Global well-posedness for the Cauchy problem of the Zakharov-Kuznetsov equation on cylindrical spaces

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博　士　論　文

Global well-posedness for the Cauchy problem of the Zakharov-Kuznetsov equation on cylindrical spaces
(円筒領域上のザハロフ・クズネツォフ方程式に対する初期値問題の大域適切性)

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Contents

Preface

In this thesis, we consider the Cauchy problem for the Zakharov-Kuznetsov equation on the cylindrical spaces. The Zakharov-Kuznetsov equation is a type of nonlinear dispersive equation which describes the propagation of ionic waves in magnetized plasma. When we expand the KdV equation to multidimensional spaces, we have the Zakharov-Kuznetsov equation. We investigate the solutions, by confirming the well-posedness to the Cauchy problem of the equation. In general, it is difficult to write down solutions of nonlinear partial differential equations as an explicit form.

The notion of well-posedness is of importance for mathematical problems. Let X be a Banach space. It is said that the Cauchy problem is well-posed in X if we confirm the existence, uniqueness of the solution and the continuous dependence on the initial data in X. In other words, the data-to-solution map $X \ni u(0) \mapsto u(t) \in X$ exists and is continuous with respect to the metric induced by X, where $u_0 = u(0)$ and $u = u(t)$ are initial data and solution of the Cauchy problem, respectively. We say the local (respectively global) well-posedness, if the well-posedness holds for a positive time interval (respectively all time).

It is natural to consider question of well-posedness in the Sobolev space $X = H^s$, since the conservation laws control Sobolev norms of solutions. Here the exponent s stands for the fractional Sobolev regularity. If $s = 1$, this is the energy space and that yields the expected global solution via energy conservation laws.

We also study the well-posedness for less regular data, namely in H^s for $s < 1$. It is worth noting that the smoothing effect of solutions has been considered in the whole space as powerful tool in the study of well-posedness theories with low-regularity initial data. The smoothing effect is crucial in allowing to recover the dispersive properties and the loss of derivatives of solutions. This effect can not be expected under the periodic boundary condition that the solutions do not decay at infinity. If we consider the cylindrical space as an unbounded manifold, an anisotropic version of the smoothing effect properties of the solutions could be expected.

There are several literatures on results for the cylindrical boundary problems [16, 8, 18, 20, 25]. In [20], we proved that the Cauchy problem of the Zakharov-Kuznetsov equation is locally well-posed in H^s for $s > 9/10$. This improved the previous result by Molinet and Pilod [18]. In this thesis, we extend the local solution to the global one by using some a priori estimates. In proofs, we employ so-called I-method. Consequently, we establish the global well-posedness in $H^s(\mathbb{R} \times \mathbb{T})$ for $s > 29/31$.

In Chapter 1, we introduce the setting of the problem and state the main result. Chapter 2 is written based on my paper [20]. We recall some harmonic analysis tools including Littlewood-Paley decompositions of functions. Moreover we prepare the smoothing effects and linear estimates in the Bourgain spaces $X^{s,b}$ [2, 3, 11]. One also reviews some preliminary results for the local well-posedness result in the above spaces. Chapters 3 and 4 are written based on the joint paper with Hideo Takaoka [21]. In Chapter 3, we follow the I-method scheme [5]. We give several lemmas of bilinear estimates in conjunction with rescaling argument in [6] and give the local well-posedness results for rescaled data. Chapter 4 is devoted to the proof of main theorem. We define modified energy functional $E[Iu]$ in term of the H^s-norm of the solution for $s < 1$. One of the key steps in the construction of global solutions is to control the increment of the modified energy. We estimate the growth of the modified energy and provide a priori estimates for the solutions.

Chapter 1

Setting of the problem and results

In this chapter, we will address the background of the problem and the statement of the results.

1.1 Setting of the problem

In this thesis, we study the Cauchy problem for the Zakharov-Kuznetsov equation on a cylinder:

$$\begin{cases} \partial_t u + \partial_x \Delta u + u\partial_x u = 0, \\ u(x,y,0) = u_0(x,y), \end{cases} \quad (x,y) \in \mathbb{R} \times \mathbb{T}, \quad t \in \mathbb{R}, \tag{1.1}$$

where $u = u(x,y,t)$ is a real-valued function, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, and $\Delta = \partial_x^2 + \partial_y^2$ is the Laplacian. This equation was introduced by Zakharov and Kuznetsov [26], as a model for the propagation of ionic-acoustic waves in magnetized plasma. This equation is one of the two-dimensional extensions of the Korteweg-de Vries (KdV) equation:

$$\partial_t u + \partial_{xxx} u + u\partial_x u = 0.$$

The KdV equation is one of the famous nonlinear dispersive equation, which describes the behavior of shallow water waves. Another two-dimensional generalization of the KdV equation is the Kadomtsev-Petviashvili (KP) equations:

$$\partial_x \left(\partial_t u + \partial_{xxx} u + u\partial_x u\right) \pm \partial_{yy} u = 0,$$

where the KP-I equation corresponds to minus sign, while the KP-II equation to plus sign.

We will establish the global well-posedness of (1.1) whose initial data belongs to $H^s(\mathbb{R}\times\mathbb{T})$. More precisely, we show that the solutions exist global-in-time for initial data in $H^s(\mathbb{R}\times\mathbb{T})$ for any $s > 29/31$.

The Zakharov-Kuznetsov equation (1.1) has at least two conserved quantities:

$$E[u](t) = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} \left(|\nabla u(x,y,t)|^2 - \frac{1}{3}u(x,y,t)^3\right) dxdy,$$

$$M[u](t) = \int_{\mathbb{R}\times\mathbb{T}} u(x,y,t)^2 \, dxdy.$$

These conservation laws identify $H^1(\mathbb{R}\times\mathbb{T})$ as the energy space.

For the Zakharov-Kuznetsov equation in the $\mathbb{R}^2$ setting, Faminskii [7] proved the local well-posedness in the energy space $H^1(\mathbb{R}^2)$. After this, Linares and Pastor [15] showed the local well-posedness for $s > 3/4$, by proving a sharp maximal function estimates. Grünrock and Herr [10] proved local well-posedness in $H^s(\mathbb{R}^2)$ for $s > 1/2$, by using the Fourier restriction norm method and the Strichartz estimates. The Fourier restriction norm method utilizing the $X^{s,b}$ spaces was introduced by Bourgain [2, 3] (also by Kenig, Ponce and Vega [11]), in order to prove the local well-posedness for nonlinear Schrödinger and KdV equations in low regularity Sobolev spaces. At the same time, Molinet and Pilod [18] showed the local well-posedness in $H^s(\mathbb{R}^2)$ for $s > 1/2$ using bilinear Strichartz estimates. Observing proofs of global well-posedness, Shan [24] obtained that the solution exists globally in time to data in $H^s(\mathbb{R}^2)$ for $s > 5/7$. Shan used the I-method based on H^1 conservation laws. The I-method was introduced for global well-posedness of the KdV equation by Colliander, Keel, Staffilani, Takaoka and Tao [5]. Recently, Kinoshita [12] showed the local well-posedness for $s > -1/4$, using Loomis-Whitney inequality and an orthogonal decomposition technique. In [12], the local ill-posedness results for $s < -1/4$ was also obtained. This means that the data-to-solution map from the unit ball in $H^s(\mathbb{R}^2)$ to $C([0,T]; H^s)$ fails to be smooth for any $T > 0$. As a corollary from the result of local well-posedness combining with the L^2 conservation law, the global well-posedness in $L^2(\mathbb{R}^2)$ follows.

In the $\mathbb{R} \times \mathbb{T}$ setting, Molinet and Pilod in the same paper as above [18] showed the global well-posedness in $H^s(\mathbb{R} \times \mathbb{T})$ for $s \geq 1$. The strategy is similar to the case of $\mathbb{R}^2$ setting, however it is more difficult to make sure the proof because of the lack of the smoothing effects. In [20], we showed the local well-posedness for $s > 9/10$ by refining the bilinear estimates (see Lemma 2.2.2 in the thesis).

In the $\mathbb{T}^2$ setting, the local well-posedness result in $H^s(\mathbb{T}^2)$ has been obtained by Linares, Panthee, Robert and Tzvetkov [14]. They showed that the initial value problem is locally well-posed in $H^s(\mathbb{T}^2)$ for $s > 5/3$. The proof used short-time Strichartz estimates. In [23], Schippa improved the local well-posedness to $s > 3/2$ by using short-time bilinear Strichartz estimates. Moreover, Kinoshita and Schippa [13] obtained the local well-posedness for $s > 1$. They estimated the nonlinear interaction term by short-time trilinear estimates.

Observing related results on the Zakharov-Kuznetsov equation. Yamazaki [25] showed the stability and instability results for line solitary waves of the Zakharov-Kuznetsov equations in $\mathbb{R} \times \mathbb{T}_L$, where $\mathbb{T}_L$ is the torus of length $2\pi L$. The solitary waves are constructed by hyperbolic functions. The long-time behavior stability of solitary waves were also studied in [4] and in [22].

1.2 Main theorem

We aim to prove that the initial value problem of the Zakharov-Kuznetsov equation in $H^s(\mathbb{R} \times \mathbb{T})$ is globally well-posed for some $s < 1$. This proof is based on bilinear estimates developed in [18] and I-method [5].

The main theorem of the thesis is the following.

Theorem 1.2.1. *The initial value problem of* (1.1) *is globally well-posed in $H^s(\mathbb{R} \times \mathbb{T})$ for $s > 29/31$. In other words, for any $u_0 \in H^s(\mathbb{R} \times \mathbb{T})$, for all $T > 0$, there exists a unique solution u of* (1.1) *such that*

$$u \in C([0,T] : H^s((\mathbb{R} \times \mathbb{T})) \cap X_T^{s,1/2+}.$$

Moreover, for all $0 < T' < T$, there exists a neighborhood $\mathcal{U}$ of u_0 in $H^s(\mathbb{R} \times \mathbb{T})$ such that the data-to-solution map

$$\mathcal{U} \ni v_0 \mapsto v(t) \in C([0,T'] : H^s((\mathbb{R} \times \mathbb{T})) \cap X_{T'}^{s,1/2+}$$

is smooth, where $v(t)$ is a unique solution of (1.1) *to the initial data v_0.*

Remark 1.2.2. Function spaces H^s and $X_T^{s,b}$ are defined in Section 2.1.

Remark 1.2.3. By time reversibility, we need to consider the existence for positive time.

Chapter 2

Preparation by some results

This chapter is written based on my paper [20]. We revisit the local well-posedness results of (1.1) below the energy norm.

2.1 Some notations of function spaces

In this section, we will introduce some function spaces that will be used throughout the thesis.

We use the absolute value of (ξ, q) as $|(\xi, q)|^2 = 3\xi^2 + q^2$ in this paper. For positive real number a and b, the notion $a \lesssim b$ means that there is a constant c such that $a \le cb$. When $a \lesssim b$ and $b \lesssim a$, we write $a \sim b$. Moreover, $b+$ means that there exists $\delta > 0$ such that $b + \delta$.

Denote $\mathbb{T}_\lambda = \lambda\mathbb{T} = \mathbb{R}/(2\pi\lambda\mathbb{Z})$ for $\lambda > 0$. Let $f(x, y)$ be a function defined on $\mathbb{R} \times \mathbb{T}_\lambda$. We denote the Fourier transform of f with respect to the variables (x, y) as $\mathcal{F}^\lambda_{xy} f(\xi, q)$:

$$\mathcal{F}^\lambda_{xy} f(\xi, q) = \int_{\mathbb{R}} \int_0^{2\pi\lambda} e^{-i(x\xi + yq)} f(x, y)\, dydx,$$

where $(\xi, q) \in \mathbb{R} \times \mathbb{Z}/\lambda$. Moreover, let $\mathcal{F}^{-1,\lambda}_{\xi q}$ be the Fourier inverse transform with respect to spacial variables by

$$\mathcal{F}^{-1,\lambda}_{\xi q} f(x, y) = \frac{1}{(2\pi)^2\lambda} \sum_{q \in \mathbb{Z}/\lambda} \int_{\mathbb{R}} e^{i(\xi x + qy)} f(\xi, q)\, d\xi$$

for $(x, y) \in \mathbb{R} \times \mathbb{T}_\lambda$. For simplicity, we abbreviate $\mathcal{F}^\lambda_{xy}$ and $\mathcal{F}^{-1,\lambda}_{\xi q}$ by $\mathcal{F}^\lambda$ and $\mathcal{F}^{-1,\lambda}$, respectively, when no confusion is likely.

Let $u(x, y, t)$ be a function of $\mathbb{R} \times \mathbb{T}_\lambda \times \mathbb{R}$, and let $\widehat{u}^\lambda(\xi, q, \tau)$ be Fourier transform of $u(x, y, t)$ in a similar manner as well

$$\widehat{u}^\lambda(\xi, q, \tau) = \int_{\mathbb{R}^2} \int_0^{2\pi\lambda} e^{-i(t\tau + ix\xi + yq)} u(x, y, t)\, dydxdt.$$

Similarly, define the Fourier inverse transform

$$\check{u}^\lambda(x, y, t) = \frac{1}{(2\pi)^3\lambda} \sum_{q \in \mathbb{Z}/\lambda} \int_{\mathbb{R}^2} e^{i(t\tau + \xi x + qy)} u(\xi, q, \tau)\, d\xi d\tau.$$

Denote $\sigma(\xi, q) = \xi^3 + \xi q^2$. Let $\eta \in C_0^\infty(\mathbb{R})$ satisfies $0 \le \eta \le 1$, $\eta_{[-1,1]} = 1$, and $\operatorname{supp} \eta \subset [-2, 2]$. For a dyadic number $N = 2^k$ with $k \in \mathbb{N}$, we denote

$$\phi(\xi) = \eta(\xi) - \eta(2\xi), \quad \phi_N(\xi, q) = \phi(N^{-1}|(\xi, q)|), \quad \psi_N(\xi, q, \tau) = \phi(N^{-1}(\tau - \sigma(\xi, q))),$$

where $(\xi, q, \tau) \in \mathbb{R} \times \mathbb{Z}/\lambda \times \mathbb{R}$. Here, we prepare notation of interval as $I_N = \operatorname{supp} \phi_N$ for a dyadic number N. We define the Littlewood-Paley decomposition as

$$P_N u = \mathcal{F}^{-1,\lambda}(\phi_N \mathcal{F}^\lambda u), \quad Q_L u = (\psi_L \widehat{u}^\lambda)^{\smallsmile\lambda}.$$

We let denote for $a, b \in \mathbb{R}$, $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

For $s \in \mathbb{R}$, we define the Sobolev spaces $H^s_\lambda = H^s_\lambda(\mathbb{R} \times \mathbb{T}_\lambda)$ equipped with the norm

$$\|f\|_{H^s_\lambda} = \left(\frac{1}{2\pi\lambda} \sum_{q \in \mathbb{Z}/\lambda} \int_{\mathbb{R}} \langle |(\xi, q)| \rangle^{2s} |\mathcal{F}^\lambda f(\xi, q)|^2 \, d\xi \right)^{1/2}, \tag{2.1}$$

where $\langle x \rangle = 1 + |x|$. We also use $L^2_\lambda = H^0_\lambda$.

