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A PROBLEM ON COMMON FIXED POINTS

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1. Introduction. The example 5 of Sastry and Naidu [1] showed that the result of Ciric [2] is not true for a pair of non-commuting selfmappings of a bounded metric space. Thus they mentioned:

Problem 1. Let f and g be selfmappings of a metric space (X, d) . What can we say about the existence of common fixed points of f and g if they satisfy

$$d(fx, gy) \leq \alpha \max \{d(x, gy), d(y, fx)\}$$

for all $x, y \in X$ and for some $\alpha \in (0, 1)$?

Hence the following problem is also open.

Problem 2. If f and g satisfy any one of the definitions 149, 174, 199 of Rhoades [3], then what additional hypotheses on f , g and X are added to guarantee the existence of common fixed point?

In this paper, we shall discuss the above Problems under weaker additional hypotheses. Our Theorems improve and generalize a great number of recently known results.

2. Definitions and Lemmas. In this paper, we shall denote by ω the set of all nonnegative integers, by N the set of all positive integers and by R_+ the set of all nonnegative real numbers.

Let f and g be selfmappings of a metric space (X, d) . For any $x \in X$, write $O_{fg}(x, 0, \infty) = \{f^i g^j x; i, j \in \omega\}$ and $O_f(x, 0, \infty) = \{f^i x; i \in \omega\}$. For any $A \subset X$, $\delta(A) = \sup \{d(x, y); x, y \in A\}$ denotes the diameter of A and $\bar{A}$ the closure of A .

Throughout the paper, we shall suppose that the function $\varphi: R_+ \rightarrow R_+$ is nondecreasing and upper semicontinuous from the right and for each $t > 0$, $\lim_{n \rightarrow \infty} \varphi^n(t) = 0$, where φ^n is n -th iteration of φ . The function φ is called a contractive gauge function.

LEMMA 1. ([4]) *Let $\varphi: R_+ \rightarrow R_+$ be nondecreasing and upper semicontinuous from the right, then $\lim_{n \rightarrow \infty} \varphi^n(t) = 0$ for every $t > 0$ if and only if $\varphi(t) < t$ for any $t > 0$.*

LEMMA 2. *Let f and g be continuous selfmappings of a complete metric*

space (X, d) and $fg = gf$. Suppose there exists a point $x_0 \in X$ such that $\delta(O_{fg}(x_0, 0, \infty)) < \infty$. If there are $p, q \in \omega$ with $p+q > 1$ such that for all $x \in O_{fg}(x_0, 0, \infty)$

$$(1) \quad \delta(O_{fg}(f^p g^q x, 0, \infty)) \leq \varphi(\delta(O_{fg}(x, 0, \infty))),$$

then $\{(fg)^n x_0\}_{n \in \omega}$ converges to a common fixed point of f and g .

PROOF. Let $h = \max\{p, q\}$. By (1)

$$(2) \quad \begin{aligned} \delta(O_{fg}(f^h g^h x, 0, \infty)) &\leq \delta(O_{fg}(f^p g^q x, 0, \infty)) \\ &\leq \varphi(\delta(O_{fg}(x, 0, \infty))). \end{aligned}$$

Hence for all $k \in \mathbb{N}$

$$(3) \quad \begin{aligned} \delta(O_{fg}((fg)^{kh} x_0, 0, \infty)) \\ &\leq \varphi(\delta(O_{fg}((fg)^{(k-1)h} x_0, 0, \infty))) \\ &\vdots \\ &\leq \varphi(\delta(O_{fg}(x_0, 0, \infty))). \end{aligned}$$

From $\delta(O_{fg}(x_0, 0, \infty)) < \infty$ and the assumption of φ , it follows that

$$\lim_{k \rightarrow \infty} \delta(O_{fg}((fg)^{kh} x_0, 0, \infty)) = 0.$$

Hence for any given $\varepsilon > 0$ there exists an integer $k_0 \in \mathbb{N}$ such that for all $i, j, r, t \geq k_0 h$

$$d(f^i g^j x_0, f^r g^t x_0) < \varepsilon.$$

This implies that for any subsequences $\{n_i\}, \{m_i\}$ of $\mathbb{N}$ ($n_i \rightarrow \infty, m_i \rightarrow \infty$) the subsequences $\{f^{n_i} g^{m_i} x_0\}_{i \in \omega}$ of $O_{fg}(x_0, 0, \infty)$ are Cauchy sequences and converge to a same limit x^* . Specially, the sequence

