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Wakaiki, Masashi

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Decay estimates for Cayley transforms and inverses of semigroup generators via the $\mathcal{B}$-calculus

Masashi Wakaiki

Abstract. Let $-A$ be the generator of a bounded C_0-semigroup $(e^{-tA})_{t\geq 0}$ on a Hilbert space. First we study the long-time asymptotic behavior of the Cayley transform $V_\omega(A) := (A-\omega I)(A+\omega I)^{-1}$ with $\omega > 0$. We give a decay estimate for $\|V_\omega(A)^n A^{-1}\|$ when $(e^{-tA})_{t\geq 0}$ is polynomially stable. Considering the case where the parameter ω varies, we estimate $\|(\prod_{k=1}^n V_{\omega_k}(A))A^{-1}\|$ for exponentially stable C_0-semigroups $(e^{-tA})_{t\geq 0}$. Next we show that if the generator $-A$ of the bounded C_0-semigroup has a bounded inverse, then $\sup_{t\geq 0}\|e^{-tA^{-1}}A^{-\alpha}\| < \infty$ for all $\alpha > 0$. We also present an estimate for the rate of decay of $\|e^{-tA^{-1}}A^{-1}\|$, assuming that $(e^{-tA})_{t\geq 0}$ is polynomially stable. To obtain these results, we use operator norm estimates offered by a functional calculus called the $\mathcal{B}$-calculus.

1. Introduction

When solving differential equations numerically, we employ time-discretization methods. The first topic is the long-time asymptotic behavior of numerical solutions. Let $-A$ be a bounded C_0-semigroup $(e^{-tA})_{t\geq 0}$ on a Banach space X, and consider the differential equation

$$\dot{x} = -Ax, \quad t \geq 0; \qquad x(0) = x_0 \in X. \tag{1}$$

The Crank–Nicolson discretization scheme with parameter $\omega > 0$ transforms the differential equation (1) into the following difference equation:

$$x_d(n+1) = -V_\omega(A)x_d(n), \quad n \in \mathbb{N}_0 := \{0,1,2,\ldots\}; \quad x_d(0) = x_0 \in X, \tag{2}$$

where $V_\omega(A)$ is defined by $V_\omega(A) := (A-\omega I)(A+\omega I)^{-1}$ and is called *the Cayley transform of A with parameter ω*. Assume that A has a bounded inverse. To obtain a uniform rate of decay of numerical solutions x_d starting in $D(A)$, we investigate the quantitative behavior of the operator norm $\|V_\omega(A)^n A^{-1}\|$ as $n \to \infty$.

For simplicity of notation, let $V(A) := V_1(A)$. The asymptotic behavior of $(V(A)^n)_{n\in\mathbb{N}_0}$ has been extensively studied; see the survey [15] and the book [10, Chapter 5]. Some particular relevant studies will be cited below. For the generator $-A$ of a bounded C_0-semigroup on a Banach space, $\|V(A)^n\| = O(\sqrt{n})$ as $n \to \infty$

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holds, i.e., there exist constants $M > 0$ and $n_0 \in \mathbb{N}$ such that $\|V(A)^n\| \leq M\sqrt{n}$ for all $n \geq n_0$, and this estimate cannot be improved in general; see [6,11,17,22]. In the Hilbert space setting, it is still unknown whether the boundedness of $(e^{-tA})_{t\geq 0}$ implies that of $(V(A)^n)_{n\in\mathbb{N}_0}$. For bounded C_0-semigroups on Hilbert spaces, the estimates

$$\|V(A)^n\| = O(\log n) \qquad (n \to \infty) \tag{3}$$

and

$$\sup_{n\in\mathbb{N}_0} \|V(A)^n(I+A)^{-\alpha}\| < \infty, \quad \alpha > 0 \tag{4}$$

remain the best so far. The former estimate (3) has been obtained in [14], and one can derive the latter estimate (4) by combining [16, Lemma 2.3] and the theory of the $\mathcal{B}$-calculus established in [4,5]. If both A and A^{-1} generate a bounded C_0-semigroup on a Hilbert space, then $(V(A)^n)_{n\in\mathbb{N}_0}$ is bounded, which has been independently proved in [1,14,19]. Under the additional assumption that $(e^{-tA})_{t\geq 0}$ is strongly stable, $(V(A)^n)_{n\in\mathbb{N}_0}$ is also strongly stable; see [19]. The approach developed in [19] is based on Lyapunov equations and has been extended in [22] from the case of constant parameters $V(A)^n$ to the case of variable parameters $\prod_{k=1}^n V_{\omega_k}(A)$, where the sequence $(\omega_k)_{k\in\mathbb{N}_0}$ satisfies $0 < \inf_{k\in\mathbb{N}} \omega_k$ and $\sup_{k\in\mathbb{N}} \omega_k < \infty$.

In this paper, we consider bounded C_0-semigroups on Hilbert spaces with certain decay properties. In particular, we focus on polynomial stability and exponential stability, which are defined as follows:

Definition 1. A C_0-semigroup $(e^{-tA})_{t\geq 0}$ on a Banach space is called

(a) *polynomially stable with parameter* $\beta > 0$ if $(e^{-tA})_{t\geq 0}$ is bounded and satisfies $\|e^{-tA}(I+A)^{-1}\| = O(t^{-1/\beta})$ as $t \to \infty$, or simply *polynomially stable* if it is polynomially stable with some parameter; and
(b) *exponentially stable* if $\lim_{t\to\infty} \|e^{\varepsilon t}e^{-tA}\| = 0$ for some $\varepsilon > 0$.

Note that if the C_0-semigroup $(e^{-tA})_{t\geq 0}$ is polynomially stable or exponentially stable, then the negative generator A has a bounded inverse; see [3, Theorem 1.1]. For more information on these stability notions, we refer to the survey article [7].

Several results on the rate of decay of $\|V(A)^n A^{-1}\|$ have been developed in the Hilbert space setting. If $(V(A)^n)_{n\in\mathbb{N}_0}$ is bounded, then

$$\|V(A)^n A^{-1}\| = O\left(\left(\frac{\log n}{n}\right)^{1/(2+\beta)}\right) \qquad (n \to \infty) \tag{5}$$

for all polynomially stable C_0-semigroups with parameter $\beta > 0$; see [23]. It has been also proved in [23] that if $-A$ is normal and generates a polynomially stable C_0-semigroup with parameter $\beta > 0$, then $\|V(A)^n A^{-1}\| = O(n^{-1/(2+\beta)})$ as $n \to \infty$ and the decay rate $n^{-1/(2+\beta)}$ cannot be replaced by a better one in general. In the case where $(e^{-tA})_{t\geq 0}$ is exponentially stable and $(e^{-tA^{-1}})_{t\geq 0}$ is bounded, the following estimate has been derived in [24]:

$$\left\| \left(\prod_{k=1}^{n} V_{\omega_k}(A) \right) A^{-1} \right\| = O\left(\sqrt{\frac{\log n}{n}} \right) \qquad (n \to \infty), \tag{6}$$

where $(\omega_k)_{k\in\mathbb{N}}$ satisfies $0 < \inf_{k\in\mathbb{N}} \omega_k$ and $\sup_{k\in\mathbb{N}} \omega_k < \infty$. In the proof of the estimates (5) and (6), Lyapunov equations on $V(A)$ and $-A^{-1}$ play an important role, as in [19,22]. Hence, the assumptions on the boundedness of $(V(A)^n)_{n\in\mathbb{N}_0}$ and $(e^{-tA^{-1}})_{t\geq 0}$ are placed.

Our aim is to estimate the decay rate of $\|V(A)^n A^{-1}\|$ without assuming that $(V(A)^n)_{n\in\mathbb{N}_0}$ or $(e^{-tA^{-1}})_{t\geq 0}$ is bounded. To this end, we apply the $\mathcal{B}$-calculus. First, we derive a new estimate for operator norms in the context of the $\mathcal{B}$-calculus, which is tailored to polynomially stable C_0-semigroups on Hilbert spaces. Using this estimate, we next show that

$$\|V(A)^n A^{-1}\| = O\left(\frac{\log n}{n^{1/(2+\beta)}} \right) \qquad (n \to \infty) \tag{7}$$

for all polynomially stable C_0-semigroups $(e^{-tA})_{t\geq 0}$ on Hilbert spaces. The estimate (7) is less sharp than (5) because it has an additional logarithmic term, and we do not know whether the logarithmic term can be omitted. However, it should be emphasized that the boundedness of $(V(A)^n)_{n\in\mathbb{N}_0}$ is not assumed. When $-A$ is the generator of an exponentially stable C_0-semigroup on a Hilbert space, we obtain the estimate

$$\left\| \left(\prod_{k=1}^{n} V_{\omega_k}(A) \right) A^{-1} \right\| = O\left(\frac{1}{\sqrt{n}} \right) \qquad (n \to \infty), \tag{8}$$

where $(\omega_k)_{k\in\mathbb{N}}$ satisfies $0 < \inf_{k\in\mathbb{N}} \omega_k$ and $\sup_{k\in\mathbb{N}} \omega_k < \infty$. This result implies that not only the assumption on the boundedness of $(e^{-tA^{-1}})_{t\geq 0}$ but also the logarithmic term $\sqrt{\log n}$ in the previous result (6) can be omitted. Moreover, we show that the estimate (8) cannot be improved in the case $A := iB + I$, where B is a self-adjoint operator with a certain spectral property. A similar improved estimate is provided for a polynomially stable C_0-semigroup on a Hilbert space such that the generator is normal.

We now turn our attention to the long-time asymptotic behavior of $(e^{-tA^{-1}})_{t\geq 0}$. Let $(e^{-tA})_{t\geq 0}$ be a bounded C_0-semigroup on a Banach space, and assume that there exists a densely defined algebraic inverse A^{-1} of A. The inverse generator problem raised by deLaubenfels [8] asks whether $-A^{-1}$ also generates a C_0-semigroup, and a survey can be found in [15]. For bounded C_0-semigroups on Banach spaces, a negative answer to this problem has been provided implicitly in [21, pp. 343–344], and other counterexamples can be found in [13,18,26]. The inverse generator problem in the Hilbert space setting remains open. It has been shown in [26] that if the answer is positive for all generators and all Hilbert spaces, then the C_0-semigroup $(e^{-tA^{-1}})_{t\geq 0}$

is bounded. For exponentially stable C_0-semigroups on Hilbert spaces, the estimate $\|e^{-tA^{-1}}\| = O(\log t)$ as $t \to \infty$ has been obtained by using Lyapunov equations in [25] and by connecting the growth bounds of $(e^{-tA^{-1}})_{t\geq 0}$ and $(V(A)^n)_{n\in\mathbb{N}_0}$ in [17]. This estimate has been extended to bounded C_0-semigroups on Hilbert spaces with invertible generators by means of the $\mathcal{B}$-calculus in [4, Corollary 5.7].

Since we are interested in the asymptotics of $e^{-tA^{-1}}x$ that is uniform with respect to $x \in D(A)$, the problem we study is to estimate $\|e^{-tA^{-1}}A^{-1}\|$. If $(e^{-tA^{-1}})_{t\geq 0}$ is an exponentially stable C_0-semigroup on a Banach space, then $\|e^{-tA^{-1}}A^{-k}\| = O(t^{-k/2+1/4})$ for all $k \in \mathbb{N}$, which has been established for the case $k = 1$ in [26] and for the case $k \geq 2$ in [9]. The $\mathcal{B}$-calculus has been applied in [24] in order to obtain the estimate

$$\|e^{-tA^{-1}}A^{-\alpha}\| = O\left(\frac{1}{t^{\alpha/2}}\right) \qquad (n \to \infty) \tag{9}$$

with $\alpha > 0$ for all exponentially stable C_0-semigroups on Hilbert spaces. If the generator $-A$ of a polynomially stable C_0-semigroup with parameter $\beta > 0$ on a Hilbert space is normal, then

$$\|e^{-tA^{-1}}A^{-\alpha}\| = O\left(\frac{1}{t^{\alpha/(2+\beta)}}\right) \qquad (n \to \infty) \tag{10}$$

for all $\alpha > 0$; see [24]. In [24], simple examples have been also provided to show that the estimates (9) and (10) cannot be improved in general. For polynomially stable C_0-semigroups with any parameters on Hilbert spaces, the estimate

$$\sup_{t\geq 0} \|e^{-tA^{-1}}A^{-1}\| < \infty \tag{11}$$

has been derived in [24] by the Lyapunov-based approach.

Our second contribution is to improve the estimate (11) in two ways by means of the $\mathcal{B}$-calculus. First, we show that $\sup_{t\geq 0} \|e^{-tA^{-1}}A^{-\alpha}\| < \infty$ for all $\alpha > 0$ when $-A$ is the generator of a bounded C_0-semigroup on a Hilbert space and has a bounded inverse. Second, we give the estimate of the decay rate

$$\|e^{-tA^{-1}}A^{-1}\| = O\left(\frac{\log t}{t^{1/(2+\beta)}}\right) \qquad (n \to \infty) \tag{12}$$

for all polynomially stable C_0-semigroups with parameter $\beta > 0$ on Hilbert spaces. For this, the operator norm estimate exploiting polynomial stability is again applied in the setting of the $\mathcal{B}$-calculus. We see from the estimate (12) that a decay estimate for $(e^{-tA})_{t\geq 0}$ is transferred to that for $(e^{-tA^{-1}})_{t\geq 0}$ as in the estimate (7) for $(V(A)^n)_{n\in\mathbb{N}_0}$. There is a question whether the logarithmic term in the estimate (12) can be omitted, and it is open as in other estimates.

