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A Schumpeterian Microfoundation of the Geometric Brownian Motion of Firm Size and Zipf's Law

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Abstract

A geometric Brownian motion is often used in dynamic economic analysis when variables of interest grow stochastically. What economic mechanisms are working behind? What economic forces contribute to shaping such stochastic processes? The existing studies leave those questions unanswered. The present paper represents an effort to answer them, focusing upon the firm size distribution. Using the otherwise standard Schumpeterian growth model, Poisson-distributed innovations in "many" sectors give rise to the geometric Brownian motion of a firm size via the Lindberg-Feller Central Limit Theorem. The resulting distribution of firm sizes is Pareto, and the Pareto exponent can take a low or high value. Local stability analysis reveals that the lower Pareto exponent, close to 1, is locally stable.

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1 Introduction

A Brownian motion and a geometric Brownian motion are often used in dynamic economic analysis. The former is applied to, e.g. short-run macroeconomic fluctuations (Ahn, Kaplan, Moll, Winberry, and Wolf (2018)), while the latter has become popular in analysis where a variable of interest grows stochastically and its distribution is examined. For example, the wealth distribution (e.g. Benhabib, Bisin, and Zhu (2011)), the income distribution (e.g. Gabaix, Lasry, Lions, and Moll (2016)), firm dynamics (e.g. Luttmer (2007)), and the city size distribution (e.g. Gabaix (1999)) in addition to finance (e.g. Duffie (2003)). This fact proved that Brownian and geometric Brownian motions are instrumental in generating valuable insights on those topics.

Having said this, however, a (geometric) Brownian motion is merely a mathematical apparatus, and hence the following questions remain. What mechanisms are working behind? What economic forces contribute to shaping such stochastic processes? The existing studies leave those questions unanswered. An attempt to answer them is important, as Steindl (1987) wrote "The stochastic models have often been criticised for their lack of economic content. Perhaps it has been overlooked that they only represent the first steps in a new and exceedingly difficult terrain." (p.810) The present paper represents an effort in the direction Steindl perceived, focusing upon the firm size distribution.

Our starting point is the Schumpeterian model of economic growth, pioneered by Aghion and Howitt (1992) and Grossman and Helpman (1991). Typically, innovation follows a Poisson process in those models. As in other studies (see below), we introduce incumbent R&D in "many" manufacturing sectors. Then, invoking the Lindberg-Feller Central Limit Theorem, we will show that the rate of firm growth in terms of the number of products is normally distributed, and hence, the size of a firm exactly follows a geometric Brownian motion (hereafter, GBM). That is, Poisson distributed innovations in many sectors give rise to the GBM of a firm size. Note that this mechanism fundamentally differs from a simple normal approximation of the sum of independent Poisson random variables. What is required for the firm size GBM is a normally distributed rate of growth in the number of products, not the level of the number of goods. A driving force for this mechanism is a positive externality which makes it possible for incumbents to generate more innovations as their size expands.

Given the GBM of an individual firm size, it is easy to establish that the firm sizes is Pareto-distributed. This is a standard result. What differentiates our analysis from others is the existence of multiple values for the Pareto exponents because of the afore-mentioned externality. A mechanism of this result can be understood by distinguishing three types of creative destruction of innovation. The first type is by entrant firms. Their new products replace the existing goods, thinning the tail distribution. Incumbent innovation without the externality constitutes the second type of creative destruction. Incumbents forge ahead with new products on average, thickening the tail distribution. The third creative destruction is realised via incumbent innovation amplified by the positive externality. The third type works against the first two types of creative destruction in terms of how they affect the Pareto exponent. Multiple values for the exponent arises due to those opposing creative destruction effects.

While this is an interesting result, stability analysis is required to determine what value the Pareto exponent takes in equilibrium. For this, we develop a full-fledged growth model, endogenising entrant and incumbent Poisson rates. Firms have to be successful in R&D first before entry, and incumbent firms invest in R&D to expand its portfolio of products. They make R&D decisions by choosing the optimal number of workers. In this framework, we conduct local stability analysis based on the Kolmogorov forward equation and establish that a lower value for the Pareto exponent is stable, and a higher value is unstable. In addition, we show that the Pareto exponent is arbitrarily close 1, i.e. Zipf's law, if the degree of the positive externality is small enough.

Our study is related to the literature of firm dynamics, pioneered by Lucas (1978) and Hopenhayn (1992) who consider a competitive industry. In the sense of monopoly profit and R&D activity, the present paper is closer to Klette and Kortum (2004) and Luttmer (2011). In particular, we borrow a valuable insight of the former study that products are countable in a continuum of industries. The latter study extends the former to show that the firm sizes are Pareto-distributed. Our study is more closely related to Luttmer (2007) who uses a GBM and establishes that the stationary distribution of firm sizes closely approximates Zipf's law, i.e. a Pareto exponent is one. A difference is that we obtain a Pareto exponent close to 1 using the fairly standard Schumpeterian endogenous growth model. In this sense, our model has an advantage in that it can be easily extended to, for example, international trade and analysis of policy instruments like patents and taxes. Di Giovanni, Levchenko, and Ranciere (2011) consider power laws in firm sizes in the context of international trade. Acemoglu and Cao (2015) is also related to our study in that entrant and incumbent innovations drive economic growth. Clementi and Palazzo (2016), Arellano, Bai, and Kehoe (2019) and Bilal, Engbom, Mongey, and Violante (2022) are more recent studies which apply firm dynamics models to shed light on business cycles and labour market issues. Luttmer (2010) gives an overview of the literature on firm dynamics, and Gabaix (2009) reviews studies on power laws in economics. In a more general setting based on Markov processes, Beare and Toda (2022) explore a rigorous mechanism of generating a Pareto distribution of sizes.

The structure of the paper is as follows. Section 2 takes Poisson arrival rates of entrant and incumbent innovations as given. It is shown that (i) the growth rate of a firm size is normally distributed, (ii) a firm size follows the GBM, (iii) the stationary distribution of firm sizes is Pareto, and (iv) there exists multiplicity of Pareto exponents. Section 3 endogenises entrant and incumbent Poisson rates in the otherwise standard Schumpeterian model. Local stability is examined and a lower Pareto exponent, closer to 1, is shown to be locally stable. Section 4 concludes.

2 Exogenous Arrival Rates of Innovation

The purpose here is to demonstrate that the GBM of a firm size arises in a model of Poisson-process innovations in a familiar framework of the Schumpeterian model of technological innovation. For this, Poisson arrival rates of innovation are taken as given in this section.

2.1 Assumptions

Time t is continuous, and we initially partition t into small time intervals Δt such that $T\Delta t = t$, $T > 0$. All variables are taken to be constant during Δt. We will $\Delta t \to 0$ in a later analysis. The production function of final output $Y(t)$ is given by

$$\ln [Y(t)] = \frac{1}{K} \sum_{k=1}^{K} \ln [Y_k(t)] \tag{1}$$

$$\ln [Y_k(t)] = \int_0^1 \ln [q_k(j,t)\, x_k(j,t)]\, dj \tag{2}$$

There are K number of what we call manufacturing sectors, denoted by $k = 1, 2, \cdots K$. (2) shows that in each sector there are a unit continuum of industries, indexed by $j \in [0,1]$. $x_k(j,t)$ denotes an intermediate product in an industry j in a sector k at time t. Its quality level is given by $q_k(j,t) \equiv \lambda^{\ell_k(j,t)}$, $\lambda > 1$, $\ell_k(j,t) = 0, 1, 2, \cdots$. In a later analysis, we will let $K \to \infty$ for a continuum of sectors.

In developing our model, we borrow an insight of Klette and Kortum (2004) that the number of goods n produced by firms is countable in the product space. That is, firms start with $n = 1$ upon successful entry, and $n = 1, 2, 3, \cdots$ for incumbent firms. Using π for profit per product, which is taken as given in this section, firms with n products earn profits $n\pi$. Countability of n has an important implication. It allows us to calculate the mean and variance of a proportionate rate of change in n in discrete numbers, which we take advantage of in invoking a Central Limit Theorem in an environment of a continuum of products.

The quality levels $q_k(j,t)$ improve through R&D activities, entrant and incumbent. Free entry prevails in R&D. Suppose that an entrant firm decides to invest in a sector k. A research success occurs with a Poisson arrival rate g_E. If successful, the entrant innovation is implemented in a randomly selected industry in the entering sector. An entrant becomes a sole producer of a single product, i.e. $n(t) = 1$, and starts R&D as an incumbent firm.

To describe incumbent R&D, consider a firm producing $n(t) \geq 1$ number of goods. It conducts R&D in *all* sectors, and g_I is used to denote a common Poisson arrival rate of a research success. If the firm succeeds in a sector k, it can generate $n(t)$ number of potentially launchable goods due to the positive externality of its past R&D successes.[1] Each of those innovations is implemented with the probability $0 < \phi \leq 1$ in industries which are randomly selected out of those in a sector k. In this sense, ϕ is the measure of the positive externality captured by n.

[1] A similar assumption is used in Garcia-Macia, Hsieh, and Klenow (2019) to quantify the extent of the creative destruction effects of innovation in the US.

We assume that g_E is the same in all sectors and so is g_I in all sectors for all incumbent firms. This symmetric feature greatly simplifies exposition. The time argument t is dropped in what follows unless ambiguity arises.

2.2 A Normal Distribution of Firm Growth Rate

Consider a firm producing $n = \sum_k^K n_k$ products where n_k denotes the number of goods that the firm produces in a sector k. Using m_{Ik} for the random number of R&D successes in a sector k during a time interval Δt, the number of new products that the firm creates via incumbent R&D in a sector k, denoted by Δn_k^{gain}, is given by

$$\Delta n_k^{\text{gain}} = \phi n m_{Ik}. \tag{3}$$

Note that the presence of n captures the positive externality, mentioned above.