$$\{x_0, g x_0, f g x_0, g(f g) x_0, \dots, (f g)^n x_0, g(f g)^n x_0, \dots\}$$

converges to x^* . From $fg = gf$, the continuity of f and g and

$$\lim_{n \rightarrow \infty} f g (f g)^n x = \lim_{n \rightarrow \infty} g (f g)^n x = \lim_{n \rightarrow \infty} (f g)^n x = x^*$$

it follows that x^* is a common fixed point f and g . This completes the proof of Lemma 2.

3. Main results.

THEOREM 1. Let f and g be continuous selfmappings of a complete metric space (X, d) and $fg = gf$. Suppose that there exists a point $x_0 \in X$ such that

$\delta(O_{fg}(x_0, 0, \infty)) < \infty$. If there exist $p, p', q, q' \in \omega$, with $p + p' \geq 1$ and $q + q' \geq 1$ such that for all $x, y \in O_{fg}(x_0, 0, \infty)$

$$(4) \quad d(f^p g^{p'} x, f^q g^{q'} y) \leq \varphi(\delta(O_{fg}(x, 0, \infty) \cup O_{fg}(y, 0, \infty))),$$

then $\{(fg)^n x_0\}_{n \in \omega}$ converges to a common fixed point x^* of f and g . Further, if (4) holds for all $x, y \in X$, then x^* is a unique common fixed point of f and g .

PROOF. Let $r = \max\{p, p', q, q'\}$. For any $\xi \in O_{fg}(x_0, 0, \infty)$ and $i, j, k, l \in \omega$, Letting $x = f^{r-p+i} g^{p-p'+j} \xi$ and $y = f^{r-q+k} g^{r-q'+l} \xi$ in (4), we obtain

$$\begin{aligned} d(f^{r+i} g^{r+j} \xi, f^{r+k} g^{r+l} \xi) \\ \leq \varphi(\delta(O_{fg}(f^{r-p+i} g^{p-p'+j} \xi, 0, \infty) \cup O_{fg}(f^{r-q+k} g^{r-q'+l} \xi, 0, \infty))) \\ \leq \varphi(\delta(O_{fg}(\xi, 0, \infty))) \end{aligned}$$

for all $\xi \in O_{fg}(x_0, 0, \infty)$ and $i, j, k, l \in \omega$. Hence for all $x \in O_{fg}(x_0, 0, \infty)$ we have

$$\delta(O_{fg}((fg)^r x, 0, \infty)) \leq \varphi(\delta(O_{fg}(x, 0, \infty))).$$

It follows from Lemma 2 that $\{(fg)^n x_0\}_{n \in \omega}$ converges to a common fixed point x^* of f and g . Now if the further assumption holds and y^* is also a common fixed point of f and g , then by Lemma 1 and (4) we have

$$\begin{aligned} d(x^*, y^*) &= d(f^p g^{p'} x^*, f^q g^{q'} y^*) \\ &\leq \varphi(\delta(O_{fg}(x^*, 0, \infty) \cup O_{fg}(y^*, 0, \infty))) \\ &= \varphi(d(x^*, y^*)) < d(x^*, y^*) \end{aligned}$$

which is a contradiction and so $x^* = y^*$, i.e. x^* is a unique common fixed point of f and g . This completes the proof of Theorem 1.

COROLLARY 1. Let f and g be continuous and commuting selfmappings of a complete, bounded metric space (X, d) . If there exist $p, p', q, q' \in \omega$ with $p + p' \geq 1$ and $q + q' \geq 1$ such that for all $x, y \in X$, the inequality (4) holds, then f and g have a unique common fixed point x^* and for each $x \in X$, $\{(fg)^n x\}_{n \in \omega}$ converges to x^* .

REMARK 1. If $\varphi(t) = \alpha t$, $\alpha \in (0, 1)$, we see easily that Theorem 2 of Fisher [5] is a special case of Corollary 1.

