.

2. Review of a decay rate result in a general constrained linear system

In this section, we review the so-called stability condition for a general linear system with constraints. This result allows us to obtain the decay rate of the solution of a constrained system that satisfies certain conditions. The following can be found in the references [26, 29, 30], but we include the main results here for a better comprehension of the present work.

Let us consider a general linear system with constraints such as the following one

$$A^0 U_t + \sum_{j=1}^{n} A^j U_{x_j} - \sum_{j,k=1}^{n} B^{jk} U_{x_j x_k} + LU = 0, \tag{2.1}$$

$$\sum_{j=1}^{n} Q^j U_{x_j} = 0, \tag{2.2}$$

with initial data

$$U(0, x) = U_0(x). \tag{2.3}$$

Here, U is an unknown vector function, and A^0, A^j, B^{jk}, L and Q^j are constant coefficient matrices, and $U_0 = U_0(x)$ is a given vector function.

Applying the Fourier transform to (2.1), (2.2), and (2.3) we obtain

$$A^0\hat{U}_t + i|\xi|A(\omega)\hat{U} + |\xi|^2B(\omega)\hat{U} + L\hat{U} = 0, \tag{2.4}$$

$$i|\xi|Q(\omega)\hat{U} = 0, \tag{2.5}$$

and

$$\hat{U}(0,\xi) = \hat{U}_0(\xi), \tag{2.6}$$

where $\xi \in \mathbb{R}^n$ is the Fourier variable, $\omega = \xi/|\xi| \in S^{n-1}$ is the unit vector for $\xi \neq 0$, and we define

$$A(\omega) := \sum_{j=1}^{n} A^j\omega_j, \quad B(\omega) = \sum_{j,k=1}^{n} B^{jk}\omega_j\omega_k, \quad Q(\omega) := \sum_{j=1}^{n} Q^j\omega_j,$$

where $\omega = (\omega_1, \cdots, \omega_n) \in S^{n-1}$. The eigenvalue problem corresponding to (2.4) is given by

$$\lambda A^0\varphi + (i|\xi|A(\omega) + |\xi|^2B(\omega) + L)\varphi = 0. \tag{2.7}$$

In order to fulfill the constraint conditions (2.5), the corresponding eigenvector φ must be in $\mathrm{Ker}(Q(\omega))$ for $\omega \in S^{n-1}$. We note that the solution of (2.4) can be written as $\hat{U}(t,\xi) = e^{t\hat{\Phi}(i\xi)}\hat{U}_0(\xi)$, where

$$\hat{\Phi}(i\xi) := -(A^0)^{-1}(i|\xi|A(\omega) + |\xi|^2B(\omega) + L) \tag{2.8}$$

and $e^{t\hat{\Phi}(i\xi)}$ denotes the matrix exponential.

We now introduce the following conditions for the coefficient matrices of (2.1)-(2.2):

Condition (A):

$$\begin{aligned} &A^0 = (A^0)^T, \quad A^j = (A^j)^T, \quad B^{jk} = (B^{jk})^T, \quad L = L^T, \\ &A^0 > 0, \quad B(\omega) \geq 0, \quad L \geq 0, \\ &\mathrm{Ker}(B(\omega)) \cap \mathrm{Ker}(L) \neq \{0\} \end{aligned}$$

for $j, k = 1, \cdots n$ and $\omega \in S^{n-1}$ (that is, (2.1) is a symmetric hyperbolic-parabolic system).

Using a standard argument, we can obtain the well-posedness for the problem (2.1),(2.3) under Condition (A) (see [23] for details).

In this article, we focus on the dissipative structures of the solution of (2.1)-(2.3) with the constraint condition (2.2). That is, for the constraint condition, we suppose the following condition.

Condition (C):

$$Q(\omega)(A^0)^{-1}A(\omega) = O, \quad Q(\omega)(A^0)^{-1}B(\omega) = O, \quad Q(\omega)(A^0)^{-1}L = O$$

for $\omega \in S^{n-1}$.

Condition (C) implies the fact that the equation (2.2) holds at an arbitrary time $t > 0$ for the solution of (2.2) if it holds initially. Therefore, it is reasonable for the Cauchy problem to assign the constraint condition (2.2) which satisfies Condition (C).

We refer the reader to [29, 30] for details. For this problem, the following stability condition (also called the Kawashima-Shizuta Condition) was derived in [26].

Stability Condition (SC): For each $\mu \in \mathbb{R}$ and $\omega \in S^{n-1}$,

$$\mathrm{Ker}(\mu A^0 + A(\omega)) \cap \mathrm{Ker}(B(\omega)) \cap \mathrm{Ker}(L) \cap \mathrm{Ker}(Q(\omega)) = \{0\}.$$

Then we have the following equivalence.

Theorem 2.1. ([26, 31]) *Suppose that the coefficient matrices of the system* (2.1) *satisfy Condition* (A). *Then the following assertions are equivalent.*

(i) *Condition* (SC) *holds.*

(ii) *Any pair of eigenvalue and eigenvector* (λ, φ) *of* (2.7) *satisfies that*

$$\mathrm{Re}\,(\lambda(i\xi)) \leq -c\frac{|\xi|^2}{1+|\xi|^2}$$

for $(|\xi|, \omega) \in \mathbb{R}_+ \times S^{n-1}$ (*that is, system* (2.1),(2.3) *is uniformly dissipative of type* $(1,1)$ *or, in other words, the system is the standard type, see* [29]) *and*

$$\varphi \in \mathrm{Ker}(Q(\omega)),$$

where c *is a certain positive constant.*

Corollary 2.2. *Suppose that the coefficient matrices of* (2.1) *and* (2.2) *satisfy Conditions* (A), (C), *and* (SC). *Then, from property* (ii) *in Theorem 2.1, the solution to* (2.4), (2.6) *satisfies the following pointwise estimate:*

$$|\hat{U}(t,\xi)| \leq C|\hat{U}_0(\xi)|e^{-c\eta(\xi)t}, \qquad \eta(\xi) = \frac{|\xi|^2}{1+|\xi|^2},$$

where (2.5) *is assigned to the corresponding initial data. Furthermore, this estimate gives the following decay estimate of the solution.*

$$\|\partial_x^k U(t)\|_{L^2(\mathbb{R}^n)} \leq C(1+t)^{-n/4-k/2}\|U_0\|_{L^1(\mathbb{R}^n)} + Ce^{-ct}\|\partial_x^k U_0\|_{L^2(\mathbb{R}^n)} \tag{2.9}$$

for $k \geq 0$. *Here,* c *and* C *are certain positive constants.*

Remark 2.3. *Typically, in a uniformly dissipative system of type* $(1,1)$ (*as in* (ii)), *the high-frequency part decays exponentially, and the low-frequency part decays polynomially with the rate of the heat kernel* (*see* [29]). *This is exactly what we have seen in Corollary 2.2.*

Remark 2.4. *Theorem 2.1 tells that there is no regularity loss phenomenon. Precisely,* (ii) *in Theorem 2.1 means that the real parts of the corresponding eigenvalues are strictly negative in the high-frequency region.*

The proofs of Theorem 2.1 and Corollary 2.2 are omitted here. We refer the reader to [31, 26, 29] for details. This approach was used previously to study the porous-elasticity in [22] and is valid for our problem.