On the other hand, the firm suffers losses due to rival innovations, entrant and incumbent. The number of goods that it loses in a sector k, denoted by Δn_k^{loss}, during Δt is given by

$$\Delta n_k^{\text{loss}} = \left(m_{Ek} + \phi \overline{n}_k \int_{s \in \mathcal{M}_k} m_{Ik}(s)\, ds \right) \times n_k. \tag{4}$$

Let us consider the terms inside the parentheses. m_{Ek} is the number of innovations by entrants in the sector. Turning to the second term, $\overline{n}_k$ denotes the average number of products produced per firm in a sector k. Given that there is a unit continuum of industries in the sector, we have $\overline{n}_k = 1/M_k$ where M_k is the number of firms operating. Next, define $\mathcal{M}_k$ is a set of firms investing in R&D in a sector k, and $\int_{s \in \mathcal{M}} m_{Ik}(s)\, ds$ is the total number of R&D successes in a sector k. Because each research success generates n number of new products for firms with n products with the probability ϕ, the mean of implemented innovation is given by $\phi \overline{n}_k$, and hence $\phi \overline{n}_k \int_{s \in \mathcal{M}_k} m_{Ik}(s)\, ds$ is equivalent to the number of implemented innovations by incumbents in a sector k. All told, the terms inside the parentheses is the total number of newly implemented products in the sector.

Given that those new products are implemented in randomly selected industries, they randomly hit products produced by the incumbent firm we are considering, and its probability is given by n_k, the last term outside the parentheses, because there is a unit continuum of industries in the sector.

Now recall that all sectors are symmetric in the sense that innovation is random, and industries are randomly selected for implementation. Indeed, equilibrium will be symmetric, as shown below, in the sense that $M_k = M$ and $\overline{n}_k = \overline{n}\ \forall k$. Taking advantage of this feature and using (3) and (4), a proportionate rate of change in n_k relative to n during Δt is given by the following random variable

$$\frac{\Delta n_k}{n} = \frac{\Delta n_k^{\text{gain}} - \Delta n_k^{\text{loss}}}{n} = \phi m_{Ik} - \left(m_{Ek} + \phi \frac{1}{M} \int_{s \in \mathcal{M}_k} m_{Ik}(s)\, ds \right) \frac{n_k}{n} \tag{5}$$

We are ready to state the following lemma:

Lemma 1. *The mean and variance of* $\dfrac{\Delta n_\kappa}{n}$ *are given by*

$$\mathbb{E}\left[\frac{\Delta n_k}{n}\right] = \mu_k, \qquad \mathbb{V}\left[\frac{\Delta n_k}{n}\right] = \sigma_k^2 \tag{6}$$

where

$$\mu_k = \phi g_I \Delta t - [g_E \Delta t + \phi g_I \Delta t]\, \frac{n_k}{n}, \tag{7}$$

$$\sigma_k^2 = \phi^2 g_I \Delta t + \left(g_E \Delta t + \frac{\phi^2}{M} g_I \Delta t\right)\left(\frac{n_k}{n}\right)^2. \tag{8}$$

Proof. Taking expectations of (5) gives

$$\mathbb{E}\left[\frac{\Delta n_k}{n}\right] = \phi \mathbb{E}\left[m_{Ik}\right] - \left[\mathbb{E}\left[m_{Ek}\right] + \phi \mathbb{E}\left[m_{Ik}\left(s\right)\right] ds\right] \frac{n_k}{n} \tag{9}$$

using $\int_{s \in \mathcal{M}} ds = M$. The first equation of (6) is obvious from $\mathbb{E}\left[m_{Ik}\right] = g_I \Delta t$ and $\mathbb{E}\left[m_{Ek}\right] = g_E \Delta t$. Similarly, the variance of (5) is given by

$$\mathbb{V}\left[\frac{\Delta n_k}{n}\right] = \phi^2 \mathbb{V}\left[m_{Ik}\right] + \left\{\mathbb{V}\left[m_{Ek}\right] + \frac{\phi^2}{M}\mathbb{V}\left[m_{Ik}\right]\right\}\left(\frac{n_k}{n}\right)^2. \tag{10}$$

$\mathbb{V}\left[m_{Ik}\right] = g_I \Delta t$ and $\mathbb{V}\left[m_{Ek}\right] = g_E \Delta t$ enable us to rewrite it as the second equation of (6). □

The lemma shows that the mean and variance of $\dfrac{\Delta n_k}{n}$ depend on the share of products $\dfrac{n_k}{n}$. This comes from the fact that the number of products lost due to rival innovations is proportional to n_k. Another aspect worth mentioning is that the variance is affected by the number of firms M. The effect is due to the average product per firm $\overline{n}$. The larger the number of firms M, the lower the average number of product $\overline{n}$. Therefore, a higher M means less number of implemented incumbent innovations on average, hence a lower variance of the proportionate change in n_k. This channel is the source of multiple equilibria, as shown below

Let$X_k = \dfrac{\Delta n_k}{n}$. Having identified the first two moments of the distribution of X_k, we turn to the distribution of the random growth rate of n. For this, define

$$S_K = \frac{1}{K}\sum_{k=1}^{K} X_k \tag{11}$$

$$Z_K = \frac{S_K - \mathbb{E}\left[S_K\right]}{\sqrt{\mathbb{V}\left[S_K\right]}} \tag{12}$$

S_K is the average of X_k and is random because X_k is random. Z_K is the standardised random variable of S_K. Though we do not know the distribution of S_K, hence Z_K, we can identify it by invoking the Lingberg-Feller Central Limit Theorem. The result is summarised in the following proposition:

Proposition 1. *As $K \to \infty$, Z_K approaches the standard normal distribution, i.e.*

$$\lim_{K\to\infty} Z_K = Z \sim \mathcal{N}(0,1) \tag{13}$$

Proof. Define $\chi_k \equiv X_k - \mu_k$. The Lindberg condition in our case is

$$\lim_{K\to\infty} \frac{1}{\mathbb{V}[S_K]} \sum_{k=1}^{K} \mathbb{E}\left\{ \left(\frac{\chi_k}{K}\right)^2 \cdot \mathbf{1}\left[\left|\frac{\chi_k}{K}\right| > \epsilon\sqrt{\mathbb{V}[S_K]}\right]\right\} = 0 \qquad \forall \epsilon > 0 \tag{14}$$

where $\mathbf{1}[.]$ is an indicator function. (14) is the necessary and sufficient condition for (13).[2] Given that Δt is a short time interval such that $m_{E\kappa} = \{0,1\}$ and $m_{I\kappa} = \{0,1\}$ and using (9), we have

$$\begin{aligned} \max[\chi_k] &= \phi + g_E \Delta t \equiv \chi_k^+ > 0, \\ \min[\chi_k] &= -(1+\phi) - \phi g_I \Delta t \equiv \chi_k^- < 0. \end{aligned}$$

It means that $|\chi_k|$ is bounded such that

$$|\chi_k| \le \max\left\{\chi_k^+, \left|\chi_k^-\right|\right\} \equiv \overline{\chi} < \infty \qquad \forall k. \tag{15}$$

Moreover, we have the following from (10)

$$\max\{\mathbb{V}[X_k]\} = \Delta t + \phi^2 \left(1 + \frac{1}{M}\right) g_I \Delta t < \infty \tag{16}$$

which is independent of K. Using these, we rewrite (14) as

$$\begin{aligned} &\lim_{K\to\infty} \frac{1}{\mathbb{V}[S_K]} \sum_{k=1}^{K} \mathbb{E}\left\{ \left(\frac{\chi_k}{K}\right)^2 \cdot \mathbf{1}\left[\left|\frac{\chi_k}{K}\right| > \epsilon\sqrt{\mathbb{V}[S_K]}\right]\right\} \\ &\qquad \le \lim_{K\to\infty} \frac{1}{\mathbb{V}[S_K]} \sum_{k=1}^{K} \left\{ \left(\frac{\overline{\chi}}{K}\right)^2 \cdot \mathbf{1}\left[\left|\frac{\chi_k}{K}\right| > \epsilon\sqrt{\mathbb{V}[S_K]}\right]\right\} \\ &\qquad \le \lim_{K\to\infty} \frac{1}{\mathbb{V}[S_K]} \sum_{k=1}^{K} \left\{ \left(\frac{\overline{\chi}}{K}\right)^2 \cdot \mathbf{1}\left[\frac{\overline{\chi}}{K} > \epsilon\sqrt{\mathbb{V}[S_K]}\right]\right\} \\ &\qquad = 0 \end{aligned}$$

The first line is due to $\mathbb{E}\left[\chi_k^2\right] \le \overline{\chi}^2$ and the second line uses (15). The last line holds because (1) $\mathbb{V}[S_K] < \infty$ due to (16), and (2) $\mathbf{1}\left[\frac{\overline{\chi}}{K} > \epsilon\sqrt{\mathbb{V}[S_K]}\right] = 0$ as $K \to \infty$. The Lindberg condition (14) is met. □

In general, the Lindberg condition implies that values in both tails have little influence on the variance. This holds in our case because the maximum value of X_k is finite and each random variable has less and less impacts as K goes to infinity.

[2]For example, see Spanos (1986).

Having established Proposition 1, we need to accommodate $K \to \infty$ in our model. For this, we assume $K\Delta k = 1$. This assumption reduces the production function (1) and (2) to

$$\ln(Y) = \int_0^1 \left(\int_0^1 \ln\left[q(j,k)\,x(j,k)\right] dj \right) dk. \tag{17}$$

$K \to \infty$ allows us to introduce an infinitely many manufacturing sectors in a unit continuum $k \in [0,1]$. In line with $K \to \infty$, therefore, the number of products, n, produced by an incumbent firm products is now defined as

$$n = \lim_{K\to\infty} \sum_{k=1}^{K} n_k \Delta k = \int_0^1 n(k)\, dk \tag{18}$$

where $n(k)$ is the number of goods in a sector k. Note that n is equivalent to the average of $n(k)$. Accordingly, the random variable $\frac{\Delta n_k}{n}$ in (5) is replaced with $\frac{\Delta n(k)}{n}$ in an infinitesimally small sector interval dk.

Based on those results, we state the next lemma.

