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# ON BCH-ALGEBRAS

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In 1966, Y. Imai and K. Iséki introduced two classes of the abstract algebras: **BCK**-algebras and **BCI**-algebras (see [1], [2]). Many mathematicians have studied these two kind of algebras and obtained results (see [3–9]). The first author of this paper with K. Iséki has studied the associative **BCI**-algebras (see [9]). In this paper, the authors introduce a wider kind of the algebras-**BCH**-algebras and do some initial discusses for it.

## 1. The concept on BCH-algebras

In 1966, Y. Imai and K. Iséki in [1] introduced the **BCK**-algebras. Let $X$ be a set with a binary operation $*$ and a constant element 0. $X$ is called a **BCK**-algebras iff it satisfies the following conditions: $1°$ $(x*y)*(x*z)\leqq z*y$, $2°$ $x*(x*y)\leqq y$, $3°$ $x\leqq x$, $4°$ $0\leqq x$, $5°$ $x\leqq y$, $y\leqq x\Rightarrow x=y$, $6°$ $x\leqq y\Leftrightarrow x*y=0$. Since $6°$, then $1°$–$5°$ are written as:

$$[(x*y)*(x*z)]*(z*y) = 0, \tag{1}$$

$$[x*(x*y)]*y = 0, \tag{2}$$

$$x*x = 0, \tag{3}$$

$$0*x = 0, \tag{4}$$

$$x*y = y*x = 0 \Longrightarrow x = y. \tag{5}$$

We easily know, a **BCK**-algebra has the following properties:

$$x*0 = x,$$

$$x*0 = 0 \Longrightarrow x = 0. \tag{6}$$

From this, in 1966, K. Iséki in [2] introduced the concept of the **BCI**-algebras. A **BCI**-algebras is an algebra $\langle X; *, 0\rangle$ of type (2, 0) satisfying the condition (1), (2), (3), (5) and (6). On the basis of deep studies on **BCK**-algebras, K. Iséki in [5–8] studied the BCI-algebras and got many results. For example, the **BCI**-algebra $\langle X; *, 0\rangle$ has the following main properties:

$$(x*y)*z = (x*z)*y, \tag{7}$$

$$x*0 = x. \tag{8}$$

The **BCK**-algebras and the **BCI**-algebras are two classes of abstract algebras and well worth studing. On the other hand, we naturally want to ask a question: Can we discuss still more wide algebraical class from the properties of the **BCI**-algebras? The authors in this paper will discuss this question and introduce the concept of the **BCH**-algebras. We first begin by saying from the following fact on **BCI**-algebras, i.e. we give the following characterization on **BCI**-algebras:

THEOREM 1. *Let* $\langle X; *, 0\rangle$ *be an algebra of type* (2, 0). *Then* $\langle X; *, 0\rangle$ *is a* **BCH**-*algebra iff it satisfies the conditions* (1), (3), (5) *and* (7).

PROOF. We only prove: (1), (3), (5), (7)$\Rightarrow$(2) and (6). We first prove (2). In fact, by (7) and (3) we have $[x*(x*y)]*y=(x*y)*(x*y)=0$. Prove (6). We know that $x*0=0$, and want to prove $x=0$. Now we count as follows: $0*x=(x*0)*x=(x*x)*0=0*0=0$, where we used (7) and (3). By (5) we know $x=0$. Q.E.D.

