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A generalization of the μ-function and its relation to integrable systems
(μ函数の一般化と可積分系との関連)

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A generalization of the μ-function and its relation to integrable systems
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Abstract

The purpose of this paper is to understand thoroughly Zwegers' μ-function in terms of q-special functions. The μ-function is an interesting object from the viewpoint of special functions, for instance, it often appears as a solution of factorized q-difference equations. In this paper, we introduce a one parameter deformation of the μ-function as the image of q-Borel and q-Laplace transformations of a divergent solution for the q-Hermite-Weber equation. We present some properties and expressions of the generalized μ-function, furthermore we provide that the generalized μ-function gives a τ-function of certain q-Painlevé equation. This fact shows that the generalized μ-function is also an important function in discrete integrable systems.

Contents

1 Introduction

1.1 Basic notations and definitions of q-hypergeometric series

Let $\mathbb{Z}$ denote the set of integers and let $\mathbb{C}$ denote the set of complex numbers, $i := \sqrt{-1}$ be the imaginary unit, $\tau \in \mathbb{C}$ be a complex number $\mathrm{Im}(\tau) > 0$. We define the q-shift factorials for $0 < |q| < 1$ and $x \in \mathbb{C}$ as

$$(x)_\infty = (x;q)_\infty := \prod_{j=0}^{\infty}(1 - xq^j), \quad (x)_\nu = (x;q)_\nu := \frac{(x;q)_\infty}{(q^\nu x;q)_\infty} \quad (\nu \in \mathbb{C}),$$

and for $u \in \mathbb{C}$, we define Jacobi theta functions as

$$\theta_q(x) := \sum_{n\in\mathbb{Z}} x^n q^{\frac{n(n-1)}{2}}, \quad \vartheta(u,\tau) = \vartheta(u) := \sum_{\nu\in\mathbb{Z}+\frac{1}{2}} e^{2\pi i\nu\left(u+\frac{1}{2}\right)+\pi i\nu^2\tau}$$

Jacobi theta functions satisfy the following pseudo-periodicities:

$$\theta_q(xq^n) = x^{-n}q^{-\frac{n(n-1)}{2}}\theta_q(x) \tag{1.1}$$

$$\theta(u+n,\tau) = (-1)^n\vartheta(u,\tau), \quad \vartheta(u+n\tau,\tau) = e^{-\pi i n^2\tau - 2\pi i n\left(u+\frac{1}{2}\right)}\vartheta(u,\tau), \tag{1.2}$$

and are expressed as the following triple products (for example, [Mu]):

$$\theta_q(x) = (-x, -q/x, q; q)_\infty, \qquad \vartheta(u,\tau) = -i\zeta^{-\frac{1}{2}}\tilde{q}^{\frac{1}{8}}(\zeta, \tilde{q}/\zeta, \tilde{q}; \tilde{q})_\infty,$$

where $n \in \mathbb{Z}, \tilde{q} = e^{2\pi i\tau}, \zeta = e^{2\pi i u}$. For appropriate complex numbers $a_1, \ldots, a_r, b_1, \ldots, b_s, x$, we define the basic q-hypergeometric series as

$${}_r\phi_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q; x\right) := \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s, q; q)_n}\left((-1)^n q^{\frac{n(n-1)}{2}}\right)^{s-r+1} x^n, \tag{1.3}$$

and define the bilateral q-hypergeometric series as

$${}_r\psi_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q; x\right) := \sum_{n\in\mathbb{Z}} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n}\left((-1)^n q^{\frac{n(n-1)}{2}}\right)^{s-r} x^n, \tag{1.4}$$

where

$$(a_1, \ldots, a_r; q)_\nu := (a_1;q)_\nu \cdots (a_r;q)_\nu.$$

In particular, the function ${}_2\phi_1$ is called "Heine's q-hypergeometric series". It satisfies the following q-difference equation:

$$[(c - abxq)T_x^2 - (c + q - (a+b)xq)T_x + (1-x)q]f(x) = 0, \quad T_x f(x) := f(xq). \tag{1.5}$$

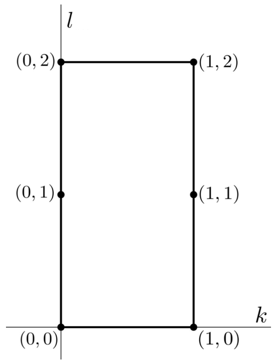
In general, the Newton-Puiseux diagram for the q-difference equation

$$[a_n(x)T_x^n + a_{n-1}(x)T_x^{n^1} + \cdots + a_0(x)]f(x) = 0 \tag{1.6}$$

is defined as the following convex hull:

$$\{(k,l) \in \mathbb{R}^2; \text{the coefficient of } x^k \text{ in } a_l(x) \text{ is not } 0 \text{ for } 1 \leq l \leq n\}.$$

For example, the Newton-Puiseux diagram for the q-difference equation (1.5) is as follows.



It is well known that Heine's q-hypergeometric series satisfies the following transformation formulas, called Heine's transformation formulas:

$$
\begin{aligned}
{}_2\phi_1\left(\begin{matrix}a, b \\ c\end{matrix}; q; x\right) &= \frac{(b, ax; q)_\infty}{(c, x; q)_\infty} {}_2\phi_1\left(\begin{matrix}c/b, x \\ ax\end{matrix}; q; b\right) && (1.7) \\
&= \frac{(c/b, bx; q)_\infty}{(c, x; q)_\infty} {}_2\phi_1\left(\begin{matrix}b, abx/c \\ c\end{matrix}; q; \frac{c}{b}\right) && (1.8) \\
&= \frac{(abx/c; q)_\infty}{(x; q)_\infty} {}_2\phi_1\left(\begin{matrix}c/b, c/a \\ c\end{matrix}; q; \frac{abx}{c}\right). && (1.9)
\end{aligned}
$$

And the function ${}_2\psi_2$ satisfies the following transformation formulas, called Bailey's transformation formulas:

$$
\begin{aligned}
{}_2\psi_2\left(\begin{matrix}a, b \\ c, d\end{matrix}; q; z\right) &= \frac{(az, c/a, d/b, qc/abz)_\infty}{(z, c, q/b, cd/abz)_\infty} {}_2\psi_2\left(\begin{matrix}a, abz/c \\ az, d\end{matrix}; q; \frac{c}{a}\right) && (1.10) \\
&= \frac{(bz, d/b, c/a, qd/abz)_\infty}{(z, d, q/a, cd/abz)_\infty} {}_2\psi_2\left(\begin{matrix}b, abz/d \\ bz, c\end{matrix}; q; \frac{d}{b}\right) && (1.11) \\
&= \frac{(az, d/a, c/b, qd/abz)_\infty}{(z, d, q/b, cd/abz)_\infty} {}_2\psi_2\left(\begin{matrix}a, abz/d \\ az, c\end{matrix}; q; \frac{d}{a}\right) && (1.12) \\
&= \frac{(bz, c/b, d/a, qc/abz)_\infty}{(z, c, q/a, cd/abz)_\infty} {}_2\psi_2\left(\begin{matrix}b, abz/c \\ bz, d\end{matrix}; q; \frac{c}{b}\right), && (1.13)
\end{aligned}
$$

which are bilateral versions of Heine's transformation formulas (for example, [GR]).