Lemma 2. *The following holds:*

$$\lim_{K\to\infty} \mathbb{E}[S_K] = \mu\Delta t, \qquad \lim_{K\to\infty} \mathbb{V}[S_K] = \sigma^2\Delta t$$

where

$$\mu = -g_E \tag{19}$$

$$\sigma^2 = \frac{1}{3} g_E + \phi^2 \left(1 + \frac{1}{3M}\right) g_I \tag{20}$$

Proof. Using (11),

$$\begin{aligned}
\lim_{K\to\infty} \mathbb{E}[S_K] &= \lim_{K\to\infty} \sum_{k=1}^{K} \mathbb{E}[X_k]\,\Delta k = \int_0^1 \mu_k dk \\
&= \phi g_I \Delta t - (g_E \Delta t + \phi g_I \Delta t) \int_0^1 \frac{n(k)}{n} dk \\
&= -g_E \Delta t
\end{aligned}$$

where the third equality uses (7). Similarly,

$$\begin{aligned}
\lim_{K\to\infty} \mathbb{V}[S_K] &= \lim_{K\to\infty} \sum_{k=1}^{K} \mathbb{V}[X_k]\,\Delta k = \int_0^1 \sigma_k^2 dk \\
&= \phi^2 g_I \Delta t + \left(g_E \Delta t + \frac{\phi^2}{M} g_I \Delta t\right) \int_0^1 \left(\frac{n(k)}{n}\right)^2 dk \\
&= \phi^2 g_I \Delta t + \left(g_E \Delta t + \frac{\phi^2}{M} g_I \Delta t\right) \int_0^1 k^2 dk
\end{aligned}$$

$$= \left[\frac{1}{3}g_E + \phi^2\left(1 + \frac{1}{3M}\right)g_I\right]\Delta t$$

where the third equality makes use of (8). The linear change of variable $\frac{n(k)}{n} = k$ is used in the third line because $n(k)$ is randomly distributed over a unit continuum of sectors $[0, 1]$. □

The lemma shows that the mean of $\frac{\Delta n}{n}$ is negative, implying that on average the firm suffers the loss of all products in all sectors, resulting in an exit from the market on average. This is required for a stationary distribution of firm sizes because it will be modelled as a diffusion process in a GBM. Conveniently μ and σ^2 are both constant, and its significance should be clear in the following proposition:

Proposition 2. *The following holds:*

$$\lim_{K\to\infty} S_K = S \sim \mathcal{N}\left(\mu\Delta t, \sigma^2\Delta t\right). \tag{21}$$

Proof. The result is clear from Proposition 1 and Lemma 2. □

We have established that the growth rate of a firm size follows a normal distribution. This result is crucial in deriving the GBM of n.

2.3 The Geometric Brownian Motion of Firm Size

Proposition 2 concerns what happens during a small time interval Δt. We turn to the whole time period up to t starting from the initial time $t_0 = 0$. Let us define

$$G_T(t) = \sum_{\tau=0}^{T} \frac{\Delta n(\tau\Delta t)}{n(\tau\Delta t)}, \qquad n(0) = 1 \tag{22}$$

where $\Delta n(t) = n(t + \Delta t) - n(t)$. Recalling $T\Delta t = t$, $G_T(t)$ is a cumulative rate of growth of n up to time t. We are ready state the next lemma:

Lemma 3. *The following holds:*

$$\lim_{T\to\infty} G_T(t) = G(t) \sim \mathcal{N}\left(\mu t, \sigma^2 t\right). \tag{23}$$

Proof. It is obvious using the fact that the sum of normally distributed random variables are normally distributed. □

The cumulative growth rate of firm size also follows a normal distribution, and this is an indispensable ingredient for the GBM of n. Given the definition of $G(t)$, the size of a firm n is given by

$$n(t) = e^{G(t)} \tag{24}$$

where $n(0) = 1$ requires $G(0) = 0$. It implies that the size of a firm is defined by a log-normal distribution. Using (23) and (24), the next proposition follows:

Proposition 3. *The number of products produced by an incumbent firm n obeys*

$$dn(t) = \theta n(t)\, dt + \sigma n(t)\, dW(t) \tag{25}$$

where

$$\theta = \mu + \frac{\sigma^2}{2} \tag{26}$$

and $W(t)$ is the Wiener process.

Proof. First define

$$W(t) = \frac{G(t) - \mu t}{\sigma} \sim \mathcal{N}(0, t) \tag{27}$$

Using (27), rewrite (24) as

$$n(t) = e^{\mu t + \sigma W(t)} \equiv \Psi(t, W(t)).$$

$\Psi_t(t, W(t)) = \mu n(t)$, $\Psi_W(t, W(t)) = \sigma n(t)$ and $\Psi_{WW}(t, W(t)) = \sigma^2 n(t)$. Using Ito's Formula[3]

$$\begin{aligned}\Psi(t, W(t)) - \Psi(0, W(0)) = &\int_0^t \left(\Psi_t(t, W(t)) + \frac{1}{2}\Psi_{WW}(t, W(t))\right) dt \\ &+ \int_0^t \Psi_W(t, W(t))\, dW(t)\end{aligned}$$

which is expressed as a differential form (25). □

Note that θ is the sum of the mean of $\dot{n}/n$ and a half of its variance. This result is due to the fact that the cumulative growth rate of n or $G(t)$ in (23) is normally distributed, and hence n itself is defined by a log-normal distribution (see (24)). The term $\sigma^2/2$ is called the Ito correction term and arises because of the convexity of the exponential function.

Proposition 3 is one of the key results of the paper. We started from the otherwise standard quality-ladder framework, and an additional assumption regarding "many" manufacturing sectors enables the model itself to serve as a microfoundation of the GBM of the size of a firm.

2.4 A Pareto Distribution of Firm Size

It is well known that if a random variable follows the GBM, its distribution is Pareto. To show this, we follow Gabaix (2009). Let the distribution of n be denoted by $f(n, t)$, which is the density function of firms producing n products at time t. Its change is governed by the Kolmogorov forward equation

$$\frac{\partial f(n, t)}{\partial t} = -\frac{\partial [\theta n f(n, t)]}{\partial n} + \frac{1}{2} \cdot \frac{\partial^2 \left[\sigma^2 n^2 f(n, t)\right]}{\partial n^2}. \tag{28}$$

[3] For example, see Theorem 7.5 of Brzezniak and Zastawniak (2000)

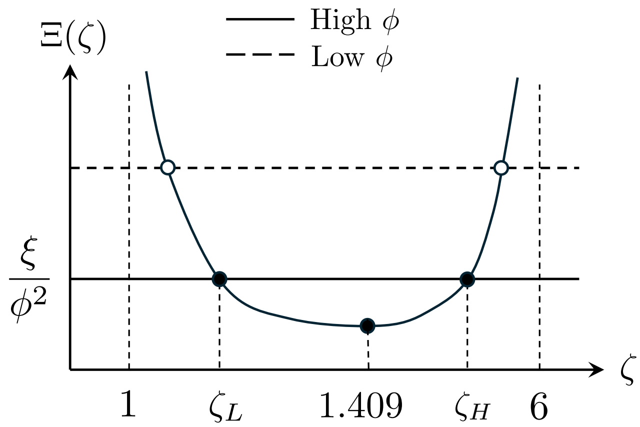

Figure 1: Determination of the Pareto exponent ζ.

We have $\dfrac{\partial f(n,t)}{\partial t} = 0$ in steady state. Using the candidate solution $f(n) = Cn^{-\zeta-1}$, it is easy to show

$$0 = \mu + \frac{\sigma^2}{2}\zeta \tag{29}$$

which defines the Pareto exponent ζ. Note that the number of incumbent firms M is given by $M = \int_1^\infty f(n)\,dn$, which allows us to pin down the value $C = M\zeta$. Therefore, the distribution of n is given by

$$f(n) = M\zeta n^{-\zeta-1}. \tag{30}$$

Recall that all sectors are symmetric, and hence (30) holds in each of all sectors $k \in [0,1]$.

To express the number of firms M in terms of the Pareto exponent, note that $\int_1^\infty nf(n)\,dn = 1$ where $nf(n)$ is equivalent to the number of goods produced by firms with n products. The LHS is equated to the total number of intermediate goods in each sector , 1 on the RHS. One can easily confirm that

$$M = 1 - \frac{1}{\zeta}. \tag{31}$$

To understand (31), recall that the number of goods n produced by a firm is countable in a continuum of $(j,k) \in [0,1] \times [0,1]$, and that firms produce one or more products. Therefore, we must have $M < 1$, and $M = 1$ holds only when each of all firms produces a single product. (31) shows that the number of firms is increasing in the Pareto exponent ζ. An intuition goes as follows. A higher ζ implies that a tail is thinner with less of larger firms, and that more firms are smaller than otherwise. Note that monopoly profits are also Pareto-distributed with the exponent ζ because they are proportional to the number of products, i.e. $n\pi$.

Now, making use of (19), (20), (26), (29) and (31), one obtains

$$\frac{\xi}{\phi^2} = \frac{\left(3 + \frac{\zeta}{\zeta-1}\right)\zeta}{6-\zeta} \equiv \Xi(\zeta) \tag{32}$$

where

$$\xi \equiv \frac{g_E}{g_I}. \tag{33}$$

This equation should be interpreted to determine the Pareto exponent ζ, taking ξ and ϕ as given. That is, (32) captures the mechanism which translates the Poisson arrival rates g_E and g_I into the distribution of firm size. Figure 1 shows (32). It takes a U shape with asymptotes at 1 and 6. (32) also shows that (i) two values of ζ, denoted by ζ_L and ζ_H, are consistent with a given value of ξ, as long as the condition in the next lemma is met ($\zeta_L = \zeta_H$ is also possible), and (ii) a given value of ζ is consistent with "many" combinations of g_E and g_I as long as their ratio is constant.

Lemma 4. *For a given value of ξ, the existence of real-valued ζ requires*

$$\phi \in \left(0, \min\left\{1, \sqrt{\left(3 - \sqrt{\frac{56}{9}}\right)\xi}\right\}\right]. \tag{34}$$

Proof. Rewrite (32) as $a(\phi)\zeta^2 + b(\phi)\zeta + c = 0$ where $a(\phi) = \xi + 4\phi^2$, $b(\phi) = -\left(7\xi + 3\phi^2\right)$ and $c = 6\xi$. Then, real-valued ζ exists for

$$h(\phi) = b(\phi)^2 - 4a(\phi)c \geq 0 \tag{35}$$

which is a fourth-degree polynomial equation in ϕ. The coefficient of ϕ^4 is positive, hence $\lim_{\phi\to-\infty} h(\phi) = \lim_{\phi\to\infty} h(\phi) = \infty$. We have $\dfrac{\partial h(\phi)}{\partial \phi} = \left[6\left(7\xi + 3\phi^2\right) - 96\xi\right]2\phi$, which means that the local maximum is achieved at $\phi = 0$ and $h(0) = 25\xi^2 > 0$. To find roots for $h(\phi) = 0$, rewrite the equality part of (35) as $\tilde{h}(\Phi) = A\Phi^2 + B\Phi + C$ where $\Phi \equiv \phi^2$, $A = 9$, $B = -54\xi$ and $C = 25\xi^2$. Then, the roots for $\tilde{h}(\Phi) = 0$ are $\Phi_1, \Phi_2 = \left(3 \pm \sqrt{\frac{56}{9}}\right)\xi$. Using the definition of $\Phi \equiv \phi^2$, the roots for $h(\phi) = 0$ are given by

$$\phi_1, \phi_2 = \pm\sqrt{\left(3 + \sqrt{\frac{56}{9}}\right)\xi}, \qquad \phi_3, \phi_4 = \pm\sqrt{\left(3 - \sqrt{\frac{56}{9}}\right)\xi}$$

Recalling $\phi \in (0, 1]$, the condition (34) follows. □

This lemma shows a meaningful value of ζ is guaranteed, as long as the strength of the positive externality ϕ is sufficiently small. An interesting implication is that the mere existence of the externality, however small it is, is a necessary and sufficient condition for an equilibrium. Given Lemma 4, the next result is obvious.

