



# Decay of operator semigroups, infinite-time admissibility, and related resolvent estimates

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(Citation)

Journal of Mathematical Analysis and Applications, 538(1):128445

(Issue Date)

2024-10-01

(Resource Type)

journal article

(Version)

Version of Record

(Rights)

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(URL)

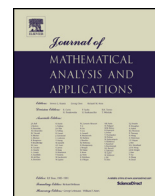
<https://hdl.handle.net/20.500.14094/0100489644>





Contents lists available at ScienceDirect

## Journal of Mathematical Analysis and Applications

journal homepage: [www.elsevier.com/locate/jmaa](http://www.elsevier.com/locate/jmaa)

## Regular Articles

## Decay of operator semigroups, infinite-time admissibility, and related resolvent estimates



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## ARTICLE INFO

## Article history:

Received 4 February 2023

Available online 18 April 2024

Submitted by H. Zwart

## Keywords:

 $C_0$ -semigroup

Infinite-dimensional system

Infinite-time admissibility

Polynomial stability

## ABSTRACT

We study decay rates for bounded  $C_0$ -semigroups from the perspective of  $L^p$ -infinite-time admissibility and related resolvent estimates. In the Hilbert space setting, polynomial decay of semigroup orbits is characterized by the resolvent behavior in the open right half-plane. A similar characterization based on  $L^p$ -infinite-time admissibility is provided for multiplication semigroups on  $L^q$ -spaces with  $1 \leq q \leq p < \infty$ . For polynomially stable  $C_0$ -semigroups on Hilbert spaces, we also give a sufficient condition for  $L^2$ -infinite-time admissibility.

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## 1. Introduction

Consider the abstract Cauchy problem

$$\begin{cases} \dot{u}(t) = Au(t), & t \geq 0, \\ u(0) = x, & x \in X, \end{cases}$$

where  $A$  is the generator of a bounded  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $X$ . Throughout this paper, we consider the domain  $D(A)$  of  $A$  to be equipped with the graph norm of  $A$ . Let us assume that  $A$  is invertible. Then the domain of  $A$  coincides with the range of  $A^{-1}$ . To obtain uniform decay rates of classical solutions, we study the quantitative behavior of the operator norm  $\|T(t)A^{-1}\|$  as  $t \rightarrow \infty$ . We concentrate mainly on the situation where  $\|T(t)A^{-1}\|$  decays polynomially, i.e.,  $\|T(t)A^{-1}\| = O(t^{-1/\alpha})$  as  $t \rightarrow \infty$  for some  $\alpha > 0$ .

The relation between polynomial rates of decay of  $\|T(t)A^{-1}\|$  and growth of  $\|(i\eta I - A)^{-1}\|$  on the imaginary axis  $i\mathbb{R}$  was studied, e.g., in [4,5,7,30]. In particular, it was shown in [7] that  $\|T(t)A^{-1}\| = O(t^{-1/\alpha})$  as  $t \rightarrow \infty$  is equivalent to  $\|(i\eta I - A)^{-1}\| = O(|\eta|^\alpha)$  as  $|\eta| \rightarrow \infty$  in the Hilbert space setting. We refer to [5,6,12,35] for more general transference between semigroup decay and resolvent growth on  $i\mathbb{R}$ .

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The long-time asymptotic behavior of semigroup orbits is related to the resolvent of  $A$  on the open right half-plane  $\mathbb{C}_+$  as well. For example, the Gearhart-Prüss theorem (see, e.g., [1, Theorem 5.2.1] and [16, Theorem V.1.11]) shows that exponential stability of a  $C_0$ -semigroup on a Hilbert space is equivalent to uniform boundedness of the resolvent on  $\mathbb{C}_+$ . Moreover, it was proved in [39] that density of the set

$$\left\{ x \in X : \lim_{\xi \rightarrow 0+} \sqrt{\xi} ((\xi + i\eta)I - A)^{-1} x = 0 \text{ for all } \eta \in \mathbb{R} \right\}$$

is sufficient for strong stability of the bounded  $C_0$ -semigroup when  $X$  is a Hilbert space. See the survey papers [11,12] for further references and recent developments on stability of  $C_0$ -semigroups. In [2,3,14,15,22,34,45], the generator  $A$  of a (not necessarily bounded)  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  was considered, and upper bounds for non-exponential growth rates of  $\|T(t)\|$  were obtained from the so-called  $\alpha$ -Kreiss condition: The spectrum  $\sigma(A)$  of  $A$  is contained in the closed left half-plane  $\overline{\mathbb{C}}_-$  and there exist  $K \geq 1$  and  $0 \leq \alpha < \infty$  such that  $\|(\lambda I - A)^{-1}\| \leq K((\operatorname{Re} \lambda)^{-\alpha} + 1)$  for all  $\lambda \in \mathbb{C}_+$ . Some variations of these results were reported in [8,41]. The estimate  $\|T(t)\| = O(t/\sqrt{\log t})$ , proved in [2], is the best obtained so far from the 1-Kreiss condition for Hilbert space semigroups.

For a bounded  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on a Hilbert space such that  $\sigma(A) \cap i\mathbb{R} = \emptyset$ ,  $\|T(t)A^{-1}\| = O(t^{-1/\alpha})$  if and only if  $\sup_{\lambda \in \mathbb{C}_+} \|(\lambda I - A)^{-1}(-A)^{-\alpha}\| < \infty$ ; see [7]. In this equivalence, the fractional power  $(-A)^{-\alpha}$  plays a role of a smoothing factor that cancels resolvent growth. When we replace it with a weaker smoothing factor  $(-A)^{-\alpha+\varepsilon}$ ,  $0 < \varepsilon < \alpha$ , the resulting norm  $\|(\lambda I - A)^{-1}(-A)^{-\alpha+\varepsilon}\|$  may grow to infinity as  $\operatorname{Re} \lambda \rightarrow 0+$ . Then it is natural to ask whether this kind of resolvent growth on  $\mathbb{C}_+$  also characterizes the decay of  $\|T(t)A^{-1}\|$  as  $t \rightarrow \infty$ .

We study a resolvent estimate on  $\mathbb{C}_+$ , sometimes called the  $p$ -Weiss condition in the community of infinite-dimensional systems. Let  $Y$  be another Banach space, and consider  $C \in \mathcal{L}(D(A), Y)$ , i.e., a linear bounded operator  $C$  from  $D(A)$  to  $Y$ . The  $p$ -Weiss condition on  $C$  is defined as follows.

**Definition 1.1.** Let  $X$  and  $Y$  be Banach spaces, let  $A$  be the generator of a bounded  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on  $X$ , and let  $1 \leq p \leq \infty$ . An operator  $C \in \mathcal{L}(D(A), Y)$  satisfies the  $p$ -Weiss condition for  $A$  if there exists  $K > 0$  such that for all  $\lambda \in \mathbb{C}_+$ ,

$$\|CR(\lambda, A)\| \leq \frac{K}{(\operatorname{Re} \lambda)^{1-1/p}},$$

where  $1/p := 0$  for  $p = \infty$ .

Following [42], we introduce the notion of admissibility. The  $p$ -Weiss condition was derived as a necessary condition for this notion by Weiss [43].

**Definition 1.2.** Let  $X$  and  $Y$  be Banach spaces, let  $A$  be the generator of a  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on  $X$ , and let  $1 \leq p < \infty$ .

- An operator  $C \in \mathcal{L}(D(A), Y)$  is  $L^p$ -infinite-time admissible for  $A$  if there exists  $M > 0$  such that

$$\int_0^\infty \|CT(t)x\|^p dt \leq M\|x\|^p \quad (1)$$

for all  $x \in D(A)$ .

- An operator  $C \in \mathcal{L}(D(A), Y)$  is  $L^\infty$ -infinite-time admissible for  $A$  if there exists  $M > 0$  such that  $\sup_{t \geq 0} \|CT(t)x\| \leq M\|x\|$  for all  $x \in D(A)$ .

Weiss conjectured in [43,44] that the 2-Weiss condition and  $L^2$ -infinite-time admissibility are equivalent when  $X$  and  $Y$  are Hilbert spaces. This conjecture was resolved negatively. Counterexamples can be found in [26,27,48]. However, positive results on the equivalence were obtained in several situations: a)  $(T(t))_{t \geq 0}$  is an exponentially stable and right-invertible  $C_0$ -semigroup [43]; b)  $(T(t))_{t \geq 0}$  is a contraction  $C_0$ -semigroup and  $Y$  is a finite-dimensional space [24]; and c)  $(T(t))_{t \geq 0}$  is a bounded analytic  $C_0$ -semigroup such that  $(-A)^{1/2}$  is  $L^2$ -infinite-time admissible for  $A$  [29]. While  $X$  and  $Y$  are Hilbert spaces in the results a) and b), they are Banach spaces in the result c). Moreover, the result c) was extended from the  $L^2$ -case to the  $L^p$ -case for  $1 \leq p \leq \infty$  in [9,18] and to Orlicz spaces in [23]. For more information on admissibility and related resolvent conditions, we refer to the survey article [25] and the books [38,40].

For  $C \in \mathcal{L}(D(A), Y)$ , the relation of the  $p$ -Weiss condition and  $L^p$ -infinite-time admissibility to the decay rate of the form

$$\|CT(t)\| \leq \frac{M}{t^{1/p}} \quad \text{as } t \rightarrow \infty \text{ for some } M > 0 \quad (2)$$

was studied in [9,17,23,47], where  $CT(t)$  extends an operator in  $\mathcal{L}(X, Y)$  (again denoted by  $CT(t)$ ) for all  $t > 0$ . For bounded analytic semigroups on Banach spaces, the decay estimate in (2) with  $1 < p \leq \infty$  holds if and only if the  $p$ -Weiss condition is satisfied [9,17]. This was extended to decay rates of a more general class of functions in [23]. It was shown in [47] that if a bounded  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on a Hilbert space commutes with  $C$  for all  $t \geq 0$ , then  $L^2$ -infinite-time admissibility implies the decay rate in (2) with  $p = 2$ .

While most previous studies on admissibility focus on  $C \in \mathcal{L}(D(A), Y)$ , we sometimes deal with  $C \in \mathcal{L}(X, Y)$  in this paper. This might seem unconventional, because  $C \in \mathcal{L}(X, Y)$  is infinite-time admissible for any generator of an exponentially stable  $C_0$ -semigroup. However, it is not trivial that when the  $C_0$ -semigroup have weaker stability properties,  $C \in \mathcal{L}(X, Y)$  satisfies  $L^p$ -infinite-time admissibility and the  $p$ -Weiss condition with  $1 \leq p < \infty$ . In particular, we are interested in the case where  $C$  is a fractional power  $(-A)^{-\delta}$  for some  $\delta > 0$ , which is not only bounded on  $X$  but also has a certain smoothing effect. This is in contrast to the situation of the above-mentioned previous studies [9,18,23,29], which discussed the infinite-time admissibility of  $(-A)^\delta$  with  $\delta > 0$  as a condition for the Weiss conjecture to be true for bounded analytic  $C_0$ -semigroups.