This paper is organized as follows: In Sect. 2, we present the notion and some properties of the $\mathcal{B}$-calculus. Section 3 is devoted to estimating the rate of decay of the Cayley transform. In Sect. 4, we study the asymptotic behavior of $\|e^{-tA^{-1}}A^{-\alpha}\|$ with $\alpha > 0$.

Notation

Let $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and let $\mathbb{R}_+ := \{\xi \in \mathbb{R} : \xi \geq 0\}$. Let $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_0$ denote the set of integers, the set of positive integers, and the set of nonnegative integers, respectively. For a real number ξ, let $\lfloor \xi \rfloor := \max\{k \in \mathbb{Z} : k \leq \xi\}$. Given functions $f, g : [t_0, \infty) \to (0, \infty)$, we write

$$f(t) = O\big(g(t)\big) \qquad (t \to \infty)$$

if there exist constants $M > 0$ and $t_1 \geq t_0$ such that $f(t) \leq Mg(t)$ for all $t \geq t_1$. When $0 < \omega_p \leq \omega_q < \infty$, we denote by $\mathcal{S}(\omega_p, \omega_q)$ the set of sequences $(\omega_k)_{k \in \mathbb{N}}$ of positive real numbers satisfying $\omega_p \leq \omega_k \leq \omega_q$ for all $k \in \mathbb{N}$.

Let X be a Banach space. We denote by $\mathcal{L}(X)$ the Banach algebra of bounded linear operators on X. For a linear operator A on X, let $D(A)$, $\sigma(A)$, and $\varrho(A)$ denote the domain, the spectrum, and the resolvent set of A, respectively. Let $-A$ be the generator of a bounded C_0-semigroup $(e^{-tA})_{t \geq 0}$ on X. We define the Cayley transform $V_\omega(A)$ of A with parameter $\omega > 0$ by $V_\omega(A) := (A - \omega I)(A + \omega I)^{-1}$. For simplicity of notation, we set $V(A) := V_1(A)$. If A is injective, the fractional power A^α of A is defined by the sectoral functional calculus for $\alpha \in \mathbb{R}$; see [20, Chapter 3].

Let H be a Hilbert space. We denote the inner product on H by $\langle \cdot, \cdot \rangle$. For a densely defined linear operator A on H, the Hilbert space adjoint of A is denoted by A^*.

2. $\mathcal{B}$-calculus

In this section, first we recall the notion of the $\mathcal{B}$-calculus and its basic properties. Then, we provide a new operator norm estimate in the $\mathcal{B}$-calculus for the negative generator of a polynomially stable C_0-semigroup on a Hilbert space.

2.1. Definition and basic facts

We provide some background material on the $\mathcal{B}$-calculus and refer the reader to [4, Sections 2 and 4] for more details. Let $\mathcal{B}$ be the algebra of holomorphic functions f on $\mathbb{C}_+$ satisfying

$$\|f\|_{\mathcal{B}_0} := \int_0^\infty \sup_{\eta \in \mathbb{R}} |f'(\xi + i\eta)| \mathrm{d}\xi < \infty.$$

For all $f \in \mathcal{B}$, the limit $f(\infty) := \lim_{\operatorname{Re} z \to \infty} f(z)$ exists in $\mathbb{C}$ and

$$\|f\|_\infty := \sup_{z \in \mathbb{C}_+} |f(z)| \leq |f(\infty)| + \|f\|_{\mathcal{B}_0}.$$

Moreover, $\mathcal{B}$ equipped with the norm

$$\|f\|_{\mathcal{B}} := \|f\|_\infty + \|f\|_{\mathcal{B}_0}, \quad f \in \mathcal{B}$$

is a Banach algebra.

Let $\mathrm{M}(\mathbb{R}_+)$ be the space of all bounded Borel measures μ on $\mathbb{R}_+$, endowed with the total variation norm $\|\mu\|_{\mathrm{M}(\mathbb{R}_+)} := |\mu|(\mathbb{R}_+)$. Then $\mathrm{M}(\mathbb{R}_+)$ is a Banach algebra with respect to the convolution product. We identify $L^1(\mathbb{R}_+)$ with a subalgebra of $\mathrm{M}(\mathbb{R}_+)$. The Laplace transform of $\mu \in \mathrm{M}(\mathbb{R}_+)$ is the function on $\mathbb{C}_+$ defined by

$$(\mathcal{L}\mu)(z) := \int_{\mathbb{R}_+} e^{-tz}\mu(\mathrm{d}t), \quad z \in \mathbb{C}_+.$$

Define $\mathcal{LM} := \{\mathcal{L}\mu : \mu \in \mathrm{M}(\mathbb{R}_+)\}$. Then $\mathcal{LM}$ equipped with the norm

$$\|\mathcal{L}\mu\|_{\mathrm{HP}} := \|\mu\|_{\mathrm{M}(\mathbb{R}_+)}, \quad \mu \in \mathrm{M}(\mathbb{R}_+)$$

is a Banach algebra. If $f \in \mathcal{LM}$, then $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \leq 2\|f\|_{\mathrm{HP}}$.

We present some examples of $\mathcal{LM}$-functions, which will be used to obtain decay estimates for the Cayley transform and the inverse of semigroup generators.

Example 1. (a) Let $c > 0$ and $d \in \mathbb{R}$. Define

$$u_{c,d}(z) := \frac{z+d}{z+c}, \quad z \in \mathbb{C}_+.$$

Then $u_{c,d}$ is the Laplace transform of $\delta_0 - (c-d)e_c \in \mathrm{M}(\mathbb{R}_+)$, where δ_0 is the Dirac measure concentrated at 0 and $e_c(t) := e^{-ct}$ for $t \geq 0$. Hence, the n-th power $(u_{c,d})^n$ belongs to $\mathcal{LM}$ for every $n \in \mathbb{N}$.

(b) Let $\alpha, c > 0$. Define

$$v_{\alpha,c}(z) := \frac{1}{(z+c)^\alpha}, \quad z \in \mathbb{C}_+.$$

By the definition of the gamma function Γ, we obtain

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\mathrm{d}t = (z+c)^\alpha \int_0^\infty t^{\alpha-1}e^{-(z+c)t}\mathrm{d}t$$

for all positive real numbers $z > 0$. Together with the uniqueness theorem for holomorphic functions, this implies that $v_{\alpha,c}$ is the Laplace transform of the function $\phi_{\alpha,c} \in L^1(\mathbb{R}_+)$ defined by

$$\phi_{\alpha,c}(t) := \frac{t^{\alpha-1}e^{-ct}}{\Gamma(\alpha)}, \quad t > 0.$$

Therefore, $v_{\alpha,c} \in \mathcal{LM}$.

(c) For a fixed $t > 0$, define

$$w_t(z) := \frac{z}{z+1}e^{-t/z}, \quad z \in \mathbb{C}_+.$$

Then, w_t is the Laplace transform of some $L^1(\mathbb{R}_+)$-function; see the proof of [9, Theorem 3.3]. Hence, $w_t \in \mathcal{LM}$.

Now we introduce a functional calculus for $\mathcal{B}$. Let $-A$ be the generator of a bounded C_0-semigroup $(e^{-tA})_{t\geq 0}$ on a Hilbert space H. We define $K := \sup_{t\geq 0}\|e^{-tA}\|$. For all $x \in H$, the Plancherel theorem gives

$$\int_{-\infty}^{\infty} \|(\xi - i\eta + A)^{-1}x\|^2 \mathrm{d}\eta = 2\pi \int_0^{\infty} e^{-2\xi t}\|e^{-tA}x\|^2 \mathrm{d}t,$$

and hence

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|(\xi - i\eta + A)^{-1}x\|^2 \mathrm{d}\eta \leq \pi K^2\|x\|^2. \tag{13}$$

Similarly,

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} \|(\xi + i\eta + A^*)^{-1}y\|^2 \mathrm{d}\eta \leq \pi K^2\|y\|^2 \tag{14}$$

for all $y \in H$. Combining these estimates with the Cauchy–Schwartz inequality, we obtain

$$\sup_{\xi>0} \xi \int_{-\infty}^{\infty} |\langle(\xi - i\eta + A)^{-2}x, y\rangle| \mathrm{d}\eta \leq \pi K^2\|x\|\,\|y\| \tag{15}$$

for all $x, y \in H$. Let $f \in \mathcal{B}$, and define the linear operator $f(A)$ on H by

$$\langle f(A)x, y\rangle := \langle f(\infty)x, y\rangle - \frac{2}{\pi}\int_0^{\infty} \xi \int_{-\infty}^{\infty} \langle(\xi - i\eta + A)^{-2}x, y\rangle f'(\xi + i\eta)\mathrm{d}\eta \mathrm{d}\xi \tag{16}$$

for $x, y \in H$. By the estimate (15), $f(A)$ is bounded on H and satisfies

$$\|f(A)\| \leq |f(\infty)| + 2K^2\|f\|_{\mathcal{B}_0} \leq 2K^2\|f\|_{\mathcal{B}}. \tag{17}$$

The map

$$\Phi_A : \mathcal{B} \to \mathcal{L}(H), \quad f \mapsto f(A)$$

is a bounded algebra homomorphism. We refer to Φ_A as *the $\mathcal{B}$-calculus for A*.

Define the map $\Pi_A : \mathcal{LM} \to \mathcal{L}(H)$ by

$$\Pi_A(f)x := \int_{\mathbb{R}_+} e^{-tA}x\mu(\mathrm{d}t)$$

for $f = \mathcal{L}\mu \in \mathcal{LM}$ and $x \in H$. The map Π_A is a bounded algebra homomorphism and is called *the Hille–Phillips calculus for A*. The $\mathcal{B}$-calculus extends the Hille–Phillips calculus in the sense that $\Phi_A(f) = \Pi_A(f)$ for all $f \in \mathcal{LM}$.

We apply the $\mathcal{B}$-calculus to the functions presented in Example 1.

Example 2. Let $-A$ be the generator of a bounded C_0-semigroup on a Hilbert space such that $0 \in \varrho(A)$. We define the functions $u_{c,d}, v_{\alpha,c}, w_t \in \mathcal{LM}$ as in Example 1.

(a) Since $\Pi_A(u_{c,d}) = (A + dI)(A + cI)^{-1}$, we have

$$(u_{c,d})^n(A) = ((A + dI)(A + cI)^{-1})^n.$$

(b) One has $\Pi_A(v_{\alpha,c}) = (A+cI)^{-\alpha}$ by [20, Proposition 3.3.5], and hence $v_{\alpha,c}(A) = (A + cI)^{-\alpha}$. Moreover, since

$$A^{-\alpha} = (A + cI)^{-\alpha}(I + cA^{-1})^{\alpha}, \tag{18}$$

the estimate (17) yields

$$\|f(A)A^{-\alpha}\| \leq 2K^2\|(I + cA^{-1})^{\alpha}\| \, \|f v_{\alpha,c}\|_{\mathcal{B}_0}$$

for all $f \in \mathcal{B}$. We shall use this estimate frequently without comment.

(c) The argument in the proof of [4, Corollary 5.8] shows that

$$w_t(A) = A(A + I)^{-1}e^{-tA^{-1}}.$$

2.2. Operator norm estimate for polynomially stable semigroups

The norm estimate (17) can be applied to every negative generator A of a bounded C_0-semigroup on a Hilbert space. Here we present a sharper norm estimate for polynomially stable C_0-semigroups on Hilbert spaces. First we recall a useful property on rates of polynomial decay for bounded C_0-semigroups on Banach spaces; see [2, Proposition 3.1] for the proof.

Proposition 1. *Let $-A$ be the generator of a bounded C_0-semigroup $(e^{-tA})_{t\geq 0}$ on a Banach space X such that $0 \in \varrho(A)$. Then, the following two statements are equivalent for a fixed $\beta > 0$:*

1. $\|e^{-tA}A^{-1}\| = O(t^{-1/\beta})$ *as* $t \to \infty$.
2. $\|e^{-tA}A^{-\beta q}\| = O(t^{-q})$ *as* $t \to \infty$ *for some/all* $q > 0$.

Next we establish a resolvent estimate analogous to (13) for polynomially stable C_0-semigroups on Hilbert spaces. The proof is similar to that of [23, Proposition 3.1], but in order to make this paper self-contained, we give a short argument.