Proposition 4. *(1) Multiple equilibria exist such that $\zeta_L < \zeta_H$ if (35) holds with a strict inequality, and (2) $\zeta_L = \zeta_H$ if the inequality is replaced with an equality in (35).*

Let us develop an intuition for multiple values for the Pareto exponent. This is basically due to the positive externality in incumbent R&D, which is captured in the form of n in (3), and in particular $\overline{n}_k$ in (4). To explain its role, let us distinguish three types of creative destruction of innovation, which affect the Pareto exponent ζ in (32). Consider entrant firms. As they enter the market, their higher-quality goods replace the existing goods. Incumbents that suffer losses become smaller in size and even exit the market. It is easy to understand that a higher g_E bring more of younger and smaller firms, tending to increase ζ. This is the first creative destruction effect.

The second and third effects are due to incumbent innovation. To isolate them, let us consider first the situation where there was no positive externality, i.e. each incumbent R&D success generates a single innovative product rather than multiple goods for $n \geq 2$. Then, a higher g_I tends to reduce the Pareto exponent ζ because incumbents grow faster and forcing "marginal" firms with a few products to exit. As a result, there are less of smaller firms and more of larger firms in the right tail. Conversely, a lower g_I raises ζ. This is the second creative destruction effect. Note that a higher g_E and a lower g_I work in the same way in that the Pareto exponent ζ and the number of firms M both increase.

Introducing the positive externality into the picture makes it possible to identify the third creative destruction effect. Now each incumbent R&D success generates ϕn number of new goods on average. Such externality seems to reinforce the second creative destruction effect, but it turns out that it affects the Pareto exponent in a different way. Remember the following relationship:

$$\overline{n} = \frac{1}{M} = \frac{\zeta}{\zeta - 1} \qquad \forall k. \tag{36}$$

$\overline{n}$ is the number of products per firm which is equivalent to the inverse of the number of firms $1/M$, which in turn negatively related to ζ (see (31)). Consider a higher g_I. The Pareto exponent ζ tends to fall due to the second creative destruction effect, as explained above. This reduces the number of firms M falls, and the resulting increase in $\overline{n}$ causes the following ramifications. Each incumbent R&D success creates a greater number of innovative goods on average, and that all incumbents become more likely to lose their own goods due to rival R&D successes than otherwise. As a result, ceteris paribus, firms grow slower, and even shrink in size, falling leftward in the distribution of firm size, i.e. less of larger firms. This is the third creative destruction, induced by the externality, and tends to raise the Pareto exponent ζ. A similar story holds for entrant innovation. A *lower* g_E reduces the pool of firms M via the first creative destruction effect, increasing the average number of products $\overline{n}$. Less entry intensifies the third creative destruction effect. Conversely, a higher g_E and/or a lower g_I weakens the third creative destruction effect and helps incumbents consolidate their monopoly positions *on average*, reducing ζ. Note that this mechanism exists due to the externality captured by n in (3), however small its strength ϕ is. Using $\xi \equiv g_E/g_I$, let us summarise the three effects:

1. In the first creative destruction effect, a higher g_E raises ξ and tends to increase ζ.

2. In the second creative destruction effect *without* the externality effect, a lower g_I raises ξ and tends to increase ζ.

3. In the third creative destruction effect *with* the externality effect, a lower g_I raises ξ and tends to decrease ζ.

In Figure 1, the third creative destruction effect dominates the other two effects at ζ_L, and the reverse holds at ζ_H. Now, let us consider the impact of a lower ϕ. It shifts up the horizontal line. If we focus on ζ_L, a sufficiently small ϕ can make the Pareto exponent ζ_L arbitrarily close to 1, which is the empirically relevant value (more on this point later). This statement, however, does not take into account how ξ responds to ϕ. In addition, the deviation of ζ_H from 1 gets larger. This highlights the importance of stability analysis of an equilibrium. To examine those issues, we next endogenise g_E and g_I to develop a full-fledged Schumpeterian model.

3 Endogenising g_E and g_I

3.1 Consumers

The number of consumers is normalised to one, and they maximise the intertemporal utility with a logarithmic felicity function. This assumption gives the Euler condition

$$\frac{\dot{E}}{E} = r - \rho \tag{37}$$

where E is equivalent to aggregate consumption expenditure, r is the interest rate and ρ is the rate of time preference. Consumers are identical except that L of them are unskilled and H are skilled. Both L and H are constant, and $L + H = 1$. The assumption of heterogeneous workers enables us to show our key results in the simplest possible setup. The appendix develops the model with homogeneous workers and establishes that the key results remain intact.

3.2 Intermediate Goods

Final output Y is competitively produced. Normalising its price $P_Y = 1$, a demand function of intermediate product is given by

$$x(j,k) = \frac{E}{p(j,k)} \tag{38}$$

where $E = Y$ and $p(j,k)$ is the price of $x(j,k)$.

One unit of intermediate goods is produced with an unskilled worker. Like other Schumpeterian models, firms charge the price with the quality step λ as a constant markup over the marginal cost, i.e. $p(j,k) = \lambda w_L$ where w_L is unskilled wage. Profit per product is

$$\pi = \Lambda E \tag{39}$$

where $\Lambda \equiv 1 - \dfrac{1}{\lambda}$. This is equivalent to π in Section 2.1.

3.3 Incumbent R&D

After entry, a firm starts incumbent R&D in all sectors, employing skilled workers, in order to increase profits further. Consider a firm producing $n \geq 1$ products. An R&D success in each sector follows a Poisson process with an arrival rate of

$$g_I(k) = \delta_I \left(\frac{R_I(k)}{n} \right)^{\gamma}, \qquad \delta_I > 0, \quad 0 < \gamma < 1 \tag{40}$$

where $R_I(k)$ is the number of skilled workers used in a sector k. The presence of n in the denominator on the RHS is due to the negative congestion externality. As the portfolio of goods expands, it gets harder to generate an additional research success. As explained above, if successful in a given sector, n number of new goods are created and ϕn are implemented on average in randomly selected industries in the sector, raising quality of those goods by a factor λ.

Let us consider an R&D decision facing an incumbent firm with n products. Let V_n denote the value of the firm, which is defined as

$$rV_n = n\pi - w_H \int_0^1 R_I(k)\, dk + \mathbb{E}\left[\frac{dV_n}{dt}\right] \tag{41}$$

where w_H is skilled wage. $\mathbb{E}\left[\frac{dV_n}{dt}\right]$ is an Ito calculus term and represents the capital gain/loss due to the GBM of n. The next lemma shows what is involved in the term.

Lemma 5. *The Ito calculus term is given by*

$$\begin{aligned} \mathbb{E}\left[\frac{dV_n}{dt}\right] = \left.\frac{\partial V_n}{\partial t}\right|_{n \text{ fixed}} &+ \left[-g_E + \frac{g_E}{6} + \frac{\phi^2 \overline{g}_I}{6M} + \frac{\phi^2}{2} \int_0^1 \delta_I \left(\frac{R_I(k)}{n}\right)^{\gamma} dk\right] \frac{\partial V_n}{\partial n} n \\ &+ \left(\frac{g_E}{6} + \frac{\phi^2 \overline{g}_I}{6} + \frac{\phi^2}{2} \int_0^1 \delta_I \left(\frac{R_I(k)}{n}\right)^{\gamma} dk\right) \frac{\partial^2 V_n}{\partial n^2} n^2 \end{aligned} \tag{42}$$

where $\overline{g}_I$ is the arrival rate for rival incumbent firms.

Proof. By Ito's Lemma, we have

$$dV_n = \frac{\partial V_n}{\partial t} dt + \frac{\partial V_n}{\partial n} \left[\theta n(t)\, dt + \sigma n(t)\, dW(t)\right] + \frac{1}{2} \frac{\partial^2 V_n}{\partial n^2} \sigma^2 n(t)^2\, dt$$

where we used (25), $(dt)^2 = dt \cdot dW(t) = 0$ and $(dW(t))^2 = dt$. Dividing both sides by dt and taking expectations, we have

$$\mathbb{E}\left[\frac{dV_n}{dt}\right] = \left.\frac{\partial V_n}{\partial t}\right|_{n \text{ fixed}} + \frac{\partial V_n}{\partial n} \theta n(t) + \frac{1}{2} \frac{\partial^2 V_n}{\partial n^2} \sigma^2 n(t)^2 \tag{43}$$

where $\mathbb{E}[dW(t)] = 0$ is used. The first term on the RHS takes n as given. Making use of (19), (20) and (26), (43) can be rearranged into (42). □

(41) and (42) show that the incumbent firm can increase the capital gain term by investing in R&D, incurring skilled wage costs. The optimal number of skilled workers can be found by

maximising the RHS of (41) by choosing $R_I(k)$. The first-order condition is

$$w_H = \left(\frac{\phi^2}{2}\gamma\delta_I R_I^{\gamma-1}\right)\left(\frac{\partial V_n}{\partial n} + \frac{\partial^2 V_n}{\partial n^2}n\right) \tag{44}$$

where

$$R_I \equiv \frac{R_I(k)}{n} \qquad \forall k. \tag{45}$$

(45) implies that R&D workers per product is the same in all sectors and for all incumbent firms. This result allows us to rewrite (40) as

$$g_I = \delta_I R_I^{\gamma} \qquad \forall k. \tag{46}$$

This is indeed equivalent to g_I in Section 2.1. Because each of n products equally contributes to the value of the firm V_n by generating a stream of profit π, equilibrium must be characterised by $V_n = nv$ where v is the contribution of a single product. Using this, it is obvious that $\dfrac{\partial V_n}{\partial n} = v$ and $\dfrac{\partial^2 V_n}{\partial n^2} = 0$. Now, the FOC (44) is reduced to

$$w_H = \frac{\phi^2\gamma\delta_I^{\frac{1}{\gamma}}}{2g_I^{\frac{1-\gamma}{\gamma}}}v. \tag{47}$$

The RHS is the expected marginal benefit of employing an additional skilled worker, and the LHS is the associated marginal cost. The RHS negatively depends on the Poisson arrival rate g_I because of the diminishing marginal product of skilled workers. This optimal condition holds for all incumbent firms.