For fixed  $\alpha > 0$  and  $1 \leq p < \infty$ , we prove that  $(-A)^{-\alpha/p}$  satisfies the  $p$ -Weiss condition for  $A$  if and only if  $\|T(t)A^{-1}\| = O(t^{-1/\alpha})$  as  $t \rightarrow \infty$ , where  $(T(t))_{t \geq 0}$  is a bounded  $C_0$ -semigroup on a Hilbert space such that  $0 \notin \sigma(A)$ . Note that the  $p$ -Weiss condition on  $(-A)^{-\alpha/p}$  gives an upper bound for the rate of growth of  $\|(\lambda I - A)^{-1}(-A)^{-\alpha/p}\|$  as  $\operatorname{Re} \lambda \rightarrow 0+$ . Since  $(-A)^{-\alpha/p}$  is bounded, the Hille-Yosida theorem guarantees a faster decay rate  $\|(\lambda I - A)^{-1}(-A)^{-\alpha/p}\| = O(1/\operatorname{Re} \lambda)$  as  $\operatorname{Re} \lambda \rightarrow \infty$  than the  $p$ -Weiss condition. We also show that non-polynomial decay of the form  $\|T(t)(-A)^{-\alpha}\| = O(t^{-\beta}(\log t)^{-\gamma})$ , where  $\alpha > 0$ ,  $0 \leq \beta < 1$ , and  $\gamma \geq 0$ , can be characterized by resolvent growth on  $\mathbb{C}_+$ . Following [6,10,31,35,37], we obtain analogous characterizations of rates of decay for  $\|T(t)A(I - A)^{-1}\|$  and  $\|T(t)A(I - A)^{-2}\|$ . They are motivated by the problem of quantifying uniform rates of decay of orbits starting in the range  $\operatorname{ran}(A)$  of  $A$  and in  $D(A) \cap \operatorname{ran}(A)$ , respectively.

In the Banach space setting, we can easily see that for a bounded  $C_0$ -semigroup  $(T(t))_{t \geq 0}$ , the decay property  $\|T(t)A^{-1}\| = O(t^{-1/\alpha})$  is almost equivalent to the  $L^p$ -infinite-time admissibility of  $(-A)^{-\alpha/p}$ . More precisely, if  $(-A)^{-\alpha/p}$  is  $L^p$ -infinite-time admissible for  $A$ , then  $\|T(t)A^{-1}\| = O(t^{-1/\alpha})$  as  $t \rightarrow \infty$ . Conversely, if this decay estimate is satisfied, then  $(-A)^{-\alpha/p-\varepsilon}$  with  $\varepsilon > 0$  is  $L^p$ -infinite-time admissible for  $A$ . We do not know whether the latter implication with  $\varepsilon = 0$  is true for general bounded  $C_0$ -semigroups. However, for multiplication semigroups on  $L^q$ -spaces with  $1 \leq q \leq p < \infty$ , it is true and hence we obtain an equivalence result.

Let  $X$  and  $Y$  be Hilbert spaces, and consider a bounded  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on  $X$  such that  $\|T(t)A^{-1}\|$  decays polynomially. The next objective is to obtain a sufficient condition for  $C \in \mathcal{L}(D(A), Y)$  to be  $L^2$ -infinite-time admissible for  $A$ . One of the motivations is that the dual notion of  $L^2$ -infinite-time

admissibility in Definition 1.1 is closely related to input-to-state stability with respect to squared integrable inputs; see [42] for duality results on admissibility. A survey on input-to-state stability of infinite-dimensional systems is given in [32]. A crucial part of the sufficient condition is that there exists  $\alpha > 0$  such that  $C(-A)^\alpha$  extends to an operator in  $\mathcal{L}(D(A), Y)$  and its extension satisfies the 2-Weiss condition for  $A$ , which is called the *strong 2-Weiss condition* for  $A$  in this paper. We show that  $C$  is  $L^2$ -infinite-time admissible if  $C$  satisfies the strong 2-Weiss condition and is  $L^2$ -finite-time admissible, in whose definition the integral over  $[0, \infty)$  in (1) is replaced by an integral over some finite interval  $[0, t_1]$ . We also examine the relation between the strong 2-Weiss condition and the decay rate of the form  $\|CT(t)\| = O(1/\sqrt{t^{1+\beta}})$  for some  $\beta > 0$  when  $Y = X$  and  $T(t)$  commutes with  $C$  for all  $t \geq 0$ .

The paper is organized as follows. In Section 2, we recall some basics on the polynomial decay of semigroup orbits and Plancherel's theorem. In Section 3, polynomial rates of orbit decay are characterized by the Weiss condition in the Hilbert space setting and by infinite-time admissibility in the Banach space setting. In Section 4, we establish an analogous resolvent characterization of the property  $\|T(t)(-A)^{-\alpha}\| = O(t^{-\beta}(\log t)^{-\gamma})$ , where  $\alpha > 0$ ,  $0 \leq \beta < 1$ , and  $\gamma \geq 0$ . In Section 5, we obtain a sufficient condition for  $L^2$ -infinite-time admissibility by using the decay estimate of  $\|T(t)(-A)^{-\alpha}\|$ .

## Notation

Let  $\mathbb{C}_+$ ,  $\mathbb{C}_-$ , and  $i\mathbb{R}$  denote the open right half-plane  $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}$ , the open left half-plane  $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\}$ , and the imaginary axis  $\{i\eta : \eta \in \mathbb{R}\}$ , respectively. Let  $X$  and  $Y$  be Banach spaces. The space of all bounded linear operators from  $X$  to  $Y$  is denoted by  $\mathcal{L}(X, Y)$ . We write  $\mathcal{L}(X) := \mathcal{L}(X, X)$ . The domain and the range of a linear operator  $A: X \rightarrow Y$  are denoted by  $D(A)$  and  $\operatorname{ran}(A)$ , respectively. When  $A$  is closed,  $D(A)$  is seen as a Banach space with the graph norm  $\|x\|_A := \|x\| + \|Ax\|$ . We denote by  $\sigma(A)$  and  $\varrho(A)$  the spectrum and the resolvent set of a linear operator  $A: D(A) \subset X \rightarrow X$ , respectively. We write  $R(\lambda, A) := (\lambda I - A)^{-1}$  for  $\lambda \in \varrho(A)$ . For linear operators  $A: D(A) \subset X \rightarrow X$  and  $B: D(B) \subset X \rightarrow X$ , we write  $A \subset B$  if  $D(A) \subset D(B)$  and  $Ax = Bx$  for all  $x \in D(A)$ , and the composite  $BA$  is defined by  $BAx := B(Ax)$  with domain  $D(BA) := \{x \in D(A) : Ax \in D(B)\}$ . Let  $\langle \cdot, \cdot \rangle$  denote the inner product in a Hilbert space.

## 2. Preliminaries

### 2.1. Fractional powers and polynomial decay

Let  $X$  be a Banach space. For a linear operator  $A: D(A) \subset X \rightarrow X$  and  $T \in \mathcal{L}(X)$ , we say that  $T$  commutes with  $A$  if  $TA \subset AT$ . We say that a densely defined, linear operator  $A$  on  $X$  is *sectorial* if  $(-\infty, 0) \subset \varrho(A)$  and if there exists  $K > 0$  such that  $\|\lambda(\lambda + A)^{-1}\| \leq K$  for all  $\lambda > 0$ . We refer to [19, Section 2.1] for background information on sectorial operators.

Let  $(T(t))_{t \geq 0}$  be a bounded  $C_0$ -semigroup on a Banach space  $X$  with generator  $A$ . The negative generator  $-A$  is sectorial, and the fractional power  $(-A)^\alpha$  is well defined for each  $\alpha > 0$ . If  $0 \in \varrho(A)$ , then  $-A^{-1}$  is also sectorial and for all  $\alpha \geq 0$ ,  $(-A)^{-\alpha}$  is well defined and satisfies  $(-A)^{-\alpha} = (-A^{-1})^\alpha = ((-A)^\alpha)^{-1}$ . Since  $R(\lambda, A)$  and  $T(t)$  commute with  $A$  for all  $\lambda \in \varrho(A)$  and  $t \geq 0$ , they also commute with  $(-A)^\alpha$  for all  $\alpha \in \mathbb{R}$ ; see, e.g., Propositions 3.1.1.f) and 3.2.1.a) of [19]. We shall use this commutative property frequently without comment. Most properties of fractional powers of sectorial operators needed in this paper can be found in [19, Chapter 3].

For a fixed  $\alpha > 0$ , the decay rate of  $\|T(t)(-A)^{-\alpha}\|$  is linked to that of  $\|T(t)A^{-1}\|$  in the Banach space setting as shown in [4, Proposition 3.1]. We here present it in a general form. The proof can be found in [6, Lemma 4.2].

**Proposition 2.1.** *Let  $(T(t))_{t \geq 0}$  be a bounded  $C_0$ -semigroup on a Banach space  $X$ , and let  $B \in \mathcal{L}(X)$  be a sectorial operator such that  $T(t)$  commutes with  $B$  for all  $t \geq 0$ . For fixed  $\alpha, \beta > 0$ , the following statements are equivalent:*

- (i)  $\|T(t)B\| = O(t^{-1/\alpha})$  as  $t \rightarrow \infty$ .
- (ii)  $\|T(t)B^\beta\| = O(t^{-\beta/\alpha})$  as  $t \rightarrow \infty$ .

Polynomial rates of decay of bounded  $C_0$ -semigroups on Hilbert spaces can be characterized by uniform boundedness of the resolvent on  $\mathbb{C}_+$ ; see Lemma 2.3 and Theorem 2.4 of [7] for the proof.

**Theorem 2.2.** *Let  $(T(t))_{t \geq 0}$  be a bounded  $C_0$ -semigroup on a Hilbert space  $X$  with generator  $A$  such that  $i\mathbb{R} \subset \varrho(A)$ . For a fixed  $\alpha > 0$ , the following statements are equivalent:*

- (i)  $\|T(t)A^{-1}\| = O(t^{-1/\alpha})$  as  $t \rightarrow \infty$ .
- (ii) *There exists  $M > 0$  such that  $\|R(\lambda, A)(-A)^{-\alpha}\| \leq M$  for all  $\lambda \in \mathbb{C}_+$ .*

## 2.2. Plancherel's theorem

Let  $X$  and  $Y$  be Hilbert spaces. Let  $(T(t))_{t \geq 0}$  be a bounded  $C_0$ -semigroup on  $X$  with generator  $A$ , and let  $C \in \mathcal{L}(D(A), Y)$ . For fixed  $x \in X$ ,  $y \in D(A)$ , and  $\xi > 0$ , define the functions  $f: \mathbb{R} \rightarrow X$  and  $g: \mathbb{R} \rightarrow Y$  by

$$f(t) := \begin{cases} e^{-\xi t} T(t)x, & t \geq 0, \\ 0, & t < 0, \end{cases} \quad g(t) := \begin{cases} te^{-\xi t} CT(t)y, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Their Fourier transforms  $\mathcal{F}f$  and  $\mathcal{F}g$  are given by  $(\mathcal{F}f)(\eta) = R(\xi + i\eta, A)x$  and  $(\mathcal{F}g)(\eta) = CR(\xi + i\eta)^2 y$  for  $\eta \in \mathbb{R}$ . Applying Hilbert-space-valued Plancherel's theorem (see, e.g., [1, Theorem 1.8.2] and [16, Theorem C.14]) to  $f$  and  $g$ , we obtain the next result, which is also called Plancherel's theorem throughout this paper.

**Theorem 2.3.** *Let  $X$  and  $Y$  be Hilbert spaces. Let  $(T(t))_{t \geq 0}$  be a bounded  $C_0$ -semigroup on a Hilbert space  $X$  with generator  $A$ , and let  $C \in \mathcal{L}(D(A), Y)$ . For all  $x \in X$ ,  $y \in D(A)$ , and  $\xi > 0$ ,*

$$\int_0^\infty \|e^{-\xi t} T(t)x\|^2 dt = \frac{1}{2\pi} \int_{-\infty}^\infty \|R(\xi + i\eta, A)x\|^2 d\eta,$$

$$\int_0^\infty \|te^{-\xi t} CT(t)y\|^2 dt = \frac{1}{2\pi} \int_{-\infty}^\infty \|CR(\xi + i\eta, A)^2 y\|^2 d\eta.$$

## 3. Polynomial decay of semigroup orbits

In this section, we study the relation of the Weiss condition and infinite-time admissibility to polynomial rates of decay of semigroup orbits.