In the proof on the sufficiency of Theorem we do not use (1). By [5] we may know that to prove (7) we have need to use [1]. But only (3), (5) and (7) do not imply (1) (see Examples 1 and 2 of next section). That is, we may give the following:

DEFINITION 1. *A* **BCH**-*algebra is an algebra* $\langle X; *, 0\rangle$ *of type* (2, 0) *satisfying the following conditions: for arbitrary* $x, y, z \in X$,

$$x*x = 0 \tag{3}$$

$$x*y = y*x = 0 \Longrightarrow x = y, \tag{5}$$

$$(x*y)*z = (x*z)*y. \tag{7}$$

Obviously, we have the following:

THEOREM 2. *Every BCI-algebra must be a BCH-algebra. Every BCH-algebra satisfying the condition* (1) *must be a BCI-algebra.*

## 2. The proper BCH-algebras

By Theorem 2 and the fact that the **BCK**-algebra must be a **BCI**-algebra we may know the following:

$$\begin{matrix}\text{the class of} & & \text{the class of} & & \text{the class of}\\ & \subset & & \subseteq & \\ \textbf{BCK}\text{-algebras} & & \textbf{BCI}\text{-algebras} & & \textbf{BCH}\text{-algebras.}\end{matrix}$$

In [5], K. Iséki gave an example of an algebra which is a **BCI**-algebra, but not a **BCK**-algebra. But, whether or not exists a **BCH**-algebra which isn't a **BCI**-algebra? For convenience, we first give the following:

DEFINITION 2. If a **BCH**-algebra isn't a **BCI**-algebra, we call it *a proper* **BCH**-*algebra*.

By the definition, the foregoing question is: whether or not exists a proper **BCH**-algebra? This is a question which must be solved. Now we first solve this question. We have the following:

THEOREM 3. *There are the proper* **BCH**-*algebras*.

PROOF. See the following examples 1 and 2. Q.E.D.

EXAMPLE 1. Let $X = \{0, 1, 2, 3\}$ and the operation $*$ given as follows:

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 3 | 3 |
| 2 | 2 | 0 | 0 | 2 |
| 3 | 3 | 0 | 0 | 0 |

,

then $\langle X; *, 0\rangle$ is a proper **BCH**-algebra. In fact, we easily prove $\langle X; *, 0\rangle$ is a **BCH**-algebra, but

$$[(2*3)*(2*1)]*(1*3) = (2*0)*3 = 2*3 = 2 \neq 0,$$

then (1) doesn't hold and $\langle X; *, 0\rangle$ is not a **BCI**-algebra.

EXAMPLE 2. Let $X = \{0, 1, 2, 3\}$ and the operation $*$ given as follows:

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 |
| 2 | 2 | 3 | 0 | 3 |
| 3 | 3 | 0 | 0 | 0 |

,

then $\langle X; *, 0\rangle$ is a preoper **BCH**-algebra. In fact, we easily prove $\langle X; *, 0\rangle$ is a **BCH**-algebra, but

$$[(1*3)*(1*2)]*(2*3) = (1*0)*3 = 1*3 = 1 \neq 0,$$

then $\langle X; *, 0\rangle$ is a proper **BCH**-algebra.

A worthy question of note is: both proper **BCH**-algebras in forgoing two examples are not **BCI**-algebras, because both they don't satisfy (1). This is not accidental. In fact, in the proof of Theorem 1 we only used (3), (5) and (7) to prove (2) and (6), but didn't use (1). Consequently, we may have the following:

LEMMA 1. *In any* **BCH**-*algebra* $\langle X; *, 0\rangle$ *for arbitrary* $x, y \in X$ *holds*

$$[x*(x*y)]*y = 0. \tag{2}$$

LEMMA 2. *In any BCH-algebra* $\langle X; *, 0\rangle$ *for arbitrary* $x, y \in X$ *holds*

$$x*0 = 0 \Longrightarrow x = 0. \tag{6}$$

By foregoing two lemmas it follows that, if we want to know whether a **BCH**-algebra is proper or not, we may use the following:

LEMMA 3. *A BCH-algebra* $\langle X; *, 0\rangle$ *is proper iff* (1) *does not hold for it.*

Next our main purpose is to show that there are infinite many proper **BCH**-algebras. For this, we also give out several related lemmas.