Lemma 1. *Let $-A$ be the generator of a bounded C_0-semigroup $(e^{-tA})_{t\geq 0}$ on a Hilbert space H such that $0 \in \varrho(A)$. Then, the following two statements are equivalent for fixed $\beta > 0$ and $q \in (0, 1/2)$:*

1. $\|e^{-tA}A^{-1}\| = O(t^{-1/\beta})$ *as* $t \to \infty$.
2. *There exists $M > 0$ such that for all $x \in H$,*

$$\sup_{0<\xi<1} \xi^{1-2q} \int_{-\infty}^{\infty} \|(\xi + i\eta + A)^{-1}A^{-\beta q}x\|^2 \mathrm{d}\eta \leq M\|x\|^2. \tag{19}$$

Proof. Let $q \in (0, 1/2)$ and let $K_1 := \sup_{t\geq 0} \|e^{-tA}\|$. By Proposition 1, the statement 1 holds if and only if there exist constants $K_2, t_0 > 0$ such that

$$\|e^{-tA}A^{-\beta q}\| \leq \frac{K_2}{t^q} \quad \text{for all } t \geq t_0. \tag{20}$$

(Proof of 1 $\Rightarrow$ 2) Let $x \in H$. The Plancherel theorem gives

$$\int_{-\infty}^{\infty} \|(\xi + i\eta + A)^{-1}A^{-\beta q}x\|^2 \mathrm{d}\eta = 2\pi \int_0^{\infty} e^{-2\xi t}\|e^{-tA}A^{-\beta q}x\|^2 \mathrm{d}t \tag{21}$$

for all $\xi > 0$. Using the estimate (20), we obtain

$$\begin{aligned}
&\int_0^{\infty} e^{-2\xi t}\|e^{-tA}A^{-\beta q}x\|^2 \mathrm{d}t \\
&\quad = \int_0^{t_0} e^{-2\xi t}\|e^{-tA}A^{-\beta q}x\|^2 \mathrm{d}t + \int_{t_0}^{\infty} e^{-2\xi t}\|e^{-tA}A^{-\beta q}x\|^2 \mathrm{d}t \\
&\quad \leq t_0 K_1^2 \|A^{-\beta q}\|^2 \, \|x\|^2 + K_2^2\|x\|^2 \int_{t_0}^{\infty} \frac{e^{-2\xi t}}{t^{2q}} \mathrm{d}t
\end{aligned}$$

for all $\xi > 0$. Since

$$\int_{t_0}^{\infty} \frac{e^{-2\xi t}}{t^{2q}} \mathrm{d}t \leq \frac{\Gamma(1-2q)}{(2\xi)^{1-2q}},$$

where Γ is a gamma function, we obtain

$$\begin{aligned}
&\sup_{0<\xi<1} \xi^{1-2q} \int_0^{\infty} e^{-2\xi t}\|e^{-tA}A^{-\beta q}x\|^2 \mathrm{d}t \\
&\qquad \leq \left(t_0 K_1^2 \|A^{-\beta q}\|^2 + K_2^2 \frac{\Gamma(1-2q)}{2^{1-2q}} \right) \|x\|^2.
\end{aligned} \tag{22}$$

From (21) and (22), we conclude that the statement 2 holds.

(Proof of 2 $\Rightarrow$ 1) Using the inverse formula given in [12, Corollary III.5.16], we have that for all $x \in D(A^2)$, $y \in H$, and $t, \xi > 0$,

$$|\langle e^{-tA}x, y\rangle| \leq \frac{e^{\xi t}}{2\pi t} \int_{-\infty}^{\infty} |\langle(\xi + i\eta + A)^{-2}x, y\rangle| \mathrm{d}\eta;$$

see also [5, Corollary 8.14] for the inversion formula. Together with (14), (19), and the Cauchy–Schwartz inequality, this estimate implies that there exists a constant $M_1 > 0$ such that

$$\|e^{-tA}A^{-\beta q}x\| \leq \frac{M_1 e^{\xi t}}{\xi^{1-q}\, t}\|x\|$$

for all $x \in D(A^2)$, $t > 0$, and $0 < \xi < 1$. Since $D(A^2)$ is dense in H, setting $\xi = 1/t$ yields

$$\|e^{-tA}A^{-\beta q}\| \leq \frac{eM_1}{t^q}$$

for all $t > 1$. Thus, $\|e^{-tA}A^{-1}\| = O(t^{-1/\beta})$ is obtained. $\square$

Let $q > 0$, and define

$$\|f\|_{\mathcal{B}_{0,q}} := \int_0^\infty \psi_q(\xi) \sup_{\eta \in \mathbb{R}} |f'(\xi + i\eta)| \mathrm{d}\xi, \quad f \in \mathcal{B},$$

where

$$\psi_q(\xi) := \begin{cases} \xi^q, & 0 < \xi < 1, \\ 1, & \xi \geq 1. \end{cases} \tag{23}$$

Since $0 < \psi_q(\xi) \leq 1$ for all $\xi > 0$, we have $\|f\|_{\mathcal{B}_{0,q}} \leq \|f\|_{\mathcal{B}_0}$ for all $f \in \mathcal{B}$. When $-A$ is the generator of a polynomially stable C_0-semigroup with parameter $\beta > 0$, the operator norm for $f(A)A^{-\beta q}$ can be upper-bounded by using $\|f\|_{\mathcal{B}_{0,q}}$.

Proposition 2. *Let $-A$ be the generator of a polynomially stable C_0-semigroup with parameter $\beta > 0$ on a Hilbert space H. Then, for all $q \in (0, 1/2)$, there exists $M > 0$ such that*

$$\|f(A)A^{-\beta q}\| \leq \|A^{-\beta q}\| \, |f(\infty)| + M\|f\|_{\mathcal{B}_{0,q}} \tag{24}$$

for all $f \in \mathcal{B}$.

Proof. Let $q \in (0, 1/2)$ and let $K := \sup_{t \geq 0} \|e^{-tA}\|$. For all $x, y \in H$ and $f \in \mathcal{B}$,

$$\begin{aligned} &\left| \int_0^\infty \xi \int_{-\infty}^\infty \langle (\xi - i\eta + A)^{-2}x, y \rangle f'(\xi + i\eta) \mathrm{d}\eta \mathrm{d}\xi \right| \\ &\quad \leq \int_0^\infty \psi_q(\xi) \sup_{\eta \in \mathbb{R}} |f'(\xi + i\eta)| \frac{\xi}{\psi_q(\xi)} \int_{-\infty}^\infty |\langle (\xi - i\eta + A)^{-2}x, y \rangle| \mathrm{d}\eta \mathrm{d}\xi. \end{aligned} \tag{25}$$

To the right-hand side, we apply Lemma 1 for $0 < \xi < 1$ and the estimate (13) for $\xi \geq 1$. For $x \in D(A^{\beta q})$, we replace x by $A^{\beta q}x$ in Lemma 1 and use the inequality $\|x\| \leq \|A^{-\beta q}\| \, \|A^{\beta q}x\|$ in (13). Then, we see that there exists $M_1 > 0$ such that

$$\sup_{\xi > 0} \frac{\xi}{\psi_q(\xi)^2} \int_{-\infty}^\infty \|(\xi - i\eta + A)^{-1}x\|^2 \mathrm{d}\eta \leq M_1^2 \|A^{\beta q}x\|^2 \tag{26}$$

for all $x \in D(A^{\beta q})$. The estimates (14) and (26) together with the Cauchy–Schwartz inequality imply

$$\sup_{\xi > 0} \frac{\xi}{\psi_q(\xi)} \int_{-\infty}^\infty |\langle (\xi - i\eta + A)^{-2}x, y \rangle| \mathrm{d}\eta \leq \sqrt{\pi} K M_1 \|A^{\beta q}x\| \, \|y\| \tag{27}$$

for all $x \in D(A^{\beta q})$ and $y \in H$. Applying the estimates (25) and (27) to the definition (16) of $f(A)$, we derive

$$|\langle f(A)x, y \rangle| \leq |f(\infty)| \, \|x\| \, \|y\| + \frac{2KM_1}{\sqrt{\pi}} \|f\|_{\mathcal{B}_{0,q}} \, \|A^{\beta q}x\| \, \|y\|$$

for all $x \in D(A^{\beta q})$, $y \in H$, and $f \in \mathcal{B}$. Thus,

$$\|f(A)A^{-\beta q}\| \le \|A^{-\beta q}\| \, |f(\infty)| + \frac{2KM_1}{\sqrt{\pi}} \|f\|_{\mathcal{B}_0,q}$$

for all $f \in \mathcal{B}$. □

In Sects. 3 and 4, we will derive operator norm estimates for the Cayley transform and the inverse of a semigroup generator from estimates of $\|f\|_{\mathcal{B}_0}$ or $\|f\|_{\mathcal{B}_0,q}$ for the corresponding $\mathcal{B}$-functions f, by using the inequalities (17) and (24). The inequality (24) will be applied in the following way.

Example 3. Let $q \in (0, 1/2)$ and let $-A$ be the generator of a polynomially stable C_0-semigroup with parameter $\beta > 0$ on a Hilbert space. For $\alpha, c > 0$, we define $v_{\alpha,c} \in \mathcal{LM}$ as in Example 1.(b). Using (18), we obtain

$$f(A)A^{-\alpha-\beta q} = (f v_{\alpha,c})(A)A^{-\beta q}(I + cA^{-1})^{\alpha}$$

for all $f \in \mathcal{B}$. Therefore, Proposition 2 shows that there exists $M > 0$ such that

$$\|f(A)A^{-\alpha-\beta q}\| \le M \|f v_{\alpha,c}\|_{\mathcal{B}_0,q}$$

for all $f \in \mathcal{B}$.

3. Estimates for Cayley transforms

In this section, first we study the asymptotic behavior of $\|V(A)^n A^{-\alpha}\|$ for $\alpha > 0$ when $-A$ is the generator of a polynomially stable C_0-semigroup on a Hilbert space. Next we turn our attention to exponentially stable C_0-semigroups on Hilbert spaces and the case of variable parameters $\|(\prod_{k=1}^{n} V_{\omega_k}(A))A^{-\alpha}\|$.

3.1. Case of constant parameters

When $-A$ is the generator of a polynomially stable C_0-semigroup on a Hilbert space, the following estimate for $\|V(A)^n A^{-\alpha}\|$ holds without the assumption that $(V(A)^n)_{n\in\mathbb{N}_0}$ is bounded. Recall that $\lfloor \xi \rfloor$ is the largest integer less than or equal to $\xi \in \mathbb{R}$.

Theorem 1. *Let $-A$ be the generator of a polynomially stable C_0-semigroup with parameter $\beta > 0$ on a Hilbert space H. Then, for all $\alpha > 0$,*

$$\|V(A)^n A^{-\alpha}\| = O\left(\frac{(\log n)^{k+1}}{n^{\alpha/(2+\beta)}}\right) \quad (n \to \infty), \quad \text{where } k := \left\lfloor \frac{2\alpha}{2+\beta} \right\rfloor. \tag{28}$$

Let $n \in \mathbb{N}$ and $p > 0$. Define

$$f_{n,2p}(z) := \frac{(z-1)^n}{(z+1)^{n+2p}}, \quad z \in \mathbb{C}_+. \tag{29}$$

Then $f_{n,2p} \in \mathcal{LM}$; see Example 1. To prove Theorem 1, we need a preliminary estimate for $\|f_{n,2p}\|_{\mathcal{B}_0,q}$.

Proposition 3. *For $n \in \mathbb{N}$ and $p > 0$, define the function $f_{n,2p}$ by (29). Then, for all $q \in (0, 1/2)$,*

$$\|f_{n,2p}\|_{\mathcal{B}_{0,q}} = \begin{cases} O\left(\dfrac{1}{n^p}\right), & p < q, \\ O\left(\dfrac{\log n}{n^p}\right), & p = q, \\ O\left(\dfrac{1}{n^q}\right), & p > q, \end{cases} \quad (n \to \infty). \tag{30}$$

Proof. Step 1: Let $n \in \mathbb{N}$. Fix $p > 0$ and $q \in (0, 1/2)$. Since

$$f'_{n,2p}(z) = -2p\frac{(z-1)^n}{(z+1)^{n+2p+1}} + 2n\frac{(z-1)^{n-1}}{(z+1)^{n+2p+1}},$$

we have

$$\sup_{\eta \in \mathbb{R}} |f'_{n,2p}(\xi + i\eta)| \le 2p \sup_{s \ge 0} g_{n,p}(\xi, s) + 2n \sup_{s \ge 0} h_{n,p}(\xi, s), \tag{31}$$

where

$$g_{n,p}(\xi, s) := \frac{\left((\xi-1)^2 + s\right)^{n/2}}{\left((\xi+1)^2 + s\right)^{(n+1)/2+p}},$$

$$h_{n,p}(\xi, s) := \frac{\left((\xi-1)^2 + s\right)^{(n-1)/2}}{\left((\xi+1)^2 + s\right)^{(n+1)/2+p}}$$

for $\xi > 0$ and $s \ge 0$. In Steps 2 and 3 below, we will show that

$$\int_0^\infty \psi_q(\xi) \sup_{s \ge 0} g_{n,p}(\xi, s) \mathrm{d}\xi = \begin{cases} O\left(\dfrac{1}{n^p}\right), & p \le q, \\ O\left(\dfrac{1}{n^q}\right), & p > q, \end{cases} \quad (n \to \infty) \tag{32}$$

and

$$\int_0^\infty \psi_q(\xi) \sup_{s \ge 0} h_{n,p}(\xi, s) \mathrm{d}\xi = \begin{cases} O\left(\dfrac{1}{n^{p+1}}\right), & p < q, \\ O\left(\dfrac{\log n}{n^{p+1}}\right), & p = q, \\ O\left(\dfrac{1}{n^{q+1}}\right), & p > q, \end{cases} \quad (n \to \infty). \tag{33}$$

Combining the estimates (31)–(33), we obtain the desired conclusion (30).