### 3.1. Decay of Hilbert space semigroups and the Weiss condition

The  $p$ -Weiss condition on  $(-A)^{-\alpha/p}$  characterizes the decay of  $\|T(t)A^{-1}\|$  in the Hilbert space context, which is the main result of this section. The case  $p = 1$  was obtained in Lemma 2.3 and Theorem 2.4 of [7], which is stated as Theorem 2.2 in this paper.

**Theorem 3.1.** *Let  $(T(t))_{t \geq 0}$  be a bounded  $C_0$ -semigroup on a Hilbert space  $X$  with generator  $A$  such that  $0 \in \varrho(A)$ . For fixed  $\alpha > 0$  and  $1 \leq p < \infty$ , the following statements are equivalent:*

- (i)  $\|T(t)A^{-1}\| = O(t^{-1/\alpha})$  as  $t \rightarrow \infty$ .
- (ii)  $(-A)^{-\alpha/p}$  satisfies the  $p$ -Weiss condition for  $A$ .

Before proving Theorem 3.1, we present related results. As a corollary of Theorem 3.1, we separately state an interpolation property for the generator of a bounded  $C_0$ -semigroup on a Hilbert space, which is the resolvent analogue of Proposition 2.1 with  $B := (-A)^{-1}$ .

**Corollary 3.2.** *Let  $(T(t))_{t \geq 0}$  be a bounded  $C_0$ -semigroup on a Hilbert space  $X$  with generator  $A$  such that  $0 \in \varrho(A)$ . For fixed  $\alpha > \beta > 0$ , the following statements are equivalent:*

- (i) There exists  $M_\alpha > 0$  such that  $\|R(\lambda, A)(-A)^{-\alpha}\| \leq M_\alpha$  for all  $\lambda \in \mathbb{C}_+$ .
- (ii) There exists  $M_{\alpha, \beta} > 0$  such that

$$\|R(\lambda, A)(-A)^{-\beta}\| \leq \frac{M_{\alpha, \beta}}{(\operatorname{Re} \lambda)^{1-\beta/\alpha}}$$

for all  $\lambda \in \mathbb{C}_+$ .

Theorem 3.1 provides a resolvent characterization of rates of decay for the orbits  $T(t)x$  with  $x \in D(A)$ . We will give analogous results on decay rates for the orbits  $T(t)x$  with  $x \in \operatorname{ran}(A)$  and  $x \in D(A) \cap \operatorname{ran}(A)$ . We begin by presenting some background materials on these orbits.

In the case  $x \in \operatorname{ran}(A)$ , we are interested in the orbits of the form  $T(t)Ay$  for  $y \in D(A)$  and hence study the decay rate of  $\|T(t)A(I - A)^{-1}\|$ . Since  $D(A) \cap \operatorname{ran}(A) = \operatorname{ran}(A(I - A)^{-2})$  by [6, Proposition 3.10], the problem we address in the case  $x \in D(A) \cap \operatorname{ran}(A)$  is to quantify the decay of  $\|T(t)A(I - A)^{-2}\|$ . By [6, Lemma 3.2],  $-A(I - A)^{-1}$  and  $-A(I - A)^{-2}$  are sectorial. Therefore, the fractional powers  $(-A(I - A)^{-1})^\alpha$  and  $(-A(I - A)^{-2})^\alpha$  are well defined for all  $\alpha > 0$ . Using the product and composition rules (see, e.g., Theorem 3.7 (iv) and Remark 3.8 (iv) of [6]), we obtain

$$(-A(I - A)^{-k})^\alpha = (-A)^\alpha (I - A)^{-k\alpha}$$

for each  $\alpha > 0$  and  $k \in \{1, 2\}$ .

When  $0 \in \varrho(A)$ , the decay of  $\|T(t)A(I - A)^{-1}\|$  to zero is equivalent to exponential stability of  $(T(t))_{t \geq 0}$ , and the decay rate of  $\|T(t)A(I - A)^{-2}\|$  is the same as that of  $\|T(t)A^{-1}\|$ . Therefore, we assume that  $0 \in \sigma(A)$ . Moreover, we concentrate on the situation where  $\|T(t)A(I - A)^{-1}\|$  and  $\|T(t)A(I - A)^{-2}\|$  decay no faster than  $t^{-1}$  as  $t \rightarrow \infty$ . If not, 0 is the eigenvalue of  $A$  and an isolated point of  $\sigma(A)$ , which is not in our interest; see Theorem 6.9 and a paragraph before Theorem 8.1 of [6] for details.

We provide a characterization of decay rates for  $\|T(t)A(I - A)^{-1}\|$  and  $\|T(t)A(I - A)^{-2}\|$  by the  $p$ -Weiss condition. This characterization in the case  $p = 1$  was obtained in Corollary 7.5 and Theorem 7.6 of [6] for  $\|T(t)A(I - A)^{-1}\|$  and in Theorem 8.4 and its proof of [6] for  $\|T(t)A(I - A)^{-2}\|$ . The proof will be given after that of Theorem 3.1.

**Theorem 3.3.** Let  $(T(t))_{t \geq 0}$  be a bounded  $C_0$ -semigroup on a Hilbert space  $X$  with generator  $A$  such that  $0 \in \sigma(A)$ . For fixed  $\alpha \geq 1$ ,  $1 \leq p < \infty$ , and  $k \in \{1, 2\}$ , the following statements are equivalent:

- (i)  $\|T(t)A(I - A)^{-k}\| = O(t^{-1/\alpha})$  as  $t \rightarrow \infty$ .
- (ii)  $(-A)^{\alpha/p}(I - A)^{-k\alpha/p}$  satisfies the  $p$ -Weiss condition for  $A$ .

Let us now turn to the proof of Theorem 3.1. First we describe how polynomial rates of decay of  $C_0$ -semigroups can be transferred to resolvent growth on  $\mathbb{C}_+$  in the Banach space setting. The following result is obtained by a slight modification of the proof of the implication (ii)  $\Rightarrow$  (i) of [9, Lemma 2.3]. Although analyticity of  $C_0$ -semigroups is assumed in [9, Lemma 2.3], the implication (ii)  $\Rightarrow$  (i) there can be proved only under a suitable boundedness condition. Recall that  $1/p := 0$  for  $p = \infty$ .

**Lemma 3.4.** Let  $X$  and  $Y$  be Banach spaces. Let  $(T(t))_{t \geq 0}$  be a bounded  $C_0$ -semigroup on  $X$  with generator  $A$ , and let  $C \in \mathcal{L}(D(A), Y)$  be such that  $CT(t)$  extends to an operator (also denoted by  $CT(t)$ ) in  $\mathcal{L}(X, Y)$  for all  $t > 0$ . If there exist  $M > 0$  and  $1 < p \leq \infty$  such that

$$\|CT(t)\| \leq \frac{M}{t^{1/p}}$$

for all  $t > 0$ , then  $C$  satisfies the  $p$ -Weiss condition for  $A$ .

Next we provide a converse result, i.e., transference from resolvents to semigroups, in the Hilbert space setting. A key technique for the proof is contained in the proof of [6, Theorem 4.7] and the paragraph after [6, Remark 4.8], which considered the case  $F(\xi) \equiv K > 0$  in the proposition below.

**Proposition 3.5.** Let  $(T(t))_{t \geq 0}$  be a bounded  $C_0$ -semigroup on a Hilbert space  $X$  with generator  $A$ . Let  $C \in \mathcal{L}(D(A), X)$  be such that  $T(t)$  commute with  $C$  for all  $t \geq 0$ . Assume that  $F: (0, \infty) \rightarrow [0, \infty)$  satisfies

$$\|CR(\lambda, A)\| \leq F(\operatorname{Re} \lambda) \quad (3)$$

for all  $\lambda \in \mathbb{C}_+$ . Then the operator  $CT(t)$  extends to a bounded linear operator (also denoted by  $CT(t)$ ) on  $X$  for all  $t > 0$ , and there exists  $M > 0$  such that

$$\|CT(t)\| \leq M \frac{F(1/t)}{t}$$

for all  $t > 0$ .

**Proof.** Let  $x \in D(A)$  be given. Using Plancherel's theorem, we obtain

$$\int_0^\infty \|te^{-\xi t}CT(t)x\|^2 dt = \frac{1}{2\pi} \int_{-\infty}^\infty \|CR(\xi + i\eta, A)^2 x\|^2 d\eta \quad (4)$$

for all  $\xi > 0$ . By the resolvent estimate (3),

$$\begin{aligned} \|CR(\lambda, A)^2 x\|^2 &\leq \|CR(\lambda, A)\|^2 \|R(\lambda, A)x\|^2 \\ &\leq F(\operatorname{Re} \lambda)^2 \|R(\lambda, A)x\|^2 \end{aligned}$$

for all  $\lambda \in \mathbb{C}_+$ . Therefore,



$$\int_{-\infty}^{\infty} \|CR(\xi + i\eta, A)^2 x\|^2 d\eta \leq F(\xi)^2 \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)x\|^2 d\eta \quad (5)$$

for all  $\xi > 0$ . Using Plancherel's theorem again, we have

$$\begin{aligned} \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)x\|^2 d\eta &= 2\pi \int_0^{\infty} \|e^{-\xi t} T(t)x\|^2 dt \\ &\leq \frac{\pi c^2}{\xi} \|x\|^2 \end{aligned} \quad (6)$$

for all  $\xi > 0$ , where  $c := \sup_{t \geq 0} \|T(t)\|$ . Combining (4)–(6), we obtain

$$\int_0^{\infty} \|te^{-\xi t} CT(t)x\|^2 dt \leq \frac{c^2 F(\xi)^2}{2\xi} \|x\|^2 \quad (7)$$

for all  $\xi > 0$ . Since

$$e^{-2} \|tCT(t)x\|^2 \leq \|te^{-t/\tau} CT(t)x\|^2$$

for  $0 \leq t \leq \tau$ , the estimate (7) with  $\xi := 1/\tau$  yields

$$\begin{aligned} \int_0^{\tau} \|tCT(t)x\|^2 dt &\leq e^2 \int_0^{\tau} \|te^{-t/\tau} CT(t)x\|^2 dt \\ &\leq \frac{e^2 c^2}{2} \tau F(1/\tau)^2 \|x\|^2 \end{aligned}$$

for all  $\tau > 0$ .