LEMMA 4. *Let* $\langle X_1; *_1, 0_1\rangle$ *and* $\langle X_2; *_2, 0_2\rangle$ *be two BCH-algebras,* $X = X_1 \times X_2$, *let* $0=(0_1, 0_2)$, *and define the following operation in* $X$:

$$(x_1, y_1)*(x_2, y_2) = (x_1 *_1 x_2, y_1 *_2 y_2), \tag{9}$$

*then* $\langle X; *, 0\rangle$ *is a BCH-algebra, and is called a product BCH-algebra.*

PROOF. 1) Let us prove (3). For arbitrary $(x, y) \in X$ we have

$$(x, y)*(x, y) = (x *_1 x, y *_2 y) = (0_1, 0_2) = 0.$$

2) Let us prove (5). If $(x_1, y_1)*(x_2, y_2) = (x_2, y_2)*(x_1, y_1) = 0 = (0_1, 0_2)$, then by (9) we have $(x_1 *_1 x_2, y_1 *_2 y_2) = (0_1, 0_2) = (x_2 *_1 x_1, y_2 *_2 y_1)$, therefore $x_1 *_1 x_2 = x_2 *_1 x_1 = 0_1$ and $y_1 *_2 y_2 = y_2 *_2 y_1 = 0_2$, consequently $x_1 = x_2$, $y_1 = y_2$, i.e. $(x_1, y_1) = (x_2, y_2)$.

3) Let us proof (7). Let $z_i = (x_i, y_i)$ $(i = 1, 2, 3)$ be three arbitrary elements in $X$. We have

$$\begin{aligned}(z_1 * z_2) * z_3 &= [(x_1, y_1)*(x_2, y_2)]*(x_3, y_3) = (x_1 *_1 x_2, y_1 *_2 y_2)*(x_3, y_3)\\ &= ((x_1 *_1 x_2) *_1 x_3, (y_1 *_2 y_2) *_2 y_3) = ((x_1 *_1 x_3) *_1 x_2, (y_1 *_2 y_3) *_2 y_2)\\ &= (x_1 *_1 x_3, y_1 *_2 y_3)*(x_2, y_2) = [(x_1, y_1)*(x_3, y_3)]*(x_2, y_2)\\ &= (z_1 * z_3) * z_2.\end{aligned}$$

Q.E.D.

LEMMA 5. *If* $\langle X; *, 0\rangle$ *is the product BCH-algebra produced by* $\langle X_1; *_1, 0_1\rangle$ *and* $\langle X_2; *_2, 0_2\rangle$ *as in Lemma* 4, *and at least one of* $\langle X_1; *_1, 0_1\rangle$

*and $\langle X_2; *_2, 0_2\rangle$ is proper BCH-algebra, then $\langle X; *, 0\rangle$ is a proper BCH-algebra.*

PROOF. By Lemma 3 we only prove that $\langle X; *, 0\rangle$ doesn't satisfy (1). We might as well let $\langle X_1; *_1, 0_1\rangle$ be a proper **BCH**-algebra, then by Lemma 3 there are three elements $a_1$, $a_2$, $a_3$ in $X$ such that

$$[(a_1*_1a_2)*_1(a_1*_1a_3)]*_1(a_3*_1a_2) \neq 0. \tag{10}$$

Let $b_1$, $b_2$, $b_3$ be three arbitrary elements in $X_2$, then $z_i=(a_i, b_i)$ $(i=1, 2, 3)$ are three elements in $X$ and

$$\begin{aligned}&[(z_1*z_2)*(z_1*z_3)]*(z_3*z_2)\\ &\quad = \{[(a_1, b_1)*(a_2, b_2)]*[(a_1, b_1)*(a_3, b_3)]\}*[(a_3, b_3)*(a_2, b_2)]\\ &\quad = ([(a_1*_1a_2)*_1(a_1*_1a_3)]*_1(a_3*_1a_2), [(b_1*_2b_2)*_2(b_1*_2b_3)]*_2(b_3*_2b_2))\\ &\quad \neq (0_1, 0_2) = 0.\end{aligned}$$

That is, **BCH**-algebra $\langle X; *, 0\rangle$ does not satisfy (1). By Lemma 3, $\langle X; *, 0\rangle$ is a proper **BCH**-algebra.