1.2 Mock thata functions and Zwegers' μ-function

Mock theta functions first appeared in Ramanujan's last letter to Hardy in 1920 [R1]. In this letter, Ramanujan told Hardy that he discovered a new class of functions which he called mock theta functions. Ramanujan introduced mock theta functions as functions having asymptotic behavior similar to theta functions at roots of unity but are not theta functions, and he gave 17 such examples. Some typical examples are as follows (see Appendix for a list of "classical" mock theta functions introduced in [R1], [R2], [Wa1], [GM1], [GM2], [BC], [Mc]):

$$
\psi(q) := \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q; q^2)_n}, \quad f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n}, \quad \phi(q) := \sum_{n=1}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q; q^2)_n}. \qquad (1.14)
$$

After Ramanujan's discovery, the functions have been actively studied in view of asymptotic behaviors at roots of unity, modular transformation laws, asymptotic expansions, etc (for example, see [A1], [D], [Wa1], [Wa2]).

In 1976, Andrews discovered a notebook written by Ramanujan. The notebook lists ten identities satisfied by mock theta functions, which have come to be known as the mock theta conjectures [R2]. Each of them involves a mock theta function, a theta function and specializations of the so-called universal mock theta functions. Definitions of universal mock theta functions are as follows:

$$g_2(x;q) := \frac{(-q)_\infty}{(q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{n(n+1)}}{1-xq^n}, \quad g_3(x;q) := \frac{1}{(q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{1-xq^n}.$$

For examples, the functions (1.14) are respectively expressed by using universal mock theta functions as follows:

$$\begin{aligned}\psi(q) &= qg_3(q;q^4),\\ f_0(q) &= -2q^2 g_3(q^2;q^10) + \frac{(q^5;q^5)_\infty(q^5;q^{10})_\infty}{(q,q^4;q^5)_\infty},\\ \phi(q) &= 2qg_2(q^2;q^5) + \frac{(-q^2,-q^3;q^5)_\infty(q^{10};q^{10})_\infty}{(q^2,q^8;q^{10})_\infty}.\end{aligned} \tag{1.15}$$

The mock theta conjectures were proved by Hickerson [H]. It is now known that all remaining mock theta functions are also expressed as a sum of universal mock theta functions and a theta function as equations (1.15) [GM3].

In 2002, the theory of mock theta functions took a new turn as a framework of modular forms with the work of Zwegers [Zw1]. One of Zwegers' achievement is the introduction of a special function called μ-function.

Definition 1.2.1 ([Zw1]). *For $u, v \in \Lambda_\tau := \{m + n\tau \in \mathbb{C}; m, n \in \mathbb{Z}\}$, define Zwegers' μ-function by*

$$\mu(u,v;\tau) = \mu(u,v) := \frac{e^{\pi iu}}{\vartheta(v)}\sum_{n\in\mathbb{Z}}\frac{(-1)^n e^{2\pi inv}e^{\pi in(n+1)\tau}}{1-e^{2\pi iu}e^{2\pi in\tau}}. \tag{1.16}$$

First, Zwegers showed that the μ-function satisfies a modular transformation property where Mordell integral appears as the non-homogeneous term. Furthermore, he showed that the μ-function satisfies a transformation law like Jacobi forms by adding an appropriate non-holomorphic function to the μ-function.

Proposition 1.2.2 ([Zw1]). *We have*

$$(1)\quad \mu(u,v;\tau) = e^{\frac{\pi i}{4}}\mu(u,v;\tau+1), \tag{1.17}$$

$$(2)\quad \mu(u,v;\tau) = -\frac{1}{\sqrt{-i\tau}}e^{\frac{\pi i(u-v)^2}{\tau}}\mu\left(\frac{u}{\tau},\frac{v}{\tau};-\frac{1}{\tau}\right) + \frac{1}{2i}h(u-v;\tau), \tag{1.18}$$

$$(1')\quad \tilde{\mu}(u,v;\tau) = e^{\frac{\pi i}{4}}\tilde{\mu}(u,v;\tau+1), \tag{1.19}$$

$$(2')\quad \tilde{\mu}(u,v;\tau) = \frac{i}{\sqrt{-i\tau}}e^{-\pi i\frac{(u-v)^2}{\tau}}\tilde{\mu}\left(\frac{u}{\tau},\frac{v}{\tau};-\frac{1}{\tau}\right), \tag{1.20}$$

where

$$h(u;\tau) := \int_{\mathbb{R}} \frac{e^{\pi i\tau x^2-2\pi ux}}{\cosh \pi x}dx, \tag{1.21}$$

$$E(x) := 2\int_0^x e^{-\pi z^2}dz, \tag{1.22}$$

$$R(u;\tau) := \sum_{\nu\in\mathbb{Z}+\frac{1}{2}} \left\{\mathrm{sgn}(\nu) - E((\nu+a)\sqrt{2t})\right\}(-1)^{\nu-\frac{1}{2}}e^{-\pi i\nu^2\tau-2\pi i\nu u}, \tag{1.23}$$

$$\tilde{\mu}(u,v;\tau) := \mu(u,v;\tau) + \frac{i}{2}R(u-v;\tau), \tag{1.24}$$

and $t = \mathrm{Im}(\tau), a = \frac{\mathrm{Im}(u)}{\mathrm{Im}(\tau)}$.

Since all Ramanujan's mock theta functions can be expressed by using the μ-function, they satisfy a classical modular transformation property by adding an appropriate non-holomorphic function to them. Zwegers' results ware later led to the interpretation that Ramanujan's mock theta functions are special elements of mock modular forms (see also [BO1], [Za]).

Further, Zwegers gave the following results.