Fix  $\tau > 0$ . Since

$$CT(\tau)x = \frac{2}{\tau^2} \int_0^{\tau} tT(\tau - t)CT(t)x dt,$$

the Cauchy-Schwarz inequality implies that for all  $y \in X$ ,

$$\begin{aligned} |\langle CT(\tau)x, y \rangle| &= \frac{2}{\tau^2} \left| \left\langle \int_0^{\tau} tT(\tau - t)CT(t)x dt, y \right\rangle \right| \\ &= \frac{2}{\tau^2} \left| \int_0^{\tau} \langle tCT(t)x, T(\tau - t)^* y \rangle dt \right| \\ &\leq \frac{2}{\tau^2} \sqrt{\int_0^{\tau} \|tCT(t)x\|^2 dt} \sqrt{\int_0^{\tau} \|T(\tau - t)^* y\|^2 dt} \\ &\leq \sqrt{2} ec^2 \frac{F(1/\tau)}{\tau} \|x\| \|y\|, \end{aligned}$$

where  $T(t)^*$  is the Hilbert space adjoint of  $T(t)$  for  $t \geq 0$ . Since  $D(A)$  is dense in  $X$ , it follows that  $CT(\tau)$  extends to a bounded linear operator on  $X$  with norm at most  $\sqrt{2}ec^2F(1/\tau)/\tau$ .  $\square$

Let  $K > 0$  and  $1 \leq p < \infty$ . If the function  $F$  in Proposition 3.5 is defined by

$$F(\xi) := \frac{K}{\xi^{1-1/p}} \quad (8)$$

for  $\xi > 0$ , then the resolvent estimate (3) coincides with that in the  $p$ -Weiss condition on  $C$ . Since

$$\frac{F(1/t)}{t} = \frac{K}{t^{1/p}},$$

Proposition 3.5 shows that if  $C$  satisfies the  $p$ -Weiss condition, then

$$\|CT(t)\| \leq \frac{M}{t^{1/p}}$$

for all  $t > 0$  and some  $M > 0$ .

**Proof of Theorem 3.1.** If the statement (i) holds, then we see from [5, Theorem 1.1] that  $i\mathbb{R} \subset \varrho(A)$ , and therefore the implication (i)  $\Rightarrow$  (ii) is true in the case  $p = 1$  by Theorem 2.2. By Proposition 2.1, the statement (i) is true if and only if  $\|T(t)(-A)^{-\alpha/p}\| = O(t^{-1/p})$  as  $t \rightarrow \infty$ . Hence, the implication (i)  $\Rightarrow$  (ii) in the case  $1 < p < \infty$  and the implication (ii)  $\Rightarrow$  (i) follow immediately from Lemma 3.4 and Proposition 3.5 with  $C := (-A)^{-\alpha/p}$ , respectively.  $\square$

**Proof of Theorem 3.3.** For each  $k \in \{1, 2\}$ , if the statement (i) holds, then  $\sigma(A) \cap i\mathbb{R} = \{0\}$  by [6, Corollary 6.2]. Hence the implication (i)  $\Rightarrow$  (ii) in the case  $p = 1$  is true for  $k = 1$  by Corollary 7.5 and Theorem 7.6 of [6] and for  $k = 2$  by Theorem 8.4 and its proof of [6]. Since Proposition 2.1 shows that the statement (i) is equivalent to

$$\|T(t)(-A)^{\alpha/p}(I - A)^{-k\alpha/p}\| = O\left(\frac{1}{t^{1/p}}\right) \quad \text{as } t \rightarrow \infty$$

for each  $k \in \{1, 2\}$ , the rest follows from Lemma 3.4 and Proposition 3.5 with

$$C := (-A)^{\alpha/p}(I - A)^{-k\alpha/p}$$

as in the proof of Theorem 3.1.  $\square$

### 3.2. Decay of Banach space semigroups and infinite-time admissibility

Next, we study the relation between polynomial decay and  $L^p$ -infinite-time admissibility. We have seen in Proposition 3.5 that the  $p$ -Weiss condition on  $C \in \mathcal{L}(D(A), X)$  implies  $\|CT(t)\| = O(t^{-1/p})$  in the Hilbert space setting if  $T(t)$  commutes with  $C$  for all  $t \geq 0$ . We obtain a similar result from the  $L^p$ -infinite-time admissibility of  $C$ , which is a stronger property than the  $p$ -Weiss condition, in the Banach space setting. This follows from the technique used in the proof given in [33] for Datko's theorem. Parts of the next result was proved in a slightly less general form (and with a slightly different proof) in [47, Theorem 2.5].

**Proposition 3.6.** *Let  $(T(t))_{t \geq 0}$  be a  $C_0$ -semigroup on a Banach space  $X$  with generator  $A$ , and let  $C \in \mathcal{L}(D(A), X)$  be such that  $T(t)$  commutes with  $C$  for all  $t \geq 0$ . Let  $1 \leq p < \infty$  and assume that  $C$  is  $L^p$ -infinite-time admissible for  $A$ . Then the following statements are true:*

- a) For all  $t > 0$ ,  $CT(t)$  extends to a bounded linear operator (also denoted by  $CT(t)$ ) on  $X$ . Moreover, for all  $t_0 > 0$ , there exists  $M > 0$  such that

$$\|CT(t)\| \leq M$$

for all  $t \geq t_0$ .

- b) If  $(T(t))_{t \geq 0}$  is a bounded  $C_0$ -semigroup, then there exists  $M > 0$  such that

$$\|CT(t)\| \leq \frac{M}{t^{1/p}}$$

for all  $t > 0$ .

**Proof.** By the  $L^p$ -infinite-time admissibility of  $C$ , there exists a constant  $M_1 > 0$  such that

$$\int_0^\infty \|CT(\tau)x\|^p d\tau \leq M_1^p \|x\|^p$$

for all  $x \in D(A)$ .

- a) There exists  $M_2 \geq 1$  and  $w > 0$  such that

$$\|T(t)\| \leq M_2 e^{wt}$$

for all  $t \geq 0$ . We have

$$\begin{aligned} \int_0^t e^{-pw\tau} d\tau \|CT(t)x\|^p &\leq \int_0^t e^{-pw\tau} \|T(\tau)CT(t-\tau)x\|^p d\tau \\ &\leq M_2^p \int_0^t \|CT(t-\tau)x\|^p d\tau \\ &\leq M_1^p M_2^p \|x\|^p \end{aligned}$$

for all  $t \geq 0$  and  $x \in D(A)$ . Therefore,

$$\|CT(t)x\| \leq \left( \frac{pw}{1 - e^{-pwt}} \right)^{1/p} M_1 M_2 \|x\|$$

for all  $t > 0$  and  $x \in D(A)$ . For all  $t > 0$ , the density of  $D(A)$  implies that  $CT(t)$  extends to a bounded linear operator on  $X$ , again denoted by  $CT(t)$ . We also have

$$\|CT(t)\| \leq \left( \frac{pw}{1 - e^{-pwt_0}} \right)^{1/p} M_1 M_2$$

for all  $t \geq t_0 > 0$ .

- b) By assumption, we have

$$c := \sup_{t \geq 0} \|T(t)\| < \infty.$$

For all  $t \geq 0$  and  $x \in D(A)$ ,

$$\begin{aligned} t\|CT(t)x\|^p &= \int_0^t \|T(t-\tau)CT(\tau)x\|^p d\tau \\ &\leq c^p \int_0^\infty \|CT(\tau)x\|^p d\tau \\ &\leq c^p M_1^p \|x\|^p. \end{aligned}$$

From the density of  $D(A)$ , it follows that  $CT(t)$  satisfies

$$\|CT(t)\| \leq \frac{cM_1}{t^{1/p}}$$

for all  $t > 0$ .  $\square$

We obtain the following proposition from a simple argument.

**Proposition 3.7.** *Let  $(T(t))_{t \geq 0}$  be a bounded  $C_0$ -semigroup on a Banach space  $X$  with generator  $A$  such that  $0 \in \varrho(A)$ . For fixed  $\alpha > 0$  and  $1 \leq p < \infty$ , the following statements are true:*

- a) *If  $(-A)^{-\alpha/p}$  is  $L^p$ -infinite-time admissible for  $A$ , then  $\|T(t)A^{-1}\| = O(t^{-1/\alpha})$  as  $t \rightarrow \infty$ .*
- b) *If  $\|T(t)A^{-1}\| = O(t^{-1/\alpha})$  as  $t \rightarrow \infty$ , then  $(-A)^{-\alpha/p-\varepsilon}$  with  $\varepsilon > 0$  is  $L^p$ -infinite-time admissible for  $A$ .*

**Proof.** a) This follows immediately from Proposition 3.6.b) with  $C := (-A)^{-\alpha/p}$  and Proposition 2.1.

b) Let  $\beta > \alpha/p$ . By Proposition 2.1, there exists a constant  $M > 0$  such that

$$\|T(t)(-A)^{-\beta}\| \leq \frac{M}{(1+t)^{\beta/\alpha}}$$

for all  $t \geq 0$ . Since  $p\beta/\alpha > 1$ , it follows that for all  $x \in X$ ,

$$\begin{aligned} \int_0^\infty \|(-A)^{-\beta}T(t)x\|^p dt &\leq \int_0^\infty \frac{M^p \|x\|^p}{(1+t)^{p\beta/\alpha}} dt \\ &\leq \frac{M^p \|x\|^p}{p\beta/\alpha - 1}. \end{aligned}$$

Hence  $(-A)^{-\beta}$  is infinite-time admissible for  $A$ .  $\square$

It remains open whether  $\|T(t)A^{-1}\| = O(t^{-1/\alpha})$  implies the  $L^p$ -infinite-time admissibility of  $(-A)^{\alpha/p}$  for all bounded  $C_0$ -semigroups  $(T(t))_{t \geq 0}$  on Banach spaces. However, it is true for multiplication  $C_0$ -semigroups on  $L^q$ -spaces with  $1 \leq q \leq p < \infty$ .

**Theorem 3.8.** *Let  $\mu$  be a  $\sigma$ -finite regular Borel measure on a locally compact Hausdorff space  $\Omega$ . Let  $\phi: \Omega \rightarrow \mathbb{C}$  be measurable with essential range  $\phi_{\text{ess}}(\Omega)$  in  $\mathbb{C}_-$ . Assume that  $A$  is the multiplication operator induced by  $\phi$  on  $L^q(\Omega, \mu)$  for a fixed  $1 \leq q < \infty$ , i.e.,  $Af = \phi f$  with domain  $D(A) := \{f \in L^q(\Omega, \mu) : \phi f \in L^q(\Omega, \mu)\}$ , and let  $(T(t))_{t \geq 0}$  be the  $C_0$ -semigroup on  $L^q(\Omega, \mu)$  generated by  $A$ . Then the following statements are equivalent for fixed  $\alpha > 0$  and  $p \in [q, \infty)$ :*

- (i)  $\|T(t)A^{-1}\| = O(t^{-1/\alpha})$  as  $t \rightarrow \infty$ .
- (ii)  $(-A)^{-\alpha/p}$  is  $L^p$ -infinite-time admissible for  $A$ .

**Proof.** We first note that  $\phi(z) \in \phi_{\text{ess}}(\Omega)$  for almost all  $z \in \Omega$ ; see, e.g., Exercise 19 of Chapter VII of [28]. Since  $\phi_{\text{ess}}(\Omega) \subset \mathbb{C}_-$  by assumption, it follows that  $(T(t))_{t \geq 0}$  is a bounded  $C_0$ -semigroup. The implication (ii)  $\Rightarrow$  (i) has been shown in Proposition 3.7. Therefore, we here prove the implication (i)  $\Rightarrow$  (ii).