By Lemma 5 we may get the following:

THEOREM 4. *There are infinite many proper BCH-lagebras.*

PROOF. Let $X$ denote the proper **BCH**-algebra in Exampl 1. Let $X^2$ be the product **BCH**-algebra by $X$ and $X$, then $X^2$ is a proper **BCH**-algebra. Then we define by induction $X^{n+1}$ as the product **BCH**-algebra by $X^n$ and $X$. By induction we may know that for any natural number $n$, $X^n$ is a proper **BCH**-algebra, Because $|X|=4$, then $|X^n|=4^n$. When $n\neq m$, then $X^n\neq X^m$. That is,

$$X^1 = X, X^2, X^3, \ldots, X^n, \ldots, \quad n\in\omega,$$

are all proper **BCH**-algebra, and they are no equal each other. Q. E. D.

### 3. The properties on BCH-algebras

In last section we discuss proper **BCH**-algebras. But the **BCH**-algebra with some conditions may be a **BCI**-algebra. We have the following:

THEOREM 5. *If a BCH-algebra $\langle X; *, 0\rangle$ has the following property: for any $x, y, z\in X$ holds*

$$(x*y)*z = x*(y*z), \tag{11}$$

*(i.e. an associative BCH-algebra), then $\langle X; *, 0\rangle$ is an associative BCI-algebra,*

*that is, it is a group in which every element is an involution.*

PROOF. By Theorem 1, we only prove (1). In fact, by (11), (7) and (5) we easily know

$$[(x*y)*(x*z)]*(z*y) = [(x*y)*x]*[(z*z)*y]$$
$$= [(x*x)*y]*[(z*z)*y] = (0*y)*(0*y) = 0.$$

and by [9] we know that second fact holds. Q.E.D.

By Lemma 2 we show an important property of **BCH**-algebras, i.e. the following:

THEOREM 6. *In any BCH-algebra $\langle X; *, 0\rangle$ for arbitrary $x \in X$ holds*

$$x*0 = x. \tag{12}$$

PROOF. First, by (7) and (3) we have

$$(x*0)*x = (x*x)*0 = 0*0 = 0.$$

Then we want to prove $x*(x*0)=0$. We compute:

$$(x*(x*0))*0 = (x*0)*(x*0) = 0.$$

By Lemma 2 we may know $x*(x*0)=0$. Then by (5) we know that (12) holds.
Q.E.D.

**BCH**-algebras have the following properties:

THEOREM 7. *Let $\langle X; *, 0\rangle$ be a BCH-algebra. Then for any $x, y \in X$ we have that*

$$(x*y)*x = 0*y. \tag{13}$$

*In particular, we have that*

$$(0*y)*0 = 0*y \tag{14}$$

*and*

$$(x*0)*x = 0. \tag{15}$$

PROOF. By (7) and (3) we have $(x*y)*x=(x*x)*y=0*y$. And in (13) we let $x=0$ and obtain (11). In (10) we let $y=0$ and obtain (11). Q.E.D.

For **BCH**-algebras we have the following characterization:

THEOREM 8. *An algebra $\langle X; *, 0\rangle$ of type* (2, 0) *is of BCH iff it satisfies* (3), (5) and

$$[(x*y)*z]*[(x*z)*y] = 0 \tag{16}$$

*for any* $x, y, z \in X$.

Proof. "$\Rightarrow$". We only prove that (16) holds. In fact by (7) and (3) we easily know:

$$[(x*y)*z]*[(x*z)*y] = [(x*y)*z]*[(x*y)*z] = 0.$$

"$\Leftarrow$". We only prove that (7) holds By (5) we only prove

$$[(x*y)*z]*[(x*z)*y] = 0$$

and

$$[(x*z)*y]*[(x*y)*z] = 0.$$

and they hold by the condition (16). Q.E.D.