Use the restriction operator R_K with respect to the x variable as

$$R_K f(x) = \int_{\mathbb{R}} \phi(\xi K^{-1}) \mathcal{F}_x f(\xi) e^{ix\xi} \, d\xi$$

for dyadic number K, where $\mathcal{F}_x$ is the Fourier transform with respect to the x variable.

We note that the following properties hold [6]:

$$\int_{\mathbb{R}} \int_0^{2\pi\lambda} f(x, y) \overline{g(x, y)} \, dx dy = \frac{1}{\lambda} \sum_{q \in \mathbb{Z}/\lambda} \int_{\mathbb{R}} \mathcal{F}^\lambda f(\xi, q) \overline{\mathcal{F}^\lambda g(\xi, q)} \, d\xi,$$

$$\mathcal{F}^\lambda (fg)(\xi, q) = \left(\mathcal{F}^\lambda f *^\lambda \mathcal{F}^\lambda g \right)(\xi, q) = \frac{1}{\lambda} \sum_{q_1 \in \mathbb{Z}/\lambda} \int_{\mathbb{R}} \mathcal{F}^\lambda f(\xi - \xi_1, q - q_1) \mathcal{F}^\lambda g(\xi_1, q_1) \, d\xi_1. \tag{2.2}$$

Next, we describe the Bourgain space via the Fourier transform. Following [11], we introduce a class of function spaces related to Bourgain space $X^{s,b}_\lambda$. Define the Bourgain space $X^{s,b}_\lambda$ as follows.

Definition 2.1.1. For $s, b \in \mathbb{R}$ and $T > 0$,

$$\|u\|_{X^{s,b}_\lambda} = \left(\frac{1}{\lambda} \sum_{q \in \mathbb{Z}/\lambda} \int_{\mathbb{R}^2} \langle \tau - \sigma(\xi, q) \rangle^{2b} \langle |(\xi, q)| \rangle^{2s} |\widehat{u}(\xi, q, \tau)|^2 \, d\xi d\tau \right)^{1/2}, \tag{2.3}$$

$$\|u\|_{X^{s,b}_{\lambda,T}} = \inf \left\{ \|\overline{u}\|_{X^{s,b}_\lambda} \mid \overline{u} : \mathbb{R} \times \mathbb{T}_\lambda \to \mathbb{C}, \ \overline{u}|_{\mathbb{R} \times \mathbb{T}_\lambda \times [0,T]} = u \right\}.$$

When a norm of u introduced in (2.3) is bounded, we denote $u \in X^{s,b}_\lambda$. We also use the same notation $L^2_\lambda = X^{0,0}_\lambda$ for functions on $\mathbb{R} \times \mathbb{T}_\lambda \times \mathbb{R}$ as defined before for one on $\mathbb{R} \times \mathbb{T}_\lambda$, if there is no confusion.

The space $X^{s,b}_{\lambda,T}$ will be used for proof of the local well-posedness result, that is under restriction of time on $[0, T]$.

If $\lambda = 1$, we denote $\mathcal{F}^\lambda$, $\mathcal{F}^{-1,\lambda}$, $\widehat{\cdot}^\lambda$, $\check{\cdot}^\lambda$, H^s_λ, $X^{s,b}_\lambda$, $X^{s,b}_{\lambda,T}$ by $\mathcal{F}$, $\mathcal{F}^{-1}$, $\widehat{\cdot}$, $\check{\cdot}$, H^s, $X^{s,b}$, $X^{s,b}_T$, respectively.

Remark 2.1.2. The space $X^{s,b}_\lambda$ will be characterized by this formula

$$\|f\|_{X^{s,b}_\lambda} = \|e^{t\partial_x \Delta} f\|_{H^b_t H^s_\lambda},$$

where $e^{-t\partial_x \Delta}$ is the operator associated linear Zakharov-Kuznetsov equation on $\mathbb{R} \times \mathbb{T}_\lambda$, described by

$$\mathcal{F}^\lambda (e^{-t\partial_x \Delta} f)(\xi, q) = e^{it\sigma(\xi, q)} \mathcal{F}^\lambda f(\xi, q).$$

We summarize some formulas which were stated in [11, 8, 18, 20].

Lemma 2.1.3. *Let $s \in \mathbb{R}$ and $b > 1/2$. Then*

$$\|\eta(t) e^{-t\partial_x \Delta} f\|_{X^{s,b}_\lambda} \lesssim \|f\|_{H^s_\lambda},$$

for all $f \in H^s_\lambda$.

Lemma 2.1.4. *Let $s \in \mathbb{R}$ and $b > 1/2$. Then*

$$\left\| \eta(t) \int_0^t e^{-(t-t')\partial_x \Delta} f(t') dt' \right\|_{X_\lambda^{s,b}} \lesssim \|f\|_{X_\lambda^{s,b-1}},$$

for all $f \in X_\lambda^{s,b-1}$.

Lemma 2.1.5. *For any $T > 0$, $s \in \mathbb{R}$ and for all $-1/2 < b' \leq b < 1/2$,*

$$\|f\|_{X_{\lambda,T}^{s,b'}} \lesssim T^{b-b'} \|f\|_{X_{\lambda,T}^{s,b}},$$

for all $f \in X_\lambda^{s,b}$.

The polynomial $\sigma(\xi, q)$ which appeared in (2.3) plays an important role in characterizing of solution. Now, we define the resonance function as

$$\mathcal{H}(\xi_1, \xi_2, q_1, q_2) = \sigma(\xi_1+\xi_2, q_1+q_2) - \sigma(\xi_1, q_1) - \sigma(\xi_2, q_2) = 3\xi_1\xi_2(\xi_1+\xi_2) + \xi_2 q_1^2 + \xi_1 q_2^2 + 2(\xi_1+\xi_2) q_1 q_2, \tag{2.4}$$

which also plays an important role in the control of the frequency instability range between nonlinear interactions.

2.2 Local well-posedness in H^s

In this section, we recall the results of the local well-posedness obtained in [20]. Let start with the bilinear estimates that are crucial for proving the local well-posedness results in [20] (see Proposition 2.2.4 below).

Lemma 2.2.1 (Proposition 3.6 in [18])**.** *Suppose that N_1, N_2, L_1, L_2 are dyadic numbers. Then*

$$\|(P_{N_1} Q_{L_1} u)(P_{N_2} Q_{L_2} v)\|_{L^2} \lesssim (N_1 \wedge N_2)(L_1 \wedge L_2)^{1/2} \|P_{N_1} Q_{L_1} u\|_{L^2} \|P_{N_2} Q_{L_2} v\|_{L^2}. \tag{2.5}$$

Moreover, when $4(N_1 \wedge N_2) \leq (N_1 \vee N_2)$, we obtain

$$\|(P_{N_1} Q_{L_1} u)(P_{N_2} Q_{L_2} v)\|_{L^2} \lesssim \frac{(N_1 \wedge N_2)^{1/2}}{N_1 \vee N_2} (L_1 \wedge L_2)^{1/2} (L_1 \vee L_2)^{1/2} \|P_{N_1} Q_{L_1} u\|_{L^2} \|P_{N_2} Q_{L_2} v\|_{L^2}.$$

We give the variant bilinear estimates, which were main ingredient for the local well-posedness below the energy norm.

Lemma 2.2.2 (Lemma 2.9 in [20])**.** *It holds that*

$$\|R_K((P_{N_1} Q_{L_1} u)(P_{N_2} Q_{L_2} v))\|_{L^2} \lesssim \frac{(N_1 \wedge N_2)^{1/2}}{K^{1/4}} (L_1 \wedge L_2)^{1/2} (L_1 \vee L_2)^{1/4} \|P_{N_1} Q_{L_1} u\|_{L^2} \|P_{N_2} Q_{L_2} v\|_{L^2},$$

where K is a dyadic number satisfying $|\xi| \in I_K$.

These two lemmas above allow to obtain the smoothing properties of a bilinear form, since by high frequency gain of the derivative term. Using these lemmas, we proved in [20] the following bilinear estimate for $\lambda = 1$.

Lemma 2.2.3 (Proposition 3.1 in [20])**.** *Let $s > 9/10$. For all $u, v \in X^{s,1/2+}$,*

$$\|\partial_x(uv)\|_{X^{s,-1/2+}} \lesssim \|u\|_{X^{s,1/2+}} \|v\|_{X^{s,1/2+}}.$$

Combining Lemma 2.2.3 with Lemmas 2.1.3, 2.1.4 and 2.1.5, we proved the following local well-posedness result in H^s for $s > 9/10$.

Proposition 2.2.4 (Theorem 1.1 in [20]). *Let $s > 9/10$. For any $u_0 \in H^s(\mathbb{R} \times \mathbb{T})$, there exists $T = T(\|u_0\|_{H^s})$ and a unique solution u of* (1.1) *such that*

$$u \in C([0,T] : H^s(\mathbb{R} \times \mathbb{T})) \cap X_T^{s,1/2+}.$$

Moreover, for all $0 < T' < T$, there exists a neighborhood $\mathcal{U}$ of u_0 in $H^s(\mathbb{R} \times \mathbb{T})$ such that the data-to-solution map

$$v_0 \in \mathcal{U} \mapsto v(t) \in C([0,T'] : H^s(\mathbb{R} \times \mathbb{T})) \cap X_{T'}^{s,1/2+}$$

is smooth, when $v(t)$ is a unique solution of (1.1) *to the initial data v_0.*

Sketch of the proof of Proposition 2.2.4. The proof was derived the implement of Lemma 2.2.3 together with a fixed point argument as follows

$$\Phi(u)(t) = \eta(t)e^{-t\partial_x\Delta}u_0 + \frac{1}{2}\eta(t)\int_0^t e^{-(t-t')\partial_x\Delta}\partial_x(u(t')^2)\,dt'.$$

We show that the map Φ defines a contraction in

$$Z_T = \left\{ u \in X_T^{s,1/2+} \mid \|u\|_{X_T^{s,1/2+}} \le 2C_0\|u_0\|_{H^s} \right\},$$

for (small) $T > 0$. In fact, from Lemmas 2.1.3, 2.1.4 and 2.1.5 and 2.2.3 with $\lambda = 1$, it follows that

$$\begin{aligned} \|\Phi(u)\|_{X_T^{s,1/2+}} &\le C_0\|u_0\|_{H^s} + C_0\|\partial_x(u^2)\|_{X_T^{s,-1/2+\varepsilon}} \\ &\le C_0\|u_0\|_{H^s} + C_0T^{\varepsilon}\|\partial_x(u^2)\|_{X_T^{s,-1/2+2\varepsilon}} \\ &\le C_0\|u_0\|_{H^s} + C_0cT^{\varepsilon}\|u\|^2_{X_T^{s,-1/2+}} \\ &\le 2C_0\|u_0\|_{H^s}, \end{aligned} \tag{2.6}$$

for $u \in Z_T$ and choosing small $T > 0$ which will depend on $\|u_0\|_{H^s}$. Similarly,

$$\|\Phi(u) - \Phi(v)\|_{X_T^{s,1/2+}} \le \frac{1}{2}\|u - v\|_{X_T^{s,1/2+}},$$

for $u, v \in Z_T$ and small $T > 0$. Then the contraction mapping theorem tells us that there is a unique solution $u = \Phi(u) \in Z_T$ to the Cauchy problem (1.1).

The persistence property $u \in C([0,T] : H^s)$ and the uniqueness in whole space $u \in X_T^{s,1/2+}$ follow in a similar way to [11, Proof of Theorem 1.5], by using a variant of (2.6), therefore we omit them.

This basically completes the outline of the proof. □

Chapter 3

Local well-posedness in $I^{-1}H^1_\lambda$

This chapter is based on the joint paper [21] with Hideo Takaoka, which devotes to the local well-posedness results in conjunction with rescaling argument in [6]. Hereafter we assume that $s < 1$.

3.1 Rescaled solutions

Throughout the thesis, assume $\lambda > 1$. Recall $\mathbb{T}_\lambda = \lambda\mathbb{T} = \mathbb{R}/(2\pi\lambda\mathbb{Z})$. By rescaling

$$u^\lambda(x, y, t) = \frac{1}{\lambda^2} u\left(\frac{x}{\lambda}, \frac{y}{\lambda}, \frac{t}{\lambda^3}\right), \tag{3.1}$$

we consider the Cauchy problem for the Zakharov-Kuznetsov equation on $\mathbb{R} \times \mathbb{T}_\lambda$:

$$\begin{cases} \partial_t u^\lambda + \partial_x \Delta u^\lambda + u^\lambda \partial_x u^\lambda = 0, \\ u^\lambda(x, y, 0) = u_0^\lambda(x, y) \in H^s_\lambda, \end{cases} \quad (x, y) \in \mathbb{R} \times \mathbb{T}_\lambda, \quad t \in \mathbb{R}. \tag{3.2}$$

If we construct the solution $u^\lambda(t)$ of (3.2) on the time interval $[0, \lambda^3 T]$, we have the solution $u(t)$ of (1.1) on $[0, T]$.