Proposition 1.2.3 ([Zw1])**.** *The μ-function satisfies the following properties:*

$$(1)\quad \mu(u+1,v) = \mu(u,v+1) = -\mu(u,v) \tag{1.25}$$

$$(2)\quad \mu(u+\tau,v) = -e^{2\pi i(u-v)+\pi i\tau}\mu(u,v) - ie^{\pi i(u-v)+\frac{3\pi i\tau}{4}} \tag{1.26}$$

$$(3)\quad \mu(u+z,v+z) - \mu(u,v) = \frac{i\eta(\tau)^3\vartheta(z)\vartheta(u+v+z)}{\vartheta(u)\vartheta(v)\vartheta(u+z)\vartheta(v+z)}, \tag{1.27}$$

$$(4)\quad \mu(u,v) = \mu(u+\tau,v+\tau) \tag{1.28}$$

$$(5)\quad \mu(u,v) = \mu(-u,-v) \tag{1.29}$$

$$(6)\quad \mu(u,v) = \mu(v,u) \tag{1.30}$$

Kang gave representations of universal mock theta functions g_2, g_3 using the μ-function by a simple algebraic manipulation [Ka]:

$$ixg_2(x,q) = xq^{-\frac{1}{4}}\mu(2u,\tau;2\tau) + \frac{\eta(2\tau)^4}{\eta(\tau)^2\vartheta(2u,2\tau)}, \tag{1.31}$$

$$iq^{-\frac{1}{24}}x^{\frac{3}{2}}g_3(x,q) = \frac{\eta(3\tau)^3}{\eta(\tau)\vartheta(3u,3\tau)} + q^{-\frac{1}{6}}x\mu(3u,\tau,3\tau) + q^{-\frac{2}{3}}x^2\mu(3u,2\tau,3\tau), \tag{1.32}$$

so Zwegers' μ-function is regarded as an extension of universal mock theta functions.

The above is a summary of well-known information on mock theta functions. On the other hand, the μ-function is also a very interesting object from the viewpoint of q-hypergeometric series. For example, let us recall the well-known Kronecker formula [We]:

$$k(x,y) := \frac{1}{1-x}{}_1\psi_1\left(\begin{matrix}x\\qx\end{matrix};q;y\right) = \sum_{n\in\mathbb{Z}}\frac{y^n}{1-q^nx} = \frac{(q,q,xy,q/xy;q)_\infty}{(x,q/x,y,q/y;q)_\infty}. \tag{1.33}$$

The second equality is the case of $a = x$, $b = xq$, $z = y$ in Ramanujan's summation formula:

$${}_1\psi_1\left(\begin{matrix}a\\b\end{matrix};q;z\right) = \sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n = \frac{(az,q/az,q,b/a;q)_\infty}{(z,b/az,b,q/a;q)_\infty}.$$

Note that from the most right hand side of (1.33), one recognizes the symmetry $k(x,y) = k(y,x)$ explicitly.

Obviously, the μ-function is an analogue of the Kronecker summation. Hence the symmetric property $\mu(u,v) = \mu(v,u)$ holds in a similar way as $k(x,y) = k(y,x)$. In fact, putting $x = e^{2\pi iu}$, $y = e^{2\pi iv}$ and $q = e^{2\pi i\tau}$, the μ-function has the following expressions:

$$\mu(u,v) = -\frac{iq^{-\frac{1}{8}}\sqrt{xy}}{(q,x,q/x;q)_\infty}\frac{1}{1-y}{}_1\psi_2\left(\begin{matrix} y \\ 0, yq \end{matrix}; q; xq\right) \tag{1.34}$$

$$= -\frac{iq^{-\frac{1}{8}}\sqrt{xy}}{(x,y;q)_\infty}{}_2\psi_2\left(\begin{matrix} x, y \\ 0, 0 \end{matrix}; q; q\right) \tag{1.35}$$

$$= -\frac{iq^{-\frac{1}{8}}\sqrt{xy}}{(q/x,q/y;q)_\infty}\frac{1}{(1-x)(1-y)}{}_0\psi_2\left(\begin{matrix} - \\ xq, yq \end{matrix}; q; xyq\right) \tag{1.36}$$

These expressions follow from some Bailey transformations (1.10)-(1.13). In fact, by taking the limit $z \to z/b$, $b \to \infty$ and $c \to 0$ in the equations (1.10)-(1.13), we derive

$${}_1\psi_2\left(\begin{matrix} a \\ 0, d \end{matrix}; q, z\right) = \frac{(z, dq/az;q)_\infty}{(d,q/a;q)_\infty}{}_1\psi_2\left(\begin{matrix} az/d \\ 0, z \end{matrix}; q; d\right), \tag{1.37}$$

$$= \frac{(d/a, dq/az;q)_\infty}{(d;q)_\infty}{}_2\psi_2\left(\begin{matrix} a, az/d \\ 0, 0 \end{matrix}; q; \frac{d}{a}\right), \tag{1.38}$$

$$= \frac{(z, d/a;q)_\infty}{(q/a;q)_\infty}{}_0\psi_2\left(\begin{matrix} - \\ z, d \end{matrix}; q; az\right), \tag{1.39}$$

and putting $a = x$, $d = qx$, $z = qy$ in the equations (1.37), (1.38), (1.39), we obtain the equations (1.34)-(1.36). In particular, the expression (1.36) was pointed out by Choi ([C, Theorem 4] and [BT, Theorem 5.1]).

Thus the μ-function is also an interesting object from the viewpoint of q-special functions. The purpose of this paper is to understand the μ-function thoroughly by considering it in view of q-special functions.

1.3 Contents of this paper

Contents of this paper is as follows. In section 2, we study the μ-function in view of q-difference equations, and introduce a function $\mu(u,v;\alpha)$ which is a generalization of the μ-function. In section 3, we present some relations between the function $\mu(u,v;\alpha)$ and the q-hypergeometric series. In section 4, we consider q-difference equations satisfied by the universal mock theta functions, and provide another proof of the equations (1.31) and (1.32). In section 5, we present a relation between the function $\mu(u,v;\alpha)$ and q-Painlevé equations.

In the following, we describe the contents of each section in more detail.