For all  $t \geq 0$  and  $f \in L^q(\Omega, \mu)$ ,

$$\begin{aligned} \|T(t)(-A)^{-\alpha/p}f\|^p &= \|e^{t\phi}(-\phi)^{-\alpha/p}f\|^p \\ &= \left( \int_{\Omega} \frac{e^{qt \operatorname{Re} \phi(z)}}{|\phi(z)|^{q\alpha/p}} |f(z)|^q d\mu(z) \right)^{p/q}. \end{aligned} \quad (9)$$

Since  $p \geq q$ , Jensen's inequality (see, e.g., Theorem 3.3 of [36]) implies that

$$\begin{aligned} \left( \int_{\Omega} \frac{e^{qt \operatorname{Re} \phi(z)}}{|\phi(z)|^{q\alpha/p}} \frac{|f(z)|^q}{\|f\|^q} d\mu(z) \right)^{p/q} &\leq \int_{\Omega} \left( \frac{e^{qt \operatorname{Re} \phi(z)}}{|\phi(z)|^{q\alpha/p}} \right)^{p/q} \frac{|f(z)|^q}{\|f\|^q} d\mu(z) \\ &= \int_{\Omega} \frac{e^{pt \operatorname{Re} \phi(z)}}{|\phi(z)|^{\alpha}} \frac{|f(z)|^q}{\|f\|^q} d\mu(z). \end{aligned} \quad (10)$$

By the assumption  $\phi_{\text{ess}}(\Omega) \subset \mathbb{C}_-$ , we obtain

$$\int_0^{\infty} \frac{e^{pt \operatorname{Re} \phi(z)}}{|\phi(z)|^{\alpha}} dt = \frac{1}{p |\operatorname{Re} \phi(z)| |\phi(z)|^{\alpha}} \quad (11)$$

for almost all  $z \in \Omega$ . Since  $\|T(t)A^{-\alpha}\| = O(t^{-1})$  as  $t \rightarrow \infty$  by Proposition 2.1, it follows from [4, Proposition 4.2] that there exist  $\Upsilon, \kappa > 0$  such that

$$\frac{1}{|\operatorname{Re} \lambda|} \leq \Upsilon |\operatorname{Im} \lambda|^{\alpha}$$

for all  $\lambda \in \sigma(A)$  satisfying  $|\operatorname{Re} \lambda| \leq \kappa$ . Therefore, if  $\lambda \in \sigma(A)$  satisfies  $|\operatorname{Re} \lambda| \leq \kappa$ , then

$$\frac{1}{|\operatorname{Re} \lambda| |\lambda|^{\alpha}} \leq \frac{\Upsilon |\operatorname{Im} \lambda|^{\alpha}}{|\lambda|^{\alpha}} \leq \Upsilon.$$

If  $\lambda \in \sigma(A)$  satisfies  $|\operatorname{Re} \lambda| > \kappa$ , then

$$\frac{1}{|\operatorname{Re} \lambda| |\lambda|^{\alpha}} \leq \frac{1}{\kappa^{1+\alpha}}.$$

Define

$$M := \max \left\{ \Upsilon, \frac{1}{\kappa^{1+\alpha}} \right\}.$$

Since  $\phi_{\text{ess}}(\Omega) = \sigma(A)$  by [16, Proposition I.4.10], we have that  $\phi(z) \in \sigma(A)$  for almost all  $z \in \Omega$ . Hence

$$\frac{1}{|\operatorname{Re} \phi(z)| |\phi(z)|^\alpha} \leq M$$

for almost all  $z \in \Omega$ . This estimate and (11) yield

$$\int_{\Omega} \int_0^{\infty} \frac{e^{pt \operatorname{Re} \phi(z)}}{|\phi(z)|^\alpha} \frac{|f(z)|^q}{\|f\|^q} dt d\mu(z) \leq \frac{M}{p} \int_{\Omega} \frac{|f(z)|^q}{\|f\|^q} d\mu(z) = \frac{M}{p}. \quad (12)$$

Combining Fubini's theorem with (9), (10), and (12), we derive

$$\int_0^{\infty} \|T(t)(-A)^{-\alpha/p} f\|^p dt \leq \frac{M}{p} \|f\|^p.$$

Thus,  $(-A)^{-\alpha/p}$  is  $L^p$ -infinite-time admissible for  $A$ .  $\square$

### 3.3. Example

We present a simple example, for which the estimates for  $\|T(t)A^{-1}\|$  in the statement (i) of Theorems 3.1 and 3.8 are optimal in the sense that the rate of decay cannot be improved.

Let  $X = \ell^2(\mathbb{N})$  and define an operator  $A: D(A) \subset X \rightarrow X$  by

$$Ax := \left( \left( -\frac{1}{n} + in \right) x_n \right)_{n \in \mathbb{N}}$$

for

$$x = (x_n)_{n \in \mathbb{N}} \in D(A) := \{\zeta = (\zeta_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}) : (n\zeta_n) \in \ell^2(\mathbb{N})\}.$$

The semigroup  $(T(t))_{t \geq 0}$  generated by  $A$  is given by

$$T(t)x = \left( e^{(-\frac{1}{n} + in)t} x_n \right)_{n \in \mathbb{N}}$$

for all  $t \geq 0$  and  $x = (x_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$ . Let  $\alpha > 0$ .

#### 3.3.1. Weiss condition

For all  $\lambda \in \mathbb{C}_+$ ,

$$\sqrt{\operatorname{Re} \lambda} \|(-A)^{-\alpha} R(\lambda, A)\| = \sup_{n \in \mathbb{N}} \frac{\sqrt{\operatorname{Re} \lambda}}{|1/n - in|^\alpha |\lambda + 1/n - in|}. \quad (13)$$

Moreover,

$$\frac{\sqrt{\operatorname{Re} \lambda}}{|1/n - in|^\alpha |\lambda + 1/n - in|} \leq \frac{\sqrt{\operatorname{Re} \lambda}}{n^\alpha (\operatorname{Re} \lambda + 1/n)} \quad (14)$$

for all  $\lambda \in \mathbb{C}_+$  and  $n \in \mathbb{N}$ . Fix  $c > 0$ , and define

$$f(\xi) := \frac{\sqrt{\xi}}{\xi + c}$$

for  $\xi > 0$ . Then

$$f'(\xi) = \frac{c - \xi}{2\sqrt{\xi}(\xi + c)^2}.$$

Therefore,  $f(\xi) \leq f(c) = 1/(2\sqrt{c})$  for all  $\xi > 0$ . Combining this with the estimate (14), we obtain

$$\frac{\sqrt{\operatorname{Re} \lambda}}{|1/n - in|^\alpha |\lambda + 1/n - in|} \leq \frac{n^{1/2-\alpha}}{2} \quad (15)$$

for all  $\lambda \in \mathbb{C}_+$  and  $n \in \mathbb{N}$ . On the other hand, since

$$|1/n - in|^\alpha \leq 2^{\alpha/2} n^\alpha,$$

it follows that for each  $n \in \mathbb{N}$ , the case  $\operatorname{Re} \lambda = 1/n$  and  $\operatorname{Im} \lambda = n$  gives

$$\frac{\sqrt{\operatorname{Re} \lambda}}{|1/n - in|^\alpha |\lambda + 1/n - in|} \geq \frac{n^{1/2-\alpha}}{2^{1+\alpha/2}}. \quad (16)$$

Substituting the estimates (15) and (16) into (13), we see that  $(-A)^{-\alpha}$  satisfies the 2-Weiss condition for  $A$  if and only if  $\alpha \geq 1/2$ .

### 3.3.2. Infinite-time admissibility

We directly obtain a similar result on  $L^2$ -infinite-time admissibility. Indeed, using the monotone convergence theorem, we have that for all  $x = (x_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$ ,

$$\begin{aligned} \int_0^\infty \|(-A)^{-\alpha} T(t)x\|^2 dt &= \int_0^\infty \sum_{n=1}^\infty \left| \left( -\frac{1}{n} + in \right)^{-\alpha} e^{(-\frac{1}{n} + in)t} x_n \right|^2 dt \\ &= \frac{1}{2} \sum_{n=1}^\infty \frac{n^{1+2\alpha}}{(1+n^4)^\alpha} |x_n|^2. \end{aligned}$$

Therefore,  $(-A)^{-\alpha}$  is  $L^2$ -infinite-time admissible for  $A$  if and only if  $\alpha \geq 1/2$ .

### 3.3.3. Polynomial decay rate

Fix  $c > 0$ . Then  $g(t) := te^{-t/c} \leq g(c) = ce^{-1}$  for  $t \geq 0$ . From this inequality, we obtain

$$t\|T(t)A^{-1}\| = \sup_{n \in \mathbb{N}} \frac{nte^{-\frac{t}{n}}}{\sqrt{1+n^4}} \leq \sup_{n \in \mathbb{N}} \frac{e^{-1}n^2}{\sqrt{1+n^4}} \leq e^{-1}$$

for all  $t \geq 0$ . Therefore,  $\|T(t)A^{-1}\| = O(1/t)$  as  $t \rightarrow \infty$ . We also have

$$n\|T(n)A^{-1}\| \geq \frac{e^{-1}n^2}{\sqrt{1+n^4}}$$

for all  $n \in \mathbb{N}$ , and hence

$$\liminf_{t \rightarrow \infty} t\|T(t)A^{-1}\| \geq e^{-1}.$$

This demonstrates the optimality of the rates of decay when the statement (ii) of Theorem 3.1 or 3.8 holds for  $p = 2$ .

#### 4. Non-polynomial decay of Hilbert space semigroups

Let  $X$  and  $Y$  be Hilbert spaces, and let  $(T(t))_{t \geq 0}$  be a bounded  $C_0$ -semigroup on  $X$  with generator  $A$ . By modifying [46, Theorem 4.3] slightly, we have that if  $C \in \mathcal{L}(D(A), Y)$  satisfies  $\sup_{\eta \in \mathbb{R}} \|CR(1 + i\eta, A)\| < \infty$  and

$$\|CR(\lambda, A)\| \leq \frac{K}{\sqrt{\operatorname{Re} \lambda} |\log \operatorname{Re} \lambda|} \quad (17)$$

for all  $\lambda \in \mathbb{C}_+$  with  $\operatorname{Re} \lambda \neq 1$  and some  $K > 0$ , then  $C$  is  $L^2$ -infinite-time admissible for  $A$ . We see from Proposition 3.5 that if  $Y = X$  and if  $T(t)$  commutes with  $C$  for all  $t \geq 0$ , then the estimate (17) implies

$$\|CT(t)\| \leq \frac{M}{\sqrt{t} |\log t|} \quad (18)$$

for all  $t > 0$  with  $t \neq 1$  and some  $M > 0$ . Note that the estimate (17) contains the condition on the decay rate of  $\|CR(\lambda, A)\|$  as  $\operatorname{Re} \lambda \rightarrow \infty$  in addition to the growth rate of  $\|CR(\lambda, A)\|$  as  $\operatorname{Re} \lambda \rightarrow 0+$ , since  $C$  may not belong to  $\mathcal{L}(X, Y)$ . Similarly, the estimate (18) includes the condition on the growth rate of  $\|CT(t)\|$  as  $t \rightarrow 0+$ . In this section, we assume that  $C \in \mathcal{L}(X, Y)$ , and then show how the decay estimate  $\|CT(t)\| = O(t^{-\beta}(\log t)^{-\gamma})$  as  $t \rightarrow \infty$  can be transferred to the growth estimate of  $\|CR(\lambda, A)\|$  as  $\operatorname{Re} \lambda \rightarrow 0+$ .