Let ζ be combination of spatial variables as $\zeta = (\xi, q) \in \mathbb{R} \times \mathbb{Z}/\lambda$. Define the Fourier multiplier operator I, which was originally introduced in [5] to consider global well-posedness for KdV equation. For $N \in 2^{\mathbb{N}}$, define the operator I by

$$\mathcal{F}^\lambda If(\zeta) = m(\zeta)\mathcal{F}^\lambda f(\zeta),$$

where m is a smooth, radially symmetric, non-increasing function satisfying

$$m(\zeta) = \begin{cases} 1, & |\zeta| \le N, \\ \left(\frac{|\zeta|}{N}\right)^{s-1}, & |\zeta| \ge 2N. \end{cases}$$

On the low frequency part $|\zeta| \le N$, the operator I is the identity operator, while on the high frequency, I is regarded as the integral operator. Remark that I maps H^s_λ functions to H^1_λ one.

3.2 Local well-posedness in $I^{-1}H^1_\lambda$

In this section, we prove the following local well-posedness results on $I^{-1}H^1_\lambda$.

Proposition 3.2.1 (A variant of local well-posedness)**.** *Let $s > 9/10$. The Cauchy problem* (3.2) *is locally well-posed in H^s_λ for data u_0^λ satisfying $u_0^\lambda \in H^s_\lambda$. Moreover, the solution exists on a time interval $[0, \delta]$ with $\delta \sim \|Iu_0^\lambda\|_{H^1_\lambda}$, and the solution $u^\lambda(t)$ satisfies the estimate*

$$\left\| Iu^\lambda \right\|_{X^{1,1/2+}_{\lambda,\delta}} \lesssim \left\| Iu_0^\lambda \right\|_{H^1_\lambda}.$$

We give several lemmas in conjunction with rescaling argument in [6]. Let us readjust several lemmas in Chapter 2 to one for rescaled functions on $\mathbb{R} \times \mathbb{T}_\lambda$.

Lemma 3.2.2 (Lemma 3.7 in [18]). *Denote a set $\Lambda \subset \mathbb{R} \times \mathbb{Z}/\lambda$. Let the projection on the q be axis contained in a set $I \subset \mathbb{Z}/\lambda$. Assume that there is a positive constant C such that for any fixed $q_0 \in I \cap \mathbb{Z}/\lambda$ and $|\Lambda \cap \{(\xi, q_0) \mid q_0 \in \mathbb{Z}/\lambda\}| \leq C$. Then, $|\Lambda| \leq \lambda C(|I| + 1)$.*

Using this lemma and the mean value theorem, we have the estimates for constructing bilinear estimates.

Lemma 3.2.3 (Lemma 3.8 in [18]). *Let I and J be two intervals on the real line and $f : J \to \mathbb{R}$ be a smooth function. Then,*

$$|\{x \in J \mid f(x) \in I\}| \lesssim \frac{|I|}{\inf_{\xi \in J} |f'(\xi)|}.$$

Lemma 3.2.4. *Let $a \neq 0,\ b,\ c$ be real numbers and I an interval on the real line. Then,*

$$\left|\{q \in \mathbb{Z}/\lambda \mid aq^2 + bq + c \in I\}\right| \lesssim \lambda \left(\frac{|I|^{1/2}}{|a|^{1/2}} + 1 \right).$$

Proof. Following Lemma 3.9 in [18], we only consider the case of $a > 0$ and $b = c = 0$. Put $I = [as^2, at^2]$ for $s, t \in \mathbb{R}$ with $s^2 < t^2$ and $s, t \geq 0$. From the shape of parabola curve and distribution of $\mathbb{Z}/\lambda$ points, we can say

$$\left|\{q \in \mathbb{Z}/\lambda \mid aq^2 + bq + c \in I\}\right| \leq 2\lambda(|t - s| + 1).$$

Since

$$|t - s| = \sqrt{(t - s)^2} \leq \sqrt{2(t^2 - s^2)} = \frac{\sqrt{2(at^2 - as^2)}}{\sqrt{a}} = \frac{\sqrt{2}|I|^{1/2}}{|a|^{1/2}},$$

we obtain

$$\left|\{q \in \mathbb{Z}/\lambda \mid aq^2 + bq + c \in I\}\right| \lesssim \lambda \left(\frac{|I|^{1/2}}{|a|^{1/2}} + 1 \right),$$

which completes the proof of lemma. $\square$

Using these lemmas, we get the following bilinear estimates.

Lemma 3.2.5. *For $u,\ v \in L^2_\lambda$,*

$$\|(P_{N_1} Q_{L_1} u)(P_{N_2} Q_{L_2} v)\|_{L^2_\lambda} \lesssim (N_1 \wedge N_2)(L_1 \wedge L_2)^{1/2} \|P_{N_1} Q_{L_1} u\|_{L^2_\lambda} \|P_{N_2} Q_{L_2} v\|_{L^2_\lambda}, \tag{3.3}$$

$$\|R_K((P_{N_1} Q_{L_1} u)(P_{N_2} Q_{L_2} v))\|_{L^2_\lambda} \lesssim \frac{(N_1 \wedge N_2)^{1/2}}{K^{1/4}} (L_1 \wedge L_2)^{1/2} (L_1 \vee L_2)^{1/4} \|P_{N_1} Q_{L_1} u\|_{L^2_\lambda} \|P_{N_2} Q_{L_2} v\|_{L^2_\lambda}. \tag{3.4}$$

Moreover, when $N_1 \wedge N_2 \ll N_1 \vee N_2$, for $0 < \theta < 1$ we have

$$\|(P_{N_1} Q_{L_1} u)(P_{N_2} Q_{L_2} v)\|_{L^2_\lambda} \lesssim \frac{(N_1 \wedge N_2)^{1/2}}{N_1 \vee N_2} (L_1 \wedge L_2)^{1/2} (L_1 \vee L_2)^{1/2} \|P_{N_1} Q_{L_1} u\|_{L^2_\lambda} \|P_{N_2} Q_{L_2} v\|_{L^2_\lambda}, \tag{3.5}$$

$$\|(P_{N_1} Q_{L_1} u)(P_{N_2} Q_{L_2} v)\|_{L^2_\lambda} \lesssim \frac{(N_1 \wedge N_2)^{(1+\theta)/2}}{(N_1 \vee N_2)^{1-\theta}} (L_1 \wedge L_2)^{1/2} (L_1 \vee L_2)^{\theta/2} \|P_{N_1} Q_{L_1} u\|_{L^2_\lambda} \|P_{N_2} Q_{L_2} v\|_{L^2_\lambda}, \tag{3.6}$$

where N_1, N_2, L_1, L_2, K are dyadic numbers.

Proof. The estimate in (3.6) follows from the interpolation argument with (3.3) and (3.5), so that we prove (3.3), (3.5) and (3.4) in this order.

Using the Plancherel's identity, convolution structure, and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\|(P_{N_1}Q_{L_1}u)\ (P_{N_2}Q_{L_2}v)\|_{L^2_\lambda} \\
&\qquad = \left(\frac{1}{\lambda}\sum_{q\in\mathbb{Z}/\lambda}\int_{\mathbb{R}^2}\left|\left(\widehat{F}^\lambda(P_{N_1}Q_{L_1}u) *^\lambda \widehat{F}^\lambda(P_{N_2}Q_{L_2}v)\right)(\xi,q,\tau)\right|^2 d\xi d\tau\right)^{1/2} \\
&\qquad \lesssim \frac{1}{\lambda^{1/2}}\sup_{(\xi,q,\tau)\in\mathbb{R}\times\mathbb{Z}/\lambda\times\mathbb{R}}|A_{\xi,q,\tau}|^{1/2}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda},
\end{aligned}$$

where

$$\begin{aligned}
A_{\xi,q,\tau} = \{(\xi_1,q_1,\tau_1)\in\mathbb{R}\times\mathbb{Z}/\lambda\times\mathbb{R} \mid |(\xi_1,q_1)|\in I_{N_1},\ |(\xi-\xi_1,q-q_1)|\in I_{N_2}, \\
|\tau_1-\sigma(\xi_1,q_1)|\in I_{L_1},\ |\tau-\tau_1-\sigma(\xi-\xi_1,q-q_1)|\in I_{L_2}\}.
\end{aligned}$$

By the definition of $A_{\xi,q,\tau}$ and Lemma 3.2.2 with $|I|\sim\lambda(N_1\wedge N_2)$, we obtain

$$|A_{\xi,q,\tau}|\lesssim\lambda(N_1\wedge N_2)^2(L_1\wedge L_2),$$

which is (3.3).

Next we prove (3.5). By the triangle inequality

$$\begin{aligned}
|\tau_1-\sigma(\xi_1,q_1)|+|\tau-\tau_1-\sigma(\xi-\xi_1,q-q_1)| &\le |\tau-\sigma(\xi_1,q_1)-\sigma(\xi-\xi_1,q-q_1)| \\
&= |\tau-\sigma(\xi,q)-\mathcal{H}(\xi_1,\xi-\xi_1,q_1,q-q_1)|,
\end{aligned}$$

we get

$$|A_{\xi,q,\tau}|\lesssim(L_1\wedge L_2)|B_{\xi,q,\tau}|,$$

where

$$\begin{aligned}
B_{\xi,q,\tau} = \{(\xi_1,q_1)\in\mathbb{R}\times\mathbb{Z}/\lambda \mid |(\xi_1,q_1)|\in I_{N_1}, |(\xi-\xi_1,q-q_1)|\in I_{N_2}, \\
|\tau-\sigma(\xi,q)-\mathcal{H}(\xi_1,\xi-\xi_1,q_1,q-q_1)|\lesssim L_1\vee L_2\}.
\end{aligned}$$

Let us focus on the resonance function $\mathcal{H}$ in (2.4). We have

$$\left|\frac{\partial\mathcal{H}}{\partial\xi_1}(\xi_1,\xi-\xi_1,q_1,q-q_1)\right| = |3\xi_1^2+q_1^2-(3(\xi-\xi_1)^2+(q-q_1)^2)|\gtrsim(N_1\vee N_2)^2. \tag{3.7}$$

If we define $\widetilde{B}_{\xi,q,\tau}(q_1)=\{\xi_1\in\mathbb{R}\mid(\xi_1,q_1)\in B_{\xi,q,\tau}\}$ for each q_1, Lemma 3.2.3 yields

$$\left|\widetilde{B}_{\xi,q,\tau}(q_1)\right|\lesssim\frac{L_1\vee L_2}{(N_1\vee N_2)^2}.$$

Hence

$$|B_{\xi,q,\tau}|\lesssim\lambda\frac{(L_1\vee L_2)(N_1\wedge N_2)}{(N_1\vee N_2)^2}.$$

Combining this with $|A_{\xi,q,\tau}|\lesssim(L_1\wedge L_2)|B_{\xi,q,\tau}|$, we obtain (3.5).

From the Cauchy-Schwarz inequality and the Plancherel's identity as before, it holds that

$$\|R_K((P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v))\|_{L^2_\lambda}\lesssim\frac{1}{\lambda^{1/2}}\sup_{(\xi,q,\tau)\in\mathbb{R}\times\mathbb{Z}/\lambda\times\mathbb{R}}|A^K_{\xi,q,\tau}|^{1/2}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda},$$

where

$$\begin{aligned}
A^K_{\xi,q,\tau} = \{(\xi_1,q_1,\tau_1)\in\mathbb{R}\times\mathbb{Z}/\lambda\times\mathbb{R} \mid |\xi|\sim K,\ |(\xi_1,q_1)|\in I_{N_1},\ |(\xi-\xi_1,q-q_1)|\in I_{N_2}, \\
|\tau_1-\sigma(\xi_1,q_1)|\in I_{L_1},\ |\tau-\tau_1-\sigma(\xi-\xi_1,q-q_1)|\in I_{L_2}\}.
\end{aligned}$$

Using the triangle inequality, we have

$$|A^K_{\xi,q,\tau}| \lesssim (L_1 \wedge L_2)|B^K_{\xi,q,\tau}|,$$

where

$$\begin{aligned} B^K_{\xi,q,\tau} = \{(\xi_1, q_1) \in \mathbb{R} \times \mathbb{Z}/\lambda \mid |\xi| \sim K,\ |(\xi_1, q_1)| \in I_{N_1},\ |(\xi - \xi_1, q - q_1)| \in I_{N_2}, \\ |\tau - \sigma(\xi, q) - \mathcal{H}(\xi_1, \xi - \xi_1, q_1, q - q_1)| \lesssim L_1 \vee L_2\}. \end{aligned}$$

For the bound of $B^K_{\xi,q,\tau}$, we calculate the second derivative of $\mathcal{H}$ as

$$\left|\frac{\partial^2 \mathcal{H}}{\partial {\xi_1}^2}(\xi_1, \xi - \xi_1, q_1, q - q_1)\right| = 6|\xi| \sim K, \tag{3.8}$$

for $(\xi_1, q_1) \in B^K_{\xi,q,\tau}$. Let $\widetilde{B}^K_{\xi,q,\tau}(q_1) = \{\xi_1 \in \mathbb{R} \mid (\xi_1, q_1) \in B^K_{\xi,q,\tau}\}$ for each q_1. Combining with Lemma 3.2.4, we get

$$|\widetilde{B}^K_{\xi,q,\tau}(q_1)| \lesssim \frac{(L_1 \vee L_2)^{1/2}}{K^{1/2}},$$

for all $q_1 \in \mathbb{Z}/\lambda$. Finally,

$$|B^K_{\xi,q,\tau}| \lesssim \lambda \frac{(N_1 \wedge N_2)(L_1 \vee L_2)}{K^{1/2}} \tag{3.9}$$

and we obtain (3.4) by $|A^K_{\xi,q,\tau}| \lesssim (L_1 \wedge L_2)|B^K_{\xi,q,\tau}|$. □

We prove the following bilinear estimates in $I^{-1}X^{1,1/2+}_\lambda$.