For the sub-algebras we have the following result:

THEOREM 9. *Let* $\langle X; *, 0\rangle$ *be a BCH-algebra,* $Y \subseteq X$, $0 \in Y$, *and* $Y$ *be closed for* $*$, *then* $\langle Y; *, 0\rangle$ *is also a BCH-algebra, which is called the sub-algebra of* $\langle X; *, 0\rangle$.

## 4. The isomorphism on BCH-algebras

For **BCH**-algebra we also may introduce an isomorphic relation. We first introduce the following:

DEFINITION 3. Let $f$ be $1-1$ mapping from **BCH**-algebra $\langle X; *_1, 0\rangle$ onto **BCH**-algebra $\langle Y; *_2, \phi\rangle$ such that

$$f(0) = \phi, \tag{17}$$

$$f(x_1 *_1 x_2) = f(x_1) *_2 f(x_2) \quad \text{for any} \quad x_1, x_2 \in X, \tag{18}$$

then $f$ is called *an isomorphic correspondence (or mapping) from* $\langle X; *_1, 0\rangle$ *onto* $\langle Y; *_2, \phi\rangle$; if for two **BCH**-algebras $\langle X; *_1, 0\rangle$ and $\langle Y; *_2, \phi\rangle$ there is such an isomorphic correspondece, then we call them to *be isomorphic*.

We have the following result:

THEOREM 10. *Let* $\langle X; *_1, 0_1\rangle$ *be a BCH-algebra,* $\langle Y; *_2, 0_2\rangle$ *be an algebra of type* (2, 0), *and* $f: X \to Y$ *is a* $1-1$ *mapping from* $X$ *to* $Y$ *satisfying* (17) *and* (18), *then* $\langle Y; *_2, 0_2\rangle$ *is a BCH-algebra, that is,* $f$ *is an isomorphic correspondence and* $X$ *is isomorphic to* $Y$.

PROOF. 1) $Y$ satisfies (3). In fact, for any $y \in Y$, let $f(x) = y$, since $f$ satisfies

(18), then by $x*_1x=0_1$ we have $y*_2y=0_2$.

2) $Y$ satisfies (5). Let $y_1$ and $y_2$ be two arbitrary elements in $Y$ satisfying $y_1*_2y_2=y_2*_2y_1=0_2$. Let $x_1=f^{-1}(y_1)$ and $x_2=f^{-1}(y_2)$, then $x_1*_1x_2=x_2*_1x_1=0_1$. Because $\langle X; *_1, 0_1\rangle$ is a **BCH**-algebra, then $x_1=x_2$. Consequently, $y_1=y_2$.

3) $Y$ satisfying (7). Let $y_1$, $y_2$ and $y_3$ be three arbitrary elements in $Y$, and $x_i=f^{-1}(y_i)$, $i=1, 2, 3$. Because $\langle X; *_1, 0_1\rangle$ is a **BCH**-algebra, then $(x_1*_1x_2)*_1x_3=(x_1*_1x_3)*_1x_2$. Consequently, under $f$ we have $(y_1*_2y_2)*_2y_3=(y_1*_2y_3)*_2y_2$.

4) By 1)–3) we may know that $\langle Y; *_2, 0_2\rangle$ is a **BCH**-algebra. By Definition 3 we have, $f$ is an isomorphic correspondence and $X$ is isomorphic to $Y$.

Q. E. D.

We also have the following result:

THEOREM 11. 1) *The isomorphic relation for BCH-algebras is an equivalent relation.*

2) *Any BCH-algebra isomorphic to a proper BCH-algebra is also a proper BCH-algebra.*

3) *All BCH-algebras and isomorphic correspondences form a category, which is called the category of BCH-algebras, and is denoted by BCH.*

4) *All proper BCH-algebras and all isomorphic correspondences between them form a full subcategory of BCH, which is denoted as proper -BCH.*

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