Note also that $V_n = nv$ means $\left.\dfrac{\partial V_n}{\partial t}\right|_{n \text{ fixed}} = n\dot{v}$ in (42). Using this result, (26), (29), (41) and (47), the value of a single product to the the firm, v, is determined by the following asset equation

$$r = \frac{\pi}{v} - \frac{w_H}{v}R_I + \frac{\dot{v}}{v} - g_E + \frac{\Sigma(\zeta, g_E, g_I)}{2} \tag{48}$$

where

$$\sigma^2 = \frac{1}{3}g_E + \phi^2\left(1 + \frac{\zeta}{3(\zeta-1)}\right)g_I \equiv \Sigma(\zeta, g_E, g_I) \tag{49}$$

is derived from (20) and (29). The RHS of (48) is the overall return from producing a single product to the firm. The first term depends on profit per product and the second term captures R&D costs. The remaining terms represent capital gains/losses. $\dot{v}/v$ reflects the cost side of R&D in equilibrium. $-g_E$ captures a risk of losing profit because of entrant innovation. The last term is due to the GBM of n, without which the model collapses to a standard quality-ladder model. Lemma 5 shows that the expected change in V_n follows a stochastic process. Given $V_n = nv$, v goes through similar stochastic changes, and such changes appear in the form of $\Sigma(\zeta, g_E, g_I)$. Its positive impact is realised via the convexity of exponentiating $G(t)$ in (23), i.e. Jensen's inequality. Intuitively, such convexity amplifies the effect of an additional product gained through R&D.

3.4 Entrant R&D

To enter a sector k, a firm has to be successful in R&D first. It employs skilled workers, and each of them brings about an R&D success with a Poisson arrival rate of $\delta_E > 0$. Let $R_E(k)$ be the total number of skilled worker employed in entrant R&D in a sector k. The Poisson rate in the sector as a whole is

$$g_E(k) = \delta_E R_E(k). \tag{50}$$

Once successful in R&D, innovation is implemented in an industry randomly selected from $[0,1]$ in a sector k.[4] The production of a higher quality intermediate good begins with profit π accruing to the firm, rendering the existing goods obsolete.

Consider a firm i employing $R_E^i(k)$ skilled workers. It generates a Poisson arrival rate of $\delta_E R_E^i(k)$, incurring costs of $w_H R_E^i(k)$. Free entry leads to

$$\delta_E v = w_H, \qquad \forall k. \tag{51}$$

Its LHS is the expected benefit of entry and its associated cost is on the RHS. This condition holds in all sections, which means $g_E(k) = g_E \; \forall k$. g_E is equivalent to g_E introduced in Section 2.1.

3.5 Labour Market

Intermediate goods are produced using unskilled workers only. Their full employment requires

$$L = \frac{E}{\lambda w_L}. \tag{52}$$

Its RHS is the unskilled labour demand. Skilled workers are used in R&D activities. Those who engage in entrant R&D is $\int_0^1 R_E(k)\,dk = \int_0^1 (g_E/\delta_E)\,dk = g_E/\delta_E$ using (50).

Turning to incumbent R&D workers, consider a sector k. An incumbent firm with n products employs $R_I(k) = \left(\frac{g_I}{\delta_I}\right)^{\frac{1}{\gamma}} n$ from (45) and (46). Summing all workers across all firms gives $\int_0^\infty R_I(k) f(n)\,dn = \int_0^\infty \left(\frac{g_I}{\delta_I}\right)^{\frac{1}{\gamma}} n f(n)\,dn = \left(\frac{g_I}{\delta_I}\right)^{\frac{1}{\gamma}}$ because $\int_0^\infty n f(n)\,dn = 1$, i.e. the total number of products in a sector k is one. Finally, summing over all sectors, the total number of skilled workers employed in incumbent R&D is given by $\int_0^1 \left(\frac{g_I}{\delta_I}\right)^{\frac{1}{\gamma}} dk = \left(\frac{g_I}{\delta_I}\right)^{\frac{1}{\gamma}}$. Therefore, skilled workers are fully employed for

$$H = \frac{g_E}{\delta_E} + \left(\frac{g_I}{\delta_I}\right)^{\frac{1}{\gamma}}. \tag{53}$$

[4] We could assume free entry in each and all industries, but there is no significant change in key results.

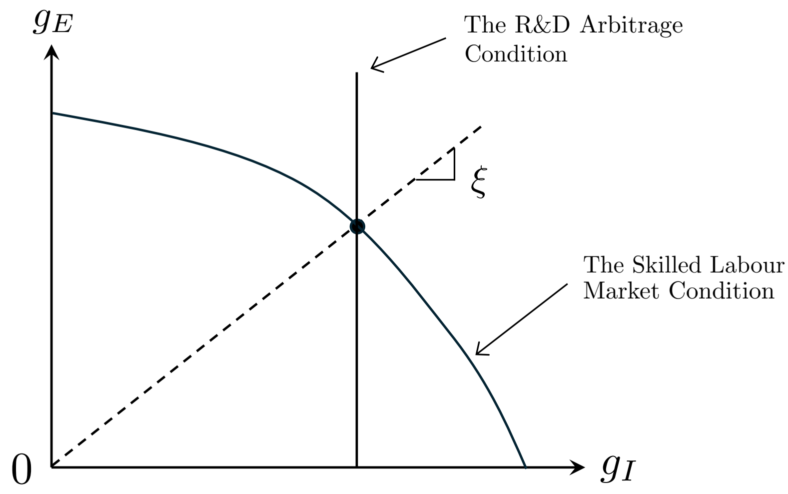

Figure 2: Determination of the Poisson arrival rates g_E and g_I.

3.6 Equilibrium

The model can be solved with four equilibrium conditions for four endogenous variables, g_E, g_I, ζ and $\omega \equiv w_L/w_H$. Use (47) and (51) to derive what we call the R&D arbitrage condition

$$\frac{2}{\phi^2 \gamma \delta_I^{\frac{1}{\gamma}}} g_I^{\frac{1-\gamma}{\gamma}} \left(= \frac{v}{w_H} \right) = \frac{1}{\delta_E}. \tag{54}$$

This condition equalises returns from incumbent and entrant R&D, making incumbent firms indifferent between incumbent and entrant R&D. We assume that incumbent firms engage in incumbent R&D only in what follows. The condition (54) defines the equilibrium value of g_E and g_I along with the skilled market labour condition (53). Figure 2 shows a unique equilibrium. The slope of the ray from the origin is equivalent to $\xi \equiv g_E/g_I$. Once ξ is found, (32) determines ζ, i.e. ζ_L and ζ_H in Figure 1.

Next we derive the condition which determines ω. Use (37), (39), (47), (51) and (48) to obtain

$$\frac{\dot{\omega}}{\omega} = \delta_E \Lambda L \lambda \omega - \frac{\phi^2}{2} \gamma g_I - g_E + \frac{\Sigma \left(g_E, g_I, \zeta \right)}{2} - \rho. \tag{55}$$

ω follows this condition, taking g_E, g_I and ζ as given.

To derive the growth rate of output, rewrite the production function (17) as $\ln(Y) = \int_0^1 \int_0^1 \{\ell(j,k) \ln[q(j,k)] + \ln(L)\} \, dj dk$, using (38) and (52). Considering a small time interval dt, the rate of growth in Y is given by $g_Y \equiv \frac{\dot{Y}}{Y} = (\lambda - 1)(m_E^e + m_I^e)$ where m_E^e and m_I^e are the expected number of innovative products created via entrant and incumbent R&D, respectively. The presence of λ is due to the fact that each innovation improves the quality of goods by a factor λ.[5] It should be clear that $m_E^e = g_E$. Regarding m_I^e, it is given by $m_I^e = \phi \overline{n} \int_{s \in \mathcal{M}} m_I(s) \, ds$, which is equivalent to the second term inside the parenthesis in (4)

[5]To be more precise, we have $m_E^e \equiv \int_0^1 \int_0^1 m_E(j,k) \, dj dk$ and $\hat{m}_I \equiv \int_0^1 \int_0^1 m_I(j,k) \, dj dk$ where $m_E(j,k)$ and $m_I(j,k)$ are the number of implemented innovations in an industry j in a sector k due to entrant and incumbent R&D, respectively.

if $\mathcal{M}_k$, a set of incumbents in a sector k, is replaced with $\mathcal{M}$, a set of firms in the economy as a whole. Using $\overline{n} = 1/M$ and (31), the growth rate is

$$g_Y = (\lambda - 1)\left(g_E + \phi \frac{\zeta}{\zeta - 1} g_I\right). \tag{56}$$

It should be intuitively clear that g_Y increases in g_E, g_I and ϕ. The average number of products produced by incumbents $\overline{n} = \dfrac{\zeta}{\zeta - 1}$ is due to the positive externality. An intuition for g_Y falling in ζ goes as follows. A higher ζ means a thinner Pareto tail, i.e. more of smaller firms and less of larger firms, which in turn implies a fall in the average number of innovation, decelerating growth.