**Proposition 4.1.** *Let  $X$  and  $Y$  be Banach spaces, and let  $(T(t))_{t \geq 0}$  be a bounded  $C_0$ -semigroup on  $X$  with generator  $A$ . For fixed  $0 \leq \beta < 1$  and  $\gamma \geq 0$ , the following statements hold.*

a) *If  $C \in \mathcal{L}(X, Y)$  satisfies*

$$\|CT(t)\| = O\left(\frac{1}{t^\beta (\log t)^\gamma}\right)$$

*as  $t \rightarrow \infty$ , then there exists  $K > 0$  such that*

$$\|CR(\lambda, A)\| \leq \frac{K}{(\operatorname{Re} \lambda)^{1-\beta} |\log \operatorname{Re} \lambda|^\gamma}$$

*for all  $\lambda \in \mathbb{C}_+$  with  $\operatorname{Re} \lambda < 1$ .*

b) *If  $C \in \mathcal{L}(X, Y)$  satisfies*

$$\|CT(t)\| = O\left(\frac{1}{t(\log t)^\gamma}\right)$$

*as  $t \rightarrow \infty$ , then, for all  $\xi_0 \in (0, e^{-1})$ , there exists  $K > 0$  such that*

$$\|CR(\lambda, A)\| \leq KF_\gamma(\operatorname{Re} \lambda)$$

*for all  $\lambda \in \mathbb{C}_+$  with  $\operatorname{Re} \lambda < \xi_0$ , where*

$$F_\gamma(\xi) := \begin{cases} |\log \xi|^{1-\gamma}, & 0 \leq \gamma < 1, \\ \log |\log \xi|, & \gamma = 1, \\ 1, & \gamma > 1 \end{cases} \quad (19)$$

*for  $0 < \xi < e^{-1}$ .*



The following lemma will be useful in the proof of Proposition 4.1.

**Lemma 4.2.** For fixed  $0 \leq \beta < 1$  and  $\gamma \geq 0$ , the following statements hold:

a) For all  $t_0 > e^{\gamma/(1-\beta)}$ , there exists  $M > 0$  such that

$$\int_{t_0}^{\infty} \frac{e^{-\xi t}}{t^{\beta}(\log t)^{\gamma}} dt \leq \frac{M}{\xi^{1-\beta} |\log \xi|^{\gamma}}$$

for all  $0 < \xi < 1/t_0$ .

b) For all  $t_0 > e$ , there exists  $M > 0$  such that

$$\int_{t_0}^{\infty} \frac{e^{-\xi t}}{t(\log t)^{\gamma}} dt \leq M F_{\gamma}(\xi)$$

for all  $0 < \xi < 1/t_0$ , where  $F_{\gamma}$  is defined by (19).

**Proof.** a) Let  $t_0 > e^{\gamma/(1-\beta)}$  and  $0 < \xi < 1/t_0$ . Then

$$\int_{1/\xi}^{\infty} \frac{e^{-\xi t}}{t^{\beta}(\log t)^{\gamma}} dt \leq \frac{1}{(1/\xi)^{\beta} |\log \xi|^{\gamma}} \int_{1/\xi}^{\infty} e^{-\xi t} dt = \frac{e^{-1}}{\xi^{1-\beta} |\log \xi|^{\gamma}}. \quad (20)$$

We also have

$$\int_{t_0}^{1/\xi} \frac{e^{-\xi t}}{t^{\beta}(\log t)^{\gamma}} dt \leq \int_{t_0}^{1/\xi} \frac{1}{t^{\beta}(\log t)^{\gamma}} dt. \quad (21)$$

Integration by parts gives

$$\begin{aligned} \int_{t_0}^{1/\xi} \frac{1}{t^{\beta}(\log t)^{\gamma}} dt &= \frac{(1/\xi)^{1-\beta}}{(1-\beta) |\log \xi|^{\gamma}} - \frac{t_0^{1-\beta}}{(1-\beta)(\log t_0)^{\gamma}} + \frac{\gamma}{1-\beta} \int_{t_0}^{1/\xi} \frac{1}{t^{\beta}(\log t)^{\gamma+1}} dt \\ &\leq \frac{(1/\xi)^{1-\beta}}{(1-\beta) |\log \xi|^{\gamma}} + \frac{\gamma}{(1-\beta) \log t_0} \int_{t_0}^{1/\xi} \frac{1}{t^{\beta}(\log t)^{\gamma}} dt. \end{aligned} \quad (22)$$

Since  $t_0 > e^{\gamma/(1-\beta)}$  is equivalent to

$$\frac{\gamma}{(1-\beta) \log t_0} < 1,$$

the estimates (21) and (22) yield

$$\int_{t_0}^{1/\xi} \frac{e^{-\xi t}}{t^{\beta}(\log t)^{\gamma}} dt \leq \frac{M_0}{\xi^{1-\beta} |\log \xi|^{\gamma}}, \quad (23)$$

where

$$M_0 := \frac{1}{1-\beta} \left( 1 - \frac{\gamma}{(1-\beta) \log t_0} \right).$$

The assertion with  $M := M_0 + e^{-1}$  then follows from the estimates (20) and (23), noting that  $M$  is independent of  $\xi$ .

b) Let  $t_0 > e$  and  $0 < \xi < 1/t_0$ . Since

$$\int_{t_0}^{1/\xi} \frac{e^{-\xi t}}{t(\log t)^\gamma} dt \leq \int_{t_0}^{1/\xi} \frac{1}{t(\log t)^\gamma} dt = \int_{\log t_0}^{\log(1/\xi)} \frac{1}{\tau^\gamma} d\tau,$$

we obtain

$$\int_{t_0}^{1/\xi} \frac{e^{-\xi t}}{t(\log t)^\gamma} dt \leq \begin{cases} \frac{|\log \xi|^{1-\gamma} - (\log t_0)^{1-\gamma}}{1-\gamma}, & \gamma \neq 1, \\ \log |\log \xi| - \log(\log t_0), & \gamma = 1. \end{cases}$$

Define

$$G_\gamma(\xi) := \begin{cases} \frac{|\log \xi|^{1-\gamma}}{1-\gamma}, & 0 \leq \gamma < 1, \\ \log |\log \xi|, & \gamma = 1, \\ \frac{(\log t_0)^{1-\gamma}}{\gamma-1}, & \gamma > 1. \end{cases}$$

Then, instead of the estimate (23), we have

$$\int_{t_0}^{1/\xi} \frac{e^{-\xi t}}{t(\log t)^\gamma} dt \leq G_\gamma(\xi).$$

On the other hand, estimating as in (20) yields

$$\int_{1/\xi}^{\infty} \frac{e^{-\xi t}}{t(\log t)^\gamma} dt \leq \frac{e^{-1}}{|\log \xi|^\gamma} < \frac{e^{-1}}{(\log t_0)^\gamma}.$$

Since  $t_0 > e$ , the function  $F_\gamma$  defined by (19) satisfies  $\inf_{0 < \xi < 1/t_0} F_\gamma(\xi) > 0$  for all  $\gamma \geq 0$ . Thus, the desired estimate is obtained for some  $M > 0$  depending only on  $t_0$  and  $\gamma$ .  $\square$

**Proof of Proposition 4.1.** a) By assumption, there exist  $M_0 > 0$  and  $t_0 > e^{\gamma/(1-\beta)}$  such that

$$\|CT(t)\| \leq \frac{M_0}{t^\beta (\log t)^\gamma}$$

for all  $t \geq t_0$ . For all  $x \in X$  and  $\lambda \in \mathbb{C}_+$ ,

$$\|CR(\lambda, A)x\| \leq \int_0^\infty e^{-t \operatorname{Re} \lambda} \|CT(t)x\| dt.$$

Define  $c := \sup_{t \geq 0} \|CT(t)\|$ . For all  $x \in X$  and  $\lambda \in \mathbb{C}_+$ ,

$$\begin{aligned} \int_0^\infty e^{-t \operatorname{Re} \lambda} \|CT(t)x\| dt &\leq \int_0^{t_0} c\|x\| dt + \int_{t_0}^\infty e^{-t \operatorname{Re} \lambda} \frac{M_0\|x\|}{t^\beta (\log t)^\gamma} dt \\ &= \left( ct_0 + M_0 \int_{t_0}^\infty \frac{e^{-t \operatorname{Re} \lambda}}{t^\beta (\log t)^\gamma} dt \right) \|x\|. \end{aligned}$$

Hence by Lemma 4.2.a), there exists  $K_0 > 0$  such that

$$\|CR(\lambda, A)\| \leq \frac{K_0}{(\operatorname{Re} \lambda)^{1-\beta} |\log \operatorname{Re} \lambda|^\gamma}$$

for all  $\lambda \in \mathbb{C}_+$  with  $\operatorname{Re} \lambda < 1/t_0$ . Combining this with the Hille-Yosida theorem, we obtain the desired estimate for all  $\lambda \in \mathbb{C}_+$  with  $\operatorname{Re} \lambda < 1$  and some  $K > 0$ .

b) By the same argument as above, we have that the statement b) is true, using Lemma 4.2.b) instead of Lemma 4.2.a).  $\square$

Let  $(T(t))_{t \geq 0}$  be a bounded  $C_0$ -semigroup on a Hilbert space with generator  $A$  such that  $i\mathbb{R} \subset \varrho(A)$ , and let  $\alpha > 0$ . From Proposition 2.1 and Theorem 2.2, we already know that  $\|T(t)(-A)^{-\alpha}\| = O(1/t)$  as  $t \rightarrow \infty$  if and only if  $\sup_{\lambda \in \mathbb{C}_+} \|R(\lambda, A)(-A)^{-\alpha}\| < \infty$ . However, Proposition 4.1.b) with  $C := (-A)^{-\alpha}$  and  $\gamma = 0$  only shows that if  $\|T(t)(-A)^{-\alpha}\| = O(1/t)$  as  $t \rightarrow \infty$ , then  $\|R(\lambda, A)(-A)^{-\alpha}\| = O(|\log \operatorname{Re} \lambda|)$  as  $\operatorname{Re} \lambda \rightarrow 0+$ . From this, we see that Proposition 4.1.b) with  $C := (-A)^{-\alpha}$  does not give a sharp bound of  $\|R(\lambda, A)(-A)^{-\alpha}\|$  in the Hilbert space setting. In contrast, Proposition 4.1.a) with  $C := (-A)^{-\alpha}$ , combined with Proposition 3.5, yields a sharp result on the relation between the rate of decay of  $\|T(t)(-A)^{-\alpha}\|$  as  $t \rightarrow \infty$  and the rate of growth of  $\|R(\lambda, A)(-A)^{-\alpha}\|$  as  $\operatorname{Re} \lambda \rightarrow 0+$ .

**Theorem 4.3.** *Let  $(T(t))_{t \geq 0}$  be a bounded  $C_0$ -semigroup on a Hilbert space  $X$  with generator  $A$  such that  $0 \in \varrho(A)$ . The following statements are equivalent for fixed  $\alpha > 0$ ,  $0 \leq \beta < 1$ , and  $\gamma \geq 0$ :*

- (i)  $\|T(t)(-A)^{-\alpha}\| = O(t^{-\beta}(\log t)^{-\gamma})$  as  $t \rightarrow \infty$ .
- (ii) *There exists  $K > 0$  such that*

$$\|R(\lambda, A)(-A)^{-\alpha}\| \leq \frac{K}{(\operatorname{Re} \lambda)^{1-\beta} |\log \operatorname{Re} \lambda|^\gamma}$$

*for all  $\lambda \in \mathbb{C}_+$  with  $\operatorname{Re} \lambda < 1$ .*

**Proof.** The implication (i)  $\Rightarrow$  (ii) has already been proved in Proposition 4.1.a) with  $C := (-A)^{-\alpha}$ . To show the converse implication (ii)  $\Rightarrow$  (i), we define the function  $F$  in Proposition 3.5 by

$$F(\xi) := \frac{K}{\xi^{1-\beta} |\log \xi|^\gamma}$$

for  $0 < \xi < 1$ . Then

$$\frac{F(1/t)}{t} = \frac{K}{t^\beta (\log t)^\gamma}$$

for all  $t > 1$ . Thus, the implication (ii)  $\Rightarrow$  (i) holds by Proposition 3.5 with  $C := (-A)^{-\alpha}$ .  $\square$

We also obtain a characterization of the same class of non-polynomial rates of decay for  $\|T(t)(-A)^\alpha(I - A)^{-\alpha-\delta}\|$  with  $\alpha > 0$  and  $\delta \geq 0$ . The proof is the same as that of Theorem 4.3 and hence is omitted.