THEOREM 2. Let f and g be commuting selfmappings of a complete metric space (X, d) and g be continuous. Suppose that there exists a point $x_0 \in X$ such that $\delta(O_{fg}(x_0, 0, \infty)) < \infty$. If there are $p', q, q' \in \omega$ with $q + q' \geq 1$ such that for all $x, y \in O_{fg}(x_0, 0, \infty)$

$$(5) \quad d(fg^{p'}x, f^qg^{q'}y) \leq \varphi(\delta(O_{fg}(x, 0, 1) \cup O_{fg}(y, 0, \infty)))$$

where $O_{fg}(x, 0, 1) = \{f^i g^j x : 0 \leq i, j \leq 1\}$. Then $\{(fg)^n x_0\}_{n \in \omega}$ converges to a unique common fixed point x^* of f and g in $\overline{O_{fg}(x_0, 0, \infty)}$. Further, if (5) holds for all $x, y \in X$, then x^* is unique in X .

PROOF. Let $r = \max\{1, p', q, q'\}$. Using similar argument as in the proof of Theorem 1, we obtain

$$\delta(O_{fg}((fg)^r x, 0, \infty)) \leq \varphi(\delta(O_{fg}(x, 0, \infty)))$$

for all $x \in O_{fg}(x_0, 0, \infty)$. By Lemma 2 we obtain

$$\lim_{k \rightarrow \infty} \delta(O_{fg}((fg)^{kr} x_0, 0, \infty)) = 0$$

and $\{(fg)^n x_0\}_{n \in \omega}$ converges to a fixed point x^* of g , since g is continuous. Hence for any given $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $d(x^*, (fg)^n x_0) \leq \varepsilon$ and $\delta(O_{fg}(f^{n-q}g^{n-q'}x_0, 0, \infty)) \leq \varepsilon$, if $\min\{n-q, n-q'\} \geq k_0 r$. It follows from (5) that

$$\begin{aligned} d(fx^*, x^*) &\leq d(fg^{p'}x^*, (fg)^n x_0) + d((fg)^n x_0, x^*) \\ &\leq \varphi(\delta(O_{fg}(x^*, 0, 1) \cup O_{fg}(f^{n-q}g^{n-q'}x_0, 0, \infty))) + d((fg)^n x_0, x^*) \\ &\leq \varphi(d(fx^*, x^*) + 2\varepsilon) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ in the above inequality we obtain

$$d(fx^*, x^*) \leq \varphi(d(fx^*, x^*))$$

which implies $fx^* = x^*$. Thus x^* is a common fixed point of f and g . The uniqueness of x^* holds easily from (5).

COROLLARY 2. Let f and g be commuting selfmappings of a complete bounded metric space (X, d) and g be continuous. If there exist $p', q, q' \in \omega$, with $q + q' \geq 1$ such that for all $x, y \in X$, (5) holds, then for each $x \in X$ $\{(fg)^n x\}_{n \in \omega}$ converges to a unique common fixed point x^* of f and g .

REMARK 2. If $\varphi(t) = \alpha t$, $\alpha \in (0, 1)$, it is easy to see that Theorem 3 of [5] is a special case of Corollary 2.

THEOREM 3. Let f and g be commuting selfmappings of a complete metric space (X, d) and g be continuous. Suppose that $\delta(O_{fg}(x, 0, \infty)) < \infty$ for all $x \in X$. If there exist $p', q, q' \in \omega$ with $q + q' \geq 1$ and $0 \leq q \leq 2$ such that for all $x, y \in X$

$$(6) \quad d(fg^{p'}x, f^qg^{q'}y) \leq \varphi(\delta(O_{fg}(x, 0, \infty) \cup O_{fg}(y, 0, \infty))),$$

then for each $x \in X$, $\{(fg)^n x\}_{n \in \omega}$ converges to a unique common fixed point x^* of f and g .