3. Dissipative structure in the 1-dimensional case

In this section, we consider problems (1.1) and (1.3) in $\mathbb{R}$. In both cases, we will see that they satisfy conditions (A), and (SC) discussed in Section 2. This will allow us to obtain the decay estimate of the corresponding solutions given in (2.9).

We start with the MGT-thermoelasticity equation (1.1) in $\mathbb{R}$, that is, we consider the following system:

$$\begin{aligned} \rho u_{tt} - \tilde{\alpha} u_{xx} + \beta(\theta_{tx} + \tau\theta_{ttx}) = 0, \\ \kappa\theta_{tt} + \kappa\tau\theta_{ttt} + \beta u_{tx} - \gamma\theta_{txx} - \gamma^*\theta_{xx} = 0, \end{aligned} \tag{3.1}$$

with the corresponding initial conditions and $x \in \mathbb{R}$. In (3.1), $\tilde{\alpha} := 2\alpha + \alpha^*$.

We rewrite these equations as a first order symmetric system. We introduce the new functions that

$$y := u_t, \quad z := u_x, \quad \phi := \theta_t + \tau\theta_{tt}, \quad \psi := (\gamma\theta_t + \gamma^*\theta)_x, \quad p := \sqrt{\gamma(\gamma - \tau\gamma^*)}\theta_{tx},$$

(3.1) is rewritten by

$$\begin{aligned} \rho y_t - \tilde{\alpha} z_x + \beta\phi_x = 0, \\ \tilde{\alpha} z_t - \tilde{\alpha} y_x = 0, \\ \kappa\phi_t + \beta y_x - \psi_x = 0, \\ \frac{1}{\gamma^*}\psi_t - \frac{\delta}{\gamma^*}p_t - \phi_x = 0, \\ \frac{1}{\gamma^*}p_t - \frac{\delta}{\gamma^*}\psi_t + \frac{1}{\gamma}p = 0, \end{aligned} \tag{3.2}$$

where $\delta := \sqrt{(\gamma - \tau\gamma^*)/\gamma}$ is positive, which comes from the dissipative condition (1.2). For the system (3.2), we suppose the following initial data

$$\begin{aligned} &y(0,x) = y_0(x), \quad z(0,x) = z_0(x), \\ &\phi(0,x) = \phi_0(x), \quad \psi(0,x) = \psi_0(x), \quad p(0,x) = p_0(x), \end{aligned} \tag{3.3}$$

where

$$y_0 := u_1, \quad z_0 := u_{0x}, \quad \phi_0 := \theta_1 + \tau\theta_2, \quad \psi_0 := (\gamma\theta_1 + \gamma^*\theta_0)_x, \quad p_0 := \sqrt{\gamma(\gamma - \tau\gamma^*)}\theta_{1x}.$$

Namely, the system (3.2) can be formulated as (2.1),(2.3), where $U := (y, z, \phi, \psi, p)^T$, $U_0 := (y_0, z_0, \phi_0, \psi_0, p_0)^T$ and

$$\begin{aligned} &A^0 = \begin{pmatrix} \rho & 0 & 0 & 0 & 0 \\ 0 & \tilde{\alpha} & 0 & 0 & 0 \\ 0 & 0 & \kappa & 0 & 0 \\ 0 & 0 & 0 & 1/\gamma^* & -\delta/\gamma^* \\ 0 & 0 & 0 & -\delta/\gamma^* & 1/\gamma^* \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & -\tilde{\alpha} & \beta & 0 & 0 \\ -\tilde{\alpha} & 0 & 0 & 0 & 0 \\ \beta & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ &B = O, \qquad L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\gamma \end{pmatrix}. \end{aligned} \tag{3.4}$$

We observe that under the dissipative condition we have $0 < \delta < 1$, these coefficient matrices satisfy Condition (A). Furthermore, as we do not have any constraints, we have

$$Q = O \tag{3.5}$$

for this system. Using the Stability Condition (SC) with the previous matrices, we derive the following result.

Theorem 3.1. *The matrices* (3.4) *and* (3.5) *satisfies Conditions* (A) *and* (SC). *Therefore, the solution of* (3.2), (3.3) *satisfies the decay estimate* (2.9).

Proof. Let $\mu \in \mathbb{R}$ and $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5)^T \in \mathbb{R}^5$. Then Condition (SC) in this case reads $\mathrm{Ker}(\mu A^0 + A(\omega)) \cap \mathrm{Ker}(L) = \{0\}$, that is

$$\begin{cases} \mu\rho\varphi_1 - \tilde{\alpha}\varphi_2 + \beta\varphi_3 = 0, \\ \mu\tilde{\alpha}\varphi_2 - \tilde{\alpha}\varphi_1 = 0, \\ \mu\kappa\varphi_3 + \beta\varphi_1 - \varphi_4 = 0, \\ \dfrac{\mu}{\gamma^*}(\varphi_4 - \delta\varphi_5) - \varphi_3 = 0, \\ \dfrac{\mu}{\gamma^*}(\varphi_5 - \delta\varphi_4) = 0, \end{cases} \quad \text{and} \quad \frac{1}{\gamma}\varphi_5 = 0. \tag{3.6}$$

Therefore, separating the cases $\mu = 0$ and $\mu \neq 0$ leads to $\varphi = 0$, and we complete the proof. □

Similarly, as before, we now would like to study the one-dimensional version of problem (1.3), which reads

$$\begin{aligned} \rho u_{tt} - \tilde{\alpha} u_{xx} + \beta\theta_{tx} = 0, \\ \kappa\theta_{tt} + \beta u_{tx} - \gamma\theta_{txx} - \gamma^*\theta_{xx} = 0, \end{aligned} \tag{3.7}$$

with the corresponding initial conditions, $x \in \mathbb{R}$ and, again, $\tilde{\alpha} := 2\alpha + \alpha^*$. Observe that we cannot use the previous argument, now with $\tau = 0$ as, in this case, $\delta = 1$. Hence, condition (A) would not be satisfied.

