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On the Existence and Uniqueness of Solutions of Volterra-Fredholm Integral Equations

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Abstract. In this paper we study the problem of existence and uniqueness of solutions of nonlinear Volterra-Fredholm integral equations of the more general type. The main tool employed in our analysis is the method of successive approximations based on the general idea of T. Ważewski.

1. INTRODUCTION. In 1960, T. Ważewski [12] has given a general method of successive approximations which is very effective and can be applied to investigate a sufficiently wide range of problems. In the present paper we apply the method of Ważewski [12] to study the problems of existence and uniqueness of the solutions of more general Volterra-Fredholm integral equation of the form

$$(1)\quad x(t) = F(t,x(t),\int_0^t f[t,s,x(s)]ds,\ \int_0^T g[t,s,x(s)]ds),\ 0 \leqslant t \leqslant T,$$

where $x(t)$ is an unknown function. The equation (1) is of more general nature and contains as special cases several types of integral equations studied by many authors in the literature. The problems of existence and uniqueness of solutions of various special forms of equation (1) have been investigated by a number of authors by using different techniques (see, [1] - [6], [9] - [11]). In section 2 we give our main hypotheses and the basic Lemma used in the paper. In section 3 we establish our main results on the existence and uniqueness of solutions of equation (1) by using Ważewski's method. Our results for equation (1) in this general form will bring the study of a great number of integral equations under one proof and the method used in this paper is very effective as well as versatile.

2. PRELIMINARIES. In this section, we first introduce the following hypotheses used in our subsequent discussion.

HYPOTHESIS A. Suppose that:

1^o E be a Banch space with norm $\|.\|$, $I = [0,T], \Delta = \{(t,s): 0 \leqslant s \leqslant t \leqslant T\}$, f, $g \varepsilon C[\Delta X E, E]$, $F \varepsilon C[I X E^3, E]$ and, if $x \varepsilon C[I,E]$ and

$$z(t) = F\Big(t, x(t), \int_0^t f[t,s,x(s)]ds, \int_0^T g[t,s,x(s)]ds\Big) ,$$

then $z \varepsilon C[I,E]$,

2^o there exist functions $w_1(t,s,r)$, $w_2(t,s,r)$ such that $w_1, w_2 \varepsilon C[\Delta X R^+, R^+]$, $R^+ = (0,\infty)$, which are nondecreasing in r and fulfil the conditions

$$\|f[t,s,x] - f[t,s,\bar{x}]\| \leqslant w_1(t,s,\|x-\bar{x}\|) ,$$

$$\|g[t,s,x] - g[t,s,\bar{x}]\| \leqslant w_1(t,s,\|x-\bar{x}\|) ,$$

for x, $\bar{x} \varepsilon C[I,E]$,

3^o there exists a function $H(t,r_1,r_2,r_3)$ defined for $t \varepsilon I$ and $0 \leqq r_1, r_2, r_3 < \infty$ such that

(a) if $u \varepsilon C[I,I]$ and

$$v(t) = H\Big(t, u(t), \int_0^t w_1(t,s,u(s))ds, \int_0^T w_2(t,s,u(s))ds\Big) ,$$

then $v \in C[I,I]$;

(b) if u , $\bar{u} \in C[I,I]$ and $u(t) \leqslant \bar{u}(t)$ for $t \in I$, then

$$H\Big(t,u(t), \int_0^t w_1(t,s,u(s))ds, \int_0^T w_2(t,s,u(s))ds\Big)$$
$$\leqslant H\Big(t,\bar{u}(t), \int_0^t w_1(t,s,\bar{u}(s))ds, \int_0^T w_2(t,s,\bar{u}(s))ds\Big) ,$$

for $t \in I$;

(c) if $u_n \in C[I,I]$, $u_{n+1} \leqslant u_n$, $n=0,1,2,\ldots$, and $\lim_{n\to\infty} u_n(t) = u(t)$, then

$$\lim_{n\to\infty} H\Big(t, u_n(t),\int_0^t w_1(t,s,u_n(s))ds, \int_0^T w_2(t,s,u_n(s))ds\Big)$$
$$= H\Big(t,u(t),\int_0^t w_1(t,s,u(s))ds, \int_0^T w_2(t,s,u(s))ds\Big) ,$$

for $t \in I$,

4^o the inequality

$$\| F(t,x_1,x_2,x_3) - F(t,\bar{x}_1,\bar{x}_2,\bar{x}_3) \| \leqslant \| H(t, \|x_1-\bar{x}_1\|,\|x_2-\bar{x}_2\|,\|x_3-\bar{x}_3\|) ,$$

holds for $x_i,\bar{x}_i \in C[I,E]$ $(i=1,2,3)$ and $t \in I$.