Lemma 3.2.6. *For $s > 9/10$,*

$$\|\partial_x I(uv)\|_{X^{1,-1/2+}_\lambda} \lesssim \|Iu\|_{X^{1,1/2+}_\lambda}\|Iv\|_{X^{1,1/2+}_\lambda}.$$

Proof. We may assume the functions $\widehat{u}^\lambda$ and $\widehat{v}^\lambda$ are nonnegative, since by the definition of $X^{s,b}_\lambda$ norm. Replacing $\langle\zeta\rangle\langle\tau - \sigma(\zeta)\rangle^{1/2+}m(\zeta)\widehat{u}(\zeta, \tau)$ and $\langle\zeta\rangle\langle\tau - \sigma(\zeta)\rangle^{1/2+}m(\zeta)\widehat{v}(\zeta, \tau)$ by $\widehat{u}(\zeta, \tau)$ and $\widehat{v}(\zeta, \tau)$, respectively, and duality argument, one has to show that

$$\begin{aligned} J =& \frac{1}{\lambda^2} \sum_{q,q_1 \in \mathbb{Z}/\lambda} \int_{\mathbb{R}^4} \Gamma^{\xi_1,q_1,\tau_1}_{\xi,q,\tau} \frac{m(\zeta)}{m(\zeta_1)m(\zeta - \zeta_1)} \widehat{u}^\lambda(\zeta_1, \tau_1)\widehat{v}^\lambda(\zeta - \zeta_1, \tau - \tau_1)\widehat{w}^\lambda(\zeta, \tau)\, d\xi d\xi_1 d\tau d\tau_1 \\ \lesssim& \|u\|_{L^2_\lambda}\|v\|_{L^2_\lambda}\|w\|_{L^2_\lambda}, \end{aligned} \tag{3.10}$$

for $w \in L^2_\lambda$ whose Fourier transform is nonnegative. Here

$$\Gamma^{\xi_1,q_1,\tau_1}_{\xi,q,\tau} = \frac{|\xi|\langle\zeta\rangle}{\langle\zeta_1\rangle\langle\zeta - \zeta_1\rangle\langle\tau - \sigma(\zeta)\rangle^{1/2-}\langle\tau_1 - \sigma(\zeta_1)\rangle^{1/2+}\langle\tau - \tau_1 - \sigma(\zeta - \zeta_1)\rangle^{1/2+}}.$$

Using dyadic decomposition, we rewrite J as the following

$$J = \sum_{\substack{N_0,N_1,N_2 \\ L_0,L_1,L_2}} J^{L_0,L_1,L_2}_{N_0,N_1,N_2},$$

where

$$\begin{aligned} J^{L_0,L_1,L_2}_{N_0,N_1,N_2} =& \frac{1}{\lambda^2} \sum_{q,q_1 \in \mathbb{Z}/\lambda} \int_{\mathbb{R}^4} \Gamma^{\xi_1,q_1,\tau_1}_{\xi,q,\tau} \frac{m(\zeta)}{m(\zeta_1)m(\zeta - \zeta_1)} \\ & \widehat{P_{N_1}Q_{L_1}u}^\lambda(\zeta_1, \tau_1)\widehat{P_{N_2}Q_{L_2}v}^\lambda(\zeta - \zeta_1, \tau - \tau_1)\widehat{P_{N_0}Q_{L_0}w}^\lambda(\xi, q, \tau)\, d\xi d\xi_1 d\tau d\tau_1. \end{aligned} \tag{3.11}$$

We decompose again J to five parts in frequencies:

$$J_{LL\to L} = \sum_{\substack{N_1\vee N_2\vee N_0\\ L_0,L_1,L_2}} J^{L_0,L_1,L_2}_{N_0,N_1,N_2},$$

$$J_{LH\to H} = \sum_{\substack{N_1\ll N_2\sim N_0\\ N_0\gtrsim N\\ L_0,L_1,L_2}} J^{L_0,L_1,L_2}_{N_0,N_1,N_2},$$

$$J_{HL\to H} = \sum_{\substack{N_2\ll N_1\sim N_0\\ N_0\gtrsim N\\ L_0,L_1,L_2}} J^{L_0,L_1,L_2}_{N_0,N_1,N_2},$$

$$J_{HH\to L} = \sum_{\substack{N_0\ll N_1\sim N_2\\ N_1\gtrsim N\\ L_0,L_1,L_2}} J^{L_0,L_1,L_2}_{N_0,N_1,N_2},$$

$$J_{HH\to H} = J - (J_{LL\to L} + J_{HL\to H} + J_{LH\to H} + J_{HH\to L}).$$

<u>Estimate of $J_{LL\to L}$</u>: In this case, we have

$$\frac{m(\zeta)}{m(\zeta_1)m(\zeta-\zeta_1)} \sim 1, \quad \Gamma^{\xi_1,q_1,\tau_1}_{\xi,q,\tau} \lesssim \frac{N_1\vee N_2}{(N_1\wedge N_2)L_0^{1/2-}L_1^{1/2+}L_2^{1/2+}}.$$

We split this case two cases, when $N_1\wedge N_2 \ll N_1\vee N_2$, and when $N_1\wedge N_2\sim N_1\vee N_2$.

First, we consider the contribution of the case when $N_1\wedge N_2\ll N_1\vee N_2$. In this case, by (3.5) and the Cauchy-Schwarz inequality, we have that the contribution of the case to $J_{LL\to L}$ is bounded by

$$\begin{aligned}
&\lesssim \sum_{\substack{N_1\vee N_2\vee N_0\ll N\\ N_1\wedge N_2\ll N_1\vee N_2\\ L_0,L_1,L_2}} \frac{N_1\vee N_2}{(N_1\wedge N_2)L_0^{1/2-}L_1^{1/2+}L_2^{1/2+}}\|(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\|_{L^2_\lambda}\|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}\\
&\lesssim \sum_{\substack{N_1\vee N_2\vee N_0\ll N\\ L_0,L_1,L_2}} \frac{1}{(N_1\wedge N_2)^{0+}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda}\|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}\\
&\lesssim \|u\|_{L^2_\lambda}\|v\|_{L^2_\lambda}\|w\|_{L^2_\lambda},
\end{aligned}$$

where we use $N_0\lesssim N_1\vee N_2$.

Second, we consider the contribution of the case when $N_1\wedge N_2\sim N_1\vee N_2$, namely $N_1\sim N_2$. Note that $P_N = P_N\widetilde{P}_N$ for $\widetilde{P}_N = P_{N/2}+P_N+P_{2N}$. In this case, the contribution of the case to $J_{LL\to L}$ is bounded by

$$\begin{aligned}
&\lesssim \sum_{\substack{N_1\vee N_2\vee N_0\ll N\\ N_1\sim N_2\\ L_0,L_1,L_2}} \frac{N_1\vee N_2}{(N_1\wedge N_2)L_0^{1/2-}L_1^{1/2+}L_2^{1/2+}}\|\widetilde{P}_{N_0}((P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v))\|_{L^2_\lambda}\|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}\\
&\lesssim \sum_{\substack{N_1\sim N_2\\ L_1,L_2}} \frac{1}{L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda}\|w\|_{L^2_\lambda}\\
&\lesssim \|u\|_{L^2_\lambda}\|v\|_{L^2_\lambda}\|w\|_{L^2_\lambda}.
\end{aligned}$$

Hence

$$J_{LL\to L}\lesssim \|u\|_{L^2_\lambda}\|v\|_{L^2_\lambda}\|w\|_{L^2_\lambda}.$$

<u>Estimate of $J_{LH\to H}$</u>: We split this case into two cases that $N_1\ll N\lesssim N_2\sim N_0$ and that $N\lesssim N_1\ll N_2\sim N_0$.

If $N_1 \ll N \lesssim N_2 \sim N_0$, we have

$$\frac{m(\zeta)}{m(\zeta_1)m(\zeta-\zeta_1)} \sim 1.$$

Applying the Cauchy-Schwarz inequality again with (3.5), we have that the contribution of this case to $J^{L_0,L_1,L_2}_{N_0,N_1,N_2}$ is bounded by

$$\begin{aligned}
&\lesssim \frac{N_0}{N_1 L_0^{1/2-} L_1^{1/2+} L_2^{1/2+}} \|(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\|_{L^2_\lambda} \|P_{N_0}Q_{L_0}w\|_{L^2_\lambda} \\
&\lesssim \frac{1}{N_1^{1/2} L_0^{1/2-} L_1^{0+} L_2^{0+}} \|P_{N_1}Q_{L_1}u\|_{L^2_\lambda} \|P_{N_2}Q_{L_2}v\|_{L^2_\lambda} \|P_{N_0}Q_{L_0}w\|_{L^2_\lambda} \\
&\lesssim \frac{1}{N_1^{0+} L_0^{0+} L_1^{0+} L_2^{0+}} \|P_{N_1}Q_{L_1}u\|_{L^2_\lambda} \|P_{N_2}Q_{L_2}v\|_{L^2_\lambda} \|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}.
\end{aligned} \tag{3.12}$$

If $N \lesssim N_1 \ll N_2 \sim N_0$, we have

$$\frac{m(\zeta)}{m(\zeta_1)m(\zeta-\zeta_1)} \sim \frac{N_1^{1-s}}{N^{1-s}}.$$

In a similar way to above, we have that the contribution of this case to $J^{L_0,L_1,L_2}_{N_0,N_1,N_2}$ is bounded by

$$\begin{aligned}
&\lesssim \frac{N_0 N_1^{1-s}}{N_1 N^{1-s} L_0^{1/2-} L_1^{1/2+} L_2^{1/2+}} \|(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\|_{L^2_\lambda} \|P_{N_0}Q_{L_0}w\|_{L^2_\lambda} \\
&\lesssim \frac{1}{N_1^{s-1/2} L_0^{1/2-} L_1^{0+} L_2^{0+}} \|P_{N_1}Q_{L_1}u\|_{L^2_\lambda} \|P_{N_2}Q_{L_2}v\|_{L^2_\lambda} \|P_{N_0}Q_{L_0}w\|_{L^2_\lambda} \\
&\lesssim \frac{1}{N_1^{0+} L_0^{0+} L_1^{0+} L_2^{0+}} \|P_{N_1}Q_{L_1}u\|_{L^2_\lambda} \|P_{N_2}Q_{L_2}v\|_{L^2_\lambda} \|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}.
\end{aligned} \tag{3.13}$$

Therefore, by (3.12) and (3.13) in conjunction with previous estimate, we have

$$J_{LH\to H} \lesssim \|u\|_{L^2_\lambda} \|v\|_{L^2_\lambda} \|w\|_{L^2_\lambda},$$

which is acceptable.

<u>Estimate of $J_{HL\to H}$</u>: The proof is same as for $J_{LH\to H}$, because of the symmetry.

<u>Estimate of $J_{HH\to L}$</u>: By symmetry, we assume $N_0 \ll N_1 \le N_2$. We separate this case into two cases that $N_0 \ll N \ll N_1 \sim N_2$ and that $N \lesssim N_0 \ll N_1 \sim N_2$.

If $N_0 \ll N \ll N_1 \sim N_2$, we have

$$\frac{m(\zeta)}{m(\zeta_1)m(\zeta-\zeta_1)} \sim \frac{N_1^{1-s} N_2^{1-s}}{N^{2(1-s)}}.$$

Applying the L^2_λ norm of functions $(\widetilde{P_{N_1}Q_{L_1}}u)(P_{N_0}Q_{L_0}w)$ and $P_{N_2}Q_{L_2}v$, we have that then the contribution of this case to $J^{L_0,L_1,L_2}_{N_0,N_1,N_2}$ is bounded by

$$\lesssim \frac{N_1^{1-s} N_2^{1-s} N_0^2}{N^{2(1-s)} N_1 N_2 L_0^{1/2-} L_1^{1/2+} L_2^{1/2+}} \left\|(\widetilde{P_{N_1}Q_{L_1}}u)(P_{N_0}Q_{L_0}w)\right\|_{L^2_\lambda} \|P_{N_2}Q_{L_2}v\|_{L^2_\lambda},$$

where $\widehat{\widetilde{f}}^\lambda(\zeta,\tau) = \widehat{f}^\lambda(-\zeta,-\tau)$. We apply (3.6) to this and have that the contribution of this case to $J^{L_0,L_1,L_2}_{N_0,N_1,N_2}$ is bounded by

$$\begin{aligned}
&\lesssim \frac{N_1^{1-s} N_2^{1-s} N_0^2}{N^{2(1-s)} N_1 N_2 L_0^{1/2-} L_1^{1/2+} L_2^{1/2+}} \frac{N_0^{(1+\theta)/2} (L_0 \wedge L_1)^{1/2} (L_0 \vee L_1)^{\theta/2}}{N_1^{1-\theta}} \\
&\qquad \|P_{N_1}Q_{L_1}u\|_{L^2_\lambda} \|P_{N_0}Q_{L_0}w\|_{L^2_\lambda} \|P_{N_2}Q_{L_2}v\|_{L^2_\lambda} \\
&\lesssim \frac{1}{N^{1-s} N_1^{s-1/2-3\theta/2} L_0^{0+} L_1^{0+} L_2^{0+}} \|P_{N_1}Q_{L_1}u\|_{L^2_\lambda} \|P_{N_0}Q_{L_0}w\|_{L^2_\lambda} \|P_{N_2}Q_{L_2}v\|_{L^2_\lambda},
\end{aligned} \tag{3.14}$$

for small $\theta > 0$.