Now, putting $v := u_t$, $w := u_x$, $\phi := \theta_t$ and $\psi := \theta_x$, then (3.7) is rewritten by

$$\begin{aligned} \rho y_t - \tilde{\alpha} z_x + \beta\phi_x = 0, \\ \tilde{\alpha} z_t - \tilde{\alpha} y_x = 0, \\ \kappa\phi_t + \beta y_x - \gamma\phi_{xx} - \gamma^*\psi_x = 0, \\ \gamma^*\psi_t - \gamma^*\phi_x = 0, \end{aligned} \tag{3.8}$$

and the initial data is assigned by

$$y(0,x) = y_0(x), \quad z(0,x) = z_0(x), \quad \phi(0,x) = \phi_0(x), \quad \psi(0,x) = \psi_0(x), \tag{3.9}$$

where $y_0 := u_1$, $z_0 := u_{0x}$, $\phi_0 := \theta_1$ and $\psi_0 := \theta_{0x}$. Then, system (3.8) can be formulated in the form of (2.1), where $U := (y, z, \phi, \psi)^T$, $U_0 := (y_0, z_0, \phi_0, \psi_0)^T$ and

$$A^0 = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \tilde{\alpha} & 0 & 0 \\ 0 & 0 & \kappa & 0 \\ 0 & 0 & 0 & \gamma^* \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -\tilde{\alpha} & \beta & 0 \\ -\tilde{\alpha} & 0 & 0 & 0 \\ \beta & 0 & 0 & -\gamma^* \\ 0 & 0 & -\gamma^* & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L = O, \quad Q = O. \tag{3.10}$$

We observe that these coefficient matrices also satisfy Condition (A). Using Stability Condition (SC) for this system, we obtain the following result.

Theorem 3.2. *The matrices* (3.10) *satisfy Conditions* (A) *and* (SC)*. Therefore, the solution of* (3.8)*,* (3.9) *satisfies the decay estimate* (2.9)*.*

Proof. Let $\mu \in \mathbb{R}$ and $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^T \in \mathbb{R}^4$. Then Condition (SC) gives

$$\begin{cases} \mu\rho\varphi_1 - \tilde{\alpha}\varphi_2 + \beta\varphi_3 = 0, \\ \mu\tilde{\alpha}\varphi_2 - \tilde{\alpha}\varphi_1 = 0, \\ \mu\kappa\varphi_3 + \beta\varphi_1 - \gamma^*\varphi_4 = 0, \\ \mu\gamma^*\varphi_4 - \gamma^*\varphi_3 = 0, \end{cases} \quad \text{and} \quad \gamma\varphi_3 = 0. \tag{3.11}$$

Therefore, we obtain $\varphi_3 = 0$, and (3.11) is reduced to

$$\mu\rho\varphi_1 - \tilde{\alpha}\varphi_2 = 0, \quad \mu\varphi_2 - \varphi_1 = 0, \quad \beta\varphi_1 - \gamma^*\varphi_4 = 0, \quad \mu\varphi_4 = 0.$$

Thus, it is not difficult to conclude that $\varphi = 0$, and we complete the proof. □

Remark 3.3. *We observe that Theorems 3.1 and 3.2 do not give the decay rate of the solution itself, but the decay rate of a certain energy of the solution. In Section 5 we will complete these results using the eigenvalues expansion method.*

4. Dissipative structure in the multi-dimensional case

In this section, we study the dissipative structure of problems (1.1) and (1.3) in $\mathbb{R}^n$, for $n > 1$. In the multi-dimensional case, problem (1.1) has not enough dissipative structure to get the decay estimate (2.9). The reason is the following. We introduce the new function

$$w^{jk} := \frac{\partial u^j}{\partial x_k} - \frac{\partial u^k}{\partial x_j}$$

for $j, k = 1, \cdots n$. Using the first equation of (1.1), we can see that the function w^{jk} (which means the rotational effect of u) satisfies the wave equation $\rho w^{jk}_{tt} - \alpha\Delta w^{jk} = 0$. As the energy of the wave equation does not decay, we conclude that problem (1.1) does not have enough dissipative structure, as we said.

We now proceed to rewrite (1.1) in a more convenient form. We introduce $v := \operatorname{div} u$ and using (1.1) we obtain

$$\begin{aligned} \rho v_{tt} - \tilde{\alpha}\Delta v + \beta(\Delta\theta_t + \tau\Delta\theta_{tt}) = 0, \\ \kappa\theta_{tt} + \kappa\tau\theta_{ttt} + \beta v_t - \gamma\Delta\theta_t - \gamma^*\Delta\theta = 0, \end{aligned} \tag{4.1}$$

where $\tilde{\alpha}$ is also defined by $\tilde{\alpha} := 2\alpha + \alpha^*$. Our purpose for this section is to apply the stability condition to this new system (4.1) and obtain the desired decay estimate for this one. To this end, we introduce the following new functions

$$\begin{aligned} &y := v_t, \quad z := \nabla v, \quad \phi := \nabla(\theta_t + \tau\theta_{tt}), \\ &\psi := \Delta(\gamma\theta_t + \gamma^*\theta), \quad p := \sqrt{\gamma(\gamma - \tau\gamma^*)}\Delta\theta_t. \end{aligned} \tag{4.2}$$