PROOF. Let $r = \max \{1, p', q, q'\}$. Using similar argument as in the proof of Theorem 1, we have

$$\delta(O_{fg}((fg)^r x, 0, \infty)) \leq \varphi(\delta(O_{fg}(x, 0, \infty)))$$

for all $x \in X$. By Lemma 2 for each $x \in X$ $\{(fg)^n x\}_{n \in \omega}$ converges to a fixed point x^* of g , since g is continuous. Moreover for all $x \in X$

$$\lim_{k \rightarrow \infty} \delta(O_{fg}((fg)^{kr} x, 0, \infty)) = 0.$$

Thus for any given $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$d(x^*, (fg)^n x) \leq \varepsilon \quad \text{and} \quad \delta(O_{fg}(f^{n-q} g^{n-q'} x, 0, \infty)),$$

if $\min \{n-q, n-q'\} \geq k_0 r$. For any $m \in \mathbb{N}$, by (6), we have

$$\begin{aligned} d(f^m x^*, x^*) &\leq d(x^*, (fg)^n x) + d(f^m g^{p'} x^*, f^n g^n x) \\ &\leq \varepsilon + \varphi(\delta(O_f(f^{m-1} x^*, 0, \infty) \cup O_{fg}(f^{n-q} g^{n-q'} x, 0, \infty))) \\ &\leq \varepsilon + \varphi(\delta(O_f(x^*, 0, \infty)) + 2\varepsilon). \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ in the above inequality, we obtain

$$(7) \quad \sup \{d(x^*, f^m x^*) : m \in \mathbb{N}\} \leq \varphi(\delta(O_f(x^*, 0, \infty))).$$

On the other hand, for any $i, j \in \mathbb{N}$ with $j > i$, we have

$$\begin{aligned} d(f^i x^*, f^j x^*) &= d(f^i g^{p'} x^*, f^j g^{q'} f^{j-q} x^*) \\ &\leq \varphi(\delta(O_{fg}(f^{i-1} x^*, 0, \infty) \cup O_{fg}(f^{j-q} x^*, 0, \infty))) \\ &\leq \varphi(\delta(O_f(x^*, 0, \infty))). \end{aligned}$$

Thus

$$(8) \quad \delta(O_f(fx^*, 0, \infty)) \leq \varphi(\delta(O_f(x^*, 0, \infty))).$$

It follows from (7) and (8) that

$$\delta(O_f(x^*, 0, \infty)) \leq \varphi(\delta(O_f(x^*, 0, \infty)))$$

which implies $\delta(O_f(x^*, 0, \infty)) = 0$ and so $x^* = fx^*$. Hence x^* is a common fixed point of f and g . The uniqueness of x^* holds easily from (6).

COROLLARY 3. *Let f be a selfmapping of a complete metric space (X, d) . Suppose that $\delta(O_f(x, 0, \infty)) < \infty$ for all $x \in X$. If there exists an integer q with $1 \leq q \leq 2$ such that for all $x, y \in X$*

$$d(fx, f^q y) \leq \varphi(\delta(O_f(x, 0, \infty) \cup O_f(y, 0, \infty))),$$

then for each $x \in X$, $\{f^n x\}_{n \in \mathbb{N}}$ converges to a unique fixed point x^* of f .

PROOF. Letting $g=I$ (the identical mapping) in Theorem 3, the conclusion of corollary 3 follows Theorem 3.

REMARK 3. Theorem 4 of [4], Theorem 2 of [7] and [8], and Theorem 7 of [9] are special cases of Corollary 3.

THEOREM 4. Let f and g be selfmappings of a complete metric space (X, d) and $fg=gf$. Suppose that there exists a point $x_0 \in X$ such that $\delta(\overline{O_{fg}(x_0, 0, \infty)}) < \infty$. If there are $p, q' \in \mathbb{N}$ such that for all $x, y \in \overline{O_{fg}(x_0, 0, \infty)}$

$$(9) \quad d(f^p x, g^{q'} y) \leq \varphi(\max\{d(x, y), d(x, f^p x), \\ d(y, g^{q'} y), d(x, g^{q'} y), d(y, f^p x)\}),$$

then $\{(fg)^n x_0\}_{n \in \mathbb{N}}$ converges to a unique common fixed point x^* of f and g in $\overline{O_{fg}(x_0, 0, \infty)}$. Further, if (9) holds for all $x, y \in X$, then x^* is unique in X .