Step 2: The aim of this step is to show the estimate (32) for $g_{n,p}$. Fix $\xi > 0$, and define

$$G_\xi(s) := g_{n,p}(\xi, s)^2 = \frac{\left((\xi-1)^2 + s\right)^n}{\left((\xi+1)^2 + s\right)^{n+2p+1}}$$

for $s \geq 0$. Then,

$$G'_\xi(s) = \frac{\big((\xi-1)^2+s\big)^{n-1}}{\big((\xi+1)^2+s\big)^{n+2p+2}}\chi(s),$$

where

$$\chi(s) := n\big((\xi+1)^2+s\big) - (n+2p+1)\big((\xi-1)^2+s\big).$$

Since

$$\chi(s) = 2(2p+1+2n)\xi - (2p+1)(s+\xi^2+1),$$

the equation $\chi(s) = 0$ has the following solution:

$$s_1 := -\xi^2 - 1 + 2\left(1+\frac{2n}{2p+1}\right)\xi.$$

Moreover, the solution s_1 satisfies $s_1 \geq 0$ if and only if $\xi_0 \leq \xi \leq \xi_1$, where

$$\begin{aligned}\xi_0 := \xi_0(n) &:= 1 + \frac{2n}{2p+1} - \frac{2\sqrt{n(n+2p+1)}}{2p+1},\\ \xi_1 := \xi_1(n) &:= 1 + \frac{2n}{2p+1} + \frac{2\sqrt{n(n+2p+1)}}{2p+1}.\end{aligned}$$

Note that $\xi_1 > 1$. Hence $\xi_0 = 1/\xi_1 < 1$. We also have

$$\sup_{s\geq 0} g_{n,p}(\xi,s) = \begin{cases}\sqrt{G_\xi(s_1)}, & \xi \in [\xi_0,\xi_1],\\ \sqrt{G_\xi(0)}, & \xi \notin [\xi_0,\xi_1].\end{cases} \tag{34}$$

Since routine calculations show that

$$\begin{aligned}(\xi-1)^2+s_1 &= \frac{4n}{2p+1}\xi,\\ (\xi+1)^2+s_1 &= \frac{4(n+2p+1)}{2p+1}\xi,\end{aligned}$$

we obtain

$$G_\xi(s_1) = \left(\frac{n}{n+2p+1}\right)^n \left(\frac{2p+1}{4(n+2p+1)}\right)^{2p+1}\frac{1}{\xi^{2p+1}}.$$

Combining this with (34), we obtain

$$\begin{aligned}&\int_{\xi_0}^{\xi_1}\psi_q(\xi)\sup_{s\geq 0} g_{n,p}(\xi,s)\mathrm{d}\xi\\ &\quad= \left(\frac{n}{n+2p+1}\right)^{n/2}\left(\frac{2p+1}{4(n+2p+1)}\right)^{p+1/2}\int_{\xi_0}^{\xi_1}\frac{\psi_q(\xi)}{\xi^{p+1/2}}\mathrm{d}\xi.\end{aligned}$$

By the definition (23) of ψ_q,

$$\int_{\xi_0}^{\xi_1} \frac{\psi_q(\xi)}{\xi^{p+1/2}} \mathrm{d}\xi = \int_{\xi_0}^{1} \frac{1}{\xi^{p-q+1/2}} \mathrm{d}\xi + \int_{1}^{\xi_1} \frac{1}{\xi^{p+1/2}} \mathrm{d}\xi.$$

If $p \neq 1/2$ and $p \neq q + 1/2$, then

$$\int_{\xi_0}^{\xi_1} \frac{\psi_q(\xi)}{\xi^{p+1/2}} \mathrm{d}\xi = \left(\frac{1}{q-p+1/2} - \frac{1}{1/2-p} \right) - \frac{\xi_0^{q-p+1/2}}{q-p+1/2} + \frac{\xi_1^{1/2-p}}{1/2-p},$$

and hence we have from $\xi_0(n) = O(n^{-1})$ and $\xi_1(n) = O(n)$ as $n \to \infty$ that

$$\int_{\xi_0}^{\xi_1} \psi_q(\xi) \sup_{s \geq 0} g_{n,p}(\xi, s) \mathrm{d}\xi = \begin{cases} O\left(\dfrac{1}{n^{2p}}\right), & p < \dfrac{1}{2}, \\ O\left(\dfrac{1}{n^{p+1/2}}\right), & \dfrac{1}{2} < p < q + \dfrac{1}{2}, \\ O\left(\dfrac{1}{n^{q+1}}\right), & p > q + \dfrac{1}{2} \end{cases} \tag{35}$$

as $n \to \infty$. Similarly, if $p = 1/2$ or $p = q + 1/2$, then

$$\int_{\xi_0}^{\xi_1} \psi_q(\xi) \sup_{s \geq 0} g_{n,p}(\xi, s) \mathrm{d}\xi = O\left(\frac{\log n}{n^{p+1/2}}\right) \tag{36}$$

as $n \to \infty$. On the other hand, we derive from (34) that

$$\int_{\xi_1}^{\infty} \psi_q(\xi) \sup_{s \geq 0} g_{n,p}(\xi, s) \mathrm{d}\xi = \int_{\xi_1}^{\infty} \frac{(\xi - 1)^n}{(\xi + 1)^{n+2p+1}} \mathrm{d}\xi \leq \int_{\xi_1}^{\infty} \frac{1}{\xi^{2p+1}} \mathrm{d}\xi$$

and

$$\int_{0}^{\xi_0} \psi_q(\xi) \sup_{s \geq 0} g_{n,p}(\xi, s) \mathrm{d}\xi = \int_{0}^{\xi_0} \frac{\xi^q (1 - \xi)^n}{(\xi + 1)^{n+2p+1}} \mathrm{d}\xi \leq \int_{0}^{\xi_0} \xi^q \mathrm{d}\xi.$$

Therefore,

$$\int_{\xi_1}^{\infty} \psi_q(\xi) \sup_{s \geq 0} g_{n,p}(\xi, s) \mathrm{d}\xi = O\left(\frac{1}{n^{2p}}\right) \tag{37}$$

and

$$\int_{0}^{\xi_0} \psi_q(\xi) \sup_{s \geq 0} g_{n,p}(\xi, s) \mathrm{d}\xi = O\left(\frac{1}{n^{q+1}}\right) \tag{38}$$

as $n \to \infty$. Thus, the estimate (32) for $g_{n,p}$ holds, and it is actually less sharp than the estimate obtained from the above argument.

Step 3: We immediately obtain the estimate (33) for $h_{n,p}$, by replacing n with $n-1$ and p with $p + 1/2$ in the estimates (35)–(38). □

We are well prepared to obtain the estimate (28) for $\|V(A)^n A^{-\alpha}\|$.

Proof of Theorem 1. Let $p \in (0, 1/2)$ and define the function $f_{n,2p}$ by (29) for $n \in \mathbb{N}$. By Example 3 in the case $\alpha = 2p$, $c = 1$, $q = p$, and

$$f(z) = \left(\frac{z-1}{z+1}\right)^n,$$

there exists a constant $M > 0$ such that

$$\|V(A)^n A^{-(2+\beta)p}\| \le M \|f_{n,2p}\|_{\mathcal{B}_{0,p}}$$

for all $n \in \mathbb{N}$. From the estimate (30) with $p = q$, we have

$$\|V(A)^n A^{-(2+\beta)p}\| = O\left(\frac{\log n}{n^p}\right) \tag{39}$$

as $n \to \infty$. Hence, the desired estimate (28) holds in the case $k = \lfloor 2\alpha/(2+\beta) \rfloor = 0$.

Next we consider the case $k = \lfloor 2\alpha/(2+\beta) \rfloor \ge 1$. To this end, choose $\alpha \ge (2+\beta)/2$ arbitrarily. Let $k \in \mathbb{N}$ and $0 \le \delta_0 < (2+\beta)/2$ satisfy

$$\alpha = \frac{k(2+\beta)}{2} + \delta_0.$$

Then there exists $\varepsilon > 0$ such that

$$0 < \delta := \delta_0 + \frac{\varepsilon k}{2} < \frac{2+\beta}{2}.$$

By definition,

$$\alpha = k\gamma + \delta, \quad \text{where } \gamma := \frac{2+\beta-\varepsilon}{2},$$

and hence

$$\|V(A)^n A^{-\alpha}\| \le \left(\max_{0 \le \ell \le k} \|V(A)^\ell\|\right) \|V(A)^{\lfloor n/(k+1) \rfloor} A^{-\gamma}\|^k \, \|V(A)^{\lfloor n/(k+1) \rfloor} A^{-\delta}\|$$

for all $n \in \mathbb{N}$. Using the estimate (39), we obtain

$$\|V(A)^{\lfloor n/(k+1) \rfloor} A^{-\gamma}\| = O\left(\frac{\log n}{n^{\gamma/(2+\beta)}}\right)$$

and

$$\|V(A)^{\lfloor n/(k+1) \rfloor} A^{-\delta}\| = O\left(\frac{\log n}{n^{\delta/(2+\beta)}}\right)$$

as $n \to \infty$. Thus, the desired estimate (28) holds. $\square$

Remark 1. In the proof of Theorem 1, we have used the estimate (30) with $p = q$ for $\|f_{n,2p}\|_{\mathcal{B}_{0,q}}$. This is because the estimate (30) with $p \neq q$ yields a worse decay rate. To see this, assume that the C_0-semigroup $(e^{-tA})_{t\geq 0}$ is polynomially stable with parameter $\beta > 0$. The estimate (30) with $p < q$ $(< 1/2)$ gives

$$\|V(A)^n A^{-2p-\beta q}\| = O\left(\frac{1}{n^p}\right) \qquad (n \to \infty). \tag{40}$$

If $2p + \beta q = 1$ and $p < q$, then

$$p < \frac{1}{2+\beta}.$$

Hence, the estimate (40) with $2p+\beta q = 1$ is worse than the estimate (28) with $\alpha = 1$. A similar argument can be applied to the estimate (30) with $p > q$.

3.2. Case of variable parameters

Here we study the case where the parameter ω of the Cayley transform $V_\omega(A)$ varies. First we improve the previous estimate developed in [24, Theorem 3.3] for exponentially stable C_0-semigroups. Next we consider polynomially stable C_0-semigroups with normal generators and present a similar improved estimate over the one given in [24, Proposition 3.6].

3.2.1. Exponentially stable semigroups

When the C_0-semigroup $(e^{-tA})_{t\geq 0}$ is exponentially stable, we give an estimate for $\|(\prod_{k=1}^n V_{\omega_k}(A))A^{-\alpha}\|$ without assuming the boundedness of $(e^{-tA^{-1}})_{t\geq 0}$. Before that, we define the function F_α on $\mathbb{N}$ by

$$F_\alpha(n) := \begin{cases} \log n, & \alpha = 0, \\ \dfrac{1}{n^{\alpha/2}}, & \alpha > 0. \end{cases} \tag{41}$$

Recall that for $0 < \omega_p \leq \omega_q < \infty$, we denote by $\mathcal{S}(\omega_p, \omega_q)$ the set of sequences $(\omega_k)_{k\in\mathbb{N}}$ of positive real numbers satisfying $\omega_p \leq \omega_k \leq \omega_q$ for all $k \in \mathbb{N}$.

Theorem 2. *Let $-A$ be the generator of an exponentially stable C_0-semigroup on a Hilbert space H. Then, for all $\alpha \geq 0$ and $0 < \omega_p \leq \omega_q < \infty$, there exist constants $M > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\left\|\left(\prod_{k=1}^n V_{\omega_k}(A)\right) A^{-\alpha}\right\| \leq M F_\alpha(n) \tag{42}$$

for all $n \geq n_0$ and $(\omega_k)_{k\in\mathbb{N}} \in \mathcal{S}(\omega_p, \omega_q)$, where the function F_α is as in (41).

When $(e^{-tA})_{t\geq 0}$ is an exponentially stable C_0-semigroup, $-A+cI$ generates a bounded C_0-semigroup for some sufficiently small constant $c>0$. To prove Theorem 2 by means of the $\mathcal{B}$-calculus, we consider the function $f_{n,\alpha,(\omega_k)}$ defined by

$$f_{n,\alpha,(\omega_k)}(z) := \frac{1}{(z+\omega_p+c)^\alpha}\prod_{k=1}^{n}\frac{z-\omega_k+c}{z+\omega_k+c},\quad z\in\mathbb{C}_+, \tag{43}$$

where $0<\omega_p\leq\omega_q<\infty$ and $(\omega_k)_{k\in\mathbb{N}}\in\mathcal{S}(\omega_p,\omega_q)$. As seen in Example 1, we have $f_{n,\alpha,(\omega_k)}\in\mathcal{LM}$. The following result gives an estimate for $\|f_{n,\alpha,(\omega_k)}\|_{\mathcal{B}_0}$.

Proposition 4. *Let $\alpha\geq 0$, $c>0$, and $0<\omega_p\leq\omega_q<\infty$. Then there exist constants $M>0$ and $n_0\in\mathbb{N}$ such that the function $f_{n,\alpha,(\omega_k)}$ defined by (43) satisfies*

$$\|f_{n,\alpha,(\omega_k)}\|_{\mathcal{B}_0}\leq MF_\alpha(n) \tag{44}$$

for all $n\geq n_0$ and $(\omega_k)_{k\in\mathbb{N}}\in\mathcal{S}(\omega_p,\omega_q)$, where the function F_α is as in (41).

The difficulty in estimating $\|f_{n,\alpha,(\omega_k)}\|_{\mathcal{B}_0}$ is that the parameter ω_k can vary on $[\omega_p,\omega_q]$, which complicates the computation of $\sup_{\eta\in\mathbb{R}}|f'_{n,\alpha,(\omega_k)}(\xi+i\eta)|$. To circumvent this difficulty, we show that the function on $[\omega_p,\omega_q]$ defined by

$$\omega\mapsto\left|\frac{(\xi-\omega+c)+i\eta}{(\xi+\omega+c)+i\eta}\right|$$

takes the maximum value at $\omega=\omega_p$ for all $\xi>0$ and $\eta\in\mathbb{R}$ under a suitable condition on the constant c.