In Section 2, we consider a q-difference equation satisfied by the μ-function. The μ-function is an important pioneer function in the study of mock modular forms, but it seems that there are many problems to be clarified further. For example, why the μ-function is a two-variable function, whereas the universal mock theta functions $g_2(x,q)$ and $g_3(x,q)$ are a one-variable function (i.e. what is the origin of the extra variable of the μ-function?). Also, why the factors $e^{\pi iu}$ in the numerator and $\vartheta(v)$ in the denominator are needed. Further, where the translation formula (1.27) comes from. Some explanation has been given for these questions. For example, Zwegers gives constructions of mock theta functions as indefinite theta series, of which μ-function is a special case, and these naturally have theta functions in denominators which are explainable from the geometry of the quadratic forms. Also Zwegers' two-variable $\tilde{\mu}$-function is decomposed into a sum of meromorphic Jacobi form and one-variable Maaß-Jacobi form [Rau] (see also [BFOR, Theorem 8.18]). However, as we will show in section 2, one can give another answer from the view of q-special functions. First, by using the pseudo-periodicity of the μ-function (1.26) and a bit of calculation, we obtain the following second-order difference equation:

$$\mu(u+v+2\tau,v)-(1-e^{2\pi i(u+\tau)})e^{\pi i\tau}\mu(u+v+\tau,v)-e^{2\pi i(u+\tau)}\mu(u+v,v)=0. \tag{1.40}$$

By considering the equation (1.40), we naturally obtain the μ-function, and clarify the above questions (Theorem 2.2.5). Furthermore, we obtain a function $\mu(u,v;\alpha)$ which is a natural extension of the μ-function by considering the following q-difference equation:

$$[T_x^2-(1-xq)\sqrt{a}T_x-xq]f(x)=0. \tag{1.41}$$

Definition 1.3.1. *Let α be a complex parameter such that $u-\alpha\tau, v\notin\Lambda_\tau$. We define the function $\mu(u,v;\alpha)$ as the following infinite series:*

$$\mu(u,v;\alpha;\tau)=\mu(u,v;\alpha):=\frac{e^{\pi i\alpha(u-v)}}{\vartheta(v)}\sum_{n\in\mathbb{Z}}(-1)^n e^{2\pi i(n+\frac{1}{2})v}q^{\frac{n(n+1)}{2}}\prod_{j=1}^{\infty}\frac{1-e^{2\pi iu}q^{n+j}}{1-e^{2\pi iu}q^{n-\alpha+j}} \tag{1.42}$$

$$=iq^{-\frac{1}{8}}\frac{e^{\pi i\alpha(u-v)}}{\theta_q(-y)}\frac{(xq;q)_\infty}{(xq/a;q)_\infty}{}_1\psi_2\left(\begin{matrix}xq/a\\0,xq\end{matrix};q;yq\right), \tag{1.43}$$

where $q=e^{2\pi i\tau}$, $x=e^{2\pi iu}$, $y=e^{2\pi iv}$ and $a=e^{2\pi i\alpha}$.

In particular, note that

$$\mu(u,v;1)=\mu(u,v),$$

and

$$\mu(u,v;0)=-iq^{-\frac{1}{8}},\quad \mu(u,v;-1)=-2iq^{-\frac{1}{8}}\cos\pi(u-v). \tag{1.44}$$

The function $\mu(u,v;\alpha)$ satisfies many interesting properties (for example, an analogue of Proposition 1.2.3, a difference equation with respect to the variable α, non-trivial expressions, etc). At the end of this section, we present the properties satisfies the function $\mu(u,v;\alpha)$ (Theorem 2.3.1, Theorem 2.4.2, Theorem 2.4.3, Theorem 2.4.5 and Theorem 2.4.6).

In Section 3, we state the relation between the function $\mu(u,v;\alpha)$ (or the μ-function) and the q-hypergeometric series. First, we present the relation to the continuous q-Hermite polynomial

$$H_k(\cos\theta;q):=\sum_{l=0}^{k}\frac{(q)_k}{(q)_l(q)_{k-l}}e^{i(2l-k)\theta} \tag{1.45}$$

$$=e^{-ik\theta}{}_2\phi_0\left(\begin{matrix}q^{-k},0\\-\end{matrix};q;e^{2i\theta}q^k\right). \tag{1.46}$$

which is a special case of the q-hypergeometric series (1.3). The continuous q-Hermite polynomial $H_k(z;q)$ satisfies the following relations ([KLS, p.541]):

$$\begin{aligned}&2zH_k(z;q)=H_{k+1}(z;q)+(1-q^k)H_{k-1}(z;q),\\&H_0(z;q)=1,\quad H_1(z;q)=2z.\end{aligned} \tag{1.47}$$

If $k \in \mathbb{Z}_{<0}$, the sum defining the continuous q-Hermite polynomial is a divergent series. Also if $k = 0$, since $(1 - q^k) = 0$, we cannot define the function $H_{-1}(z; q)$ in terms of the relations (1.47). Therefore, in general, the continuous q-Hermite polynomial is defined only on $k \in \mathbb{Z}_{\geq 0}$. But for $k \in \mathbb{Z}_{\geq 0}$, the function $\mu(u, v; -k)$ is closely related to the continuous q-Hermite polynomial, so the function $\mu(u, v; \alpha)$ gives a continuous parameter extension of the continuous q-Hermite polynomial. In particular, the original μ-function is regarded as a continuous q-Hermite polynomial of "-1 degree" (Theorem 3.1.1).

Next, we present some relations to the bilateral q-hypergeometric series (1.4). From the equation (1.43), we expect that the function $\mu(u, v; \alpha)$ is related to the bilateral q-hypergeometric series and the rich transformation formulas of the bilateral q-hypergeometric series give us a mysterious relation of the function $\mu(u, v; \alpha)$. Here, we present some transformation formulas and non-trivial expressions of the function $\mu(u, v; \alpha)$ using Bailey's transformation formulas and Slater's transformation formulas (Proposition 3.2.1, Theorem 3.2.2, and Theorem 3.2.4).

In Section 4, we consider q-difference equations satisfied by the universal mock theta functions g_2 and g_3. Universal mock theta functions g_2 and g_3 satisfy the following relations:

$$g_2(xq, q) + x^2 g_2(x, q) = -x \tag{1.48}$$
$$(T_x - q)(T_x + x^2) g_2(x, q) = 0 \tag{1.49}$$
$$g_3(xq, q) + x^3 g_3(x, q) = -x(1 + x) \tag{1.50}$$
$$(T_x - q^2)(T_x - q)(T_x + x^3) g_3(x, q) = 0 \tag{1.51}$$

Putting $x = e^{\pi i \left(u + \frac{\tau}{2}\right)}$, $q = e^{\pi i \tau}$ in the equation (1.49), we obtain a difference equation that coincides with the equation (1.40). Similarly for equation (1.51), we obtain difference equations that coincide with the equation (1.40), to apply appropriate transformations in the equations $(T_x - q^2)(T_x + x^3)$ and $(T_x - q)(T_x + x^3)$. Thus, the μ-function appears as a solution of factorized q-difference equations which have some solutions of theta functions such as (1.49) and (1.51). Here, we present fundamental solutions of a factorized q-difference equations which have theta function solutions (Theorem 4.2.2).