If $N \lesssim N_0 \ll N_1 \sim N_2$, we have

$$\frac{m(\zeta)}{m(\zeta_1)m(\zeta-\zeta_1)} \sim \frac{N_1^{1-s}N_2^{1-s}}{N^{1-s}N_0^{1-s}}.$$

Similarly, the contribution of this case to $J_{N_0,N_1,N_2}^{L_0,L_1,L_2}$ is bounded by

$$\lesssim \frac{1}{N_1^{s-1/2-3\theta/2}N^{1-s}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda}, \tag{3.15}$$

for small $\theta > 0$.

Summing up with respect to $N_0, N_1, N_2, L_0, L_1, L_2$, we obtain

$$J_{HH\to L} \lesssim \|u\|_{L^2_\lambda}\|v\|_{L^2_\lambda}\|w\|_{L^2_\lambda}.$$

Estimate of $J_{HH\to H}$: In this case, we may assume $N \ll N_0 \sim N_1 \sim N_2$ and have

$$\frac{m(\zeta)}{m(\zeta_1)m(\zeta-\zeta_1)} \sim \frac{N_0^{1-s}}{N^{1-s}}, \quad \Gamma_{\xi,q,\tau}^{\xi_1,q_1,\tau_1} \sim \frac{|\xi|}{N_0L_0^{1/2-}L_1^{1/2+}L_2^{1/2+}}.$$

Now, we separate this case into five subcases:

(i) $|\xi| \lesssim 1$,

(ii) $|\xi_1| \wedge |\xi-\xi_1| \lesssim 1 \ll |\xi|$,

(iii) $1 \ll |\xi_1|\wedge|\xi-\xi_1|\wedge|\xi|$ and $||\zeta|^2-|\zeta_1|^2|\vee||\zeta_1|^2-|\zeta-\zeta_1|^2|\vee||\zeta-\zeta_1|^2-|\zeta|^2| \gtrsim N_0^{6/5}(L_0\vee L_1\vee L_1)^{0+}$,

(iv) $1 \ll |\xi_1|\wedge|\xi-\xi_1|\wedge|\xi|$, $||\zeta|^2-|\zeta_1|^2|\wedge||\zeta_1|^2-|\zeta-\zeta_1|^2|\wedge||\zeta-\zeta_1|^2-|\zeta|^2| \ll N_0^{6/5}(L_0\vee L_1\vee L_1)^{0+}$ and $(|\xi|\vee|\xi_1|\vee|\xi-\xi_1|)(|\xi|\wedge|\xi_1|\wedge|\xi-\xi_1|) \gg N_0^{6/5}(L_0\vee L_1\vee L_1)^{0+}$,

(v) $1 \ll |\xi_1|\wedge|\xi-\xi_1|\wedge|\xi|$, $||\zeta|^2-|\zeta_1|^2|\wedge||\zeta_1|^2-|\zeta-\zeta_1|^2|\wedge||\zeta-\zeta_1|^2-|\zeta|^2| \ll N_0^{6/5}(L_0\vee L_1\vee L_1)^{0+}$ and $(|\xi|\vee|\xi_1|\vee|\xi-\xi_1|)(|\xi|\wedge|\xi_1|\wedge|\xi-\xi_1|) \lesssim N_0^{6/5}(L_0\vee L_1\vee L_1)^{0+}$.

Subcase (i): Denote

$$J_{N_0,N_1,N_2}^{L_0,L_1,L_2} = \sum_{k\in\mathbb{N}} J_{N_0,N_1,N_2}^{L_0,L_1,L_2}(k),$$

where

$$\begin{aligned}J_{N_0,N_1,N_2}^{L_0,L_1,L_2}(k) = \frac{1}{\lambda^2}\sum_{q,q_1\in\mathbb{Z}/\lambda}\int_{Y_k}\Gamma_{\xi,q,\tau}^{\xi_1,q_1,\tau_1}\frac{m(\zeta)}{m(\zeta_1)m(\zeta-\zeta_1)}\\ \widehat{P_{N_1}Q_{L_1}u}^\lambda(\zeta_1,\tau_1)\widehat{P_{N_2}Q_{L_2}v}^\lambda(\zeta-\zeta_1,\tau-\tau_1)\widehat{P_{N_0}Q_{L_0}w}^\lambda(\zeta,\tau)\,d\xi d\xi_1 d\tau d\tau_1,\end{aligned}$$

$$Y_k = \{(\xi,\xi_1,\tau,\tau_1)\in\mathbb{R}^4 \mid |\xi| \sim 2^{-k}\}.$$

We apply the Cauchy-Schwarz inequality, and we have that the contribution of this case to $J_{N_0,N_1,N_2}^{L_0,L_1,L_2}(k)$ is bounded by

$$\lesssim \frac{2^{-k}}{N^{1-s}N_0^sL_0^{1/2-}L_1^{1/2+}L_2^{1/2+}}\|R_{2^{-k}}\left((P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\right)\|_{L^2_\lambda}\|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}.$$

For $|\xi| \sim 2^{-k}$, we obtain by (2.4)

$$\left|\frac{\partial^2\mathcal{H}}{\partial\xi_1^2}(\xi_1,\xi-\xi_1,q_1,q-q_1)\right| = 6|\xi| \sim 2^{-k}.$$

Using (3.4), we get

$$\|R_{2^{-k}}((P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v))\|_{L^2_\lambda} \lesssim 2^{k/4}N_0^{1/2}(L_1\vee L_2)^{1/4}(L_1\wedge L_2)^{1/2}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda}.$$

Then

$$J^{L_0,L_1,L_2}_{N_0,N_1,N_2}(k) \lesssim \frac{2^{-3k/4}}{N^{1-s}N_0^{s-1/2}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda}\|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}. \tag{3.16}$$

Summing with respect to dyadic numbers $N_0, N_1, N_2, L_0, L_1, L_2$ and $k \in \mathbb{N}$, we have the desired bound for the contribution of this case to $J_{HH\to H}$.

Subcase (ii): By symmetry, we may suppose $|\xi-\xi_1| \le |\xi_1|\wedge 1$. First, we calculate the resonance function (2.4) as

$$\frac{\partial\mathcal{H}}{\partial\xi_1}(\xi_1,\xi-\xi_1,q,q-q_1) = 3\xi(\xi-2\xi_1)+q(q-2q_1).$$

First we consider the case when $|\partial\mathcal{H}/\partial\xi_1| \gtrsim \xi^2$. We shall use the dyadic decomposition $|\xi|\sim K$. By the Cauchy-Schwarz inequality such as Case (i) above, we have that the contribution of this case to $J^{L_0,L_1,L_2}_{N_0,N_1,N_2}$ is bounded by

$$\lesssim \sum_{K\in 2^{\mathbb{N}}} \frac{K}{N^{1-s}N_0^sL_0^{1/2-}L_1^{1/2+}L_2^{1/2+}}\|R_K((P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v))\|_{L^2_\lambda}\|R_KP_{N_0}Q_{L_0}w\|_{L^2_\lambda}.$$

Recall the proof of Lemma 3.2.5, we have

$$\|R_K((P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v))\|_{L^2_\lambda} \lesssim \frac{1}{\lambda^{1/2}}\sup_{(\xi,q,\tau)\in\mathbb{R}\times\mathbb{Z}/\lambda\times\mathbb{R}}|A_{\xi,q,\tau}(K)|^{1/2}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda},$$

where

$$\begin{aligned}A_{\xi,q,\tau}(K) = \{(\xi_1,q_1,\tau_1)\in\mathbb{R}\times\mathbb{Z}/\lambda\times\mathbb{R}\ ||\xi|\sim K,\ |\zeta_1|\in I_{N_1},\ |\zeta-\zeta_1|\in I_{N_2},\\ |\tau_1-\sigma(\zeta_1)|\in I_{L_1}, |\tau-\tau_1-\sigma(\zeta-\zeta_1)|\in I_{L_2}\}.\end{aligned}$$

From the triangle inequality,

$$|A_{\xi,q,\tau}(K)| \lesssim (L_1\wedge L_2)|B_{\xi,q,\tau}(K)|,$$

where

$$\begin{aligned}B_{\xi,q,\tau}(K) = \{(\xi_1,q_1)\in\mathbb{R}\times\mathbb{Z}/\lambda\ ||\xi|\sim K,\ |\zeta_1|\in I_{N_1}, |\zeta-\zeta_1|\in I_{N_2},\\ |\tau-\sigma(\zeta)-\mathcal{H}(\xi_1,\xi-\xi_1,q_1,q-q_1)|\lesssim L_1\vee L_2\}.\end{aligned}$$

Let $\widetilde{B}_{\xi,q,\tau}(K,q_1) = \{\xi_1\in\mathbb{R} \mid \zeta_1\in B_{\xi,q,\tau}(K)\}$ for each q_1. Then combining Lemma 3.2.3 and the bound $|\partial\mathcal{H}/\partial\xi_1| \gtrsim \xi^2\sim K^2$, we get

$$|\widetilde{B}_{\xi,q,\tau}(K,q_1)| \lesssim \frac{L_1\vee L_2}{K^2}.$$

Hence, we obtain

$$|B_{\xi,q,\tau}| \lesssim \lambda\frac{(L_1\vee L_2)(N_1\wedge N_2)}{K^2}.$$

Then,

$$\|R_K((P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v))\|_{L^2_\lambda} \lesssim \frac{N_0^{1/2}(L_1\vee L_0)^{1/2}(L_1\wedge L_0)^{1/2}}{K}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}.$$

Combining the Cauchy-Schwarz inequality and bilinear estimate, we obtain that the contribution of this case to $J^{L_0,L_1,L_2}_{N_0,N_1,N_2}$ is bounded by

$$\begin{aligned}&\lesssim \sum_{K\lesssim N_0}\frac{1}{N^{1-s}N_0^{s-1/2}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda}\|R_KP_{N_0}Q_{L_0}w\|_{L^2_\lambda}\\ &\lesssim\frac{1}{N^{1-s}N_0^{s-1/2-}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda}\|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}.\end{aligned} \tag{3.17}$$

Again summing with respect to dyadic numbers $N_0, N_1, N_2, L_0, L_1, L_2$, we have the desired bound for the contribution of this case to $J_{HH\to H}$.

Next, we consider the case when $|\partial\mathcal{H}/\partial\xi_1| \ll \xi^2$. In this case, we have

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial\xi_1}(\xi_1, \xi-\xi_1, q_1, q-q_1) =& 3\xi(\xi-\xi_1) - 3\xi\xi_1 + q(q-q_1) - qq_1\\ =& 6\xi(\xi-\xi_1) - 3\xi^2 + q(q-q_1) - qq_1\\ =& O(\xi) - 3\xi^2 + q(q-q_1) - qq_1.\end{aligned}$$

To satisfy the hypothesis of resonance function $|\partial\mathcal{H}/\partial\xi_1| \ll \xi^2$, $N_0 \sim N_1 \sim N_2$ and $|\xi-\xi_1| \lesssim 1 \ll |\xi|$, we need $|q-q_1| \sim N_0$, $|\xi| \lesssim |q| \wedge |q_1|$ to set $q(q-q_1) - qq_1 \sim \xi^2$. Furthermore we have

$$|q| \sim |q_1| \sim |q-q_1| \sim |\xi| \sim N_0, \quad 3\xi\xi_1(\xi-\xi_1) \sim O(\xi^2), \quad (\xi-\xi_1)(q^2-(q-q_1)^2) \sim O(\xi^2).$$

Let us recall the resonance function $\mathcal{H}$ in (2.4)

$$\mathcal{H}(\xi_1, \xi-\xi_1, q_1, q-q_1) = 3\xi\xi_1(\xi-\xi_1) + (\xi-\xi_1)(q^2-(q-q_1)^2) + (q-q_1)\xi_1(q+q_1),$$

If $|\mathcal{H}(\xi_1, \xi-\xi_1, q_1, q-q_1)| \gtrsim \xi^2$, then $L_0 \vee L_1 \vee L_2 \gtrsim \xi^2$ by (2.4). Hence we have $|\xi| \lesssim N_0^{0+}(L_0 \vee L_1 \vee L_2)^{1/2-}$. The same estimate as in (3.13) implies that the contribution of this case to $J_{HH\to H}$ is bounded by

$$\begin{aligned}&\lesssim \frac{N_0^{0+}N_0^{1-s}}{N_0 N^{1-s} L_0^{0+} L_1^{0+} L_2^{0+}} \|(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}v)\|_{L^2_\lambda} \|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}\\ &\sim \frac{1}{N^{1-s}N_0^{s-}L_0^{0+}L_1^{0+}L_2^{0+}} \|P_{N_1}Q_{L_1}u\|_{L^2_\lambda} \|P_{N_2}Q_{L_2}v\|_{L^2_\lambda} \|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}.\end{aligned} \tag{3.18}$$

Summing in dyadic numbers $N_0, N_1, N_2, L_0, L_1, L_2$, we have the bound of this case to $J_{HH\to H}$ by

$$\lesssim \|u\|_{L^2_\lambda} \|v\|_{L^2_\lambda} \|w\|_{L^2_\lambda},$$

as desired.