**Theorem 4.4.** Let  $(T(t))_{t \geq 0}$  be a bounded  $C_0$ -semigroup on a Hilbert space  $X$  with generator  $A$ . The following statements are equivalent for fixed  $\alpha > 0$ ,  $0 \leq \beta < 1$ ,  $\gamma \geq 0$ , and  $\delta \geq 0$ :

- (i)  $\|T(t)(-A)^\alpha(I - A)^{-\alpha-\delta}\| = O(t^{-\beta}(\log t)^{-\gamma})$  as  $t \rightarrow \infty$ .
- (ii) There exists  $K > 0$  such that

$$\|R(\lambda, A)(-A)^\alpha(I - A)^{-\alpha-\delta}\| \leq \frac{K}{(\operatorname{Re} \lambda)^{1-\beta} |\log \operatorname{Re} \lambda|^\gamma}$$

for all  $\lambda \in \mathbb{C}_+$  with  $\operatorname{Re} \lambda < 1$ .

## 5. Sufficient condition for $L^2$ -infinite-time admissibility

Let  $X$  and  $Y$  be Hilbert spaces. Let  $A$  be the generator of a bounded  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on  $X$ . In this section, we derive a sufficient condition for  $C \in \mathcal{L}(D(A), Y)$  to be  $L^2$ -infinite-time admissible. Before proceeding to the details, we recall the definitions of polynomial stability of  $C_0$ -semigroups and  $L^2$ -finite-time admissibility.

**Definition 5.1.** A  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $X$  with generator  $A$  is *polynomially stable* if the following two conditions are satisfied:

- a)  $(T(t))_{t \geq 0}$  is bounded.
- b) There exists  $\alpha > 0$  such that  $\|T(t)(I - A)^{-1}\| = O(t^{-1/\alpha})$  as  $t \rightarrow \infty$ .

Let  $A$  be the generator of a polynomially stable  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on a Banach space. Then  $i\mathbb{R} \cap \sigma(A) = \emptyset$  by [5, Theorem 1.1]. Therefore, when  $\|T(t)(I - A)^{-1}\| = O(t^{-1/\alpha})$  as  $t \rightarrow \infty$  for some  $\alpha > 0$ , the estimate  $\|T(t)A^{-1}\| = O(t^{-1/\alpha})$  as  $t \rightarrow \infty$  is also satisfied.

**Definition 5.2.** Let  $X$  and  $Y$  be Banach spaces, and let  $A$  be the generator of a  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on  $X$ . An operator  $C \in \mathcal{L}(D(A), Y)$  is  *$L^2$ -finite-time admissible for  $A$*  if there exist  $M_1 > 0$  and  $t_1 > 0$  such that

$$\int_0^{t_1} \|CT(t)x\|^2 dt \leq M_1 \|x\|^2$$

for all  $x \in D(A)$ .

From the semigroup property, it follows that if  $C \in \mathcal{L}(D(A), Y)$  is  $L^2$ -finite-time admissible for  $A$ , then for each  $t_2 > 0$ , there exists  $M_2 > 0$  such that for all  $x \in D(A)$ ,

$$\int_0^{t_2} \|CT(t)x\|^2 dt \leq M_2 \|x\|^2.$$

By definition, every  $C \in \mathcal{L}(X, Y)$  is  $L^2$ -finite-time admissible.

Let  $X$  and  $Y$  be Hilbert spaces, and let  $\mathbb{C}_a := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > a\}$  for  $a \in \mathbb{R}$ . If  $A$  is the generator of a  $C_0$ -semigroup on  $X$  such that  $T(t_0)$  is surjective for some  $t_0 > 0$  and if  $C \in \mathcal{L}(D(A), Y)$  satisfies  $\sup_{\lambda \in \mathbb{C}_a} \|CR(\lambda, A)\| < \infty$  for some  $a \in \mathbb{R}$ , then  $C$  is  $L^2$ -finite-time admissible for  $A$ ; see [43, Theorem 4.1]

and [46, Theorem 2.2]. Other sufficient conditions for  $L^2$ -finite-time admissibility in the Hilbert space setting can be found, e.g., in [46, Theorem 3.3] and [13, Lemma 2.6].

Combining  $L^2$ -finite-time admissibility with the logarithmic decay of  $\|T(t)(-A)^{-\alpha}\|$  and the 2-Weiss condition of  $C(-A)^\alpha$ , we obtain a sufficient condition for  $L^2$ -infinite-time admissibility. It can be proved by the technique used in the proof of [46, Theorem 4.2]. Note that the extension of  $C(-A)^\alpha$  to  $\mathcal{L}(D(A), Y)$  is unique if it exists, since  $D((-A)^{1+\alpha})$  is dense in  $D(A)$  with respect to the graph norm  $\|\cdot\|_A$ .

**Theorem 5.3.** *Let  $X$  and  $Y$  be Hilbert spaces. Let  $(T(t))_{t \geq 0}$  be a bounded  $C_0$ -semigroup on  $X$  with generator  $A$  such that  $0 \in \varrho(A)$ , and let  $C \in \mathcal{L}(D(A), Y)$ . Assume that the following conditions hold for fixed  $\alpha > 0$  and  $\beta > 1/2$ :*

- a)  $C$  is  $L^2$ -finite-time admissible for  $A$ ;
- b)  $\|T(t)(-A)^{-\alpha}\| \leq O((\log t)^{-\beta})$  as  $t \rightarrow \infty$ ; and
- c)  $C(-A)^\alpha$  extends to an operator in  $\mathcal{L}(D(A), Y)$ , and its extension satisfies the 2-Weiss condition for  $A$ .

*Then  $C$  is  $L^2$ -infinite-time admissible for  $A$ .*

**Proof.** Using Plancherel's theorem, we have that for all  $\xi > 0$  and  $x \in X$ ,

$$\int_{-\infty}^{\infty} \|R(\xi + i\eta, A)(-A)^{-\alpha}x\|^2 d\eta = 2\pi \int_0^{\infty} \|e^{-\xi t}T(t)(-A)^{-\alpha}x\|^2 dt. \quad (24)$$

By the assumption b), there exist constants  $M_0 > 0$  and  $t_0 > e^{2\beta}$  such that

$$\|T(t)(-A)^{-\alpha}\| \leq \frac{M_0}{(\log t)^\beta}$$

for all  $t > t_0$ . Lemma 4.2 shows that there exists a constant  $M_1 > 0$  such that

$$\int_{t_0}^{\infty} \frac{e^{-\xi t}}{(\log t)^{2\beta}} dt \leq \frac{M_1}{\xi |\log \xi|^{2\beta}}$$

for all  $0 < \xi < 1/t_0$ . Put  $c := \sup_{t \geq 0} \|T(t)(-A)^{-\alpha}\|$ . Then

$$\begin{aligned} \int_0^{\infty} \|e^{-\xi t}T(t)(-A)^{-\alpha}x\|^2 dt &\leq c^2 t_0 \|x\|^2 + M_0^2 \int_{t_0}^{\infty} \frac{e^{-2\xi t}}{(\log t)^{2\beta}} dt \|x\|^2 \\ &\leq \left( c^2 t_0 + \frac{M_0^2 M_1}{2\xi |\log(2\xi)|^{2\beta}} \right) \|x\|^2 \end{aligned} \quad (25)$$

for all  $0 < \xi < 1/(2t_0)$  and  $x \in X$ . Let the extension of  $C(-A)^\alpha$  in the assumption c) be also denoted by  $C(-A)^\alpha$ . There exists a constant  $K > 0$  such that

$$\|C(-A)^\alpha R(\lambda, A)\| \leq \frac{K}{\sqrt{\operatorname{Re} \lambda}} \quad (26)$$

for all  $\lambda \in \mathbb{C}_+$ . From (24)–(26), it follows that for all  $0 < \xi < 1/(2t_0)$  and  $x \in X$ ,

$$\begin{aligned}
& \int_{-\infty}^{\infty} \|CR(\xi + i\eta, A)^2 x\|^2 d\eta \\
& \leq \sup_{\eta \in \mathbb{R}} \|C(-A)^\alpha R(\xi + i\eta, A)\|^2 \int_{-\infty}^{\infty} \|R(\xi + i\eta, A)(-A)^{-\alpha} x\|^2 d\eta \\
& \leq \frac{2\pi K^2}{\xi} \left( c^2 t_0 + \frac{M_0^2 M_1}{2\xi |\log(2\xi)|^{2\beta}} \right) \|x\|^2.
\end{aligned} \tag{27}$$

Using Plancherel's theorem again, we obtain that for all  $\xi > 0$  and  $x \in D(A)$ ,

$$\int_0^\infty \|te^{-\xi t} CT(t)x\|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|CR(\xi + i\eta, A)^2 x\|^2 d\eta.$$

This and (27) show that there exists a constant  $M_2 > 0$  such that

$$\int_0^\infty \|\xi te^{-\xi t} CT(t)x\|^2 dt \leq \frac{M_2}{|\log(2\xi)|^{2\beta}} \|x\|^2 \tag{28}$$

for all  $0 < \xi < 1/(2t_0)$  and  $x \in D(A)$ .

Take  $\tau_1 < 1 < \tau_2$  such that  $\tau_1 e^{-\tau_1} = 1/(2e) = \tau_2 e^{-\tau_2}$ . Set

$$\mu_n := \left( \frac{\tau_1}{\tau_2} \right)^{n-1}, \quad \tau_n := \frac{\tau_1}{\mu_n}.$$

Then  $\mu_n \rightarrow 0$  and  $\tau_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Moreover, for all  $n \in \mathbb{N}$ ,

$$\mu_n t e^{-\mu_n t} > \frac{1}{2e} \tag{29}$$

whenever  $t \in (\tau_n, \tau_{n+1})$ . Let  $m \in \mathbb{N}$  satisfy  $\mu_m < 1/(2t_0)$ . Combining the estimates (28) and (29), we have that for all  $x \in D(A)$ ,

$$\begin{aligned}
\int_{\tau_m}^\infty \|CT(t)x\|^2 dt &= \sum_{n=m}^\infty \int_{\tau_n}^{\tau_{n+1}} \|CT(t)x\|^2 dt \\
&\leq (2e)^2 \sum_{n=m}^\infty \int_{\tau_n}^{\tau_{n+1}} \|\mu_n t e^{-\mu_n t} CT(t)x\|^2 dt \\
&\leq (2e)^2 M_2 \|x\|^2 \sum_{n=m}^\infty \frac{1}{|\log(2\mu_n)|^{2\beta}}.
\end{aligned}$$

Since

$$\frac{1}{|\log(2\mu_n)|} = \frac{1}{(\log(\tau_2) - \log(\tau_1))(n-1) - \log 2}$$

for all  $n \geq m$ , we have from  $\beta > 1/2$  that

$$M_3 := (2e)^2 M_2 \sum_{n=m}^{\infty} \frac{1}{|\log(2\mu_n)|^{2\beta}} \in (0, \infty),$$

and then

$$\int_{\tau_m}^{\infty} \|CT(t)x\|^2 dt \leq M_3 \|x\|^2 \quad (30)$$

for all  $x \in D(A)$ . Combining the estimate (30) with the finite-time admissibility of  $C$  in the assumption a), we conclude that  $C$  is infinite-time admissible for  $A$ .  $\square$

The assumption c) of Theorem 5.3 leads us to the following strong version of the Weiss condition.