Then, (4.1) is rewritten as

$$\begin{aligned} \rho y_t - \tilde{\alpha}\mathrm{div} z + \beta \mathrm{div}\phi &= 0, \\ \tilde{\alpha} z_t - \tilde{\alpha}\nabla y &= 0, \\ \kappa\phi_t + \beta\nabla y - \nabla\psi &= 0, \\ \frac{1}{\gamma^*}\psi_t - \frac{\delta}{\gamma^*}p_t - \mathrm{div}\phi &= 0, \\ \frac{1}{\gamma^*}p_t - \frac{\delta}{\gamma^*}\psi_t + \frac{1}{\gamma}p &= 0, \end{aligned} \tag{4.3}$$

where $\delta := \sqrt{(\gamma - \tau\gamma^*)/\gamma}$, and the initial data is assigned by

$$\begin{aligned} &y(0,x) = y_0(x), \quad z(0,x) = z_0(x), \\ &\phi(0,x) = \phi_0(x), \quad \psi(0,x) = \psi_0(x), \quad p(0,x) = p_0(x), \end{aligned} \tag{4.4}$$

where

$y_0 := v_1,\ z_0 := \nabla v_0,\ \phi_0 := \nabla(\theta_1 + \tau\theta_2),\ \psi_0 := \Delta(\gamma\theta_1 + \gamma^*\theta_0),\ p_0 := \sqrt{\gamma(\gamma - \tau\gamma^*)}\Delta\theta_1.$

Furthermore, because of (4.2), observe that z and ϕ should satisfy

$$\partial_{x_j} z^k - \partial_{x_k} z^j = 0, \qquad \partial_{x_j}\phi^k - \partial_{x_k}\phi^j = 0 \tag{4.5}$$

for $1 \le j < k \le n$, where z^j and ϕ^j denote the jth components of the vectors z and ϕ. Thus, we assign the constraint condition (4.5) to the problem (4.3)-(4.4) (which, then, is equivalent to (4.1)). We remark that the constraint condition (4.5) is the same as $\mathrm{rot}\, z = 0$ and $\mathrm{rot}\,\phi = 0$ in $\mathbb{R}^3$.

The system (4.3) can be formulated in the form (2.1), with $U := (y, z, \phi, \psi, p)^T$, $U_0 := (y_0, z_0, \phi_0, \psi_0, p_0)^T$ and

$$\begin{aligned} &A^0 = \begin{pmatrix} \rho & 0 & 0 & 0 & 0 \\ 0 & \tilde{\alpha} I_n & 0 & 0 & 0 \\ 0 & 0 & \kappa I_n & 0 & 0 \\ 0 & 0 & 0 & 1/\gamma^* & -\delta/\gamma^* \\ 0 & 0 & 0 & -\delta/\gamma^* & 1/\gamma^* \end{pmatrix}, \quad A(\omega) = \begin{pmatrix} 0 & -\tilde{\alpha}\omega & \beta\omega & 0 & 0 \\ -\tilde{\alpha}\omega^T & 0 & 0 & 0 & 0 \\ \beta\omega^T & 0 & 0 & -\omega^T & 0 \\ 0 & 0 & -\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ &B(\omega) = O, \qquad L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\gamma \end{pmatrix}. \end{aligned} \tag{4.6}$$

Observe that A^0, A^j, L, B are matrices of dimension $(2n+3) \times (2n+3)$.

We can see that these coefficient matrices satisfy Condition (A). Also, the constraint condition (4.5) can be expressed in the form (2.2) with

$$Q(\omega) = \begin{pmatrix} 0 & Q_n(\omega) & 0 & 0 & 0 \\ 0 & 0 & Q_n(\omega) & 0 & 0 \end{pmatrix}, \tag{4.7}$$

where $Q_n(\omega)$ is a $n(n-1)/2 \times n$ matrix defined by induction by $Q_2(\omega) = (-\omega_2 \; \omega_1)$ and

$$Q_n(\omega) = \left(\begin{array}{cccccc|c} 0 & 0 & \cdots & 0 & -\omega_n & & \omega_{n-1} \\ 0 & 0 & \cdots & -\omega_n & 0 & & \omega_{n-2} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & -\omega_n & \cdots & 0 & 0 & & \omega_2 \\ -\omega_n & 0 & \cdots & 0 & 0 & & \omega_1 \\ \hline & & Q_{n-1}(\omega) & & & & 0 \end{array}\right) \tag{4.8}$$

for $\omega \in S^{n-1}$ and $n \geq 3$.

As $Q_n(\omega)\omega^T = 0$, it is not difficult to check that $Q(\omega)$ satisfies Condition (C) that tells that the solution satisfies the constraint condition if the initial data satisfy the constraint condition. Also, we can check the Condition (SC), for this problem and we obtain the following result.

Theorem 4.1. *The matrices* (4.6) *and* (4.7) *with* (4.8) *satisfy Conditions* (A)*,* (C) *and* (SC)*. Therefore, the solution of* (4.3)-(4.4)*, whose initial data hold the constraint condition* (4.5)*, satisfies the decay estimate* (2.9)*.*

Proof. Let $\mu \in \mathbb{R}$ and $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5)^T \in \mathbb{R}^{2n+3}$ with $\varphi_2 = (\varphi_{2_1}, \cdots, \varphi_{2_n}) \in \mathbb{R}^n$ and $\varphi_3 = (\varphi_{3_1}, \cdots, \varphi_{3_n}) \in \mathbb{R}^n$. Then Condition (SC) gives

$$\begin{cases} \mu\rho\varphi_1 - \tilde{\alpha}(\omega\cdot\varphi_2) + \beta(\omega\cdot\varphi_3) = 0, \\ \mu\tilde{\alpha}\varphi_2 - \tilde{\alpha}\varphi_1\omega^T = 0, \\ \mu\kappa\varphi_3 + \beta\varphi_1\omega^T - \varphi_4\omega^T = 0, \\ \dfrac{\mu}{\gamma^*}(\varphi_4 - \delta\varphi_5) - (\omega\cdot\varphi_3) = 0, \\ \dfrac{\mu}{\gamma^*}(\varphi_5 - \delta\varphi_4) = 0, \end{cases} \quad \text{and} \quad \frac{1}{\gamma}\varphi_5 = 0, \tag{4.9}$$

and

$$\omega_j\varphi_{2_k} - \omega_k\varphi_{2_j} = 0, \qquad \omega_j\varphi_{3_k} - \omega_k\varphi_{3_j} = 0 \tag{4.10}$$

for $1 \leq j < k \leq n$. Therefore, we obtain $\varphi_5 = 0$ and $(\omega\cdot\varphi_3) = 0$, and (4.9) is reduced to

$$\mu\rho\varphi_1 - \tilde{\alpha}(\omega\cdot\varphi_2) = 0, \quad \mu\varphi_2 - \varphi_1\omega^T = 0, \quad \beta\varphi_1 - \varphi_4 = 0, \quad \mu\varphi_4 = 0. \tag{4.11}$$