HYPOTHESIS B. Suppose that:

1^o there exists a nonnegative continuous function $\bar{u}:I \to R^+$ being the solution of the inequality

$$(2) \quad H\Big(t,u(t),\int_0^t w_1(t,s,u(s))ds, \int_0^T w_2(t,s,u(s))ds\Big) + h(t) \leqslant u(t) ,$$

where

$$h(t) = \sup_{t\in I} \| F(t,o,\int_0^t f[t,s,o]ds, \int_0^T g[t,s,o]ds) \|,$$

2^o in the class of functions satisfying the condition $0 \leqslant u(t) \leqslant \bar{u}(t)$, $t \in I$, the function $u(t) \equiv o$, $t \in I$, is the only solution of the equation

(3) $u(t) = H\left(t, u(t), \int_0^t w_1(t,s,U(s))ds, \int_0^T w_2(t,s,u(s))ds\right)$,

for $t \in I$.

In order to prove the existence of a solution of equation (1), we define the sequence

$x_0(t) \equiv 0$,

(4) $x_{n+1}(t) = F\left(t, x_n(t), \int_0^t f[t,s,x_n(s)]ds, \int_0^T g[t,s,x_n(s)]ds\right)$,

for $n=0,1,2,\ldots\ldots$.

To prove the convergence of the sequence $\{x_n\}$ to the solution $\bar{x}$ of the equation (1), we define the sequence $\{u_n\}$ by the relations

$u_0(t) = \bar{u}(t)$,

(5) $u_{n+1}(t) = H\left(t, u_n(t), \int_0^t f[t,s,u_n(s)]ds, \int_0^T g[t,s,u_n(s)]ds\right)$,

for $n=0,1,2,\ldots\ldots$, where the function $\bar{u}(t)$ is from Hypothesis B.

Now we establish the following basic Lemma needed in our subsequent discussion.

LEMMA. If condition 3^o of Hypothesis A and Hypothesis B are satisfied, then

(6) $0 \leqslant u_{n+1}(t) \leqslant u_n(t) \leqslant \bar{u}(t)$, $t \in I$, $n=0,1,2,\ldots\ldots$,

$$\lim_{n\to\infty} u_n(t) = 0 \ , \quad t \in I \ ,$$

and the convergence is uniform in each bounded set.

Proof. From (5) and (2) we have

$$u_1(t) = H\Big(t, u_0(t), \int_0^t f[t,s,u_0(s)]ds, \int_0^T g[t,s,u_0(s)]ds\Big)$$
$$H\Big(t, \bar{u}(t), \int_0^t f[t,s,\bar{u}(s)]ds, \int_0^T g[t,s,\bar{u}(s)]ds\Big) + h(t)$$
$$\bar{u}(t) = u_0(t) ,$$

for $t \in I$. Further, we obtain (6) by induction. But (6) implies the convergence of the sequence $\{u_n(t)\}$ to some nonnegative function $\phi(t)$ for $t \in I$. By Lebesgue's theorem and the continuity of H it follows that the function $\phi(t)$ satisfies equation (3). Now from Hypothesis B we have $\phi(t) \equiv 0, \ t \in I$. The uniform convergence of the sequence $\{u_n\}$ in I follows from Dini's theorem. Thus the proof of Lemma is complete.

3. MAIN RESULTS.

In this section we establish our main results on the existence and uniqueness of the solutions of equation (1).

THEOREM 1. If Hypotheses A and B are satisfied, then there exists a continuous solution $\bar{x}$ of equation (1). The sequence $\{x_n\}$ defined by (4) converges uniformly on I to $\bar{x}$, and the following estimates

(7) $\quad \|\bar{x}(t) - x_n(t)\| \leqslant u_n(t) , \quad t \in I , \quad n=0,1,2,\ldots\ldots ,$

and

(8) $\quad \|\bar{x}(t)\| \leqslant \bar{u}(t) , \quad t \in I ,$

hold. The solution $\bar{x}$ of equation (1) is unique in the class of functions satisfying the condition (8).