On the other hand, if $|\mathcal{H}(\xi_1, \xi-\xi_1, q_1, q-q_1)| \ll \xi^2$, it follows that $|(q-q_1)\xi_1(q+q_1)| \lesssim O(\xi^2)$ and then $|q+q_1| \lesssim 1$. In this case, we use the form

$$q_1 = [q_1] + (q_1 - [q_1])$$

and then

$$q - q_1 = [q] + (q-[q]) - q_1 = [q] - [q_1] + (q - [q] - q_1 + [q_1]),$$

where $[a]$ denotes integer part of $a \in \mathbb{R}$. Note that $q_1 - [q_1],\ q - q_1 - [q-q_1] \in \mathbb{Z}/\lambda \cap [0,1)$. The restriction $|q+q_1| \lesssim 1$ implies $|2[q_1]+[q]| \lesssim 1$. By performing the same calculation as [20, Proof of Proposition 3.1] and [11, Proof of Theorem 2.1], we use the Cauchy-Schwarz inequality to have that the contribution of this case to $J_{HH\to H}$ is bounded by

$$\lesssim \sup_{(\zeta,\tau)\in\mathbb{R}\times\mathbb{Z}/\lambda\times\mathbb{R}} \mathcal{I}(\zeta,\tau)\|u\|_{L^2_\lambda}\|v\|_{L^2_\lambda}\|w\|_{L^2_\lambda},$$

where

$$\mathcal{I}(\zeta,\tau) = \frac{N_0^{1-s}}{\lambda^{1/2}L_0^{1/2-}N^{1-s}} \left(\sum_{|2[q_1]+[q]|\lesssim 1} \int_{|\xi-\xi_1|\lesssim 1} \frac{d\xi_1}{\langle \tau - \sigma(\zeta) - \mathcal{H}(\xi_1, \xi-\xi_1, q_1, q-q_1)\rangle^{1+}} \right)^{1/2},$$

where we assume $|\mathcal{H}(\xi_1, \xi-\xi_1, q_1, q-q_1)| \ll \xi^2$ in the integration of region. Then it suffices to show

$$\sup_{(\zeta,\tau)\in\mathbb{R}\times\mathbb{Z}/\lambda\times\mathbb{R}} \mathcal{I}(\zeta,\tau)^2 < \infty. \tag{3.19}$$

The resonance function is reformulated as

$$\mathcal{H} = 3\xi\xi_1(\xi - \xi_1) + (\xi - \xi_1)(q^2 - (q - q_1)^2) + (q - q_1)\xi_1(q + q_1),$$

which is equivalent to a quadratic equation in ξ_1,

$$3\xi\xi_1^2 - ((2q - q_1)q + 3\xi^2)\xi_1 - (q^2 - (q - q_1)^2)\xi + \mathcal{H} = 0.$$

The roots of the quadratic equation are the values of ξ_1 as $\xi_1^\pm$, where

$$\xi_1^\pm = \frac{(2q_1 - q)q_1 - 3\xi^2 \pm \sqrt{((2q_1 - q)q - 3\xi^2)^2 + 12\xi^2 (q^2 - (q - q_1)^2) - 12\xi\mathcal{H}}}{6\xi}.$$

Using the change of variables, we have

$$d\xi_1 = \pm\frac{d\mathcal{H}}{\sqrt{((2q_1 - q)q - 3\xi^2)^2 + 12\xi^2 (q^2 - (q - q_1)^2) - 12\xi\mathcal{H}}}.$$

We evaluate integrals using the change of variables as

$$\begin{aligned}
&\int_{|\xi-\xi_1|\lesssim 1} \frac{d\xi_1}{\langle \tau - \sigma(\zeta) - \mathcal{H}(\xi_1, \xi - \xi_1, q_1, q - q_1)\rangle^{1+}} \\
\lesssim &\int_{\mathbb{R}} \frac{d\mathcal{H}}{\langle \tau - \sigma(\zeta) - \mathcal{H}\rangle^{1+} \left|((2q_1 - q)q - 3\xi^2)^2 + 12\xi^2 (q^2 - (q - q_1)^2) - 12\xi\mathcal{H}\right|^{1/2}} \\
\lesssim &\frac{1}{|\xi|^{1/2} + \left|\xi(\tau - \sigma(\zeta)) + ((2q_1 - q)q - 3\xi^2)^2/12 + \xi^2 (q^2 - (q - q_1)^2)\right|^{1/2}}.
\end{aligned}$$

Then we have the bound

$$\int_{|\xi-\xi_1|\lesssim 1} \frac{d\xi_1}{\langle \tau - \sigma(\zeta) - \mathcal{H}(\xi_1, \xi - \xi_1, q_1, q - q_1)\rangle^{1+}} \lesssim \frac{1}{N_0^{1/2}}.$$

Therefore

$$\mathcal{I}(\zeta, \tau) \lesssim \frac{N_0^{1-s}}{\lambda^{1/2} L_0^{1/2-} N^{1-s}} \left(\sum_{|2[q_1]+[q]|\lesssim 1} \frac{1}{N_0^{1/2}}\right)^{1/2} \lesssim \frac{1}{L_0^{1/2-} N^{1-s} N_0^{s-3/4}} \lesssim 1, \tag{3.20}$$

which is acceptable for (3.19).

Subcase (iii): The proof follows from the same as one in [18] (also [20, Proposition 3.1]). By symmetry, we assume $||\zeta|^2 - |\zeta_1|^2| \gtrsim N_0^{6/5} L_0^{0+}$. Then

$$\left|\frac{\partial\mathcal{H}}{\partial\xi_1}(\xi_1, \xi - \xi_1, q_1, q - q_1)\right| = \left||\zeta|^2 - |\zeta_1|^2\right| \gtrsim N_0^{6/5} L_0^{0+},$$

which modifies the computation appeared in (3.7) for the proof of (3.5). Hence the contribution of this case to $J_{N_0,N_1,N_2}^{L_0,L_1,L_2}$ is bounded by

$$\begin{aligned}
&\lesssim \frac{N_0^{1-s}}{N^{1-s} L_0^{1/2-} L_1^{1/2+} L_2^{1/2+}} \frac{N_0^{1/2} (L_1 \vee L_2)^{1/2}}{N_0^{3/5} L_0^{0+}} \|P_{N_1} Q_{L_1} u\|_{L_\lambda^2} \|P_{N_1} Q_{L_2} v\|_{L_\lambda^2} \|P_{N_0} Q_{L_0} w\|_{L_\lambda^2} \\
&\lesssim \frac{1}{N^{1-s} N_0^{s-9/10} L_0^{0+} L_1^{0+} L_2^{0+}} \|P_{N_1} Q_{L_1} u\|_{L_\lambda^2} \|P_{N_1} Q_{L_2} v\|_{L_\lambda^2} \|P_{N_0} Q_{L_0} w\|_{L_\lambda^2},
\end{aligned} \tag{3.21}$$

which is acceptable after taking the sum in dyadic numbers $N_0, N_1, N_2, L_0, L_1, L_2$.

Subcase (iv): We can rewrite resonance function $\mathcal{H}$ as

$$\begin{aligned}\mathcal{H}(\xi_1,\xi-\xi_1,q_1,q-q_1) =& 3\xi\xi_1(\xi-\xi_1)+\xi_1 q^2-\xi q_1^2-2\xi_1 q q_1+2\xi q q_1\\ =& -3\xi\xi_1(\xi-\xi_1)+P(\xi,\xi_1,q,q_1),\end{aligned}$$

where

$$P(\xi,\xi_1,q,q_1)=6\xi\xi_1\xi_2+\xi_1 q^2-\xi q_1^2-2\xi_1 q q_1+2\xi q q_1.$$

By the same argument as in [18] (also [20, Proposition 3.1]), we have in this case

$$\begin{aligned}|\mathcal{H}(\xi_1,\xi-\xi_1,q_1,q-q_1)| &\gtrsim (|\xi|\vee|\xi_1|\vee|\xi-\xi_1|)^2(|\xi|\wedge|\xi_1|\wedge|\xi-\xi_1|)\\ &\gtrsim (|\xi|\vee|\xi_1|\vee|\xi-\xi_1|)(|\xi|\wedge|\xi_1|\wedge|\xi-\xi_1|)|\xi|\\ &\gg N_0^{6/5}L_0^{0+}|\xi|\gtrsim|\xi|^{11/5}L_0^{0+}.\end{aligned}$$

By symmetry we assume $L_0=L_0\vee L_1\vee L_2$, which implies $|\mathcal{H}(\xi_1,\xi-\xi_1,q_1,q-q_1)|\lesssim L_0$. Combining these estimates in this case, we obtain $|\xi|\lesssim L_0^{5/11-}$. We use (3.4) to have that the contribution of this case to $J^{L_0,L_1,L_2}_{N_0,N_1,N_2}$ is bounded by

$$\begin{aligned}&\lesssim \sum_{K\lesssim L_0^{1/2--}}\frac{K}{N^{1-s}N_0^sL_0^{1/2-}L_1^{1/2+}L_2^{1/2+}}\frac{N_0^{1/2}}{K^{1/4}}(L_1\wedge L_2)^{1/2}(L_1\vee L_2)^{1/2}\\ &\qquad \|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_1}Q_{L_2}v\|_{L^2_\lambda}\|R_KP_{N_0}Q_{L_0}w\|_{L^2_\lambda}\\ &\lesssim\frac{1}{N^{1-s}N_0^{s-9/10}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_1}Q_{L_2}v\|_{L^2_\lambda}\|R_KP_{N_0}Q_{L_0}w\|_{L^2_\lambda},\end{aligned}\tag{3.22}$$

which is acceptable after taking the sum in dyadic numbers N_0,N_1,N_2,L_0,L_1,L_2.

Subcase (v): Finally, we consider this case. We repeat the argument in [18] (also [20, Proposition 3.1]). Introducing dyadic numbers K_j, we have that the contribution this case to $J^{L_0,L_1,L_2}_{N_0,N_1,N_2}$ is bounded by

$$\lesssim\frac{1}{N^{1-s}N_0^sL_0^{1/2-}L_1^{1/2+}L_2^{1/2+}}\sum_{K_0,K_1,K_2}K_0\|R_{K_0}((R_{K_1}P_{N_1}Q_{L_1}u)(R_{K_2}P_{N_2}Q_{L_2}v))\|_{L^2_\lambda}\|R_{K_0}P_{N_0}Q_{L_0}w\|_{L^2_\lambda}.\tag{3.23}$$

In the proof of (3.4), we modify that in (3.9)

$$|B^K_{\xi,q,\tau}|\lesssim\lambda\frac{(K_1\wedge K_2)(L_1\vee L_2)}{K^{\frac12}},$$

which shows

$$\begin{aligned}&\|R_{K_0}((R_{K_1}P_{N_1}Q_{L_1}u)(R_{K_2}P_{N_2}Q_{L_2}v))\|_{L^2_\lambda}\\ \lesssim&\frac{(K_1\wedge K_2)^{1/2}(L_1\wedge L_2)^{1/2}(L_1\vee L_2)^{1/4}}{K_0^{1/4}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda}.\end{aligned}$$

Then the left hand-side of (3.23) is controlled by

$$\lesssim\sum_{K_0,K_1,K_2}\frac{K_0^{3/4}(K_1\wedge K_2)^{1/2}(L_1\wedge L_2)^{1/2}(L_1\vee L_2)^{1/4}}{N^{1-s}N_0^sL_0^{1/2-}L_1^{1/2+}L_2^{1/2+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda}\|P_{N_0}Q_{L_0}w\|_{L^2_\lambda},$$

where the sum is based on the region $(K_0\vee K_1\vee K_2)(K_0\wedge K_1\wedge K_2)\lesssim N_0^{6/5}(L_0\vee L_1\vee L_1)^{0+}$. Since $K_0^{3/4}(K_1\wedge K_2)^{1/2}\lesssim N_0^{9/10}(L_0\vee L_1\vee L_2)^{0+}$, we have that the contribution of this case to $J^{L_0,L_1,L_2}_{N_0,N_1,N_2}$ is bounded by

$$\lesssim\frac{1}{N^{1-s}N_0^{s-9/10}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}R_{K_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}R_{K_2}v\|_{L^2_\lambda}\|R_{K_0}P_{N_0}Q_{L_0}w\|_{L^2_\lambda},\tag{3.24}$$

which is acceptable.

We finish the proof of $J_{HH\to H}$, and hence Lemma 3.2.6. □

We are now in position to prove Proposition 3.2.1.

Proof of Proposition 3.2.1. The proof is a simple variant of one of Proposition 2.2.4. Define

$$\Psi(u^\lambda)(t) = \eta(t)e^{-t\partial_x\Delta}u_0^\lambda + \frac{1}{2}\eta(t)\int_0^t e^{-(t-t')\partial_x\Delta}\partial_x\left(u^\lambda(t')^2\right)\,dt'.$$

We show that the map Ψ defines a contraction in

$$Y_{\lambda,\delta} = \left\{u^\lambda \in I^{-1}X^{1,1/2+}_{\lambda,\delta} \mid \|Iu^\lambda\|_{X^{1,1/2+}_{\lambda,\delta}} \le 2C_0\|Iu_0^\lambda\|_{H^1_\lambda}\right\},$$

for (small) $\delta > 0$, where the norm on $Y_{\lambda,\delta}$ is induced by $\|Iu^\lambda\|_{X^{1,1/2+}_{\lambda,\delta}}$, In fact, from Lemmas 2.1.3, 2.1.4 and 2.1.5 and 3.2.6, it follows that

$$\begin{aligned}
\|I\Psi(u^\lambda)\|_{X^{1,1/2+}_{\lambda,\delta}} &\le C\|Iu_0^\lambda\|_{H^1_\lambda} + C\left\|\partial_x I(u^\lambda)^2\right\|_{X^{1,-1/2+\varepsilon}_{\lambda,\delta}}\\
&\le C\|Iu_0^\lambda\|_{H^1_\lambda} + C\delta^\varepsilon\left\|\partial_x I(u^\lambda)^2\right\|_{X^{1,-1/2+2\varepsilon}_{\lambda,\delta}}\\
&\le C\|Iu_0^\lambda\|_{H^1_\lambda} + C\delta^\varepsilon\left\|Iu^\lambda\right\|^2_{X^{1,-1/2+}_{\lambda,\delta}}\\
&\le 2C\|Iu_0^\lambda\|_{H^1_\lambda},
\end{aligned}$$

for $u^\lambda \in Y_{\lambda,\delta}$ and choosing small $\delta > 0$ which will depend on $\|Iu_0^\lambda\|_{H^s_\lambda}$. Similarly,

$$\left\|I\Psi(u^\lambda) - I\Psi(v^\lambda)\right\|_{X^{1,1/2+}_{\lambda,\delta}} \le \frac{1}{2}\left\|Iu^\lambda - Iv^\lambda\right\|_{X^{1,1/2+}_{\lambda,\delta}},$$

for $u^\lambda, v^\lambda \in Y_{\lambda,\delta}$ and small $\delta > 0$. Then the contraction mapping theorem tells us that there is a unique solution $u^\lambda = \Psi(u^\lambda) \in Y_{\lambda,\delta}$ to the Cauchy problem (3.2).