Furthermore, the second equation in (4.10) and $(\omega \cdot \varphi_3) = 0$ give

$$\begin{aligned}
|\varphi_3|^2 &= \sum_{j,k=1}^{n} \omega_j^2 |\varphi_{3_k}|^2 \\
&= \sum_{j=1}^{n} \omega_j^2 |\varphi_{3_j}|^2 + \sum_{j,k=1, j\neq k}^{n} \omega_j^2 |\varphi_{3_k}|^2 \\
&= \sum_{j=1}^{n} \omega_j^2 |\varphi_{3_j}|^2 + \sum_{j,k=1, j\neq k}^{n} (\omega_j \varphi_{3_j})(\omega_k \varphi_{3_k}) \\
&= |(\omega \cdot \varphi_3)|^2 = 0.
\end{aligned}$$

Then, (4.11) lead to $\varphi_1 = \varphi_2 = \varphi_4 = 0$ if $\mu \neq 0$, and $\varphi_1 = \varphi_4 = 0$ and $(\omega \cdot \varphi_2) = 0$ if $\mu = 0$. Using the same arguments before, we also obtain $|\varphi_2| = |(\omega \cdot \varphi_2)| = 0$ under (4.10). Therefore, it is not difficult to conclude that $\varphi = 0$, and we complete the proof. □

We now want to study the dissipative structure of (1.3) in the multidimensional case. Formally, we substitute $\tau = 0$ into (4.1) and obtain

$$\begin{aligned}
\rho v_{tt} - \tilde{\alpha} \Delta v + \beta \Delta \theta_t = 0, \\
\kappa \theta_{tt} + \beta v_t - \gamma \Delta \theta_t - \gamma^* \Delta \theta = 0,
\end{aligned} \tag{4.12}$$

where $v := \operatorname{div} u$. As before, it is convenient to rewrite (4.12) in the following new variables and derive the decay estimate of solutions. We consider

$$y := v_t, \quad z := \nabla v, \quad \phi := \nabla \theta_t, \quad \psi := \Delta \theta, \tag{4.13}$$

and, then, (4.12) is rewritten by

$$\begin{aligned}
\rho y_t - \tilde{\alpha} \mathrm{div} z + \beta \mathrm{div} \phi = 0, \\
\tilde{\alpha} z_t - \tilde{\alpha} \nabla y = 0, \\
\kappa \phi_t + \beta \nabla y - \gamma \Delta \phi - \gamma^* \nabla \psi = 0, \\
\gamma^* \psi_t - \gamma^* \mathrm{div} \phi = 0.
\end{aligned} \tag{4.14}$$

The initial data is assigned by

$$y(0,x) = y_0(x), \quad z(0,x) = z_0(x), \quad \phi(0,x) = \phi_0(x), \quad \psi(0,x) = \psi_0(x), \tag{4.15}$$

where $y_0 := v_1$, $z_0 := \nabla v_0$, $\phi_0 := \nabla \theta_1$ and $\psi_0 := \Delta \theta_0$. Furthermore, because of (4.13), z and ϕ should satisfy

$$\partial_{x_j} z^k - \partial_{x_k} z^j = 0, \qquad \partial_{x_j} \phi^k - \partial_{x_k} \phi^j = 0 \tag{4.16}$$

for $1 \leq j < k \leq n$, where z^j and ϕ^j denote the jth components of the vectors z and ϕ. Thus, we also assign the constraint condition (4.16) to the problem (4.14)-(4.15).

We write system (4.14) in the form (2.1), with $U := (y, z, \phi, \psi)^T$, $U_0 := (y_0, z_0, \phi_0, \psi_0)^T$ and

$$A^0 = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \tilde{\alpha} I_n & 0 & 0 \\ 0 & 0 & \kappa I_n & 0 \\ 0 & 0 & 0 & \gamma^* \end{pmatrix}, \quad A(\omega) = \begin{pmatrix} 0 & -\tilde{\alpha}\omega & \beta\omega & 0 \\ -\tilde{\alpha}\omega^T & 0 & 0 & 0 \\ \beta\omega^T & 0 & 0 & -\gamma^*\omega^T \\ 0 & 0 & -\gamma^*\omega & 0 \end{pmatrix},$$
$$B(\omega) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L = O. \tag{4.17}$$

We remark that these coefficients also satisfy Condition (A). Also, the constraint condition (4.16) can be expressed (2.2) with

$$Q(\omega) = \begin{pmatrix} 0 & Q_n(\omega) & 0 & 0 \\ 0 & 0 & Q_n(\omega) & 0 \end{pmatrix}, \tag{4.18}$$

where $Q_n(\omega)$ is defined by (4.8). In this situation, using Stability Condition (SC) for this system, we also obtain the following result.

Theorem 4.2. *The matrices* (4.17) *and* (4.18) *with* (4.8) *satisfy Conditions* (A), (C) *and* (SC). *Therefore, the solution of* (4.14)-(4.15), *whose initial data hold the constraint condition* (4.16), *satisfies the decay estimate* (2.9).

Proof. Let $\mu \in \mathbb{R}$ and $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^T \in \mathbb{R}^{2n+2}$ with $\varphi_2 = (\varphi_{2_1}, \cdots \varphi_{2_n}) \in \mathbb{R}^n$ and $\varphi_3 = (\varphi_{3_1}, \cdots \varphi_{3_n}) \in \mathbb{R}^n$. Then Condition (SC) gives

$$\begin{cases} \mu\rho\varphi_1 - \tilde{\alpha}(\omega \cdot \varphi_2) + \beta(\omega \cdot \varphi_3) = 0, \\ \mu\tilde{\alpha}\varphi_2 - \tilde{\alpha}\varphi_1\omega^T = 0, \\ \mu\kappa\varphi_3 + \beta\varphi_1\omega^T - \gamma^*\varphi_4\omega^T = 0, \\ \mu\gamma^*\varphi_4 - \gamma^*(\omega \cdot \varphi_3) = 0, \end{cases} \quad \text{and} \quad \gamma\varphi_3 = 0, \tag{4.19}$$

and

$$\omega_j\varphi_{2_k} - \omega_k\varphi_{2_j} = 0, \qquad \omega_j\varphi_{3_k} - \omega_k\varphi_{3_j} = 0 \tag{4.20}$$

for $1 \le j < k \le n$. Therefore, we obtain $\varphi_3 = 0$, and (4.19) is reduced that

$$\mu\rho\varphi_1 - \tilde{\alpha}(\omega \cdot \varphi_2) = 0, \quad \mu\varphi_2 - \varphi_1\omega^T = 0, \quad \beta\varphi_1 - \gamma^*\varphi_4 = 0, \quad \mu\varphi_4 = 0$$