In Section 5, we state the relation between the function $\mu(u, v; \alpha)$ and discrete integrable systems. The following simultaneous q-difference equation for unknown functions f_0, f_1, f_2 is called the q-P_{IV} equation:

$$\begin{gathered} T(f_j) = a_j a_{j+1} f_{j+1} \frac{1 + a_{j+2} f_{j+2} + a_{j+2} a_j f_{j+2} f_j}{1 + a_j f_j + a_j a_{j+1} f_j f_{j+1}}, \quad (j \in \mathbb{Z}/3\mathbb{Z}) \\ a_0 a_1 a_2 = q, \quad f_0 f_1 f_2 = q x^2 \quad T : (a_0, a_1, a_2, x) \mapsto (a_0, a_1, a_2, qx). \end{gathered} \tag{1.52}$$

If $a_2 = 1, a_1 = -1, f_2 = -1, f_1 = \frac{g(xq)}{g(x)}, q = e^{\pi i \tau}, x = e^{\pi i u}$, the function $g(e^{\pi i u})$ satisfies a difference equation that coincides with the equation (1.40). Thus, we expect that mock Jacobi forms are related to discrete integrable systems. Here, we present that the function $\mu(u, v; \alpha)$ introduced in section 2 is a τ-function of the type $(A_2 + A_1)^{(1)}$ q-Painlevé equation (Theorem 5.1.1).

2 Generalization of the μ-function

In this section, we put $x = e^{2\pi i u}$, $y = e^{2\pi i v}$ and $q = e^{2\pi i \tau}$.

2.1 A q-difference equation satisfied by the Zwegers' μ-function

First, we provide a factorized q-difference equation satisfied by the μ-function.

Proposition 2.1.1. *The μ-function defined by the equation* (1.16) *satisfies the following factorized q-difference equation:*

$$(T_x - q^{\frac{1}{2}})(T_x + xq^{\frac{1}{2}})f(x) = \left[T_x^2 - (1-xq)\, q^{\frac{1}{2}} T_x - xq\right] f(x) = 0. \tag{2.1}$$

Proof. We rewrite the pseudo-periodicity of the μ-function (1.26) multiplicatively, we have

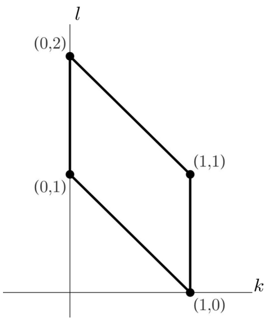
$$(T_x + xq^{\frac{1}{2}})\mu(u+v, v) = -ix^{\frac{1}{2}}q^{\frac{3}{8}}.$$

Since $(T_x - q^{\frac{1}{2}})x^{\frac{1}{2}} = 0$, the μ-function satisfies the q-difference equation (2.1). □

This q-difference equation (2.1) coincides with the specialization $a = q$ of the equation (1.41). The q-difference equation (1.41) coincides with transformations $a \mapsto q/a$, $u(xq) = \theta_q\left(-\frac{\sqrt{a}}{xq}\right) f(x)$ in the q-Hermite-Weber equation:

$$[axT_x^2 + (1-x)T_x - 1]u(x) = 0. \tag{2.2}$$

The Newton-Puiseux diagram for the q-difference equations (2.1) and (1.41) is as follows.



The equation (1.41)

$$[T_x^2 - (1-xq)\sqrt{a}T_x - xq]f(x) = 0,$$

which we also call the "q-Hermite-Weber equation", is a typical example of the second-order q-difference equation of the Laplace type:

$$[(a_0 + b_0x)T_x^2 + (a_1 + b_1x)T_x + (a_2 + b_2x)]u(x) = 0. \tag{2.3}$$

The q-difference equations of the Laplace type have long been studied. For example, the q-difference equation satisfied by Heine's q-hypergeometric series (1.5), which is the most popular and master class in the following hierarchy of the second order q-difference equations, was discovered in the 19th century. The q-Hermite-Weber equation (1.41) is one of some equations in the hierarchy obtained by taking some limits of q-difference equation (1.5) (for example, see [O1, Figure 2]).

However, a systematic study of the global analysis of degenerate Laplace types had to wait for Ramis J. P., Sauloy J. and Zhang C.'s work [RSZ]. They introduced the q-Borel and q-Laplace transformations.

Definition 2.1.2. *We define the transformations $\mathcal{B}^+$ and $\mathcal{L}^+$ as follows:*

$$\mathcal{B}^+(f)(\xi) := \sum_{n\geq 0} a_n q^{\frac{n(n-1)}{2}} \xi^n, \ \mathcal{L}^+(f)(x,\lambda) := \sum_{n\in\mathbb{Z}} \frac{f(\lambda q^n)}{\theta_q(x^{-1}\lambda q^n)}, \tag{2.4}$$

weher $f(x) = \sum_{n\geq 0} a_n x^n \in \mathbb{C}[\![x]\!]$.

By a simple calculation, it follows that

$$\mathcal{B}^+(x^m T_x^n x^k) = x^m q^{\frac{m(m-1)}{2}} T_x^{n+m} \mathcal{B}^+(x^k), \quad \mathcal{L}^+(x^m T_x^n x^k) = x^m q^{-\frac{m(m-1)}{2}} T_x^{n-m} \mathcal{L}^+(x^k),$$

and consequently

$$\mathcal{B}^+ \circ \mathcal{L}^+(x^m T_x^n x^k) = x^m T_x^n \mathcal{B}^+ \circ \mathcal{L}^+(x^k).$$

In particular, Zhang C. [Zh1] obtained some connection formulas for the q-convergent hypergeometric series ${}_2\phi_0(a,b;q,x)$ by applying these transformations. By considering the degeneration of Zhang's result, Zhang C. and Ohyama Y. [O2] gave some connection formulas for the q-Hermite series ${}_1\phi_1(a;0;q,x)$.

If we calculate formal solutions of the q-difference equation (2.1) satisfied by the μ-function in a careless manner, divergent solutions often appear. Actually, the solution constructed by the image of the composition of the q-Borel transformation and q-Laplace transformation of the divergent solution is Zwegers' μ-function (S. Garoufalidis and C. Wheeler [GW] pointed out this fact independently of us). In view of the story discussed above, we introduce the function $\mu(u,v;\alpha)$ (see Definition 1.3.1) which is an extension of the μ-function as an image of the composition of the q-Borel transformation and q-Laplace transformation of a divergent solution of the q-Hermite-Weber equation (1.41) (see also Theorem 2.2.5).