Lemma 2. *Let $c>0$ and $0<\omega_p\leq\omega_q<\infty$ satisfy $c^2\geq\omega_p\omega_q$. If $\omega_p\leq\omega\leq\omega_q$, then*

$$\frac{(\xi-\omega+c)^2+s}{(\xi+\omega+c)^2+s}\leq\frac{(\xi-\omega_p+c)^2+s}{(\xi+\omega_p+c)^2+s}$$

for all $\xi>0$ and $s\geq 0$.

Proof. Fix $\xi>0$ and $s\geq 0$. Set $\zeta:=\xi+c$. Define

$$\phi_{\zeta,s}(\omega):=\frac{(\zeta-\omega)^2+s}{(\zeta+\omega)^2+s},\quad\omega>0.$$

Then

$$\phi'_{\zeta,s}(\omega)=\frac{4\zeta(\omega^2-\zeta^2-s)}{\big((\zeta+\omega)^2+s\big)^2}.$$

Therefore, $\omega_0:=\sqrt{\zeta^2+s}$ satisfies $\phi'_{\zeta,s}(\omega_0)=0$, and $\phi_{\zeta,s}$ decreases on the interval $(0,\omega_0)$ and increases on the interval (ω_0,∞). This implies that for all $\omega\in[\omega_p,\omega_q]$,

$$\phi_{\zeta,s}(\omega)\leq\max\{\phi_{\zeta,s}(\omega_p),\ \phi_{\zeta,s}(\omega_q)\}.$$

It remains to show that if $c^2 \geq \omega_p \omega_q$, then

$$\phi_{\zeta,s}(\omega_p) \geq \phi_{\zeta,s}(\omega_q). \tag{45}$$

This can be proved by a direct calculation. We have

$$\phi_{\zeta,s}(\omega_p) - \phi_{\zeta,s}(\omega_q) = \frac{\chi_{\zeta,s}}{\big((\zeta+\omega_p)^2 + s\big)\big((\zeta+\omega_q)^2+s\big)},$$

where

$$\chi_{\zeta,s} := \big((\zeta-\omega_p)^2+s\big)\big((\zeta+\omega_q)^2+s\big) - \big((\zeta-\omega_q)^2+s\big)\big((\zeta+\omega_p)^2+s\big).$$

One can rewrite $\chi_{\zeta,s}$ as

$$\begin{aligned}\chi_{\zeta,s} = &\big((\zeta-\omega_p)^2(\zeta+\omega_q)^2 - (\zeta-\omega_q)^2(\zeta+\omega_p)^2\big)\\ &+ \big((\zeta-\omega_p)^2 + (\zeta+\omega_q)^2 - (\zeta-\omega_q)^2 - (\zeta+\omega_p)^2\big)s.\end{aligned}$$

The coefficient of s satisfies

$$(\zeta-\omega_p)^2 + (\zeta+\omega_q)^2 - (\zeta-\omega_q)^2 - (\zeta+\omega_p)^2 = 4(\omega_q-\omega_p)\zeta \geq 0.$$

Moreover, a routine calculation shows that

$$(\zeta-\omega_p)^2(\zeta+\omega_q)^2 - (\zeta-\omega_q)^2(\zeta+\omega_p)^2 = 4(\omega_q-\omega_p)\zeta(\zeta^2-\omega_p\omega_q).$$

Substituting $\zeta = \xi + c$, we have from the condition $c^2 \geq \omega_p\omega_q$ that

$$\zeta^2 - \omega_p\omega_q = \xi^2 + 2c\xi + (c^2-\omega_p\omega_q) \geq 0.$$

Thus, the inequality (45) holds. □

Next we give auxiliary estimates to be used after Lemma 2 is applied.

Lemma 3. *Let $n \in \mathbb{N}$, $\alpha \geq 0$, and $\omega, c > 0$. Define*

$$g_{n,\alpha,\omega}(\xi,s) := \frac{\big((\xi-\omega+c)^2+s\big)^{n/2}}{\big((\xi+\omega+c)^2+s\big)^{(n+\alpha+1)/2}}, \tag{46}$$

$$h_{n,\alpha,\omega}(\xi,s) := \frac{\big((\xi-\omega+c)^2+s\big)^{(n-1)/2}}{\big((\xi+\omega+c)^2+s\big)^{(n+\alpha+1)/2}} \tag{47}$$

for $\xi > 0$ and $s \geq 0$. Then

$$\int_0^\infty \sup_{s\geq 0} g_{n,\alpha,\omega}(\xi,s)\mathrm{d}\xi = \begin{cases} O\left(\dfrac{1}{n^\alpha}\right), & 0 \leq \alpha < 1, \\ O\left(\dfrac{\log n}{n}\right), & \alpha = 1, \\ O\left(\dfrac{1}{n^{(\alpha+1)/2}}\right), & \alpha > 1, \end{cases} \qquad (n\to\infty) \tag{48}$$

and

$$\int_0^\infty \sup_{s\geq 0} h_{n,\alpha,\omega}(\xi,s)\mathrm{d}\xi = \begin{cases} O\left(\dfrac{\log n}{n}\right), & \alpha = 0, \\ O\left(\dfrac{1}{n^{\alpha/2+1}}\right), & \alpha > 0, \end{cases} \qquad (n \to \infty). \tag{49}$$

Proof. It suffices to show the estimate (48) for $g_{n,\alpha,\omega}$. Indeed, the estimate (49) for $h_{n,\alpha,\omega}$ is obtained by replacing n with $n-1$ and α with $\alpha+1$ in the estimate (48) for $g_{n,\alpha,\omega}$.

Let $n \in \mathbb{N}$, $\alpha \geq 0$, and $\omega > 0$. To obtain the estimate for $g_{n,\alpha,\omega}$, fix $\xi > 0$ and define

$$G_\xi(s) := g_{n,\alpha,\omega}(\xi,s)^2 = \frac{\big((\xi-\omega+c)^2+s\big)^n}{\big((\xi+\omega+c)^2+s\big)^{n+\alpha+1}}$$

for $s \geq 0$. Then,

$$G_\xi'(s) = \frac{\big((\xi-\omega+c)^2+s\big)^{n-1}}{\big((\xi+\omega+c)^2+s\big)^{n+\alpha+2}}\chi(s),$$

where

$$\chi(s) := n\big((\xi+\omega+c)^2+s\big) - (n+\alpha+1)\big((\xi-\omega+c)^2+s\big).$$

We have

$$\chi(s) = (\alpha+1)\left(-\xi^2 + 2\left(\frac{2\omega n}{\alpha+1}+\omega-c\right)\xi + \frac{4\omega c n}{\alpha+1} - (\omega-c)^2 - s\right).$$

Therefore, $\chi(s_1) = 0$, where s_1 is defined by

$$s_1 := -\xi^2 + 2\left(\frac{2\omega n}{\alpha+1}+\omega-c\right)\xi + \frac{4\omega c n}{\alpha+1} - (\omega-c)^2.$$

Since $\omega, c > 0$, we obtain

$$\frac{4\omega c n}{\alpha+1} - (\omega-c)^2 > 0$$

for all sufficiently large $n \in \mathbb{N}$. Hence, there exists $n_1 \in \mathbb{N}$ such that the equation

$$-\xi^2 + 2\left(\frac{2\omega n}{\alpha+1}+\omega-c\right)\xi + \frac{4\omega c n}{\alpha+1} - (\omega-c)^2 = 0$$

has a unique positive solution $\xi_1 = \xi_1(n)$ for all $n \geq n_1$, and

$$\xi_1(n) = \left(\frac{2\omega n}{\alpha+1}+\omega-c\right) + \frac{2\omega\sqrt{n(n+\alpha+1)}}{\alpha+1}.$$

Let $n \geq n_1$. We obtain

$$\sup_{s\geq 0} g_{n,\alpha,\omega}(\xi,s) = \begin{cases} \sqrt{G_\xi(s_1)}, & \xi \leq \xi_1, \\ \sqrt{G_\xi(0)}, & \xi > \xi_1. \end{cases} \tag{50}$$

Routine calculations show that

$$\begin{aligned}(\xi-\omega+c)^2+s_1 &= \frac{4\omega n}{\alpha+1}(\xi+c),\\ (\xi+\omega+c)^2+s_1 &= \frac{4\omega(n+\alpha+1)}{\alpha+1}(\xi+c).\end{aligned}$$

Hence we obtain

$$G_\xi(s_1) = \left(\frac{n}{n+\alpha+1}\right)^n \left(\frac{\alpha+1}{4\omega(n+\alpha+1)}\right)^{\alpha+1} \frac{1}{(\xi+c)^{\alpha+1}}.$$

This and (50) give

$$\begin{aligned}&\int_0^{\xi_1} \sup_{s\geq 0} g_{n,\alpha,\omega}(\xi,s)\mathrm{d}\xi \\ &\quad = \left(\frac{n}{n+\alpha+1}\right)^{n/2} \left(\frac{\alpha+1}{4\omega(n+\alpha+1)}\right)^{(\alpha+1)/2} \int_0^{\xi_1} \frac{1}{(\xi+c)^{(\alpha+1)/2}}\mathrm{d}\xi.\end{aligned}$$

Noting that $\xi_1(n) = O(n)$ as $n \to \infty$, we have

$$\int_0^{\xi_1} \sup_{s\geq 0} g_{n,\alpha,\omega}(\xi,s)\mathrm{d}\xi = \begin{cases} O\left(\frac{1}{n^\alpha}\right), & \alpha < 1, \\ O\left(\frac{\log n}{n}\right), & \alpha = 1, \\ O\left(\frac{1}{n^{(\alpha+1)/2}}\right), & \alpha > 1 \end{cases} \tag{51}$$

as $n \to \infty$. We also see from (50) that

$$\int_{\xi_1}^\infty \sup_{s\geq 0} g_{n,\alpha,\omega}(\xi,s)\mathrm{d}\xi = \int_{\xi_1}^\infty \frac{(\xi-\omega+c)^n}{(\xi+\omega+c)^{n+\alpha+1}}\mathrm{d}\xi \leq \int_{\xi_1}^\infty \frac{1}{(\xi+\omega+c)^{\alpha+1}}\mathrm{d}\xi.$$

Hence,

$$\int_{\xi_1}^\infty \sup_{s\geq 0} g_{n,\alpha,\omega}(\xi,s)\mathrm{d}\xi = O\left(\frac{1}{n^\alpha}\right) \tag{52}$$

as $n \to \infty$. Combining the estimates (51) and (52), we obtain the estimate (48) for g_n. □

Now we are in a position to prove Proposition 4 and Theorem 2.

Proof of Proposition 4. Define $\tilde{\omega}_p := \min\{\omega_p, c^2/\omega_q\}$. Then $\mathcal{S}(\omega_p, \omega_q) \subset \mathcal{S}(\tilde{\omega}_p, \omega_q)$ and $c^2 \geq \tilde{\omega}_p \omega_q$. Replacing ω_p with $\tilde{\omega}_p$ in the definition (43) of $f_{n,\alpha,(\omega_k)}$, we define the function $\tilde{f}_{n,\alpha,(\omega_k)}$ by

$$\tilde{f}_{n,\alpha,(\omega_k)}(z) := \frac{1}{(z+\tilde{\omega}_p+c)^\alpha} \prod_{k=1}^n \frac{z-\omega_k+c}{z+\omega_k+c}, \quad z \in \mathbb{C}_+,$$

where $(\omega_k)_{k\in\mathbb{N}} \in \mathcal{S}(\tilde{\omega}_p, \omega_q)$. Then, for all $(\omega_k)_{k\in\mathbb{N}} \in \mathcal{S}(\omega_p, \omega_q)$,

$$f_{n,\alpha,(\omega_k)} = r_\alpha \tilde{f}_{n,\alpha,(\omega_k)},$$

where

$$r_\alpha(z) := \left(\frac{z+\tilde{\omega}_p+c}{z+\omega_p+c}\right)^\alpha, \quad z \in \mathbb{C}_+.$$

Since $r_1 \in \mathcal{LM}$ (see Example 1.(a)), we have from [4, Proposition 2.2] that $r_\alpha \in \mathcal{B}$.