Hence, it is not difficult to conclude that $\varphi = 0$, and we complete the proof. □

Remark 4.3. *We observe that Theorems 4.1 and 4.2 do not give the decay rate of the solution itself, but the decay rate of a certain energy of the solution. In Section 5 we will complete these results using the eigenvalues expansion method.*

5. Eigenvalue problem

In this last section, we consider the eigenvalue problem related to our system to discuss the optimality of the decay estimates and the regularity given in Theorems 3.1, 3.2, 4.1, 4.2. We also recall that, in these theorems, we have not given the decay rate of u, θ, but a certain norm of the solution. The results in the present section will allow us to state these kind of results for the solution of the corresponding systems.

The characteristic equation of the eigenvalue problem (2.7) is given by

$$\det(\lambda I - \hat{\Phi}(i\xi)) = 0, \tag{5.1}$$

where $\hat{\Phi}(i\xi)$ is defined by (2.8). For this equation, we have to consider the effect of the constraint condition. The characteristic equation for the system (4.3) in $\mathbb{R}^N$ is rewritten as

$$\begin{aligned} \lambda^{2(n-1)}\{\tau\rho\kappa\lambda^5 + \rho\kappa\lambda^4 + (\rho\gamma + \tau(\tilde{\alpha}\kappa + \beta^2))|\xi|^2\lambda^3 \\ + (\rho\gamma^* + \tilde{\alpha}\kappa + \beta^2)|\xi|^2\lambda^2 + \tilde{\alpha}\gamma|\xi|^4\lambda + \tilde{\alpha}\gamma^*|\xi|^4\} = 0, \end{aligned} \tag{5.2}$$

where δ is replaced by $\sqrt{(\gamma - \tau\gamma^*)/\gamma}$. The equation (5.2) means that $\hat{\Phi}(i\xi)$ has $2(n-1)$ repeated zero eigenvalues. The eigenspace W_ξ of zero eigenvalues is described as

$$W_\xi = \{\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) \in \mathbb{C}^{2n+3} \,|\, \xi \cdot \varphi_2 = 0,\, \xi \cdot \varphi_3 = 0,\, \varphi_1 = \varphi_4 = \varphi_5 = 0\},$$

and the orthogonal complement V_ξ of W_ξ is characterized by

$$V_\xi = \{\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) \in \mathbb{C}^{2n+3} \,|\, \xi_j\varphi_\ell^k - \xi_k\varphi_\ell^j = 0,\, 1 \le j, k \le n,\, \ell = 2, 3\},$$

where $\varphi_\ell = (\varphi_\ell^1, \cdots, \varphi_\ell^n)$ for $\ell = 2, 3$ (see constraint condition (4.5)). Consequently, because the initial data of our problem belong to V_ξ, we conclude that our eigenvalues are not equal to zero under the constraint condition. Hence, it is enough to focus on the eigenvalues which satisfy

$$\begin{aligned} \tau\rho\kappa\lambda^5 + \rho\kappa\lambda^4 + (\rho\gamma + \tau(\tilde{\alpha}\kappa + \beta^2))|\xi|^2\lambda^3 \\ + (\rho\gamma^* + \tilde{\alpha}\kappa + \beta^2)|\xi|^2\lambda^2 + \tilde{\alpha}\gamma|\xi|^4\lambda + \tilde{\alpha}\gamma^*|\xi|^4 = 0. \end{aligned} \tag{5.3}$$

The detailed derivation of the characteristic equation under constraint conditions is described in [22, 29].

We remark that the characteristic equations of (3.2) and (4.3) are equivalent because the corresponding coefficient matrices (3.4) and (4.6) have the same structure (but in (4.3) we also impose the constraint condition (4.5)).

Let $\lambda_j(i\xi)$ be a non-zero eigenvalue of $\hat{\Phi}(i\xi)$. The asymptotic expansions of $\lambda_j(i\xi)$ when $|\xi| \to 0$ and when $|\xi| \to \infty$ are related to the decay rate of the solutions, and the regularity-loss phenomenon, respectively. In order to prove the optimality of the decay rate of the solutions, we first consider the asymptotic expansion for $|\xi| \to 0$. That is, we consider the expression

$$\lambda_j(i\xi) = \sum_{k=0}^{\infty} \lambda_j^{(k)}|\xi|^k$$

and substitute it into (5.3). After lengthy but straightforward calculations, we obtain the following leading order terms:

$$
\begin{aligned}
\lambda_{1\pm}(i\xi) &= \sqrt{\frac{1}{2\rho\kappa}\left(\rho\gamma^* + \tilde{\alpha}\kappa + \beta^2 \mp \sqrt{(\rho\gamma^* + \tilde{\alpha}\kappa + \beta^2)^2 - 4\rho\kappa\tilde{\alpha}\gamma^*}\right)}\, i|\xi| \\
&\quad - \frac{\gamma - \tau\gamma^*}{4\kappa}\left\{1 \pm \frac{\tilde{\alpha}\kappa - (\rho\gamma^* + \beta^2)}{\sqrt{(\rho\gamma^* + \tilde{\alpha}\kappa + \beta^2)^2 - 4\rho\kappa\tilde{\alpha}\gamma^*}}\right\}|\xi|^2 + O(|\xi|^3), \\
\lambda_{2\pm}(i\xi) &= -\sqrt{\frac{1}{2\rho\kappa}\left(\rho\gamma^* + \tilde{\alpha}\kappa + \beta^2 \mp \sqrt{(\rho\gamma^* + \tilde{\alpha}\kappa + \beta^2)^2 - 4\rho\kappa\tilde{\alpha}\gamma^*}\right)}\, i|\xi| \\
&\quad - \frac{\gamma - \tau\gamma^*}{4\kappa}\left\{1 \pm \frac{\tilde{\alpha}\kappa - (\rho\gamma^* + \beta^2)}{\sqrt{(\rho\gamma^* + \tilde{\alpha}\kappa + \beta^2)^2 - 4\rho\kappa\tilde{\alpha}\gamma^*}}\right\}|\xi|^2 + O(|\xi|^3), \\
\lambda_3(i\xi) &= -\frac{1}{\tau} + O(|\xi|^2).
\end{aligned} \tag{5.4}
$$