2.2 Derivation of the functions $\mu(u,v;\alpha)$

First we obtain formal solutions of the q-Hermite-Weber equation (1.41). Since the equation (1.41) satisfies the following Lemma, it is sufficient to obtain formal solutions around $x=0$.

Lemma 2.2.1. *If a function $f(x)$ is a solution of the q-difference equation* (1.41)*, then $f(x^{-1})$ is also a solution of the q-difference equation* (1.41).

Proof. By the assumption, we have

$$f(xq^2) - (1-xq)\sqrt{a} f(xq) - xq f(x) = 0.$$

Putting $x \mapsto \frac{1}{xq^2}$, we obtain

$$\begin{aligned} 0 &= f\left(\frac{1}{x}\right) - \left(1 - \frac{1}{xq}\right)\sqrt{a} f\left(\frac{1}{xq}\right) - \frac{1}{xq} f\left(\frac{1}{xq^2}\right) \\ &= -\frac{1}{xq}[T_x^2 - (1-xq)\sqrt{a}T_x - xq] f(x^{-1}), \end{aligned}$$

so the function $f(x^{-1})$ is also a solution of the q-difference equation (1.41). $\square$

From a simple calculation, the formal series $x^{\frac{\alpha}{2}}{}_2\phi_0\left(\begin{matrix}a,0\\ -\end{matrix};q;\frac{x}{a}\right)$ is a formal solution of the q-difference equation (1.41). Furthermore putting $f(x)=\frac{1}{\theta_q(-x)}g(x)$ in the equation (1.41), the function $g(x)$ satisfies the following q-difference equation:

$$\left[xT_x^2+(1-xq)\frac{\sqrt{a}}{q}T_x-1\right]g(x)=0. \tag{2.5}$$

From a simple calculation, we obtain a solution $x^{1-\frac{\alpha}{2}}{}_1\phi_1\left(\begin{matrix}q/a\\ 0\end{matrix};q;xq\right)$ of the q-difference equation (2.5). Some connection problems of the q-difference equation (2.5) ware given by unpublished results by Ohyama and Zhang. Here, we prove them according to [O2].

Lemma 2.2.2 ([O2])**.** *A fundamental solution of the q-difference equation* (1.41) *around* $x=0$ *is*

$$f_0(a,x,\lambda):=x^{\frac{\alpha}{2}}\mathcal{L}^+\circ\mathcal{B}^+(\tilde{f}_0)(a,x,\lambda),\quad g_0(a,x):=\frac{x^{1-\frac{\alpha}{2}}}{\theta_q(-x)}{}_1\phi_1\left(\begin{matrix}q/a\\ 0\end{matrix};q;xq\right),$$

where $\tilde{f}_0$ *is a formal solution around* $x=0$*:*

$$\tilde{f}_0(a,x):={}_2\phi_0\left(\begin{matrix}a,0\\ -\end{matrix};q;\frac{x}{a}\right).$$

And that of around $x=\infty$ *is*

$$f_\infty(a,x,\lambda):=f_0(a,x^{-1},\lambda),\quad g_\infty(a,x):=g_0(a,x^{-1}).$$

In this case, the connection formals for $f_0(a,x,\lambda)$*,* $f_\infty(x,\lambda)$*, and* $g_0(a,x)$*,* $g_\infty(a,x)$ *are as follows:*

$$\begin{pmatrix}f_0(a,x,\lambda)\\ f_\infty(a,x,\lambda)\end{pmatrix}=-\frac{(q)_\infty}{(q/a)_\infty}\begin{pmatrix}\frac{\theta_q(\lambda)\theta_q(ax/\lambda)x^\alpha}{\theta_q(\lambda/a)\theta_q(x/\lambda)a} & 1\\ 1 & \frac{\theta_q(\lambda)\theta_q(x\lambda/a)}{\theta_q(\lambda/a)\theta_q(x\lambda)}x^{-\alpha}\end{pmatrix}\begin{pmatrix}g_0(a,x)\\ g_\infty(a,x)\end{pmatrix}. \tag{2.6}$$

To prove the Lemma 2.2.2, we use the following Lemma.

Lemma 2.2.3 ([Zh1] p.18, Theorem 2.2.1)**.** *The q-difference equation*

$$[(1-abxq)T_x^2-(1-(a+b)xq)T_x-xq]f(x)=0 \tag{2.7}$$

has the following fundamental solution around $x=0$

$$\mathcal{L}^+\circ\mathcal{B}^+(F_1)(a,b,x,\lambda),\quad F_2(a,b,x)=\frac{(abx)_\infty}{\theta_q(-xq)}{}_2\phi_1\left(\begin{matrix}q/a,q/b\\ 0\end{matrix};q,abx\right),$$

and around $x=\infty$

$$G_1(a,b,x)=\frac{\theta_q(-axq)}{\theta_q(-xq)}{}_2\phi_1\left(\begin{matrix}a,0\\ aq/b\end{matrix};q,\frac{q}{abx}\right),\quad G_2(a,b,x)=\frac{\theta_q(-bxq)}{\theta_q(-xq)}{}_2\phi_1\left(\begin{matrix}b,0\\ bq/a\end{matrix};q,\frac{q}{abx}\right),$$

where the series F_1 *is a formal series around* $x=0$*,*

$$F_1=F_1(a,b,x):={}_2\phi_0\left(\begin{matrix}a,b\\ -\end{matrix}q,x\right).$$

Then, the connection formulas for $\mathcal{L}^+\circ\mathcal{B}^+(F_1)(a,b,x,\lambda)$, $F_2(a,b,x)$, and $G_1(a,b,x)$, $G_2(a,b,x)$ are as follows:

$$\begin{aligned}\mathcal{L}^+\circ\mathcal{B}^+(F_1)(a,b,x,\lambda)=&\frac{(b)_\infty\theta_q(a\lambda)\theta_q(axq/\lambda)\theta_q(-xq)}{(b/a)_\infty\theta_q(\lambda,xq/\lambda,-axq)}G_1(a,b,x)\\&+\frac{(a)_\infty\theta_q(b\lambda)\theta_q(bxq/\lambda)\theta_q(-xq)}{(a/b)_\infty\theta_q(\lambda)\theta_q(xq/\lambda)\theta_q(-bxq)}G_2(a,b,x),\end{aligned}\tag{2.8}$$

$$F_2(a,b,x)=\frac{(q/a)_\infty}{(b/a,q)_\infty}G_1(a,b,x)+\frac{(q/b)_\infty}{(a/b,q)_\infty}G_2(a,b,x).\tag{2.9}$$