Now we apply Lemmas 2 and 3 to $\tilde{f}_{n,\alpha,(\omega_k)}$. The derivative of $\tilde{f}_{n,\alpha,(\omega_k)}$ is given by

$$\begin{aligned}\tilde{f}'_{n,\alpha,(\omega_k)}(z) &= \frac{-\alpha}{(z+\tilde{\omega}_p+c)^{\alpha+1}} \prod_{k=1}^n \frac{z-\omega_k+c}{z+\omega_k+c} \\ &\quad + \frac{2}{(z+\tilde{\omega}_p+c)^\alpha} \sum_{\ell=1}^n \frac{\omega_\ell}{(z+\omega_\ell+c)^2} \prod_{k=1,k\neq\ell}^n \frac{z-\omega_k+c}{z+\omega_k+c}.\end{aligned}$$

Let $\xi > 0$, $\eta \in \mathbb{R}$, and $s := \eta^2 \geq 0$. Since $c^2 \geq \tilde{\omega}_p \omega_q$, Lemma 2 shows that

$$|\tilde{f}'_{n,\alpha,(\omega_k)}(\xi+i\eta)| \leq \alpha g_{n,\alpha,\tilde{\omega}_p}(\xi, s) + 2\omega_q n h_{n,\alpha,\tilde{\omega}_p}(\xi, s)$$

for all $(\omega_k)_{k\in\mathbb{N}} \in \mathcal{S}(\tilde{\omega}_p, \omega_q)$, where $g_{n,\alpha,\tilde{\omega}_p}$ and $h_{n,\alpha,\tilde{\omega}_p}$ are defined by (46) and (47) with $\omega = \tilde{\omega}_p$, respectively. From Lemma 3, we see that there exist constants $\tilde{M} > 0$ and $n_0 \in \mathbb{N}$ such that

$$\|\tilde{f}_{n,\alpha,(\omega_k)}\|_{\mathcal{B}_0} \leq \tilde{M} F_\alpha(n)$$

for all $n \geq n_0$ and $(\omega_k)_{k\in\mathbb{N}} \in \mathcal{S}(\tilde{\omega}_p, \omega_q)$. Since $\mathcal{S}(\omega_p, \omega_q) \subset \mathcal{S}(\tilde{\omega}_p, \omega_q)$ and since

$$\|f_{n,\alpha,(\omega_k)}\|_{\mathcal{B}} \leq \|r_\alpha\|_{\mathcal{B}} \|\tilde{f}_{n,\alpha,(\omega_k)}\|_{\mathcal{B}},$$

for all $(\omega_k)_{k\in\mathbb{N}} \in \mathcal{S}(\omega_p, \omega_q)$, the constant $M := 2\|r_\alpha\|_{\mathcal{B}} \tilde{M}$ satisfies

$$\|f_{n,\alpha,(\omega_k)}\|_{\mathcal{B}_0} \leq M F_\alpha(n)$$

for all $n \geq n_0$ and $(\omega_k)_{k\in\mathbb{N}} \in \mathcal{S}(\omega_p, \omega_q)$. $\square$

Proof of Theorem 2. Since the C_0-semigroup $(e^{-tA})_{t\geq 0}$ is exponentially stable, there exist constants $K \geq 1$ and $c > 0$ such that

$$\|e^{-tA}\| \leq Ke^{-ct}$$

for all $t \geq 0$. Define $B := A - cI$. Then $-B$ is the generator of a bounded C_0-semigroup $(e^{-tB})_{t\geq 0}$, and $\sup_{t\geq 0} \|e^{-tB}\| \leq K$.

Define the function $f_{n,\alpha,(\omega_k)}$ by (43). Then,

$$\begin{aligned} f_{n,\alpha,(\omega_k)}(B) &= \left(\prod_{k=1}^{n}(B-\omega_k I + cI)(B+\omega_k I + cI)^{-1}\right)(B+\omega_p I + cI)^{-\alpha} \\ &= \left(\prod_{k=1}^{n} V_{\omega_k}(A)\right)(A+\omega_p I)^{-\alpha}. \end{aligned}$$

The inequality (17) gives

$$\|f_{n,\alpha,(\omega_k)}(B)\| \leq 2K^2 \|f_{n,\alpha,(\omega_k)}\|_{\mathcal{B}_0}.$$

Combining this with Proposition 4, we see that there exist constants $M > 0$ and $n_0 \in \mathbb{N}$ such that the desired estimate (42) holds for all $n \geq n_0$ and $(\omega_k)_{k\in\mathbb{N}} \in \mathcal{S}(\omega_p, \omega_q)$. □

We provide some remarks on the optimality of the estimates given in Proposition 4 and Theorem 2.

Remark 2. The estimate (44) for $\|f_{n,\alpha,(\omega_k)}\|_{\mathcal{B}_0}$ with $\alpha > 0$ cannot be improved in general even in the constant case $\omega_k \equiv 1$. To see this, fix $c > 0$, and define

$$f_{n,\alpha}(z) := \frac{(z-1+c)^n}{(z+1+c)^{n+\alpha}}, \quad z \in \mathbb{C}_+, \tag{53}$$

where $n \in \mathbb{N}$ and $\alpha > 0$. Then, we have

$$\|f_{n,\alpha}\|_{\mathcal{B}_0} \geq \|f_{n,\alpha}\|_\infty = \sup_{\eta\in\mathbb{R}} \left|f_{n,\alpha}(i\eta)\right|. \tag{54}$$

Since

$$|f_{n,\alpha}(i\sqrt{n})|^2 = \frac{1}{\big((1+c)^2+n\big)^\alpha}\left(\frac{(1-c)^2+n}{(1+c)^2+n}\right)^n,$$

there exist $M_1 > 0$ and $n_1 \in \mathbb{N}$ such that

$$|f_{n,\alpha}(i\sqrt{n})| \geq \frac{M_1}{n^{\alpha/2}} \tag{55}$$

for all $n \geq n_1$. The estimates (54) and (55) yield

$$\|f_{n,\alpha}\|_{\mathcal{B}_0} \geq \frac{M_1}{n^{\alpha/2}}$$

for all $n \geq n_1$.

Remark 3. One cannot in general improve the operator norm estimate (42) for $\left\|\left(\prod_{k=1}^{n} V_{\omega_k}(A)\right) A^{-\alpha}\right\|$ with $\alpha > 0$ either. Let B be a self-adjoint operator on a Hilbert space H, and assume that

$$\sigma(B) \supset \{\sqrt{n} : n \in \mathbb{N} \text{ and } n \geq n_2\}$$

for some $n_2 \in \mathbb{N}$. Fix $c > 0$ and define $A := iB + cI$. Then $-A$ is the generator of an exponentially stable C_0-semigroup on H. For $n \in \mathbb{N}$ and $\alpha > 0$, define the function $f_{n,\alpha} \in \mathcal{B}$ by (53). Since

$$V(A) = (iB - I + cI)(iB + I + cI)^{-1},$$

the spectral mapping theorem (see, e.g., [20, Theorem 2.7.8]) shows that

$$\|V(A)^n (A + I)^{-\alpha}\| = \|f_{n,\alpha}(iB)\| = \sup_{\lambda \in \sigma(iB)} |f_{n,\alpha}(\lambda)|.$$

By the estimate (55),

$$\|V(A)^n A^{-\alpha}\| \geq \frac{M_2}{n^{\alpha/2}}$$

for all $n \geq \max\{n_1, n_2\}$, where $M_2 := M_1 / \|(I + A^{-1})^{-\alpha}\|$.

Remark 4. The argument in the proof of Proposition 4 gives a non-sharp constant $M > 0$ for the estimate (44), although the decay rate $n^{-\alpha/2}$ cannot be improved in general. In the proof, we have replaced ω_p by c^2/ω_q when $c^2/\omega_q < \omega_p$. Consequently, the constant M there may allow a unnecessary large variation of $(\omega_k)_{k \in \mathbb{N}}$ when c is small and ω_q is large. Finding a sharp constant M for given constants ω_p and ω_q is left as a topic for further research.

3.2.2. Polynomially stable semigroups with normal generators

Using spectral properties of normal operators, one can also obtain an estimate of $\|(\prod_{k=1}^{n} V_{\omega_k}(A)) A^{-\alpha}\|$ for the normal generator $-A$ of a polynomially stable C_0-semigroup. A similar result holds for multiplication operators on L^p-spaces and spaces of continuous functions that vanish at infinity.

Proposition 5. *Consider the following two cases:*

(a) *Let $(e^{-tA})_{t \geq 0}$ be a bounded C_0-semigroup on a Hilbert space H such that A is normal.*

(b) *Let Ω be a locally compact Hausdorff space, and let μ be a σ-finite regular Borel measure on Ω. Assume that either*

  (i) *$X := L^p(\Omega, \mu)$ for $1 \leq p < \infty$ and $\phi \colon \Omega \to \mathbb{C}$ is measurable with essential range in $\mathbb{C}_+$; or that*

  (ii) *$X := C_0(\Omega)$ and $\phi \colon \Omega \to \mathbb{C}$ is continuous with range in $\mathbb{C}_+$.*

  Let A be the multiplication operator induced by ϕ on X, i.e, $Af = \phi f$ with domain $D(A) := \{f \in X : \phi f \in X\}$.

In both cases (a) *and* (b)*, assume that* $\|e^{-tA}(I+A)^{-1}\| = O(t^{-1/\beta})$ *as* $t \to \infty$ *for some* $\beta > 0$*. Then, for all* $\alpha > 0$ *and* $0 < \omega_p \le \omega_q < \infty$*, there exist constants* $M > 0$ *and* $n_0 \in \mathbb{N}$ *such that*

$$\left\| \left(\prod_{k=1}^{n} V_{\omega_k}(A) \right) A^{-\alpha} \right\| \le \frac{M}{n^{\alpha/(2+\beta)}} \tag{56}$$

for all $n \ge n_0$ *and* $(\omega_k)_{k\in\mathbb{N}} \in \mathcal{S}(\omega_p, \omega_q)$*.*

Proof. Step 1: Let $n \in \mathbb{N}$, $\alpha > 0$, and $(\omega_k)_{k\in\mathbb{N}} \in \mathcal{S}(\omega_p, \omega_q)$. Define

$$f_{n,\alpha,(\omega_k)}(z) := \left(\prod_{k=1}^{n} \frac{z-\omega_k}{z+\omega_k} \right) z^{-\alpha}$$

for $z \in \mathbb{C}_+$. The estimate $\|e^{-tA}(I+A)^{-1}\| = O(t^{-1/\beta})$ as $t \to \infty$ implies that $\sigma(A)$ does not intersect with the imaginary axis $i\mathbb{R}$; see [3, Theorem 1.1]. Hence $\sigma(A) \subset \mathbb{C}_+$ and A has a bounded inverse in both cases (a) and (b). Moreover,

$$\left\| \left(\prod_{k=1}^{n} V_{\omega_k}(A) \right) A^{-\alpha} \right\| = \sup_{\lambda\in\sigma(A)} |f_{n,\alpha,(\omega_k)}(\lambda)|, \tag{57}$$

where we used the spectral mapping theorem (see, e.g., [20, Theorem 2.7.8]) for the case (a) and [12, Propositions I.4.2 and I.4.10] for the case (b); see also the proof of [23, Proposition 4.5], which provides a detailed derivation of (57) in the case $\omega_k \equiv 1$.

By (57), it suffices to show that there exist constants $M > 0$ and $n_0 \in \mathbb{N}$ such that

$$\sup_{\lambda\in\sigma(A)} |f_{n,\alpha,(\omega_k)}(\lambda)| \le \frac{M}{n^{\alpha/(2+\beta)}} \tag{58}$$

for all $n \ge n_0$ and $(\omega_k)_{k\in\mathbb{N}} \in \mathcal{S}(\omega_p, \omega_q)$. We have from Proposition 1 that $\|e^{-tA}A^{-\beta}\| = O(t^{-1})$ as $t \to \infty$. Hence, in both cases (a) and (b), there exist $\delta, C > 0$ such that $|\operatorname{Im}\lambda| \ge C(\operatorname{Re}\lambda)^{-1/\beta}$ for all $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda \le \delta$; see [2, Propositions 4.1 and 4.2]. Define

$$\Omega_1 := \{\lambda \in \sigma(A) : \operatorname{Re}\lambda \le \delta\}, \quad \Omega_2 := \sigma(A)\backslash\Omega_1.$$

We divide the proof of the estimate (58) into two cases: $\lambda \in \Omega_1$ and $\lambda \in \Omega_2$. The following observation will be useful for later purposes: For all $\lambda \in \sigma(A)$ and $\omega > 0$,

$$\left| \frac{\lambda-\omega}{\lambda+\omega} \right|^2 = \frac{(\operatorname{Re}\lambda-\omega)^2 + |\operatorname{Im}\lambda|^2}{(\operatorname{Re}\lambda+\omega)^2 + |\operatorname{Im}\lambda|^2} = 1 - \frac{4\omega\operatorname{Re}\lambda}{|\lambda+\omega|^2}. \tag{59}$$

Step 2: First we assume that $\lambda \in \Omega_1$. We have

$$|\lambda+\omega_q|^2 \le C_0 |\operatorname{Im}\lambda|^2$$

for some constant $C_0 \geq 1$ independent of λ. Combining this estimate with $\operatorname{Re}\lambda \geq C^\beta/|\operatorname{Im}\lambda|^\beta$, we obtain

$$\frac{4\omega \operatorname{Re}\lambda}{|\lambda+\omega|^2} \geq \frac{4C^\beta \omega_p}{C_0|\operatorname{Im}\lambda|^{\beta+2}}$$

for all $\omega \in [\omega_p, \omega_q]$. Then (59) gives that for all $\omega \in [\omega_p, \omega_q]$,

$$\left|\frac{\lambda-\omega}{\lambda+\omega}\right|^2 \leq 1 - \frac{C_1}{|\operatorname{Im}\lambda|^{\beta+2}}, \quad \text{where } C_1 := \frac{4C^\beta \omega_p}{C_0}.$$

Hence,

$$|f_{n,\alpha,(\omega_k)}(\lambda)| \leq \frac{1}{|\operatorname{Im}\lambda|^\alpha}\left(1 - \frac{C_1}{|\operatorname{Im}\lambda|^{\beta+2}}\right)^{n/2} \tag{60}$$

for all $(\omega_k)_{k\in\mathbb{N}} \in \mathcal{S}(\omega_p, \omega_q)$.