The persistence property $u^\lambda \in C([0,\delta] : H^s_\lambda)$ and the uniqueness in whole space $Iu^\lambda \in X^{1,1/2+}_{\lambda,\delta}$ follow in a similar way to the proof of Proposition 2.2.4, therefore we complete the proof. $\square$

Chapter 4

Global well-posedness in H^s

This chapter is concerns the global well-posedness in H^s for $s < 1$, Theorem 1.2.1, which is based on the joint paper [21] with Hideo Takaoka.

4.1 Modified energy

The conservation quantities associated with (3.2) are

$$E^\lambda[u^\lambda](t) = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}_\lambda} \left(|\nabla u^\lambda(x,y,t)|^2 - \frac{1}{3}u^\lambda(x,y,t)^3 \right) dxdy,$$

$$M^\lambda[u^\lambda](t) = \int_{\mathbb{R}\times\mathbb{T}_\lambda} u^\lambda(x,y,t)^2\, dxdy.$$

From $\mathcal{F}^\lambda u_0^\lambda(\xi,q) = \mathcal{F}u_0(\lambda\xi,\lambda q)$, it is easy to see

$$\left\| Iu_0^\lambda \right\|_{L^2_\lambda} \le \left\| u_0^\lambda \right\|_{L^2_\lambda} \le \frac{\|u_0\|_{L^2}}{\lambda} = o(1) \tag{4.1}$$

for $\lambda \gg 1$. Here the choice of the large parameter N will be made latter, but $\lambda > 1$ is chosen by

$$\frac{N^{1-s}}{\lambda^{1+s}} = o(1), \tag{4.2}$$

in which

$$\left\| \nabla Iu_0^\lambda \right\|_{L^2_\lambda} \lesssim \frac{N^{1-s}}{\lambda^{1+s}}\|u_0\|_{H^s} = o(1), \quad \|Iu_0^\lambda\|_{H^1_\lambda} = o(1). \tag{4.3}$$

Moreover, we have

$$\left\| u\left(\frac{t}{\lambda^3}\right) \right\|_{H^s} \lesssim \lambda^{1+s}\left\| Iu^\lambda(t) \right\|_{H^s_\lambda} \le \lambda^{1+s}\left\| Iu^\lambda(t) \right\|_{H^1_\lambda}. \tag{4.4}$$

Remark 4.1.1. By Gagliardo-Nirenberg inequality, the conservation law of L^2_λ-norm and (4.1), the solution $u^\lambda(t)$ satisfies

$$\left\| \nabla Iu^\lambda(t) \right\|_{L^2_\lambda}^2 - C_1\left\| u_0^\lambda \right\|_{L^2_\lambda}^4 \le C_2 E^\lambda[Iu^\lambda](t), \tag{4.5}$$

where constants C_1 and C_2 are independent of λ.

Let us introduce the modified energy for proving global well-posedness. Using the Fundamental Theorem of Calculus, we get

$$E^\lambda[Iu^\lambda](\delta) - E^\lambda[Iu^\lambda](0) = \int_0^\delta \frac{dE^\lambda[Iu^\lambda](t)}{dt}\, dt$$

for $\delta > 0$. We continue calculation of the integrand in the right hand. Then

$$
\begin{aligned}
\frac{dE^\lambda[Iu^\lambda]}{dt}(t) &= \int_{\mathbb{R}\times\mathbb{T}_\lambda} \left(\frac{d}{dt}\nabla Iu^\lambda \cdot \nabla Iu^\lambda - \frac{1}{2}(Iu^\lambda)^2 \frac{d}{dt} Iu^\lambda \right) dxdy \\
&= \int_{\mathbb{R}\times\mathbb{T}_\lambda} \left(-I\partial_t u^\lambda \Delta Iu^\lambda - \frac{1}{2} I\partial_t u^\lambda (Iu^\lambda)^2 \right) dxdy \\
&= \int_{\mathbb{R}\times\mathbb{T}_\lambda} \partial_x \left(I\Delta u^\lambda + \frac{1}{2} I(u^\lambda)^2 \right) \left(\frac{1}{2}(Iu^\lambda)^2 + \Delta Iu^\lambda \right) dxdy \\
&= -\frac{1}{2} \int_{\mathbb{R}\times\mathbb{T}_\lambda} \left(I\Delta u^\lambda \partial_x \left((Iu^\lambda)^2 - I(u^\lambda)^2 \right) + \frac{1}{2} I(u^\lambda)^2 \partial_x \left((Iu^\lambda)^2 - I(u^\lambda)^2 \right) \right) dxdy
\end{aligned}
\tag{4.6}
$$

where we use the equation in (3.2). Taking the integral on $[0, \delta]$, we have

$$
\begin{aligned}
E^\lambda[Iu^\lambda](\delta) - E^\lambda[Iu^\lambda](0) = &-\frac{1}{2} \int_0^\delta \int_{\mathbb{R}\times\mathbb{T}_\lambda} I\Delta u^\lambda \partial_x \left((Iu^\lambda)^2 - I(u^\lambda)^2 \right) dxdydt \\
&- \frac{1}{4} \int_0^\delta \int_{\mathbb{R}\times\mathbb{T}_\lambda} I(u^\lambda)^2 \partial_x \left((Iu^\lambda)^2 - I(u^\lambda)^2 \right) dxdydt
\end{aligned}
\tag{4.7}
$$

We will estimate two terms on the right hand of (4.7) with $\|Iu^\lambda\|_{X^{1,1/2+}_{\lambda,\delta}}$.

4.2 Extra smoothing effects

In this section, we prove multilinear estimates associated with the transition of energy in (4.7).

Lemma 4.2.1. *For $s > 9/10$,*

$$
\|\partial_x (IuIv - I(uv))\|_{X^{1,-1/2+}_\lambda} \lesssim N^{-1/10+} \|Iu\|_{X^{1,1/2+}_\lambda} \|Iv\|_{X^{1,1/2+}_\lambda}. \tag{4.8}
$$

Remark 4.2.2. Comparing to the estimate in Lemma 3.2.6, we have the small factor at the front of the right-hand, which corresponding to properties such as dispersion or the smoothing effects.

Proof. We recast the proof of Lemma 3.2.6 in the form of formula (4.8). Here we used the same symbol over as in the proof of Lemma 3.2.6. By Plancherel identity, it suffices to show that

$$
\begin{aligned}
J =& \frac{1}{\lambda^2} \sum_{q,q_1 \in \mathbb{Z}/\lambda} \int_{\mathbb{R}^4} \Gamma^{\xi_1,q_1,\tau_1}_{\xi,q,\tau} \frac{m(\zeta)}{m(\zeta_1)m(\zeta-\zeta_1)} \Xi(\zeta,\zeta_1,\zeta-\zeta_1) \\
& \qquad \widehat{u}^\lambda(\zeta_1,\tau_1) \widehat{v}^\lambda(\zeta-\zeta_1,\tau-\tau_1) \widehat{w}^\lambda(\zeta,\tau)\, d\xi d\xi_1 d\tau d\tau_1 \\
\lesssim& \|u\|_{L^2_\lambda} \|v\|_{L^2_\lambda} \|w\|_{L^2_\lambda},
\end{aligned}
\tag{4.9}
$$

for $u, v, w \in L^2_\lambda$ whose Fourier transform are nonnegative. In (4.9),

$$
\Gamma^{\xi_1,q_1,\tau_1}_{\xi,q,\tau} = \frac{|\xi|\langle\zeta\rangle}{\langle\zeta_1\rangle\langle\zeta-\zeta_1\rangle\langle\tau-\sigma(\zeta)\rangle^{1/2-}\langle\tau_1-\sigma(\zeta_1)\rangle^{1/2+}\langle\tau-\tau_1-\sigma(\zeta-\zeta_1)\rangle^{1/2+}},
$$

$$
\Xi(\zeta,\zeta_1,\zeta-\zeta_1) = \frac{N^{1/10-}}{m(\zeta)} \left| m(\zeta_1)m(\zeta-\zeta_1) - m(\zeta) \right|.
$$

Note the trivial bound

$$
\Xi(\zeta,\zeta_1,\zeta-\zeta_1) \lesssim \frac{N^{1/10-}}{m(\zeta)}. \tag{4.10}
$$

Using dyadic decomposition in a similar way to the proof of Lemma 3.2.6, we rewrite J as the following

$$
J = \sum_{\substack{N_0,N_1,N_2 \\ L_0,L_1,L_2}} J^{L_0,L_1,L_2}_{N_0,N_1,N_2},
$$

where

$$J^{L_0,L_1,L_2}_{N_0,N_1,N_2} = \frac{1}{\lambda^2} \sum_{q,q_1\in\mathbb{Z}/\lambda} \int_{\mathbb{R}^4} \Gamma^{\xi_1,q_1,\tau_1}_{\xi,q,\tau} \frac{m(\zeta)}{m(\zeta_1)m(\zeta-\zeta_1)} \Xi(\zeta,\zeta_1,\zeta-\zeta_1)$$
$$\widehat{P_{N_1}Q_{L_1}u}^{\lambda}(\zeta_1,\tau_1)\widehat{P_{N_2}Q_{L_2}v}^{\lambda}(\zeta-\zeta_1,\tau-\tau_1)\widehat{P_{N_0}Q_{L_0}w}^{\lambda}(\xi,q,\tau)\, d\xi d\xi_1 d\tau d\tau_1. \tag{4.11}$$

We repeat the same procedure as one of Lemma 3.2.6. Decompose again J to five parts in frequencies:

$$J_{LL\to L} = \sum_{\substack{N_1\vee N_2\vee N_0 \ll N\\ L_0,L_1,L_2}} J^{L_0,L_1,L_2}_{N_0,N_1,N_2},$$

$$J_{LH\to H} = \sum_{\substack{N_1\ll N_2\sim N_0\\ N_0\gtrsim N\\ L_0,L_1,L_2}} J^{L_0,L_1,L_2}_{N_0,N_1,N_2},$$

$$J_{HL\to H} = \sum_{\substack{N_2\ll N_1\sim N_0\\ N_0\gtrsim N\\ L_0,L_1,L_2}} J^{L_0,L_1,L_2}_{N_0,N_1,N_2},$$

$$J_{HH\to L} = \sum_{\substack{N_0\ll N_1\sim N_2\\ N_1\gtrsim N\\ L_0,L_1,L_2}} J^{L_0,L_1,L_2}_{N_0,N_1,N_2},$$

$$J_{HH\to H} = J - (J_{LL\to L} + J_{HL\to H} + J_{LH\to H} + J_{HH\to L}).$$

We estimate each of them by case analysis.

Estimate of $J_{LL\to L}$: In this region, we have $\Xi(\zeta,\zeta_1,\zeta-\zeta_1) = 0$, since by $m(\zeta) = m(\zeta_1) = m(\zeta-\zeta_1) = 1$. Hence $J_{LL\to L} = 0$.

Estimate of $J_{LH\to H}$: We split the case into two cases, when $N_1 \gtrsim N$ and when $N_1 \ll N$. When $N_1 \gtrsim N$, we use the bound (4.10). On the other hand, when $N_1 \ll N$, the mean value theorem gives

$$|m(\zeta_1)m(\zeta-\zeta_1) - m(\zeta)| = |m(\zeta-\zeta_1) - m(\zeta)| \lesssim m'(N_0)N_1 \sim \frac{m(N_0)N_1}{N_0}. \tag{4.12}$$

By making use of the estimates in (3.12) and (3.13), we easily have the desired bounds for the contribution of this case to $J_{LH\to H}$. Indeed, when $N_1 \ll N$, the computation in (3.12) with (4.12) leads the bound

$$\lesssim \frac{N_1}{N_0} \frac{N^{1/10-}}{N_1^{1/2} L_0^{1/2-} L_1^{0+} L_2^{0+}} \|P_{N_1}Q_{L_1}u\|_{L^2_\lambda} \|P_{N_2}Q_{L_2}v\|_{L^2_\lambda} \|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}$$
$$\lesssim \frac{1}{N_1^{0+} L_0^{1/2-} L_1^{0+} L_2^{0+}} \|P_{N_1}Q_{L_1}u\|_{L^2_\lambda} \|P_{N_2}Q_{L_2}v\|_{L^2_\lambda} \|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}.$$

While $N_1 \gtrsim N$, we have $|m(\zeta_1)m(\zeta-\zeta_1) - m(\zeta)| \lesssim m(N_0)$ and then $\Xi(\zeta,\zeta_1,\zeta-\zeta_1) \lesssim N^{1/10-}$. We estimate the contribution by

$$\lesssim N^{1/10-} \frac{1}{N_1^{s-1/2} L_0^{1/2-} L_1^{0+} L_2^{0+}} \|P_{N_1}Q_{L_1}u\|_{L^2_\lambda} \|P_{N_2}Q_{L_2}v\|_{L^2_\lambda} \|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}$$
$$\lesssim \frac{1}{N_1^{0+} L_0^{0+} L_1^{0+} L_2^{0+}} \|P_{N_1}Q_{L_1}u\|_{L^2_\lambda} \|P_{N_2}Q_{L_2}v\|_{L^2_\lambda} \|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}.$$

Then the desired estimate follows.

Estimate of $J_{HL\to H}$: We use symmetry arguments as for $J_{LH\to H}$ to conclude the proof.

Estimate of $J_{HH\to L}$: In this case, we use the bound (4.10). By employing the same estimates as in (3.14) and (3.15), we have the desired bounds. Actually, the computation in (3.14) will be changed into

$$\begin{aligned}
&\lesssim \frac{N^{1/10-}}{N^{1-s}N_1^{s-1/2-3\theta/2}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda}\\
&\lesssim \frac{1}{N_1^{2/5-3\theta/2}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda},
\end{aligned}$$

for small $\theta>0$, while the formula in (3.15) is changed by the same.