Observe that, under the dissipative condition (1.2), we have

$$
(\rho\gamma^* + \tilde{\alpha}\kappa + \beta^2)^2 - 4\rho\kappa\tilde{\alpha}\gamma^* = (\tilde{\alpha}\kappa - \rho\gamma^* - \beta^2)^2 + 4\tilde{\alpha}\kappa\beta^2
$$

and, hence

$$
\frac{\tilde{\alpha}\kappa - (\rho\gamma^* + \beta^2)}{\sqrt{(\rho\gamma^* + \tilde{\alpha}\kappa + \beta^2)^2 - 4\rho\kappa\tilde{\alpha}\gamma^*}} < 1. \tag{5.5}
$$

Therefore, $\mathrm{Re}(\lambda_{j\pm}(i\xi)) < 0$ as $|\xi| \to 0$ for $j = 1, 2$.

We now study the asymptotic expansion of the eigenvalues for $|\xi| \to \infty$, which is related to the regularity-loss phenomenon. That is, we consider the expression

$$
\lambda_j(i\xi) = \lambda_j^{(-1)}|\xi| + \sum_{k=0}^{\infty} \lambda_j^{(k)}|\xi|^{-k}
$$

and substitute it into (5.3). Similar calculation as before leads to the following expansion:

$$
\begin{aligned}
\lambda_{1\pm}(i\xi) &= \sqrt{\frac{1}{2\tau\rho\kappa}\left(\rho\gamma + \tau(\tilde{\alpha}\kappa + \beta^2) \mp \sqrt{(\rho\gamma + \tau(\tilde{\alpha}\kappa + \beta^2))^2 - 4\tau\rho\kappa\tilde{\alpha}\gamma}\right)}\, i|\xi| \\
&\quad - \frac{\gamma - \tau\gamma^*}{4\tau\gamma}\left\{1 \pm \frac{\tau(\tilde{\alpha}\kappa + \beta^2) - \rho\gamma}{\sqrt{(\rho\gamma + \tau(\tilde{\alpha}\kappa + \beta^2))^2 - 4\tau\rho\kappa\tilde{\alpha}\gamma}}\right\} + O(|\xi|^{-1}), \\
\lambda_{2\pm}(i\xi) &= -\sqrt{\frac{1}{2\tau\rho\kappa}\left(\rho\gamma + \tau(\tilde{\alpha}\kappa + \beta^2) \mp \sqrt{(\rho\gamma + \tau(\tilde{\alpha}\kappa + \beta^2))^2 - 4\tau\rho\kappa\tilde{\alpha}\gamma}\right)}\, i|\xi| \\
&\quad - \frac{\gamma - \tau\gamma^*}{4\tau\gamma}\left\{1 \pm \frac{\tau(\tilde{\alpha}\kappa + \beta^2) - \rho\gamma}{\sqrt{(\rho\gamma + \tau(\tilde{\alpha}\kappa + \beta^2))^2 - 4\tau\rho\kappa\tilde{\alpha}\gamma}}\right\} + O(|\xi|^{-1}), \\
\lambda_3(i\xi) &= -\frac{\gamma^*}{\gamma} + O(|\xi|^{-1}).
\end{aligned} \tag{5.6}
$$

Here, we also remark that

$$(\tau(\tilde{\alpha}\kappa+\beta^2)-\rho\gamma)^2+4\tau\rho\gamma\beta^2=(\rho\gamma+\tau(\tilde{\alpha}\kappa+\beta^2))^2-4\tau\rho\kappa\tilde{\alpha}\gamma,$$

and this gives $\mathrm{Re}(\lambda_{j\pm}(i\xi))<0$ as $|\xi|\to\infty$ for $j=1,2$.

On the other hand, using the same previous argument, the characteristic equations of (4.12) and (4.14) are described by

$$\rho\kappa\tilde{\lambda}^4+\rho\gamma|\xi|^2\tilde{\lambda}^3+(\rho\gamma^*+\tilde{\alpha}\kappa+\beta^2)|\xi|^2\tilde{\lambda}^2+\tilde{\alpha}\gamma|\xi|^4\tilde{\lambda}+\tilde{\alpha}\gamma^*|\xi|^4=0. \tag{5.7}$$

Then, using the same argument, we arrive at the following leading order terms in the asymptotic expansions of the eigenvalues. When $|\xi|\to 0$ we have:

$$\begin{aligned}
\tilde{\lambda}_{1\pm}(i\xi)&=\sqrt{\frac{1}{2\rho\kappa}\left(\rho\gamma^*+\tilde{\alpha}\kappa+\beta^2\mp\sqrt{(\rho\gamma^*+\tilde{\alpha}\kappa+\beta^2)^2-4\rho\kappa\tilde{\alpha}\gamma^*}\right)}i|\xi|\\
&\quad-\frac{\gamma}{4\kappa}\left\{1\pm\frac{\tilde{\alpha}\kappa-(\rho\gamma^*+\beta^2)}{\sqrt{(\rho\gamma^*+\tilde{\alpha}\kappa+\beta^2)^2-4\rho\kappa\tilde{\alpha}\gamma^*}}\right\}|\xi|^2+O(|\xi|^3),\\
\tilde{\lambda}_{2\pm}(i\xi)&=-\sqrt{\frac{1}{2\rho\kappa}\left(\rho\gamma^*+\tilde{\alpha}\kappa+\beta^2\mp\sqrt{(\rho\gamma^*+\tilde{\alpha}\kappa+\beta^2)^2-4\rho\kappa\tilde{\alpha}\gamma^*}\right)}i|\xi|\\
&\quad-\frac{\gamma}{4\kappa}\left\{1\pm\frac{\tilde{\alpha}\kappa-(\rho\gamma^*+\beta^2)}{\sqrt{(\rho\gamma^*+\tilde{\alpha}\kappa+\beta^2)^2-4\rho\kappa\tilde{\alpha}\gamma^*}}\right\}|\xi|^2+O(|\xi|^3),
\end{aligned} \tag{5.8}$$

and when $|\xi|\to\infty$ we have

$$\begin{aligned}
\tilde{\lambda}_{\pm}(i\xi)&=\pm\sqrt{\frac{\tilde{\alpha}}{\rho}}i|\xi|-\frac{\beta^2}{2\rho\gamma}+O(|\xi|^{-1}),\\
\tilde{\lambda}_1(i\xi)&=-\frac{\gamma^*}{\gamma}+O(|\xi|^{-1}),\qquad \tilde{\lambda}_2(i\xi)=-\frac{\gamma}{\kappa}|\xi|^2+O(1).
\end{aligned} \tag{5.9}$$

For the same reason as in the MGT eigenvalues (see (5.5)), we have $\mathrm{Re}(\tilde{\lambda}_{j\pm}(i\xi))<0$ as $|\xi|\to 0$ for $j=1,2$ in the case $|\xi|\to 0$. So, the real parts of the leading order terms of all the eigenvalues are strictly negative, both when $|\xi|\to 0$ and $|\xi|\to\infty$.