Let $a, b, c > 0$, and define the function g_n by

$$g_n(s) := \frac{1}{s^a}\left(1 - \frac{c}{s^b}\right)^{n/2}, \quad s \geq c^{1/b}.$$

Then,

$$g_n'(s) = \frac{1}{s^{a+1}}\left(1 - \frac{c}{s^b}\right)^{n/2-1} \frac{ac + bcn/2 - as^b}{s^b}.$$

for all $s > c^{1/b}$. Since g_n takes the maximum value at

$$s = \left(c + \frac{bc}{2a}n\right)^{1/b} > c^{1/b},$$

we obtain

$$\sup_{s\geq c^{1/b}} g_n(s) = \left(c + \frac{bc}{2a}n\right)^{-a/b}\left(1 - \frac{2a}{2a+bn}\right)^{n/2}. \tag{61}$$

This and the estimate (60) show that there exist $M_1 > 0$ and $n_1 \in \mathbb{N}$ such that

$$\sup_{\lambda\in\Omega_1} |f_{n,\alpha,(\omega_k)}(\lambda)| \leq \frac{M_1}{n^{\alpha/(\beta+2)}} \tag{62}$$

for all $n \geq n_1$ and $(\omega_k)_{k\in\mathbb{N}} \in \mathcal{S}(\omega_p, \omega_q)$.

Step 3: Next we assume that $\lambda \in \Omega_2$. Since $\operatorname{Re}\lambda > \delta$, we obtain

$$\frac{4\omega\operatorname{Re}\lambda}{|\lambda+\omega|^2} \geq \frac{4\delta\omega}{(|\lambda|+\omega)^2} \geq \frac{4\delta\omega_p}{(1+\omega_q/\delta)^2|\lambda|^2}$$

for all $\omega \in [\omega_p, \omega_q]$. Together with (59), this implies that for all $\omega \in [\omega_p, \omega_q]$,

$$\left|\frac{\lambda-\omega}{\lambda+\omega}\right|^2 \leq 1 - \frac{C_2}{|\lambda|^2}, \quad \text{where } C_2 := \frac{4\delta\omega_p}{(1+\omega_q/\delta)^2}.$$

From this estimate, we derive

$$|f_{n,\alpha,(\omega_k)}(\lambda)| \leq \frac{1}{|\lambda|^\alpha}\left(1-\frac{C_2}{|\lambda|^2}\right)^{n/2}$$

for all $(\omega_k)_{k\in\mathbb{N}} \in \mathcal{S}(\omega_p, \omega_q)$. Using the estimate (61) again, we see that there exist $M_2 > 0$ and $n_2 \in \mathbb{N}$ such that

$$\sup_{\lambda\in\Omega_2} |f_{n,\alpha,(\omega_k)}(\lambda)| \leq \frac{M_2}{n^{\alpha/2}} \tag{63}$$

for all $n \geq n_2$ and $(\omega_k)_{k\in\mathbb{N}} \in \mathcal{S}(\omega_p, \omega_q)$. Thus, the desired estimate (58) is obtained from (62) and (63). □

4. Estimates for inverse generators

In this section, we are interested in the asymptotic behavior of $\|e^{-tA^{-1}}A^{-\alpha}\|$ for $\alpha > 0$, which has been studied in the Banach space case [9,26] and the Hilbert space case [24] as mentioned in Sect. 1. First we show that $\|e^{-tA^{-1}}A^{-\alpha}\|$ is uniformly bounded for $t \geq 0$ if $-A$ is the generator of a bounded C_0-semigroup on a Hilbert space and has a bounded inverse. Next we estimate the decay rate of $\|e^{-tA^{-1}}A^{-\alpha}\|$ for a polynomially stable C_0-semigroup $(e^{-tA})_{t\geq 0}$ on a Hilbert space.

4.1. Bounded semigroups

We present an estimate of $\|e^{-tA^{-1}}A^{-\alpha}\|$ for a bounded C_0-semigroup $(e^{-tA})_{t\geq 0}$ such that the generator $-A$ has a bounded inverse.

Theorem 3. *Let $-A$ be the generator of a bounded C_0-semigroup on a Hilbert space H such that $0 \in \varrho(A)$. Then, for all $\alpha > 0$,*

$$\sup_{t\geq 0} \|e^{-tA^{-1}}A^{-\alpha}\| < \infty. \tag{64}$$

Let $t > 0$ and $\alpha \geq 0$. To prove Theorem 3, we consider the function $f_{t,\alpha}$ on $\mathbb{C}_+$ defined by

$$f_{t,\alpha}(z) := \frac{ze^{-t/z}}{(z+1)^{\alpha+1}}, \quad z \in \mathbb{C}_+. \tag{65}$$

One has $f_{t,\alpha} \in \mathcal{LM}$; see Example 1. In [5, Lemma 5.6], the estimate in the case $\alpha = 0$,

$$\|f_{t,0}\|_{\mathcal{B}_0} = O(\log t) \qquad (n \to \infty), \tag{66}$$

has been obtained. The next proposition gives an estimate for $\|f_{t,\alpha}\|_{\mathcal{B}_0}$ in the case $\alpha > 0$.

Proposition 6. *Let $t, \alpha > 0$ and define the function $f_{t,\alpha}$ by (65). Then,*

$$\sup_{t>0} \| f_{t,\alpha} \|_{\mathcal{B}_0} < \infty. \tag{67}$$

The estimate (67) can be proved in a way similar to the estimate (66). We need the following auxiliary result.

Lemma 4. *Let $p > 0$, and define*

$$g_{t,p}(\xi, s) := \frac{e^{-t\xi/(\xi^2+s)}}{\big((\xi+1)^2+s\big)^{p+1/2}\sqrt{\xi^2+s}} \tag{68}$$

for $t, \xi > 0$ and $s \geq 0$. Then

$$\sup_{t>0} t \int_0^\infty \sup_{s\geq 0} g_{t,p}(\xi, s)\mathrm{d}\xi < \infty. \tag{69}$$

Proof. Define $\zeta := t\xi > 0$ and

$$G_\zeta(\tau) := \frac{e^{-\zeta/\tau}}{(\tau+1)^{p+1/2}\sqrt{\tau}}, \quad \tau > 0.$$

Then,

$$g_{t,p}(\xi, s) \leq G_\zeta(\xi^2 + s) \tag{70}$$

for all $t, \xi > 0$ and $s \geq 0$. We have

$$G'_\zeta(\tau) = \frac{-2(p+1)\tau^2 + (2\zeta - 1)\tau + 2\zeta}{2(\tau+1)^{p+3/2}\tau^{5/2}} e^{-\zeta/\tau}.$$

There exists a unique positive solution $\tau_1 = \tau_1(\zeta)$ of the equation

$$-2(p+1)\tau^2 + (2\zeta - 1)\tau + 2\zeta = 0,$$

and

$$\tau_1(\zeta) = \frac{(2\zeta - 1) + \sqrt{(2\zeta-1)^2 + 16\zeta(p+1)}}{4(p+1)}.$$

Since

$$\sqrt{(2\zeta-1)^2 + 16\zeta(p+1)} \geq \sqrt{(2\zeta-1)^2 + 16\zeta} \geq 1,$$

we have

$$\tau_1 \geq c_1\zeta, \quad \text{where } c_1 := \frac{1}{2(p+1)}.$$

This yields

$$\sup_{s\geq 0} G_\zeta(\xi^2+s) \leq \sup_{\tau\geq 0} G_\zeta(\tau) = G_\zeta(\tau_1) \leq \frac{1}{(c_1\zeta+1)^{p+1/2}\sqrt{c_1\zeta}} \tag{71}$$

for all $t, \xi > 0$. This together with (70) implies that

$$\begin{aligned} t\int_0^\infty \sup_{s\geq 0} g_{t,p}(\xi,s)\mathrm{d}\xi &\leq t\int_0^\infty \frac{1}{(c_1 t\xi+1)^{p+1/2}\sqrt{c_1 t\xi}}\mathrm{d}\xi \\ &= \frac{1}{c_1}\int_0^\infty \frac{1}{(\xi+1)^{p+1/2}\sqrt{\xi}}\mathrm{d}\xi < \infty \end{aligned}$$

for all $t > 0$. Thus, the estimate (69) holds. □

Proof of Proposition 6. Let $t, \alpha > 0$ and let $f_{t,\alpha}$ be defined as in (65). The derivative of $f_{t,\alpha}$ is given by

$$f'_{t,\alpha}(z) = \left(\frac{t}{z(z+1)^{\alpha+1}} - \frac{\alpha z}{(z+1)^{\alpha+2}} + \frac{1}{(z+1)^{\alpha+2}}\right)e^{-t/z}.$$

Therefore,

$$\begin{aligned} &\sup_{\eta\in\mathbb{R}} |f'_{t,\alpha}(\xi+i\eta)| \\ &\leq t\sup_{\eta\in\mathbb{R}} \frac{e^{-t\xi/(\xi^2+\eta^2)}}{\big((\xi+1)^2+\eta^2\big)^{(\alpha+1)/2}\sqrt{\xi^2+\eta^2}} + \alpha\sup_{\eta\in\mathbb{R}} \frac{e^{-t\xi/(\xi^2+\eta^2)}\sqrt{\xi^2+\eta^2}}{\big((\xi+1)^2+\eta^2\big)^{\alpha/2+1}} \\ &\quad + \sup_{\eta\in\mathbb{R}} \frac{e^{-t\xi/(\xi^2+\eta^2)}}{\big((\xi+1)^2+\eta^2\big)^{\alpha/2+1}} \\ &=: \phi_{t,1}(\xi)+\phi_{t,2}(\xi)+\phi_{t,3}(\xi) \end{aligned}$$

for all $\xi > 0$. To the term $\phi_{t,1}$, we apply Lemma 4 with $p = \alpha/2$ and $s = \eta^2$. Then we obtain

$$\sup_{t>0}\int_0^\infty \phi_{t,1}(\xi)\mathrm{d}\xi < \infty.$$

The term $\phi_{t,2}$ satisfies

$$\phi_{t,2}(\xi) \leq \alpha\sup_{\eta\in\mathbb{R}} \frac{\sqrt{(\xi+1)^2+\eta^2}}{\big((\xi+1)^2+\eta^2\big)^{\alpha/2+1}} = \frac{\alpha}{(\xi+1)^{\alpha+1}}$$

for all $\xi > 0$. We also have that

$$\phi_{t,3}(\xi) \leq \frac{1}{(\xi+1)^{\alpha+2}}$$

for all $\xi > 0$. Therefore,

$$\sup_{t>0}\int_0^\infty \phi_{t,2}(\xi)+\phi_{t,3}(\xi)\mathrm{d}\xi < \infty.$$

Thus, the desired estimate (67) is obtained. □

We are now ready to prove Theorem 3.

Proof of Theorem 3. Let $t, \alpha > 0$, and let the function $f_{t,\alpha}$ be defined as in (65). As seen in Example 2, we have

$$f_{t,\alpha}(A) = A(A+I)^{-1} e^{-tA^{-1}} (A+I)^{-\alpha}.$$

Hence

$$\|e^{-tA^{-1}}(A+I)^{-\alpha}\| \leq \|I + A^{-1}\| \, \|f_{t,\alpha}(A)\|. \tag{72}$$

By the estimate (17),

$$\|f_{t,\alpha}(A)\| \leq 2K^2 \|f_{t,\alpha}\|_{\mathcal{B}_0}, \tag{73}$$

where $K := \sup_{t\geq 0} \|e^{-tA}\|$. Combining the estimates (72) and (73) with Proposition 6, we obtain the desired conclusion (64). $\square$

4.2. Polynomially stable semigroups

Here we consider a polynomially stable C_0-semigroup $(e^{-tA})_{t\geq 0}$ on a Hilbert space. The next result shows how a decay estimate for $(e^{-tA})_{t\geq 0}$ can be transferred to that for $(e^{-tA^{-1}})_{t\geq 0}$. As in Theorem 1, we use the notation $\lfloor \xi \rfloor := \max\{k \in \mathbb{Z} : k \leq \xi\}$ for $\xi \in \mathbb{R}$.

Theorem 4. *Let $-A$ be the generator of a polynomially stable C_0-semigroup with parameter $\beta > 0$ on a Hilbert space H. Then, for all $\alpha > 0$,*

$$\|e^{-tA^{-1}} A^{-\alpha}\| = O\left(\frac{(\log t)^{k+1}}{t^{\alpha/(2+\beta)}}\right) \quad (t \to \infty), \quad \text{where } k := \left\lfloor \frac{2\alpha}{2+\beta} \right\rfloor. \tag{74}$$

To prove Theorem 4, we need the following estimate for $\|f_{t,2p}\|_{\mathcal{B}_0,q}$ with $0 < q < 1/2$, where $f_{t,2p}$ is defined by (65) with $\alpha = 2p$.