Estimate of $J_{HH\to H}$: In this case, we can assume $N \ll N_0 \sim N_1 \sim N_2$ and then

$$\Xi(\zeta,\zeta_1,\zeta-\zeta_1)\lesssim N^{1/10-}.$$

We divide the region into five regions as in the proof of Lemma 3.2.6. On each case, we recall the estimates in (3.16), (3.17), (3.18), (3.20), (3.21), (3.22) and (3.24), by multiplying $\Xi(\zeta,\zeta_1,\zeta-\zeta_1)$. We see that $J_{HH\to H}\lesssim \|u\|_{L^2_\lambda}\|v\|_{L^2_\lambda}\|w\|_{L^2_\lambda}$ holds provided $s>9/10$.

In fact, in subcase (i), the estimate corresponding to (3.16) will be

$$\begin{aligned}
&\lesssim N^{1/10-}\frac{2^{-3k/4}}{N^{1-s}N_0^{s-1/2}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda}\|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}\\
&\lesssim \frac{2^{-3k/4}}{N_0^{2/5-}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda}\|P_{N_0}Q_{L_0}w\|_{L^2_\lambda},
\end{aligned}$$

which is acceptable after summing over N_0,N_1,N_2,L_0,L_1,L_2.

In subcase (ii), we revisit the estimates in (3.17), (3.18) and (3.20). We require the following estimate corresponding to (3.17)

$$\begin{aligned}
&\lesssim N^{1/10-}\frac{1}{N^{1-s}N_0^{s-1/2-}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda}\|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}\\
&\lesssim \frac{1}{N_0^{2/5-}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda}\|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}.
\end{aligned}$$

The corresponding estimate to (3.18) is

$$\begin{aligned}
&\lesssim N^{1/10-}\frac{1}{N^{1-s}N_0^{s-}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda}\|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}\\
&\sim \frac{1}{N_0^{9/10-}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}v\|_{L^2_\lambda}\|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}.
\end{aligned}$$

The corresponding estimate to (3.20) is

$$\mathcal{I}(\zeta,\tau)\lesssim N^{1/10-}\frac{1}{L_0^{1/2-}N^{1-s}N_0^{s-3/4}}\lesssim 1.$$

By a similar proof to one of Lemma 3.2.6, we have a desired bound.

In subcase (iii), it suffices to show the following estimate corresponding to (3.21)

$$\begin{aligned}
&\lesssim N^{1/10-}\frac{N_0^{1-s}}{N^{1-s}L_0^{1/2-}L_1^{1/2+}L_2^{1/2+}}\frac{N_0^{1/2}(L_1\vee L_2)^{1/2}}{N_0^{3/5}L_0^{0+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_1}Q_{L_2}v\|_{L^2_\lambda}\|P_{N_0}Q_{L_0}w\|_{L^2_\lambda}\\
&\lesssim \frac{1}{N_0^{0+}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_1}Q_{L_2}v\|_{L^2_\lambda}\|P_{N_0}Q_{L_0}w\|_{L^2_\lambda},
\end{aligned}$$

which holds for $s>9/10$.

In subcase (iv), the corresponding estimate to (3.22) is

$$\begin{aligned}
&\lesssim N^{1/10-}\frac{1}{N^{1-s}N_0^{s-9/10}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_1}Q_{L_2}v\|_{L^2_\lambda}\|R_KP_{N_0}Q_{L_0}w\|_{L^2_\lambda}\\
&\lesssim\frac{1}{N_0^{0+}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}u\|_{L^2_\lambda}\|P_{N_1}Q_{L_2}v\|_{L^2_\lambda}\|R_KP_{N_0}Q_{L_0}w\|_{L^2_\lambda},
\end{aligned}$$

which is acceptable for $s>9/10$.

Finally, in subcase (v), we need the following estimate corresponding estimate to (3.24)

$$\begin{aligned}
&\lesssim N^{1/10-}\frac{1}{N^{1-s}N_0^{s-9/10}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}R_{K_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}R_{K_2}v\|_{L^2_\lambda}\|R_{K_0}P_{N_0}Q_{L_0}w\|_{L^2_\lambda}\\
&\lesssim\frac{1}{N_0^{0+}L_0^{0+}L_1^{0+}L_2^{0+}}\|P_{N_1}Q_{L_1}R_{K_1}u\|_{L^2_\lambda}\|P_{N_2}Q_{L_2}R_{K_2}v\|_{L^2_\lambda}\|R_{K_0}P_{N_0}Q_{L_0}w\|_{L^2_\lambda},
\end{aligned}$$

which is also acceptable for $s>9/10$.

Hence we complete the proof. □

By Lemma 4.2.1, we have the following lemma.

Lemma 4.2.3. *For $s>9/10$,*

$$\left|\int_0^\delta\int_{\mathbb{R}\times\mathbb{T}_\lambda} I\Delta u\partial_x\left((Iu)^2-I(u)^2\right)\,dxdydt\right|\lesssim\frac{1}{N^{1/10-}}\|Iu\|^3_{X^{1,1/2+}_{\lambda,\delta}}.$$

Proof. We apply the Parseval's identity and the Cauchy-Schwartz inequality to get the bound of the left-hand side by

$$\lesssim\|I\Delta u\|_{X^{-1,1/2+}_{\lambda,\delta}}\left\|\partial_x(Iu)^2-I(u)^2)\right\|_{X^{1,1/2-}_{\lambda,\delta}}\lesssim\|Iu\|_{X^{1,1/2+}_{\lambda,\delta}}\left\|\partial_x(Iu)^2-I(u)^2)\right\|_{X^{1,1/2-}_{\lambda,\delta}}.$$

Employing Lemma 4.2.1, we have the desired bound. □

Lemma 4.2.4. *For $s>31/40$,*

$$\left|\int_{\mathbb{R}\times\mathbb{T}_\lambda\times\mathbb{R}} Iu^2\partial_x\left((Iu)^2-I(u)^2\right)\,dxdydt\right|\lesssim\frac{1}{N^{1/10+}}\|Iu\|^4_{X^{1,1/2+}_\lambda}.$$

Proof. We apply Parseval's identity to the left-hand side and use the same argument as in the proof of Lemma 3.2.6.

It suffices to show that

$$J=\frac{1}{\lambda^3}\sum_*\int_*|\xi_3+\xi_4|\Gamma^{\tau_1,\tau_2,\tau_3,\tau_4}_{\zeta_1,\zeta_2,\zeta_3,\zeta_4}\Xi(\zeta_1,\zeta_2,\zeta_3,\zeta_4)\prod_{j=1}^4\widehat{u}^\lambda(\zeta_j,\tau_j)\lesssim\|u\|^4_{L^2_\lambda},\tag{4.13}$$

for $u\in L^2_\lambda$ whose Fourier transform is nonnegative. In (4.13), $\sum_*\int_*$ stands for the sum-integral of $(\zeta_j,\tau_j)\in\mathbb{R}\times\mathbb{Z}/\lambda\times\mathbb{R}$ $(1\le j\le 4)$ over $\xi_1+\xi_2+\xi_3+\xi_4=\eta_1+\eta_2+\eta_3+\eta_4=\tau_1+\tau_2+\tau_3+\tau_4=0$, and

$$\Gamma^{\tau_1,\tau_2,\tau_3,\tau_4}_{\zeta_1,\zeta_2,\zeta_3,\zeta_4}=\prod_{j=1}^4\frac{1}{m(\zeta_j)\langle\zeta_j\rangle\langle\tau_j-\sigma(\zeta_j)\rangle^{1/2+}},$$

$$\Xi(\zeta_1,\zeta_2,\zeta_3,\zeta_4)=m(\zeta_1+\zeta_2)\left|m(\zeta_3)m(\zeta_4)-m(\zeta_3+\zeta_4)\right|N^{1/10+}.$$

Note the trivial bound

$$\Xi(\zeta_1,\zeta_2,\zeta_3,\zeta_4)\lesssim N^{1/10+}.\tag{4.14}$$

By dyadic decomposition in a similar way to the proof of Lemma 3.2.6, we use $\langle \zeta_j \rangle \sim N_j$ and $\langle \tau_j - \sigma(\zeta_j)\rangle \sim L_j$, as before:

$$J = \sum_{\substack{N_1,N_2,N_3,N_4 \\ L_1,L_2,L_3,L_4}} J^{L_1,L_2,L_3,L_4}_{N_1,N_2,N_3,N_4},$$

where

$$J^{L_1,L_2,L_3,L_4}_{N_1,N_2,N_3,N_4} = \frac{1}{\lambda^3}\sum_{*}\int_{*} |\xi_3+\xi_4| \Gamma^{\tau_1,\tau_2,\tau_3,\tau_4}_{\zeta_1,\zeta_2,\zeta_3,\zeta_4} \Xi(\zeta_1,\zeta_2,\zeta_3,\zeta_4) \prod_{j=1}^{4} \widehat{P_{N_j}Q_{L_j}u}^{\lambda}(\zeta_j,\tau_j).$$

We will show (4.13) by the case-by-case analysis.

In case $N_3 \vee N_4 \ll N$, we have $\Xi(\zeta_1,\zeta_2,\zeta_3,\zeta_4) = 0$, which is canceled out when evaluating (4.13). Then we may assume $N_3 \vee N_4 \gtrsim N$ in (4.13).

Note that $|\xi_3+\xi_4| = |\xi_1+\xi_2| \lesssim (N_1 \vee N_2) \wedge (N_3 \vee N_4)$. We take $(P_{N_1}Q_{L_1}u)(P_{N_2}Q_{L_2}u)$ and $(P_{N_3}Q_{L_3}u)(P_{N_4}Q_{L_4}u)$ in L^2_λ by using (3.3), respectively. By (4.14), the contribution of this case to $J^{L_1,L_2,L_3,L_4}_{N_1,N_2,N_3,N_4}$ is bounded by

$$\begin{aligned}
&\lesssim \frac{(N_1\wedge N_2)(N_1\vee N_2)^{11/20}}{N_1 m(N_1) N_2 m(N_2) L_1^{0+}L_2^{0+}} \frac{(N_3\wedge N_4)(N_3\vee N_4)^{9/20}}{N_3 m(N_3) N_4 m(N_4) L_3^{0+}L_4^{0+}} N^{1/10+} \prod_{j=1}^{4} \|P_{N_j}Q_{L_j}u\|_{L^2_\lambda} \\
&\lesssim \prod_{j=1}^{4} \frac{1}{N_j^{9/40-} m(N_j) L_j^{0+}} \|P_{N_j}Q_{L_j}u\|_{L^2_\lambda} \\
&\lesssim \prod_{j=1}^{4} \frac{1}{N_j^{0+} L_j^{0+}} \|P_{N_j}Q_{L_j}u\|_{L^2_\lambda},
\end{aligned}$$

since the function $f(x) = x^{9/40-}m(x)$ is non-decreasing on $[1,\infty)$ provided $s > 31/40$. This is acceptable after taking the dyadic sum on N_j and L_j. ☐

The following lemma is not difficult to prove using Lemma 4.2.4 as in [5, 6].

Lemma 4.2.5. *For $s > 31/40 (< 29/31)$,*

$$\left| \int_0^\delta \int_{\mathbb{R}\times\mathbb{T}_\lambda} I(u)^2 \partial_x \left((Iu)^2 - I(u)^2 \right) dxdydt \right| \lesssim \frac{1}{N^{1/10+}} \|Iu\|^4_{X^{1,1/2+}_{\lambda,\delta}}.$$

4.3 Proof of main theorem

In this section, we will give the proof of Theorem 1.2.1, by the local well-posedness results of Proposition 3.2.1 in conjunction with Lemmas 4.2.3 and 4.2.5. The proof follows the strategy described in [5]. We prove the global well-posedness by the local well-posedness theory for solution Iu obtained in Proposition 3.2.1 and by the upper bound of increment of modified energy from (4.7).

Proof of Theorem 1.2.1. Consider the Cauchy problem (1.1) on an arbitrary time interval $[0,T]$. We rescale solutions by scaling (3.1) and consider the rescaled problem in (3.2) with initial data

$$u_0^\lambda(x,y) = \frac{1}{\lambda^2} u_0\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right).$$

The goal is to construct rescaled solution $u^\lambda(t)$ to the Cauchy problem (3.2) on the time interval $[0, \lambda^3 T]$.

By the local well-posedness result of Proposition 3.2.1 along with (4.3), our solution $u^\lambda(t)$ satisfies $\|Iu^\lambda\|_{X^{1,1/2+}_{\lambda,\delta}} \ll 1$. Applying Lemmas 4.2.3 and 4.2.5 to the computation of (4.7), we have

$$E^\lambda[Iu^\lambda](\delta) \le E^\lambda[Iu^\lambda](0) + \frac{o(1)}{N^{1/10-}} = o(1).$$

By using (4.1) and (4.5), we have

$$\left\| Iu^{\lambda}(\delta) \right\|_{H^1_\lambda} = o(1).$$

We can iterate this process up to $N^{1/10-}$ times before doubling $\|Iu^{\lambda}(\delta)\|_{H^1_\lambda}$. This process extends the local solution obtained by Proposition 3.2.1 to the time $O(N^{1/10-}\delta)$. We choose N such that by (4.2)

$$N^{1/10-}\delta > \lambda^3 T \sim N^{3(1-s)/(1+s)}T,$$

which may be done for $s > 29/31$, hence complete the claim of Theorem 1.2.1.

Furthermore, replacing $\lambda \sim N^{(1-s)/(1+s)}$ and $N \sim T^{10(1+s)/(31s-29)+}$ in (4.4), the growth of Sobolev norm of the solution $u(t)$ to the Cauchy problem (1.1) has a bound at least

$$\|u(T)\|_{H^s} \lesssim \lambda^{1+s} \left\| Iu^{\lambda}(\lambda^3 T) \right\|_{H^1_\lambda} \lesssim T^{10(1-s)/(31s-29)+}.$$

This finishes the proof. □

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