**Definition 5.4.** Let  $X$  and  $Y$  be Banach spaces, and let  $A$  be the generator of a bounded  $C_0$ -semigroup on  $X$ . An operator  $C \in \mathcal{L}(D(A), Y)$  satisfies the *strong 2-Weiss condition* for  $A$  if there exists  $\alpha > 0$  such that  $C(-A)^\alpha$  extends to an operator in  $\mathcal{L}(D(A), Y)$  and its extension satisfies the 2-Weiss condition for  $A$ .

By Proposition 2.1, if  $A$  is the generator of a polynomially stable  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on a Banach space, then for all  $\alpha > 0$ , there exists  $\beta > 0$  such that  $\|T(t)(-A)^{-\alpha}\| = O(t^{-\beta})$  as  $t \rightarrow \infty$ , which is a faster decay rate than the one of the assumption b) in Theorem 5.3. Therefore, we obtain the following result as a direct consequence of Theorem 5.3.

**Corollary 5.5.** Let  $X$  and  $Y$  be Hilbert spaces, and let  $A$  be the generator of a polynomially stable  $C_0$ -semigroup on  $X$ . If  $C \in \mathcal{L}(D(A), Y)$  is  $L^2$ -finite-time admissible for  $A$  and satisfies the strong 2-Weiss condition for  $A$ , then  $C$  is  $L^2$ -infinite-time admissible for  $A$ .

The case  $X = Y = \ell^2(\mathbb{N})$  is simple but practically important as explained in [21, Remark 2.7]. Consider a diagonal operator  $A$  on  $\ell^2(\mathbb{N})$ . The resolvent condition  $\|C(-A)^\alpha R(\lambda, A)\| \leq K/\sqrt{\operatorname{Re} \lambda}$  in the strong 2-Weiss condition is transformed into the operator Carleson measure criterion introduced in [20, Definition 1.1]. To describe the equivalence more precisely, let  $(\lambda_n)_{n \in \mathbb{N}}$  be the eigenvalues of the diagonal operator  $A$  on  $\ell^2(\mathbb{N})$ . We assume that  $\operatorname{Re} \lambda_n < 0$  for all  $n \in \mathbb{N}$ . Let  $C \in \mathcal{L}(D(A), \ell^2(\mathbb{N}))$  and put  $c_n := Ce_n \in \ell^2(\mathbb{N})$  for  $n \in \mathbb{N}$ , where  $e_n$  is the  $n$ th unit vector in  $\ell^2(\mathbb{N})$ . Define  $c_n^*: \ell^2(\mathbb{N}) \rightarrow \mathbb{C}$  by  $c_n^*x = \langle x, c_n \rangle$  for  $x \in \ell^2(\mathbb{N})$ . When the operator  $C$  is represented by an infinite matrix,  $c_n$  is the  $n$ th column of  $C$ , and  $c_n c_n^*$  is an infinite matrix of rank one. For  $h > 0$  and  $\omega \in \mathbb{R}$ , the rectangle  $Q(h, \omega)$  is given by

$$Q(h, \omega) := \{\lambda \in \mathbb{C} : 0 \leq \operatorname{Re} \lambda \leq h, |\operatorname{Im} \lambda - \omega| \leq h\}.$$

For a fixed  $\alpha > 0$ , [21, Proposition 5.3] shows that  $C(-A)^\alpha$  extends to an operator in  $\mathcal{L}(D(A), \ell^2(\mathbb{N}))$  and there exists  $K > 0$  such that  $\|C(-A)^\alpha R(\lambda, A)\| \leq K/\sqrt{\operatorname{Re} \lambda}$  for all  $\lambda \in \mathbb{C}_+$  if and only if there exists  $M > 0$  such that for all  $h > 0$  and  $\omega \in \mathbb{R}$ ,

$$\left\| \sum_{-\lambda_n \in Q(h, \omega)} |\lambda_n|^{2\alpha} c_n c_n^* \right\| \leq Mh,$$

where the norm on the left-hand side is the operator norm on  $\ell^2(\mathbb{N})$ .

We have seen in Lemma 3.4 and Proposition 3.5 that when  $T(t)$  commutes with  $C \in \mathcal{L}(D(A), X)$  for all  $t \geq 0$ , the estimate  $\|CT(t)\| = O(1/\sqrt{t})$  as  $t \rightarrow \infty$  is equivalent to the 2-Weiss condition on  $C$  in the Hilbert space setting. The next proposition shows that the strong 2-Weiss condition implies a slightly better decay rate  $\|CT(t)\| = O(1/\sqrt{t^{1+\beta}})$  for some  $\beta > 0$  if  $(T(t))_{t \geq 0}$  is polynomially stable.

**Proposition 5.6.** *Let  $A$  be the generator of a polynomially stable  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on a Hilbert space  $X$ . Let  $C \in \mathcal{L}(D(A), X)$  be such that  $T(t)$  commutes with  $C$  for all  $t \geq 0$ . If  $C$  satisfies the strong 2-Weiss condition for  $A$ , then the operator  $CT(t)$  extends to a bounded linear operator (also denoted by  $CT(t)$ ) on  $X$  for all  $t > 0$ , and there exist  $M > 0$  and  $\beta > 0$  such that*

$$\|CT(t)\| \leq \frac{M}{\sqrt{t^{1+\beta}}} \quad (31)$$

for all  $t > 0$ .

**Proof.** Since  $C$  satisfies the strong 2-Weiss condition for  $A$ , there exists  $\alpha > 0$  such that  $C(-A)^\alpha$  extends to an operator in  $\mathcal{L}(D(A), X)$  and its extension satisfies the 2-Weiss condition for  $A$ . By assumption,  $T(t)$  commutes  $C(-A)^\alpha$  and hence its extension for all  $t \geq 0$ . Then it follows from Proposition 3.5 that there is  $M_1 > 0$  such that

$$\|C(-A)^\alpha T(t)x\| \leq \frac{M_1}{\sqrt{t}} \|x\| \quad (32)$$

for all  $t > 0$  and  $x \in D((-A)^{1+\alpha})$ . Since  $(T(t))_{t \geq 0}$  is polynomially stable, Proposition 2.1 shows that there exist  $M_2 > 0$  and  $\beta > 0$  such that

$$\|T(t)(-A)^{-\alpha}\| \leq \frac{M_2}{\sqrt{t^\beta}} \quad (33)$$

for all  $t > 0$ . Combining the inequalities (32) and (33), we obtain

$$\begin{aligned} \|CT(t)x\| &= \|C(-A)^\alpha T(t/2)T(t/2)(-A)^{-\alpha}x\| \\ &\leq \frac{M_1}{\sqrt{t/2}} \|T(t/2)(-A)^{-\alpha}x\| \\ &\leq \frac{\sqrt{2^{1+\beta}} M_1 M_2}{\sqrt{t^{1+\beta}}} \|x\| \end{aligned}$$

for all  $t > 0$  and  $x \in D(A)$ . The result follows by the density of  $D(A)$  in  $X$ .  $\square$

We also show that the decay property  $\|CT(t)\| = O(1/\sqrt{t^{1+\beta}})$  for some  $\beta > 0$  leads to the strong 2-Weiss condition under some additional assumption on boundedness.

**Proposition 5.7.** *Let  $A$  be the generator of a bounded  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $X$ . Let  $C \in \mathcal{L}(D(A), X)$  be such that  $T(t)$  commutes with  $C$  for all  $t \geq 0$ . If there exist constants  $M_1, M_2, \alpha, \beta > 0$  such that for all  $t > 0$ ,*

$$\|C(-A)^\alpha T(t)x\| \leq M_1 \|x\|, \quad x \in D((-A)^{1+\alpha}) \quad (34)$$

and

$$\|CT(t)x\| \leq \frac{M_2}{\sqrt{t^{1+\beta}}} \|x\|, \quad x \in D(A), \quad (35)$$

then  $C$  satisfies the strong 2-Weiss condition for  $A$ .



**Proof.** By assumption,  $\tilde{C} := C(I - A)^{-1} \in \mathcal{L}(X)$  commutes with  $A$ . Let  $\delta > 0$ . It follows from [19, Proposition 3.1.1.f)] that  $\tilde{C}$  also commutes with  $(-A)^\delta$ . For all  $x \in D((-A)^{1+\delta}) = D((I - A)^{1+\delta})$ , there exists  $y \in D((-A)^\delta) = D((I - A)^\delta)$  such that  $x = (I - A)^{-1}y$ . Then

$$C(-A)^\delta x = \tilde{C}(-A)^\delta y = (-A)^\delta \tilde{C}y = (-A)^\delta Cx. \quad (36)$$

Define  $\gamma := \alpha\beta/(1 + \beta) \in (0, \alpha)$ . The moment inequality (see, e.g., [19, Proposition 6.6.4]) shows that there exists a constant  $c > 0$  such that for all  $t \geq 0$  and  $x \in D((-A)^{1+\alpha})$ .

$$\|(-A)^\gamma CT(t)x\| \leq c\|(-A)^\alpha CT(t)x\|^{\gamma/\alpha} \|CT(t)x\|^{1-\gamma/\alpha}. \quad (37)$$

Since

$$\frac{1 + \beta}{2} \left(1 - \frac{\gamma}{\alpha}\right) = \frac{1}{2},$$

we have from (34)–(37) that

$$\|C(-A)^\gamma T(t)x\| \leq \frac{M_3}{\sqrt{t}} \|x\| \quad (38)$$

for all  $t > 0$  and  $x \in D((-A)^{1+\alpha})$ , where  $M_3 := cM_1^{\gamma/\alpha} M_2^{1-\gamma/\alpha}$ .

The estimate (38) yields that for all  $\lambda \in \mathbb{C}_+$  and  $x \in D((-A)^{1+\alpha})$ ,

$$\begin{aligned} \|C(-A)^\gamma R(\lambda, A)x\| &\leq \int_0^\infty e^{-t \operatorname{Re} \lambda} \|C(-A)^\gamma T(t)x\| dt \\ &\leq M_3 \int_0^\infty \frac{e^{-t \operatorname{Re} \lambda}}{\sqrt{t}} dt \|x\| \\ &= \frac{M_3 \Gamma(1/2)}{\sqrt{\operatorname{Re} \lambda}} \|x\|, \end{aligned} \quad (39)$$

where  $\Gamma$  is the gamma function. By [19, Proposition 3.1.1.h)],  $D((-A)^{1+\alpha})$  is a core for  $(-A)^\gamma$ , and hence the estimate (39) holds for all  $x \in D((-A)^\gamma)$ . Since  $D((-A)^\gamma)$  is dense in  $X$ , it follows that  $C(-A)^\gamma R(\lambda, A)$  extends to a bounded linear operator on  $X$  for a fixed  $\lambda \in \mathbb{C}_+$ . Therefore,  $C(-A)^\gamma$  extends to an operator in  $\mathcal{L}(D(A), X)$ . We also see from the estimate (39) that the extension of  $C(-A)^\gamma$  satisfies the 2-Weiss condition for  $A$ . Thus,  $C$  satisfies the strong 2-Weiss condition for  $A$ .  $\square$

## Acknowledgments

This work was supported in part by JSPS KAKENHI Grant Number JP20K14362.

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