3.7 Local Stability

In the above analysis, the Pareto exponent can take either ζ_L or ζ_H, ignoring the case of $\zeta_L = \zeta_H$. This section explores their stability property to investigate which Pareto exponent is likely to prevail in equilibrium. (32) shows that ζ depends on g_E and g_I only, which in turn are pinned down by (53) and (54). An important property is that g_E and g_I are constant in as well as off steady state. Taking advantage of this, we follow a heuristic approach to examine stability. Assume that the economy is not in steady state, but sufficiently close to it such that the distribution of n is given by

$$f(n,t) = [\zeta(t) - 1]\, n^{-\zeta(t)-1} \tag{57}$$

where $\zeta(t)$ is a function of time and differs from ζ_L and ζ_H. This function collapses to a steady state Pareto distribution, defined by (30) and (31). In a nutshell, we are considering the economy in the neighbourhood of a steady state where n is Pareto-distributed but with an exponent different from ζ_L and ζ_H. Note that $f(n,t)$ in (57) still follows the Kolmogorov forward equation (28). Given this observation, we state the next lemma:

Lemma 6. *The following differential equation dictates the evolution of $\zeta(t)$:*

$$\dot{\zeta}(t) = \left[-g_E + \frac{\Sigma(\zeta(t), g_E, g_I)}{2}\zeta(t)\right][\zeta(t) - 1]\,\zeta(t)^2 \equiv \Psi(\zeta(t)). \tag{58}$$

Proof. Evaluating both sides of (28) with the use of (57), we have

$$\dot{\zeta}(t)\left(\frac{1}{\zeta(t) - 1} - \log(n)\right) f(n,t) = \left[-g_E + \frac{\Sigma(\zeta(t), g_E, g_I)}{2}\zeta(t)\right]\zeta(t)\, f(n,t).$$

where (19) and (49) are used. Integrating both sides w.r.t. n from 1 to infinity gives (58) after using $\int_1^\infty f(n,t)\,dn = M(t) = 1 - 1/\zeta(t)$ and $\int_1^\infty \left[\log(n)\, n^{-\zeta(t)-1}\right] dn = 1/\zeta(t)^2$. □

A convenient feature of (58) is that it alone determines the nature of local stability of $\zeta(t)$, given g_E and g_I. To explain (58) further, consider what if σ^2 or $\Sigma(.)$ in (49) was independent of ζ. In this hypothetical case, ζ would be fixed by $\zeta = 1 - 2\theta/\sigma^2$ (where the RHS is independent

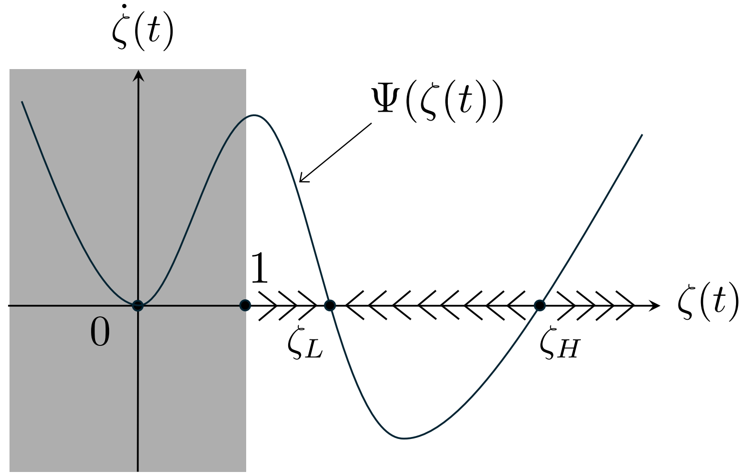

Figure 3: The local dynamics of $\zeta(t)$.

of ζ), which would replace (32). Obviously the "gradual" dynamics of ζ does not arise. In this sense, the differential equation (58) makes sense due to the dependence of σ^2 or $\Sigma(.)$ on ζ in the (49).

We are ready to state the next proposition.

Proposition 5. *ζ_L is locally stable, and ζ_H is locally unstable.*

To explain this proposition, consider $\dot{\zeta}(t) = 0$ or

$$0 = \Psi(\zeta(t)). \tag{59}$$

Basically it is a fourth-degree polynomial equation. It is easy to confirm the following properties:

1. the coefficient of ζ^2 is positive, i.e. $\lim_{\zeta \to -\infty} \Psi(\zeta) = \lim_{\zeta \to \infty} \Psi(\zeta) = \infty$.

2. there are two repeated roots of 0, corresponding to $0 = \zeta^2$.

3. The remaining two roots are defined by $0 = \left[-g_E + \frac{\Sigma(\zeta, g_E, g_I)}{2}\zeta\right](\zeta - 1)$, which is in fact equivalent to (32).[6] It means that those roots are ζ_L and ζ_H.

These observations allow us to draw Figure 3. The shaded area is irrelevant for our analysis. The figure confirms Proposition 5

3.8 Key Comparative Statics in Steady State

Table 1 summarises key comparative static results in steady state with $\dot{\zeta} = \dot{\omega} = 0$. An increase in ϕ encourages incumbent R&D, and hence, skilled workers are diverted from entrant R&D with a drop in g_E. In Figure 2, the R&D arbitrage condition shifts rightward with a lower ξ. Consequently, ζ_L rises, as can be confirmed in Figure 1. The effect of changing δ_E and δ_I involves general equilibrium effects. Consider a higher δ_I. It makes incumbent R&D relatively

[6] $(\zeta - 1)$ does not give a root because of the presence of $(\zeta - 1)$ in the denominator in σ^2 or $\Sigma(\zeta, g_E, g_I)$.

	ϕ	δ_I	δ_E	H	L	ρ	λ
g_I	$+$	$+$	$-$	0	0	0	0
g_E	$-$	$-$	$+$	$+$	0	0	0
ξ	$-$	$-$	$+$	$+$	0	0	0
ζ_L	$+$	$+$	$-$	$-$	0	0	0
g_Y	$\pm$	$\pm$	$\pm$	$+$	0	0	0

Table 1: Comparative statics in steady state. The sign $\pm$ indicates an ambiguous change, and 0 means no change.

more attractive compared with entrant R&D, shifting the R&D arbitrage condition rightward in Figure 2. At the same time, incumbents become more productive in the sense that a given g_I can be achieved with less workers. It makes more skilled worker available to both entrant and incumbent R&D. This is represented by an anti-clockwise pivot of the skilled labour market condition around the vertical intercept in Figure 2. g_I increases more than in the case of an increase in ϕ, while g_E falls because the general equilibrium effects on g_E is not sufficiently large. As a result, ξ unambiguously decreases. Its effect on ζ_L is the same as in ϕ. Similar mechanisms work for a higher δ_E, but reversing the effect on g_I, g_E, ξ and ζ_L. Regarding g_Y, the impacts of ϕ, δ_I and δ_E on it are ambiguous because g_I and g_E move in opposite directions. On the other hand, a greater H unambiguously raises output growth partly because g_I is independent of H. Table 1 shows that L, ρ and λ have no effects on the Poisson rates, the Pareto exponent and output growth. This result is an artefact of the assumption that skilled workers only are used in R&D activities.

3.9 Zipf's Law

A firm size distribution is often found to follow a Pareto distribution with an exponent being slightly above 1. Measuring firm sizes by employment, Axtell (2001) shows that the Pareto exponent is 1.059 for the US data of 5.5 million firms in 1997. Toda (2017) also reports 1.0967 using the 2011 US data and slightly higher values (e.g. around 1.1) over the period 1992–2011 [7] Let us consider how our model can be used to explain such data regularity.

Suppose that the externality weakens, i.e. ϕ decreases. Table 1 shows that ζ_L falls, and this result is represented by an upward shift of the horizontal line in Figure 1 because it ξ/ϕ increases. If we reduce ϕ further, ζ_L can be made arbitrarily close to 1 as long as $\phi > 0$. In this sense, Zipf's law emerges with a sufficiently weak positive externality.

Then, how small ϕ should be? To answer this question using a numerical example based on estimates, we slightly generalises the assumption regarding the quality improvement caused by innovation. Let us assume that the quality of a product increases by a factor $\lambda_E > 1$ if innovation is generated by entrants, and by a factor λ_I in the case of incumbents. Following Acemoglu and Cao (2015), assume $\lambda_E > \lambda_I$ to capture the "radical" nature of entrant innovation. To keep the equilibrium conditions in tact for simplicity, let us further assume that

[7] There are studies which propose non-Zipf distributions. For example, see Kondo, Lewis, and Stella (2023) for a recent study.

part of entrants' innovative ideas is available in public, so that it can be used by competitive copycat firms. To be more precise, if $q_{-1}(j,k)$ is the quality of the previous product, and the fringe firms can produce the goods with quality $\frac{\lambda_E}{\lambda_I} q_{-1}(j,k)$ whenever entrant innovation occurs. Entrants' monopoly position is eroded and they charge the price $\lambda_I w_L$, as incumbents do. If we let $\lambda = \lambda_I$, all equations derived above unchange except the growth rate (56), which is now given by

$$g_Y = (\lambda_E - 1) g_E + (\lambda_I - 1) \phi \frac{\zeta}{\zeta - 1} g_I. \tag{60}$$

We take advantage of this equation to calibrate the values of ϕ and ξ. For this, define the contribution of entrant and incumbent innovation to growth as $S_E \equiv \frac{(\lambda_E - 1) g_E}{g_Y}$ and $S_I \equiv \frac{(\lambda_I - 1) \phi \frac{\zeta}{\zeta - 1} g_I}{g_Y}$, which in turn give

$$\xi \equiv \frac{g_E}{g_I} = \frac{\lambda_I - 1}{\lambda_E - 1} \cdot \frac{S_E}{S_I} \cdot \frac{\zeta}{\zeta - 1} \phi. \tag{61}$$

This equation relates ξ and ϕ. In particular, (32) and (61) constitute the system of two equations for (ϕ, ξ), taking the values of ζ, λ_E, λ_I, S_I and S_E as given. We use three values $(1.01, 1.059, 1.1)$ for ζ as the target Pareto exponent (1.059 is the value reported in Axtell (2001)). In setting the quality steps, we adopt $\lambda_I = 1.2$ and $\lambda_E = 2$ used in Acemoglu and Cao (2015). They also consider $\lambda_E = 3$, and we briefly mention that case later. Regarding S_E and S_I, we take values from Garcia-Macia, Hsieh, and Klenow (2019) who decompose the contribution of entrant and incumbent innovation to aggregate TFP growth. In the period of 1983-1993, for example, 32.3% of TFP growth is attributed to entrants, and the complementary fraction 67.7% comes from incumbents.

Table 2 reports the result. We take $\zeta = 1.059$ as the benchmark case. First note that the contribution of entrants to growth becomes smaller in more recent years, while that of incumbents gets larger, as S_E and S_I show. This is interpreted as the declining business dynamism (see Akcigit and Ates (2021) for example). This fact is reflected in a decline in both ξ and ϕ over time. Second, all values of ϕ are less than 0.5. Note that ϕ does not need to be extremely low for a reasonable value of the Pareto exponent. Third, in the benchmark case, the average of ϕ is approximately 0.27, meaning that only about a quarter of incumbent innovation succeeds to be implemented. Fourth, the table is based on $\lambda_I = 1.2$ and $\lambda_E = 2$. It means $\frac{\lambda_E - 1}{\lambda_I - 1} = 5$, i.e. productivity gains from entrant innovation is five times as large as incumbent innovation. Acemoglu and Cao (2015) use $\lambda_E = 3$, i.e. $\frac{\lambda_E - 1}{\lambda_I - 1} = 10$. In that case, the value of ϕ halves, and ξ also moderately gets lower.

