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A FIXED POINT THEOLEM IN REFLEXIVE BANACH SPACES

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In a recent paper [1] Hwang and Shin proved a fixed point theorem (see Corollary 1 below). The proof therein is rather long and computational. In this note, we present a short proof of infact a more general result.

For a subset A of a normed space, let $cl\,\mathrm{co}\,(A)$ denote the closed convex hull of A and $\delta(A)$ its diameter.

THEOREM 1. *Let S be a nonempty closed convex subset of a reflexive Banach space E. Let $T: S\to S$ be a mapping such that TS is a bounded subset of S. If for each weakly compact, convex T-invariant subset M of S with $\delta(TM)>0$, $\delta(TM)<\delta(M)$ then T has a fixed point.*

PROOF. Let $\Delta=\{A\subseteq S,\ A\neq\phi$ *weakly compact, convex and T-invariant*$\}$. Then $cl\,\mathrm{co}\,(TS)\in\Delta$. Let Δ be partially ordered with the set inclusion. It then follows by Zorn lemma that Δ has a minimal element M. Let $K=cl\,\mathrm{co}\,(TM)$. Then, $K\neq\phi$, weakly compact, convex and $K\subseteq M$. Further, $TK\subseteq TM\subseteq K$. Consequently $K\in\Delta$ and hence $K=M$. This implies that $\delta(M)=\delta(TM)$ and hence by hypothesis $\delta(M)=0$. If $M=\{x\}$ for some $x\in S$, then $Tx=x$.

Several authors have considered mappings $T: S\to S$ which satisfy the following condition: there exist nonnegative constants a and b with $a+b\leq 1$ such that for all x and $y\in S$

$$\|Tx-Ty\|\leq a\max\{\|x-y\|,\ \|x-Ty\|,\ \|y-Tx\|\}+b\max\{\|x-Tx\|,\ \|y-Ty\|\}. \tag{1}$$

For a more detailed discussion of such mappings, see Rhodes [2].

COROLLARY 1. *Let S be a nonempty closed, bounded, convex subset of a reflexive Banach space E. Suppose for each weakly compact convex, T-invariant subset M of S with $\delta(M)>0$, $\sup\{\|x-Tx\|: x\in M\}<\delta(M)$. If T satisfies* (1) *with $b>0$, then T has a fixed point.*

PROOF. Since TS is bounded, it suffices to show that $\delta(TM)<\delta(M)$ for each weakly compact, convex, T-invariant subset M of S with $\delta(M)>0$. In fact, for

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such M, it follows by (1) that
$\delta(TM)=a\delta(M)+b \sup\{\|x-Tx\|: x\in M\}<\delta(M)$. Thus, by Theorem 1, T has a fixed point.

References

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