Proposition 7. *Let $p > 0$ and $q \in (0, 1/2)$. Then the function $f_{t,2p}$ defined by (65) with $\alpha = 2p$ satisfies*

$$\|f_{t,2p}\|_{\mathcal{B}_0,q} = \begin{cases} O\left(\dfrac{1}{t^p}\right), & p < q, \\[2ex] O\left(\dfrac{\log t}{t^p}\right), & p = q, \\[2ex] O\left(\dfrac{1}{t^q}\right), & p > q, \end{cases} \qquad (t \to \infty). \tag{75}$$

Proof. Step 1: Fix $p > 0$ and $q \in (0, 1/2)$. The derivative $f'_{t,2p}$ is given by

$$f'_{t,2p}(z) = \left(\frac{t}{z(z+1)^{2p+1}} - \frac{2pz}{(z+1)^{2p+2}} + \frac{1}{(z+1)^{2p+2}}\right) e^{-t/z}.$$

Since

$$\left|\frac{1}{(z+1)^{2p+2}}\right| \leq \left|\frac{1}{z(z+1)^{2p+1}}\right|$$

for all $z \in \mathbb{C}_+$, we have

$$\sup_{\eta\in\mathbb{R}} |f'_{t,2p}(\xi+i\eta)| \leq (t+1)\sup_{s\geq 0} g_{t,p}(\xi,s) + 2p\sup_{s\geq 0} h_{t,p}(\xi,s), \tag{76}$$

where $g_{t,p}$ is defined by (68) and

$$h_{t,p}(\xi,s) := \frac{e^{-t\xi/(\xi^2+s)}\sqrt{\xi^2+s}}{\big((\xi+1)^2+s\big)^{p+1}}. \tag{77}$$

for $\xi > 0$ and $s \geq 0$. We will prove in Steps 2 and 3 below that

$$\int_0^\infty \psi_q(\xi)\sup_{s\geq 0} g_{t,p}(\xi,s)\mathrm{d}\xi = \begin{cases} O\left(\dfrac{1}{t^{p+1}}\right), & p<q, \\ O\left(\dfrac{\log t}{t^{p+1}}\right), & p=q, \\ O\left(\dfrac{1}{t^{q+1}}\right), & p>q, \end{cases} \qquad (t\to\infty) \tag{78}$$

and

$$\int_0^\infty \psi_q(\xi)\sup_{s\geq 0} h_{t,p}(\xi,s)\mathrm{d}\xi = \begin{cases} O\left(\dfrac{1}{t^{2p}}\right), & p<\dfrac{1}{2}, \\ O\left(\dfrac{\log t}{t}\right), & p=\dfrac{1}{2}, \\ O\left(\dfrac{1}{t}\right), & p>\dfrac{1}{2}, \end{cases} \qquad (t\to\infty). \tag{79}$$

The desired conclusion (75) follows immediately from the estimates (76)–(79).

Step 2: In this step, we will prove the estimate (78) for $g_{t,p}$. From the estimates (70) and (71) in the proof of Lemma 4, we obtain

$$\sup_{s\geq 0} g_{t,p}(\xi,s) \leq \frac{1}{(c_1 t\xi+1)^{p+1/2}\sqrt{c_1 t\xi}} \tag{80}$$

for all $t, \xi > 0$ and some constant $c_1 > 0$ depending only on p.

Suppose that $t > 1$. Since the function ψ_q, defined as in (23), satisfies $\psi_q(\xi) = \xi^q$ for all $\xi \in (0,1)$, the estimate (80) gives

$$\begin{aligned} \int_0^1 \psi_q(\xi)\sup_{s\geq 0} g_{t,p}(\xi,s)\mathrm{d}\xi &\leq \int_0^1 \frac{\xi^q}{(c_1 t\xi+1)^{p+1/2}\sqrt{c_1 t\xi}}\mathrm{d}\xi \\ &= \frac{1}{(c_1 t)^{q+1}}\int_0^{c_1 t} \frac{\xi^q}{(\xi+1)^{p+1/2}\sqrt{\xi}}\mathrm{d}\xi \\ &\leq \frac{1}{(c_1 t)^{q+1}}\left(\int_0^{c_1} \frac{1}{\xi^{1/2-q}}\mathrm{d}\xi + \int_{c_1}^{c_1 t}\frac{1}{\xi^{p-q+1}}\mathrm{d}\xi\right). \end{aligned}$$

Hence,

$$\int_0^1 \psi_q(\xi) \sup_{s\geq 0} g_{t,p}(\xi,s)\mathrm{d}\xi = \begin{cases} O\left(\dfrac{1}{t^{p+1}}\right), & p<q, \\ O\left(\dfrac{\log t}{t^{p+1}}\right), & p=q, \\ O\left(\dfrac{1}{t^{q+1}}\right), & p>q \end{cases} \tag{81}$$

as $t\to\infty$. Recalling that $\psi_q(\xi)=1$ for all $\xi \geq 1$, we also have

$$\int_1^\infty \psi_q(\xi) \sup_{s\geq 0} g_{t,p}(\xi,s)\mathrm{d}\xi \leq \frac{1}{c_1 t}\int_{c_1 t}^\infty \frac{1}{(\xi+1)^{p+1/2}\sqrt{\xi}}\mathrm{d}\xi \leq \frac{1}{c_1 t}\int_{c_1 t}^\infty \frac{1}{\xi^{p+1}}\mathrm{d}\xi,$$

which yields

$$\int_1^\infty \psi_q(\xi) \sup_{s\geq 0} g_{t,p}(\xi,s)\mathrm{d}\xi = O\left(\frac{1}{t^{p+1}}\right) \tag{82}$$

as $t\to\infty$. From (81) and (82), we obtain the estimate (78) for $g_{t,p}$.

Step 3: It remains to prove the estimate (79) for $h_{t,p}$. Define $\zeta := t\xi$ and

$$H_\zeta(\tau) := \frac{e^{-\zeta/\tau}}{(\tau+1)^{p+1/2}}, \quad \tau>0.$$

Then H_ζ has similar properties to G_ζ used in the proof of Lemma 4. In fact,

$$h_{t,p}(\xi,s) \leq H_\zeta(\xi^2+s) \tag{83}$$

for all $t,\xi>0$ and $s\geq 0$. There exists a unique positive solution $\tau_2=\tau_2(\zeta)$ of $H'_\zeta(\tau)=0$, and

$$\sup_{\tau\geq 0} H_\zeta(\tau) = H_\zeta(\tau_2).$$

Moreover, $\tau_2 \geq c_2\zeta$ for some constant $c_2>0$ depending only on p. Therefore,

$$\sup_{s\geq 0} H_\zeta(\xi^2+s) \leq H_\zeta(\tau_2) \leq \frac{1}{(c_2\zeta+1)^{p+1/2}} \tag{84}$$

for all $t,\xi>0$.

By definition, $\psi_q(\xi)\leq 1$ for all $\xi>0$. Combining this with the estimates (83) and (84), we obtain

$$\int_0^t \psi_q(\xi) \sup_{s\geq 0} h_{t,p}(\xi,s)\mathrm{d}\xi \leq \int_0^t \frac{1}{(c_2 t\xi+1)^{p+1/2}}\mathrm{d}\xi = \frac{1}{c_2 t}\int_0^{c_2 t^2} \frac{1}{(\xi+1)^{p+1/2}}\mathrm{d}\xi$$

for all $t > 0$. Hence,

$$\int_0^t \psi_q(\xi) \sup_{s \geq 0} h_{t,p}(\xi, s) \mathrm{d}\xi = \begin{cases} O\left(\dfrac{1}{t^{2p}}\right), & p < \dfrac{1}{2}, \\ O\left(\dfrac{\log t}{t}\right), & p = \dfrac{1}{2}, \\ O\left(\dfrac{1}{t}\right), & p > \dfrac{1}{2} \end{cases} \tag{85}$$

as $t \to \infty$. On the other hand, we have from the definition (77) of $h_{t,p}$ that

$$h_{t,p}(\xi, s) \leq \frac{1}{\xi^{2p+1}}$$

for all $t, \xi > 0$ and $s \geq 0$. This gives

$$\int_t^\infty \psi_q(\xi) \sup_{s \geq 0} h_{t,p}(\xi, s) \mathrm{d}\xi \leq \int_t^\infty \frac{1}{\xi^{2p+1}} \mathrm{d}\xi = O\left(\frac{1}{t^{2p}}\right) \tag{86}$$

as $t \to \infty$. From (85) and (86), we obtain the estimate (79) for $h_{t,p}$. □

Now we are in a position to give the proof of Theorem 4.

Proof of Theorem 4. Let $p \in (0, 1/2)$ and define $f_{t,2p} \in \mathcal{B}$ by (65) with $\alpha = 2p$ for $t > 0$. Using Example 3 in the case $\alpha = 2p$, $c = 1$, $q = p$, and

$$f(z) = \frac{z}{z+1} e^{-t/z},$$

we see that there exists a constant $M > 0$ such that

$$\|e^{-tA^{-1}} A^{-(2+\beta)p}\| \leq M \|f_{t,2p}\|_{\mathcal{B}_{0,p}}$$

for all $t > 0$. The estimate (75) with $p = q$ yields

$$\|e^{-tA^{-1}} A^{-(2+\beta)p}\| = O\left(\frac{\log t}{t^p}\right)$$

as $t \to \infty$. Therefore, the desired estimate (74) holds in the case $k = \lfloor 2\alpha/(2+\beta) \rfloor = 0$. The case $k \geq 1$ follows as in the proof of Theorem 1. □

Remark 5. The estimate (75) with $p = q$ for $\|f_{t,2p}\|_{\mathcal{B}_{0,q}}$ has been used in the proof of Theorem 4. This is because the case $p \neq q$ yields a less sharp estimate, which can be seen from the same argument as in Remark 1.

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REFERENCES

[1] Azizov, T.Y., Barsukov, A.I., Dijksma, A.: Decompositions of a Krein space in regular subspaces invariant under a uniformly bounded C_0-semigroup of bi-contractions. J. Funct. Anal. **211**, 324–354 (2004)

[2] Bátkai, A., Engel, K.J., Prüss, J., Schnaubelt, R.: Polynomial stability of operator semigroups. Math. Nachr. **279**, 1425–1440 (2006)

[3] Batty, C.J.K., Duyckaerts, T.: Non-uniform stability for bounded semi-groups on Banach spaces. J. Evol. Equations **8**, 765–780 (2008)

[4] Batty, C.J.K., Gomilko, A., Tomilov, Yu.: A Besov algebra calculus for generators of operator semigroups and related norm-estimates. Math. Ann. **379**, 23–93 (2021)

[5] Batty, C.J.K., Gomilko, A., Tomilov, Yu.: The theory of Besov functional calculus: developments and applications to semigroups. J. Funct. Anal. **281**, Art. no. 109089, 60 pp. (2021)

[6] Brenner, P., Thomée, V.: On rational approximations of semigroups. SIAM J. Numer. Anal. **16**, 683–694 (1979)

[7] Chill, R., Seifert, D., Tomilov, Yu.: Semi-uniform stability of operator semigroups and energy decay of damped waves. Philos. Trans. Roy. Soc. A **378**, Art. no. 20190614, 24 pp. (2020)

[8] deLaubenfels, R.: Inverses of generators. Proc. Amer. Math. Soc. **104**, 443–448 (1988)

[9] deLaubenfels, R.: Inverses of generators of nonanalytic semigroups. Studia Math. **191**, 11–38 (2009)

[10] Eisner, T.: Stability of Operators and Operator Semigroups. Basel: Birkhäuser (2010)

[11] Eisner, T., Zwart, H.: The growth of a C_0-semigroup characterised by its cogenerator. J. Evol. Equations **8**, 749–764 (2008)

[12] Engel, K.J., Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations. New York: Springer (2000)

[13] Fackler, S.: A short counterexample to the inverse generator problem on non-Hilbertian reflexive L^p-spaces. Arch. Math. **106**, 383–389 (2016)

[14] Gomilko, A.: Cayley transform of the generator of a uniformly bounded C_0-semigroup of operators. Ukrainian Math. J. **56**, 1212–1226 (2004)

[15] Gomilko, A.: Inverses of semigroup generators: a survey and remarks. In: Études opératorielles, Banach Center Publ., vol. 112, Polish Acad. Sci., Warsaw, pp. 107–142 (2017)

[16] Gomilko, A., Tomilov, Yu.: Rational approximation of operator semigroups via the $\mathcal{B}$-calculus. To appear in J. Funct. Anal. **287**, Art. no. 110426, 39 pp. (2024)

[17] Gomilko, A., Zwart, H., Besseling, N.: Growth of semigroups in discrete and continuous time. Studia Math. **206**, 273–292 (2011)

[18] Gomilko, A., Zwart, H., Tomilov, Yu.: Inverse operator of the generator of a C_0-semigroup. Sb. Math. **198**, 1095–1110 (2007)

[19] Guo, B.Z., Zwart, H.: On the relation between stability of continuous- and discrete-time evolution equations via the Cayley transform. Integral Equations Operator Theory **54**, 349–383 (2006)

[20] Haase, M.: The Functional Calculus for Sectorial Operators. Basel: Birkhäuser (2006)

[21] Komatsu, H.: Fractional powers of operators. Pacific J. Math. **19**, 285–346 (1966)

[22] Piskarev, S., Zwart, H.: Crank–Nicolson scheme for abstract linear systems. Numer. Funct. Anal. Optim. **28**, 717–736 (2007)

[23] Wakaiki, M.: The Cayley transform of the generator of a polynomially stable C_0-semigroup. J. Evol. Equations **21**, 4575–4597 (2021)

[24] Wakaiki, M.: Decay rate of $\exp(A^{-1}t)A^{-1}$ on a Hilbert space and the Crank–Nicolson scheme with smooth initial data. Integral Equations Operator Theory **95**, Art. no. 28, 24 pp. (2023)

[25] Zwart, H.: Growth estimates for $\exp(A^{-1}t)$ on a Hilbert space. Semigroup Forum **74**, 487–494 (2007)

[26] Zwart, H.: Is A^{-1} an infinitesimal generator? In: Perspectives in operator theory, Banach Center Publ., vol. 75, Polish Acad. Sci., Warsaw, pp. 303–313 (2007)

Masashi Wakaiki
Graduate School of System Informatics
Kobe University
Nada, Kobe Hyogo 657-8501
Japan
E-mail: wakaiki@ruby.kobe-u.ac.jp

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