We now see the relation between the eigenvalues coming from the characteristic equations (5.3) and (5.7) as $\tau\to 0$. In the low-frequency region ($|\xi|\to 0$), it is easy to check that $\lambda_{j\pm}(i\xi)$ tends to $\tilde{\lambda}_{j\pm}(i\xi)$ as $\tau\to 0$ for $j=1,2$ (see (5.4) and (5.8)). And that $\lambda_3(i\xi)$ blows up when $\tau\to 0$ (so, this eigenvalue disappears). On the other hand, in the high-frequency region ($|\xi|\to\infty$), using the facts that

$$\begin{aligned}
&\frac{1}{\tau}\left(\rho\gamma+\tau(\tilde{\alpha}\kappa+\beta^2)-\sqrt{(\rho\gamma+\tau(\tilde{\alpha}\kappa+\beta^2))^2-4\tau\rho\kappa\tilde{\alpha}\gamma}\right)\\
&=\frac{4\rho\kappa\tilde{\alpha}\gamma}{\rho\gamma+\tau(\tilde{\alpha}\kappa+\beta^2)+\sqrt{(\rho\gamma+\tau(\tilde{\alpha}\kappa+\beta^2))^2-4\tau\rho\kappa\tilde{\alpha}\gamma}}\longrightarrow 2\kappa\tilde{\alpha},
\end{aligned}$$

and

$$\frac{1}{\tau}\left(\sqrt{(\rho\gamma+\tau(\tilde{\alpha}\kappa+\beta^2))^2-4\tau\rho\kappa\tilde{\alpha}\gamma}+\tau(\tilde{\alpha}\kappa+\beta^2)-\rho\gamma\right)$$
$$=\frac{4\rho\gamma\beta^2}{\sqrt{(\rho\gamma+\tau(\tilde{\alpha}\kappa+\beta^2))^2-4\tau\rho\kappa\tilde{\alpha}\gamma}+\rho\gamma-\tau(\tilde{\alpha}\kappa+\beta^2)}\longrightarrow 2\beta^2$$

as $\tau\to 0$, we find that $\lambda_{1+}(i\xi)$ and $\lambda_{2+}(i\xi)$ approach to $\tilde{\lambda}_+(i\xi)$ and $\tilde{\lambda}_-(i\xi)$ as $\tau\to 0$, respectively (see (5.6) and (5.9)), and that $\lambda_3(i\xi)$ has the same leading terms as $\tilde{\lambda}_1(i\xi)$.

However, because of the facts that

$$\frac{1}{\tau}\left(\rho\gamma+\tau(\tilde{\alpha}\kappa+\beta^2)+\sqrt{(\rho\gamma+\tau(\tilde{\alpha}\kappa+\beta^2))^2-4\tau\rho\kappa\tilde{\alpha}\gamma}\right)\longrightarrow\infty,$$

and

$$\frac{1}{\tau}\left(\sqrt{(\rho\gamma+\tau(\tilde{\alpha}\kappa+\beta^2))^2-4\tau\rho\kappa\tilde{\alpha}\gamma}-(\tau(\tilde{\alpha}\kappa+\beta^2)-\rho\gamma)\right)\longrightarrow\infty$$

as $\tau\to 0$, $\lambda_{1-}(i\xi)$ and $\lambda_{2-}(i\xi)$ blow up as $\tau\to 0$. Thus it is an open question that $\tilde{\lambda}_2(i\xi)$ is a limit of some superposition of eigenvalues.

Remark 5.1. *The decay rates in Theorems 3.1, 3.2, 4.1 and 4.2 come from the exponent* $|\xi|^2/(1+|\xi|^2)$ *of the real part of the eigenvalue in part* (*ii*) *in Theorem 2.1. Observe that this exponent behaves as* $|\xi|^2$ *when* $|\xi|\to 0$ *and as* 1 *when* $|\xi|\to\infty$*. This is the asymptotic behavior we have obtained in all the previous cases for the slowest eigenvalues when* $|\xi|\to 0$ *and* $|\xi|\to\infty$*, correspondingly* (*see* (5.4), (5.6), (5.8) *and* (5.9)). *We recall that the asymptotic expansions of the eigenvalues when* $|\xi|\to 0$ *and when* $|\xi|\to\infty$ *are related to the decay rate of the solutions and the regularity-loss phenomenon, respectively. Consequently, we deduce that the decay estimates in Theorem 3.1, 3.2, 4.1 and 4.2 are optimal. And so it is the no regularity loss of the initial conditions.*

6. Conclusions

In this work we study the asymptotic behavior of two thermoelastic systems in $\mathbb{R}^n$, $n\geq 1$: the system of Moore–Gibson–Thompson thermoelasticity, and the system of type III Green–Naghdi thermoelasticity (that can be seen as a limit of the previous one). In summary, we have shown that the given systems can be written in a symmetric hyperbolic form with constraint conditions, both in the one-dimensional and multi-dimensional cases. This means that the stability condition with the constraint condition in Theorem 2.1 can be applied. This allows us to conclude that the solutions to our systems satisfy the decay estimate (2.9) in Corollary 2.2 (a decay of a certain norm of the solution), and that there is no regularity-loss of the initial conditions. Furthermore, by considering the eigenvalue problem related to our systems, we have also analyzed that the decay estimate (2.9) and the no regularity-loss phenomenon are both optimal. Finally, we have compared the asymptotic expansions of the eigenvalues for the two problems: while in the low frequency regime the eigenvalues behave as expected when comparing one problem to the limit one, this remains as an open problem in the high frequency regime.

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