Proof. We first prove that the sequence $\{x_n(t)\}$, $t \in I$, fulfils the condition

(9) $\quad \|x_n(t)\| \leqslant \bar{u}(t) , \quad t \in I , \quad n=0,1,2,\ldots\ldots .$

Evidently, we see that

$$\|x_0(t)\| \equiv 0 \leqslant \bar{u}(t) \ , \quad t\varepsilon I \ .$$

Further, if we suppose that the inequality (9) is true for $n \geqslant 0$, then

$$\begin{aligned}\|x_{n+1}(t)\| &= \Big\| F\Big(t,x_n(t),\int_0^t f[t,s,x_n(s)]ds,\ \int_0^T g[t,s,x_n(s)]ds\Big) \\ &\qquad - F\Big(t,o,\int_0^t f[t,s,o]ds,\ \int_0^T g[t,s,o]ds\Big) \\ &\qquad + F\Big(t,o,\int_0^t f[t,s,o]ds,\ \int_0^T g[t,s,o]ds\Big)\Big\| \\ &\leqslant H\Big(t,\|x_n(t)\|,\int_0^t w_1(t,s,\|x_n(s)\|)ds,\int_0^T w_2(t,s,\|x_n(s)\|)ds\Big)+h(t) \\ &\leqslant H\Big(t,\bar{u}(t),\int_0^t w_1(t,s,\bar{u}(s))ds,\int_0^T w_2(t,s,\bar{u}(s))ds\Big)+h(t) \\ &\leqslant \bar{u}(t)\end{aligned}$$

for $t\varepsilon I$. Now we obtain (9) by induction.

Next we prove that

(10) $\|x_{n+r}(t) - x_n(t)\| \leqslant u_n(t)$, $t\varepsilon I$, $n=0,1,2,\dots$, $r=0,1,2,\dots$.

By (9) we have

$$\|x_r(t)-x_0(t)\| = \|x_r(t)\| \leqslant \bar{u}(t) = u_0(t) \ , \quad t\varepsilon I \ , \quad r=0,1,2,\dots \ .$$

Suppose that (10) is true for n , $r \geqslant 0$, then

$$\begin{aligned}\|x_{n+r+1}(t) - x_{n+1}(t)\| &= \Big\| F\Big(t,x_{n+r}(t),\int_0^t f[t,s,x_{n+r}(s)]ds,\int_0^T g[t,s,x_{n+r}(s)]ds\Big) \\ &\qquad - F\Big(t,x_n(t),\int_0^t f[t,s,x_n(s)]ds,\int_0^T [t,s,x_n(s)]ds\Big)\Big\| \\ &\leqslant H\Big(t,\|x_{n+r}(t)-x_n(t)\|,\ \int_0^t w_1(t,s,\|x_{n+r}(s)-x_n(s)\|)ds\ , \\ &\qquad \int_0^T w_2(t,s,\|x_{n+r}(s)-x_n(s)\|)ds \\ &\leqslant H\Big(t,u_n(t),\int_0^t w_1(t,s,u_n(s))ds,\int_0^T w_2(t,s,u_n(s))ds\Big) \\ &= u_{n+1}(t)\end{aligned}$$

for $t \in I$. Now we obtain (10) by induction.

Because of Lemma, $\lim_{n\to\infty} u_n(t) = 0$ in I , we have from (10) $x_n \to \bar{x}$ in I . The continuity of $\bar{x}$ follows from the uniform convergence of the sequence $\{x_n\}$ and the continuity of all functions x_n . If $r \to \infty$, then (10) gives estimation (7). Estimation (8) is implied by (9). It is obvious that $\bar{x}$ is a solution of equation (1).

To prove that the solution $\bar{x}$ is a unique solution of equation (1) in the class of functions satisfying the condition (8), let us suppose that there exists another solution $\hat{x}$ defined in I and such that $\bar{x}(t) \not\equiv \hat{x}(t)$ for $t \in I$ and $\|\hat{x}(t)\| \leqslant \bar{u}(t)$ for $t \in I$. From (7) we get

$$\|\hat{x}(t) - x_n(t)\| \leqslant u_n(t) \ , \quad t \in I \ , \quad n=0,1,2,\ldots\ldots \ .$$

and it follows that $\bar{x}(t) = \hat{x}(t)$ for $t \in I$. This contradiction proves the uniqueness of $\bar{x}$ in the class of functions satisfying relation (8). This completes the proof of the theorem.