Proof of Lemma 2.2.2. It is sufficient to prove the following equation

$$f_0(a,x,\lambda)=-\frac{(q)_\infty}{(q/a)_\infty}\frac{\theta_q(\lambda)\theta_q(ax/\lambda)x^\alpha}{\theta_q(\lambda/a)\theta_q(x/\lambda)a}g_0(a,x)-\frac{(q)_\infty}{(q/a)_\infty}g_\infty(a,x),\tag{2.10}$$

since the q-difference equation (1.41) is symmetric under $x\mapsto\frac{1}{x}$. We take the limit of the Heine's transformation formula (1.9) as $a\to 0$, then we have

$${}_2\phi_1\left(\begin{matrix}0,b\\c\end{matrix};q;x\right)=\frac{1}{(x)_\infty}{}_1\phi_1\left(\begin{matrix}q/b\\c\end{matrix};q;bx\right).$$

Hence, we can rewrite the solution $G_2(a,b,x)$ in Lemma 2.2.3 as

$$G_2(a,b,x)=(q,abx)_\infty\frac{\theta_q(-bxq)}{\theta_q(-abx)\theta_q(-xq)}{}_1\phi_1\left(\begin{matrix}q/a\\bq/a\end{matrix};q;\frac{q}{ax}\right).$$

Therefore the connection formula (2.8) is given by

$$\mathcal{L}^+\circ\mathcal{B}^+(F_1)\left(a,b,\frac{x}{a},\frac{\lambda}{a}\right)=C_1(x)F_2\left(a,b,\frac{x}{a}\right)+C_2(x)\frac{(bx)_\infty}{\theta_q(-x)}{}_1\phi_1\left(\begin{matrix}q/a\\bq/a\end{matrix};q;\frac{q}{x}\right),\tag{2.11}$$

where

$$\begin{aligned}C_1(x)&=\frac{(b,q)_\infty}{(q/a)_\infty}\frac{\theta_q(\lambda)\theta_q(axq/\lambda)\theta_q(-xq/a)}{\theta_q(\lambda/a)\theta_q(xq/\lambda)\theta_q(-xq)},\\C_2(x)&=\frac{(q)_\infty\theta_q(-x)}{\theta_q(\lambda/a)\theta_q(\frac{xq}{\lambda})}\left\{(a,bq/a,q)_\infty\frac{\theta_q(b\lambda/a)\theta_q(bxq/\lambda)}{\theta_q(-bq/a)\theta_q(-bx)}\right.\\&\qquad\left.-\frac{(bq/a)_\infty}{(q/a)_\infty}\frac{\theta_q(-b)\theta_q(\lambda)\theta_q(axq/\lambda)\theta_q(-bxq/a)}{\theta_q(-bq/a)\theta_q(-xq)\theta_q(-bx)}\right\}.\end{aligned}$$

By replacing $x\mapsto a^{-1}x$, $\lambda\mapsto a^{-1}\lambda$, $b\mapsto-\lambda q^n$ in the equation (2.11) and taking the limit of each term of the equation (2.11) as $n\to\infty$, we have

$$\begin{aligned}\mathcal{L}^+\circ\mathcal{B}^+(F_1)\left(a,b,\frac{x}{a},\frac{\lambda}{a}\right)&\to x^{\frac{\alpha}{2}}f_0(a,x,\lambda)\\F_2\left(a,b,\frac{x}{a}\right)&\to\frac{\theta_q(-x)}{\theta_q(-xq/a)}x^{\frac{\alpha}{2}-1}g_0(a,x)\\\frac{(bx)_\infty}{\theta_q(-x)}{}_1\phi_1\left(\begin{matrix}q/a\\bq/a\end{matrix};q;\frac{q}{x}\right)&\to-x^{-\frac{\alpha}{2}}g_\infty(a,x).\end{aligned}$$

Therefore, we obtain the equation (2.10). □

Note that the parameter λ arising from the q-Laplace transformation is an extra parameter that does not depend on the q-difference equation (1.41). That is, for a different parameter λ', the function $f_0(a,x,\lambda')$ is also a solution of the q-difference equation (1.41), and is represented by a linear combination of the functions $f_\infty(a,x,\lambda)$ and $g_\infty(a,x)$.

Theorem 2.2.4. *The function $f_0(a,x,\lambda')$ is expressed using the functions $f_\infty(a,x,\lambda)$ and $g_\infty(a,x)$ as follows:*

$$\begin{aligned} f_0(a,x,\lambda') &= \frac{\theta_q(\lambda')\theta_q(ax/\lambda')x^\alpha}{\theta_q(\lambda'/a)\theta_q(x/\lambda')a} f_\infty(a,x,\lambda) \\ &\quad - (a)_\infty (q)_\infty^2 \frac{\theta_q(-\lambda'/x\lambda)\theta_q(-x)\theta_q(-\lambda\lambda'/a)}{\theta_q(x\lambda)\theta_q(\lambda'/x)\theta_q(\lambda'/a)\theta_q(a/\lambda)} g_\infty(a,x). \end{aligned} \tag{2.12}$$

Proof. We replace the variable λ by λ' in the connection formula (2.6) of the case of $f_0(x,\lambda)$:

$$\begin{pmatrix} f_0(a,x,\lambda') \\ f_\infty(a,x,\lambda) \end{pmatrix} = -\frac{(q)_\infty}{(q/a)_\infty} \begin{pmatrix} \frac{\theta_q(\lambda')\theta_q(ax/\lambda')x^\alpha}{\theta_q(\lambda'/a)\theta_q(x/\lambda')a} & 1 \\ 1 & \frac{\theta_q(\lambda)\theta_q(x\lambda/a)}{\theta_q(\lambda/a)\theta_q(x\lambda)} x^{-\alpha} \end{pmatrix} \begin{pmatrix} g_0(x) \\ g_\infty(x) \end{pmatrix}. \tag{2.13}$$

By eliminating the function $g_0(x)$ in the equation (2.13), we have

$$\begin{aligned} &f_0(a,x,\lambda') - \frac{\theta_q(\lambda')\theta_q(ax/\lambda')x^\alpha}{\theta_q(\lambda'/a)\theta_q(x/\lambda')a} f_\infty(a,x,\lambda) \\ &= \frac{(q)_\infty}{(q/a)_\infty} \left\{ \frac{\theta_q(\lambda)\theta_q(\lambda')\theta_q(x\lambda/a)\theta_q(ax/\lambda')}{a\theta_q(\lambda/a)\theta_q(\lambda'/a)\theta_q(x\lambda)\theta_q(x/\lambda')} - 1 \right\} g_\infty(x) \\ &= \frac{(a)_\infty(q)_\infty^2}{\theta_q(-a)} \frac{\theta_q(-\lambda'/x\lambda)\theta_q(-x)}{\theta_q(x\lambda)\theta_q(\lambda'/x)} C(x) g_\infty(x), \end{aligned}$$

where

$$C(x) = \frac{\lambda'}{ax} \frac{\theta_q(\lambda)\theta_q(\lambda')\theta_q(x\lambda/a)\theta_q(ax/\lambda')}{\theta_q(-\lambda'/x\lambda)\theta_q(-x)\theta_q(\lambda/a)\theta_q(\lambda'/a)} - \frac{\theta_q(x\lambda)\theta_q(\lambda'/x)}{\theta_q(-\lambda'/x\lambda)\theta_q(-x)}.$$