Let us compare our mechanism to generate Zipf's law with the existing explanations using the GBM in the literature. A typical account goes as follows. Consider a GBM $dm(t) = \eta m(t) dt + \nu m(t) W(t)$ where $m(t)$ is the size of units, e.g. firms or cities, with the minimum

$\lambda_I = 1.2, \quad \lambda_E = 2$	Periods	$1983 - 1993$	$1993 - 2003$	$2003 - 2013$
	(S_E, S_I)	$(0.323, 0.677)$	$(0.224, 0.776)$	$(0.198, 0.802)$
$\zeta = 1.010$	ϕ	0.458	0.277	0.237
	ξ	4.412	1.615	1.181
$\zeta = 1.059$	ϕ	0.381	0.231	0.197
	ξ	0.653	0.239	0.175
$\zeta = 1.100$	ϕ	0.334	0.202	0.173
	ξ	0.351	0.128	0.094

Table 2: Calibrated values of (ϕ, ξ) required for different values of ζ, λ_E, S_E and S_I for λ_I. The values of λ_E and λ_I follow Acemoglu and Cao (2015). The values of S_E and S_I for the three periods are taken from Table V of Garcia-Macia, Hsieh, and Klenow (2019).

being $m_{\min}$. The stationary distribution is given by $\tilde{f}(m) = \left(\frac{m}{m_{\min}}\right)^{-\zeta}$ where $\zeta = 1 - \frac{2\eta}{\nu^2}$. Then, if the mean growth rate η is sufficiently close to 0, we obtain $\zeta \approx 1$. The result can also be expressed in terms of the mean of m, denoted by $\overline{m} = \frac{\zeta}{\zeta - 1} m_{\min}$. This gives $\zeta = \frac{1}{1 - m_{\min}/\overline{m}}$. If the $m_{\min}$ is sufficiently small relative to the mean, we have again $\zeta \approx 1$. Alternatively, introducing Poisson birth/death of a unit at a given size without a minimum size gives rise to a double-Pareto distribution (see (Gabaix (2009))). No externality plays a role in these explanations.

4 Conclusion

The paper is an attempt to provide a microfoundation of the GBM of firm size in the sense that we have *derived* it in the standard Schumpeterian endogenous growth model where Poisson-distributed innovation drives growth. The key driving force in our analysis is the positive externality which makes larger firms more productive. It was established that the distribution of firm size is Pareto, and the Pareto exponent can have a low or high value because of the externality. In a locally stable equilibrium, the Pareto exponent takes a low value where the incumbent creative destruction effect dominates the entrant counterpart. In particular, the stable value of the Pareto exponent can be arbitrarily close 1 if the degree of the externality is small. Our model is based on the fairly familiar endogenous growth model, and in this sense, it can be easily extended to analyse, for example, international trade and the effect of policy such as patents and taxes on long-run growth and firm size distribution.

References

Acemoglu, D., and D. Cao (2015): "Innovation by Entrants and Incumbents," *Journal of Economic Theory*, 157, 255–294.

Aghion, P., and P. Howitt (1992): "A Model of Growth Through Creative Desctruction," *Econometrica*, 60(2), 323–351.

Ahn, S., G. Kaplan, B. Moll, T. Winberry, and C. Wolf (2018): "When inequality matters for macro and macro matters for inequality," *NBER macroeconomics annual*, 32(1), 1–75.

Akcigit, U., and S. T. Ates (2021): "Ten facts on declining business dynamism and lessons from endogenous growth theory," *American Economic Journal: Macroeconomics*, 13(1), 257–298.

Arellano, C., Y. Bai, and P. J. Kehoe (2019): "Financial frictions and fluctuations in volatility," *Journal of Political Economy*, 127(5), 2049–2103.

Axtell, R. L. (2001): "Zipf distribution of US firm sizes," *science*, 293(5536), 1818–1820.

Beare, B. K., and A. A. Toda (2022): "Determination of Pareto exponents in economic models driven by Markov multiplicative processes," *Econometrica*, 90(4), 1811–1833.

Benhabib, J., A. Bisin, and S. Zhu (2011): "The distribution of wealth and fiscal policy in economies with finitely lived agents," *Econometrica*, 79(1), 123–157.

Bilal, A., N. Engbom, S. Mongey, and G. L. Violante (2022): "Firm and worker dynamics in a frictional labor market," *Econometrica*, 90(4), 1425–1462.

Brzezniak, Z., and T. Zastawniak (2000): *Basic stochastic processes: a course through exercises*. Springer Science & Business Media.

Clementi, G. L., and B. Palazzo (2016): "Entry, exit, firm dynamics, and aggregate fluctuations," *American Economic Journal: Macroeconomics*, 8(3), 1–41.

Di Giovanni, J., A. A. Levchenko, and R. Ranciere (2011): "Power laws in firm size and openness to trade: Measurement and implications," *Journal of International Economics*, 85(1), 42–52.

Duffie, D. (2003): "Intertemporal Asset Pricing Theory," in *Handbook of the Economics of Finance*, ed. by G. Constantinides, M. Harris, and R. Stulz, vol. 1B, chap. 11, pp. 639–742. Elsevier, Amsterdam.

Gabaix, X. (1999): "Zipf's law for cities: an explanation," *The Quarterly journal of economics*, 114(3), 739–767.

——— (2009): "Power laws in economics and finance," *Annual Review of Economics*, 1(1), 255–294.

Gabaix, X., J.-M. Lasry, P.-L. Lions, and B. Moll (2016): "The Dynamics of Inequality," *Econometrica*, 84(6), 2071–2111.

Garcia-Macia, D., C.-T. Hsieh, and P. J. Klenow (2019): "How destructive is innovation?," *Econometrica*, 87(5), 1507–1541.

Grossman, G., and E. Helpman (1991): "Quality Ladders in the Theory of Growth," *Review of Economic Studies*, 58(1), 43–61.

Hopenhayn, H. A. (1992): "Entry, exit, and firm dynamics in long run equilibrium," *Econometrica*, pp. 1127–1150.

Klette, T. J., and S. Kortum (2004): "Innovating Firms and Aggregate Innovation," *Journal of Political Economy*, 112(5), 986–1018.

Kondo, I. O., L. T. Lewis, and A. Stella (2023): "Heavy tailed but not Zipf: Firm and establishment size in the United States," *Journal of Applied Econometrics*, 38(5), 767–785.

Lucas, R. E. (1978): "On the size distribution of business firms," *The Bell Journal of Economics*, pp. 508–523.

Luttmer, E. G. (2007): "Selection, growth, and the size distribution of firms," *The Quarterly Journal of Economics*, 122(3), 1103–1144.

——— (2010): "Models of growth and firm heterogeneity," *Annu. Rev. Econ.*, 2(1), 547–576.

Luttmer, E. G. J. (2011): "On the Mechanics of Firm Growth," *The Review of Economic Studies*, 78(3), 1042–1068.

Spanos, A. (1986): *Statistical Foundations of Econometric Modelling*. Cambridge University Press.

Steindl, J. (1987): "Pareto Distribution," in *New Palgrave: A Dictionary of Economics*, ed. by J. Eatwell, M. Milgate, and P. Newman, vol. 3, pp. 809–811. Macmillan, London and Basingstoke.

Toda, A. A. (2017): "Zipf's Law: A Microfoundation," *Available at SSRN 2808237*.

A The Case of Homogenous Labour

A.1 Equilibrium Conditions

In this appendix, we develop the model assuming no distinction between skilled and unskilled workers, i.e. there are homogeneous workers who are employed to produce intermediate goods and conduct entrant and incumbent R&D. The full employment of workers is now given by

$$L = \frac{Z}{\lambda} + \frac{g_E}{\delta_E} + \left(\frac{g_I}{\delta_I}\right)^{\frac{1}{\gamma}} \tag{62}$$

which replaces (52) and (53). L is the total number of workers, and w is used to denote wage.

Regarding R&D activities, the incumbent FOC (47) and the free entry condition (51) still hold with w_H being replaced with w. The R&D arbitrage condition (54) unchanged, while the asset equation (55) is now

$$\frac{\dot{Z}}{Z}=\delta_E\Lambda Z-\frac{\phi_I^2}{2}\gamma g_I-g_E+\frac{\Sigma\left(g_E,g_I,\zeta\right)}{2}-\rho \tag{63}$$

where $Z\equiv\dfrac{E}{w}$. The evolution of ζ still follows (58).

Using those equations, the model can be reduced to the system of two differential equations with two unknowns (Z,ζ):

$$\dot{Z}=Z\left\{\delta_E\Lambda Z-\frac{\phi_I^2}{2}\gamma g_I-\frac{5}{6}\Gamma\left(Z\right)+\left(1+\frac{\zeta}{3\left(\zeta-1\right)}\right)\frac{\phi^2 g_I}{2}-\rho\right\} \tag{64}$$

$$\dot{\zeta}=\left\{-\left(1-\frac{\zeta}{6}\right)\Gamma\left(Z\right)+\frac{\phi^2 g_I}{2}\left(1+\frac{\zeta}{3\left(\zeta-1\right)}\right)\zeta\right\}\left(\zeta-1\right)\zeta^2 \tag{65}$$

where $\Gamma\left(Z\right)\equiv\delta_E\left(\tilde{L}-\dfrac{Z}{\lambda}\right)$, $\tilde{L}\left(g_I\right)\equiv L-\left(\dfrac{g_I}{\delta_I}\right)^{\frac{1}{\gamma}}$ and g_I is determined in (47).