PROOF. Clearly, (9) implies (4) with $p'=q=0$. From the proof of Theorem 1 and Lemma 2 it follows that

$$(10) \quad \{x_0, g^{q'} x_0, fg x_0, g^{q'} fg x_0, \dots, (fg)^n x_0, g^{q'} (fg)^n x_0, \dots\},$$

and

$$(11) \quad \{x_0, f^p x_0, fg x_0, f^p fg x_0, \dots, (fg)^n x_0, f^p (fg)^n x_0, \dots\}$$

converge to same limit x^* . By (9) and (10) we have

$$\begin{aligned} d(f^p x^*, x^*) &\leq d(f^p x, g^{q'} (fg)^n x_0) + d(g^{q'} (fg)^n x_0, x^*) \\ &\leq d(g(fg)x, x^*) + \varphi(\max\{d(x^*, (fg)^n x_0), d(x^*, f^p x^*), \\ &\quad d((fg)^n x_0, g^{q'} (fg)^n x_0), d(x^*, g^{q'} (fg)^n x_0), d((fg)^n x_0, f^p x^*)\}). \end{aligned}$$

Letting $n \rightarrow \infty$ in the above inequality we obtain

$$d(fx^*, x^*) \leq \varphi(d(f^p x^*, x^*))$$

which yields $f^p x^* = x^*$. Similarly, using (9) and (11), we obtain $g^{q'} x^* = x^*$.

Now suppose that y^* is also a fixed point of f^p in $\overline{O_{fg}(x_0, 0, \infty)}$. By (9) we have

$$\begin{aligned} d(y^*, x^*) &= d(f^p y^*, g^{q'} x^*) \\ &\leq \varphi(\max\{d(y^*, x^*), 0, 0, d(y^*, x^*), d(y^*, x^*)\}) \\ &\leq \varphi(d(y^*, x^*)) \end{aligned}$$

which implies $y^* = x^*$ and so x^* is the unique fixed point of f^p . But $x^* = f^p x^*$

implies $fx^* = f^pfx^*$ and fx^* is also a fixed point of f . It follows from the uniqueness of x^* that $x^* = fx^*$ and hence x^* is the unique fixed point of f . Similarly, x^* is also the unique fixed point of g . Thus x^* is a unique common fixed point of f and g in $\overline{O_{fg}(x_0, 0, \infty)}$. If (9) holds for all $x, y \in X$, using same argument as above, it is easy to see that x^* is unique in X .

COROLLARY 4. *Let f and g be commuting selfmappings of a complete bounded metric space (X, d) . If there exist $p, q' \in \mathbb{N}$ such that for all $x, y \in X$ (9) holds, then for each $x \in X$ $\{(fg)^n x\}_{n \in \mathbb{N}}$ converges to a unique common fixed point x^* of f and g .*

REMARK 4. Theorem 4 of [5] is a special case of Corollary 4. Theorem 4 and Corollary 4 locally answer the problem 1 and 2 mentioned in Introduction. The example 5 of [1] shows that the assumption $fg = gf$ and $\delta(O_{fg}(x_0, 0, \infty)) < \infty$ for some $x_0 \in X$ cannot be dropped simultaneously in Theorem 4 and Corollary 4.

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References

- [1] K. P. R. Sastry & S. V. R. Naidu: Fixed point theorems for generalized contraction mappings, *Yokohama Math. J.*, **28** (1980), 15–29.
- [2] L. B. Ćirić: A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.*, **42** (1974), 267–273.
- [3] B. E. Rhoades: A comparison of various definitions of contractive mappings, *Trans. Amer. Math. Soc.*, **226** (1977), 257–290.
- [4] Xieping Ding: Fixed point theorems of orbitally contraction mappings, *Sichuan Shiyuan Xuebao*, A special Issue of *Math.*, (1981), 10–14.
- [5] B. Fisher: Results on common fixed points on bounded metric spaces, *Math. Semi. Notes*, **7** (1979), 73–81.
- [6] B. Fisher: Common fixed point mappings on complete and compact metric spaces, *Math. Japonica*, **22** (1978), 565–569.
- [7] W. Walter: Remarks on a paper by F. Browder about contraction, *Nonlinear Anal. TMA.*, **5** (1981), 21–25.
- [8] S. Park & B. E. Rhoades: Some general fixed point theorems, *Acta Sci. Math.*, **42** (1980), 299–304.
- [9] Xieping Ding: On some results of fixed points, *Chin. Ann. of Math.*, **4B4** (1983).

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