We next establish a theorem which give conditions under which equation (1) has atmost one solution, these conditions do not guarantee existence.

THEOREM 2. If Hypothesis A is satisfied and the function $m(t) \equiv 0,\ t \in I$ is the only nonnegative continuous solution of the inequality

$$(11) \quad m(t) \leqslant H\Big(t, m(t), \int_0^t w_1(t,s,m(s))ds, \int_0^T w_2(t,s,m(s))ds\Big) \ , \quad 0 \leqslant t \leqslant T \ ,$$

then equation (1) has atmost one solution in I .

Proof. Let us suppose that there exist two solutions $\bar{x}$ and $\hat{x}$ of equation (1) such that $\bar{x}(t) \not\equiv \hat{x}(t)$, $t \varepsilon I$. Put $m(t) = \|\bar{x}(t)-\hat{x}(t)\|$, $t \varepsilon I$, then

$$
\begin{aligned}
m(t) &= \Big\| F\Big(t,\bar{x}(t),\int_0^t f[t,s,\bar{x}(s)]ds,\ \int_0^T g[t,s,\bar{x}(s)]ds\Big) \\
&\quad - F\Big(t,\hat{x}(t),\ \int_0^t f[t,s,\hat{x}(s)]ds,\ \int_0^T g[t,s,\hat{x}(s)]ds\Big) \Big\| \\
&\leqslant H\Big(t,\|\bar{x}(t)-\hat{x}(t)\|,\int_0^t w_1(t,s,\|\bar{x}(s)-\hat{x}(s)\|)ds,\int_0^T w_2(t,s,\|\bar{x}(s)-\hat{x}(s)\| ds\Big) \\
&= H\Big(t,m(t),\int_0^t w_1(t,s,m(s))ds,\int_0^T w_2(t,s,m(s))ds\Big) ,
\end{aligned}
$$

and by (11) we conclude that $m(t) \equiv 0$ for $t \varepsilon I$, i.e. $\bar{x}(t) = \hat{x}(t)$, $t \varepsilon I$. This contradiction proves our Theorem 2.

REMARK 1. When the right side in equation (1) is of the form

$$
\Psi\Big(t,\int_0^t f[t,s,x(s)]ds,\ \int_0^T g[t,s,x(s)]ds\Big) , \quad 0 \leqslant t \leqslant T .
$$

the equation (1) is studied by Asirov and Mamedov [1] and Mamedov and Musaev [9]. A slightly different from of Volterra-Fredholm equation considered by the authors in [1,9] is studied by Bihari [2] and Miller, Nohel and Wong [10]. Many interesting results on various special forms of equation (1) can be found in the recent book of C.Corduneanu [5] and his paper [3].

REMARK 2. The analysis of Theorems 1 and 2 can be very easily extended to study the existence and uniqueness of solutions of a more general Volterra-Fredholm integral equation of the form

$$(12)\quad x(t) = F\Big(t, x(t), \int_0^t f_1[t,s,x(s)]ds, \ldots \int_0^t f_n[t,s,x(s)]ds, \int_0^T g_1[t,s,x(s)]ds, \ldots, \int_0^T g_n[t,s,x(s)]ds\Big) ,$$

under some suitable conditions on the functions involved in (12). Equation (12) in turn can be considered as a further generalization of the non-linear Volterra integral equation studied by Grossman [6].

REMARK 3. It is easily seen that the method employed in this paper can be used to study the existence and uniqueness of the solutions of integral equation of the form

$$(13)\quad x(t) = F\Big(t, x(t), \int_R k[t,s,x(s)]ds\Big) , \quad t\epsilon R ,$$

where $R = [a,b]$ and $x(t)$ is an unknown function. The different forms of equation (13) have been studied by C.Corduneanu [4,5] by using fixed point technique.

For other interesting recent applications of Ważewski's method to functional equations and integral-functional equations we refer the interested readers to [7,8] and some of the references given therein.

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