A.2 Steady State: 3 Cases

Consider steady state. (65) with $\dot{\zeta}=0$ gives

$$Z=\lambda\left[\tilde{L}\left(g_I\right)-\frac{\phi_I^2 g_I}{\delta_E}\Xi\left(\zeta\right)\right]\equiv\Theta\left(\zeta\right). \tag{66}$$

where $\tilde{L}\left(g_I\right)>\dfrac{\phi_I^2 g_I}{\delta_E}\Xi\left(\zeta\right)$. There is $\Xi\left(\zeta\right)$ with the minus sign on the RHS. Therefore, $\Theta\left(\zeta\right)$ takes an inverted U shape, as in Figure 4. Two horizontal intercepts are defined by

$$\frac{\delta_E\tilde{L}\left(g_I\right)}{\phi_I^2 g_I}=\Xi\left(\zeta\right), \tag{67}$$

and hence, we can reuse Lemma 4 to define the two roots, replacing ξ with $\dfrac{\delta_E\tilde{L}\left(g_I\right)}{g_I}$ in (34). Let us use $\underline{\zeta}<\overline{\zeta}$, to denote those roots. (64) with $\dot{Z}=0$ gives

$$Z=\left[\left(\lambda-\frac{1}{6}\right)\frac{\delta_E}{\lambda}\right]^{-1}\left\{\rho+\frac{5}{6}\delta_E\tilde{L}\left(g_I\right)-\left(1-\gamma+\frac{\zeta}{3\left(\zeta-1\right)}\right)\frac{\phi^2 g_I}{2}\right\}\equiv\Omega\left(\zeta\right) \tag{68}$$

It is upward sloping with the horizontal intercept

$$\hat{\zeta}=\frac{3\left[\left(\rho+\frac{5}{6}\delta_E\tilde{L}\left(g_I\right)\right)\frac{2}{\phi^2 g_I}-\left(1-\gamma\right)\right]}{3\left[\left(\rho+\frac{5}{6}\delta_E\tilde{L}\left(g_I\right)\right)\frac{2}{\phi^2 g_I}-\left(1-\gamma\right)\right]-1}>1 \tag{69}$$

We require

$$\Omega\left(\zeta\right)>0\qquad\Longleftrightarrow\qquad\left(\rho+\frac{5}{6}\delta_E\tilde{L}\left(g_I\right)\right)\frac{2}{\phi^2 g_I}>\left(1-\gamma\right). \tag{70}$$

There are three cases, as illustrated in Figure 4 where three values of $\hat{\zeta}=\left\{\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3\right\}$ are

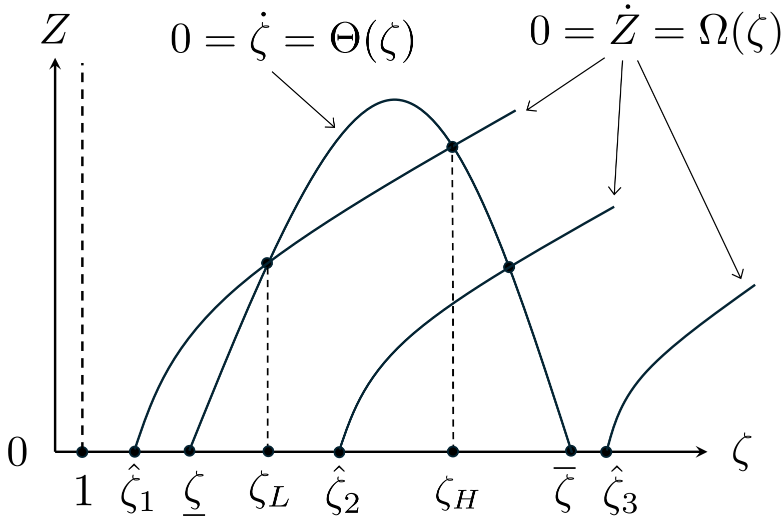

Figure 4: Three cases.

used for expositional purposes.[8] In the case of $\hat{\zeta}_2$ in the figure, $\underline{\zeta} < \hat{\zeta} < \overline{\zeta}$ ensures the existence of an interior equilibrium. A sufficiently large L or/and a sufficiently small ρ leads to $\hat{\zeta}_1$ with multiple values of ζ in steady state, as in the main model. However, $\hat{\zeta} < \underline{\zeta}$ does not necessarily guarantee the existence of equilibrium values of ζ, which we denote $\zeta_L \leq \zeta_H$. The following lemma summarises the condition required for it:

Lemma 7. *Define*

$$A \equiv \frac{\phi_I^2 g_I}{2\delta_E} > 0, \qquad B \equiv \frac{\delta_E}{3} > 0$$
$$C \equiv \rho + \left(\frac{5}{6}\tilde{L} - (1-\gamma)\frac{\phi^2 g_I}{2\delta_E}\right)\delta_E > 0$$
$$D \equiv \left(\frac{\lambda - 1}{1 - s_E} + \frac{5}{6}\right)\delta_E > 0$$

where $C > 0$ due to (70). Rearranging $\Omega(\zeta) = \Theta(\zeta)$, one obtains

$$A_1\zeta^2 + A_2\zeta + A_3 = 0$$

where

$$A_1 = 8AD + AB + \left(D\tilde{L} - C\right) > 0, \tag{71}$$
$$A_2 = -6AB - 6AD - 7\left(D\tilde{L} - C\right) < 0, \tag{72}$$
$$A_3 = 6\left(D\tilde{L} - C\right) > 0. \tag{73}$$

Assume L is large enough such that $D\tilde{L} - C > 0$, implying that the inequalities in (71), (72)

[8] If $\hat{\zeta}$ changes its position, then so do $\underline{\zeta}$ and $\overline{\zeta}$. But we ignore this to illustrate the three cases in the same diagram.

and (73) hold. Then, ζ_L *and* ζ_H *are given by* $\zeta_L, \zeta_H = \dfrac{-A_2 \pm \sqrt{A_2^2 - 4A_1A_3}}{2A_1}$, *and exist if and only if* $A_2^2 \geq 4A_1A_3$.

We can solve for g_E, using the above equations:

$$g_E = \left(\lambda - \frac{1}{6}\right)^{-1} \Upsilon_E\left(\zeta\right) \tag{74}$$

where

$$\Upsilon_E\left(\zeta\right) \equiv \delta_E\left(\lambda - 1\right)\tilde{L} - \rho + \left(1 - \gamma + \frac{\zeta}{3\left(\zeta - 1\right)}\right)\frac{\phi^2 g_I}{2} > 0. \tag{75}$$

$\Upsilon_E\left(\zeta\right)$ is decreasing in ζ. Intuitively, a smaller ζ thickens the Pareto tail, i.e. the likelihood that entrants grow big gets larger in the sense that the average number of products per firm increases. One can easily show that $A_2 < 0$ means $\Upsilon_E\left(\zeta\right) > 0$.

A.3 Stability Analysis

Linearise (64) and (65) around the steady state (Z^*, ζ^*):

$$\begin{pmatrix} \dot{Z} \\ \dot{\zeta} \end{pmatrix} = \underbrace{\begin{pmatrix} \Omega_Z\left(Z^*, \zeta^*\right) & \Omega_\zeta\left(Z^*, \zeta^*\right) \\ \Theta_Z\left(Z^*, \zeta^*\right) & \Theta_\zeta\left(Z^*, \zeta^*\right) \end{pmatrix}}_{J} \begin{pmatrix} Z - Z^* \\ \zeta - \zeta^* \end{pmatrix} \tag{76}$$

where

$$\begin{aligned}
\Omega_Z\left(Z^*, \zeta^*\right) &= Z^* \frac{\delta_E}{\lambda}\left(\lambda - \frac{1}{6}\right) > 0, \\
\Omega_\zeta\left(Z^*, \zeta^*\right) &= -Z^* \frac{\phi_I^2 g_I}{6} \frac{1}{\left(\zeta^* - 1\right)^2} < 0, \\
\Theta_Z\left(Z^*, \zeta^*\right) &= \frac{\delta_E}{\lambda}\left(1 - \frac{\zeta^*}{6}\right)\left(\zeta^* - 1\right)\left(\zeta^*\right)^2 > 0, \\
\Theta_\zeta\left(Z^*, \zeta^*\right) &= \left\{\Gamma\left(Z^*\right) + \frac{\left(2\zeta - 3\right)\left(2\zeta - 1\right)}{\left(\zeta - 1\right)^2}\phi^2 g_I\right\}\frac{\left(\zeta - 1\right)\zeta^2}{6} > 0.
\end{aligned}$$

We focus on the case of a large L such that $\Theta_\zeta > 0$. In this case, the determinant and trace of the Jacobian, J in (76), are given by

$$\begin{aligned}
\mathrm{Det}\left(J\right) &= \overbrace{\Omega_Z}^{(+)}\overbrace{\Theta_Z}^{(+)}\left\{-\overbrace{\left(-\frac{\overbrace{\Theta_\zeta}^{(+)}}{\underbrace{\Theta_Z}_{(+)}}\right)}^{\text{Slope of }\Theta(\zeta)} + \overbrace{\left(-\frac{\overbrace{\Omega_\zeta}^{(-)}}{\underbrace{\Omega_Z}_{(+)}}\right)}^{\text{Slope of }\Omega(\zeta)}\right\}, \\
\mathrm{Tr}\left(J\right) &= \overbrace{\Omega_Z}^{(+)} + \overbrace{\Theta_\zeta}^{(+)}.
\end{aligned}$$

Let us first consider the case of $\hat{\zeta}_2$ in Figure 4. Because of the assumption $\Theta_\zeta > 0$, the slope of $\Theta(\zeta)$ is negative, as in the figure. This means $\mathrm{Det}(J) > 0$ and $\mathrm{Tr}(J) > 0$, hence (ζ^*, Z^*) is an unstable source. Next, let us turn to the case of $\hat{\zeta}_1$ in Figure 4. Regarding (ζ_H, Z), its stability property is the same as in the case of $\hat{\zeta}_2$ in the figure. Consider ζ_L. The slopes of $0 = \Omega(\zeta)$ and $0 = \Theta(\zeta)$ are both positive, and the former is less steeper than the latter. It means $\mathrm{Det}(J) < 0$, hence (ζ_H, Z^*) is a saddle point.

A.4 Zipf's Law

Consider ζ_L in the case of $\hat{\zeta}_1$ in Figure 4. A smaller ϕ reduces $\underline{\zeta}$. This should be clear from the fact that (67) and (32) essentially takes the same form, and the LHS of (67) indeed increases with a smaller ϕ because $\dfrac{\partial \phi^2 g_I}{\partial \phi} > 0$ from (47). For the same reason, $\Omega(\zeta)$ increases as ϕ falls (see (68)) and $\hat{\zeta}$ decreases (see (69)), implying that the $0 = \dot{Z} = \Omega(\zeta)$ curve in Figure 4 shifts leftward. Therefore, ζ_L can be made arbitrarily close to 1, as ϕ decreases.