The function $C(x)$ has a simple pole at the points $x = q^j, \frac{\lambda'}{\lambda}q^j$ for some $j \in \mathbb{Z}$, and satisfies $C(xq) = C(x)$. We calculate the residue at the pole $x = 1$,

$$\begin{aligned} \lim_{x\to 1} \theta_q(-x)C(x) &= \frac{\lambda'}{a} \frac{\theta_q(\lambda)\theta_q(\lambda')\theta_q(\lambda/a)\theta_q(a/\lambda')}{\theta_q(-\lambda'/\lambda)\theta_q(\lambda/a)\theta_q(\lambda'/a)} - \frac{\theta_q(\lambda)\theta_q(\lambda')}{\theta_q(-\lambda'/\lambda)} \\ &= \frac{\theta_q(\lambda)\theta_q(\lambda')}{\theta_q(-\lambda'/\lambda)} \left\{ \frac{\lambda'}{a} \frac{\theta_q(a/\lambda')}{\theta_q(\lambda'/a)} - 1 \right\} = 0. \end{aligned}$$

It follows that the doubly periodic function $C(x)$ is constant in x. Hence, we have

$$C(x) = C(-a/\lambda) = -\frac{\theta_q(-a)\theta_q(-\lambda\lambda'/a)}{\theta_q(\lambda'/a)\theta_q(a/\lambda)}.$$

□

Finally, we provide that the function $\mu(u,v;\alpha)$ defined in Definition 1.3.1 coincides with the solution $f_0(a,x,\lambda)$ of the q-difference equation (1.41), and in particular when $\alpha = 0$, the original μ-function is naturally derived from the image of the composition of the q-Borel transformation and q-Laplace transformation of a divergent solution of the q-difference equation (2.1).

Theorem 2.2.5. *We have*

$$-iq^{-\frac{1}{8}}f_0(q^\alpha, e^{2\pi i(u-v)}, -e^{2\pi iu}) = \mu(u,v;\alpha). \tag{2.14}$$

Proof. From the definitions of the q-Borel transformation and the q-hypergeometric series, we have

$$f_0(a,x,\lambda) = x^{\frac{\alpha}{2}}\mathcal{L}^+\circ\mathcal{B}^+(\tilde{f}_0)(x,\lambda) = x^{\frac{\alpha}{2}}\mathcal{L}^+\left({}_1\phi_0\left(\begin{matrix}a\\ -\end{matrix};q;-\frac{\xi}{a}\right)\right).$$

Since the series ${}_1\phi_0$ has a factorized form by the q-binomial theorem (see [GR, p.8, equation (1.3.2)])

$$\sum_{n=0}^{\infty}\frac{(a)_n}{(q)_n}x^n = \frac{(ax)_\infty}{(x)_\infty},$$

we have

$$\begin{aligned}f_0(a,x,\lambda) &= x^{\frac{\alpha}{2}}\mathcal{L}^+\left(\frac{(-\xi)_\infty}{(-\xi/a)_\infty}\right) = x^{\frac{\alpha}{2}}\sum_{n\in\mathbb{Z}}\frac{1}{\theta_q(\lambda q^n/x)}\frac{(-\lambda q^n)_\infty}{(-\lambda q^n/a)_\infty}\\ &= \frac{x^{\frac{\alpha}{2}}}{\theta_q(\lambda/x)}\sum_{n\in\mathbb{Z}}\left(\frac{\lambda}{x}\right)^{n+1}q^{\frac{n(n+1)}{2}}\frac{(-\lambda q^{n+1})_\infty}{(-\lambda q^{n+1}/a)_\infty}.\end{aligned}$$

The second equality followos from the equation (1.1). By substituting $x = e^{2\pi i(u-v)}$ and $\lambda = e^{2\pi iu}$, we have

$$\begin{aligned}f_0(q^\alpha, e^{2\pi i(u-v)}, -e^{2\pi iu}) &= e^{\pi i\alpha(u-v)}\mathcal{L}^+\circ\mathcal{B}^+\left(\tilde{f}_0\right)(e^{2\pi i(u-v)}, -e^{2\pi iv})\\ &= -\frac{e^{\pi i\alpha(u-v)}}{\theta_q(-e^{2\pi iv})}\sum_{n\in\mathbb{Z}}(-1)^n e^{2\pi i(n+1)v}q^{\frac{n(n+1)}{2}}\frac{(e^{2\pi iu}q^{n+1})_\infty}{(e^{2\pi iu}q^{n-\alpha+1})_\infty}\\ &= \frac{ie^{\pi i\alpha(u-v)}}{\vartheta_{11}(v)}\sum_{n\in\mathbb{Z}}(-1)^n e^{2\pi i\left(n+\frac{1}{2}\right)v}q^{\frac{\left(n+\frac{1}{2}\right)^2}{2}}\frac{(e^{2\pi iu}q^{n+1})_\infty}{(e^{2\pi iu}q^{n-\alpha+1})_\infty}\\ &= iq^{\frac{1}{8}}\mu(u,v;\alpha).\end{aligned}$$

□

2.3 Some properties of the function $\mu(u,v;\alpha)$

First, we provide basic properties of the function $\mu(u,v;\alpha)$ similar to Proposition 1.2.3.

Theorem 2.3.1. *Let*

$$\Phi_\alpha(u,v) = \Phi_\alpha := \frac{\vartheta(v-\alpha\tau)\vartheta(u)}{\vartheta(u-\alpha\tau)\vartheta(v)}e^{2\pi i\alpha(u-v)}$$

and

$$F_{\alpha+1}(u;\tau) := \frac{\vartheta(u)(q^{\alpha+1})_\infty}{\vartheta(u-\alpha\tau)(q)_\infty}e^{-\pi i\alpha(u-\tau)}{}_1\phi_1\left(\begin{matrix}q^{-\alpha}\\ 0\end{matrix};q;e^{-2\pi iu}q\right).$$

Then the function $\mu(u,v;\alpha)$ satisfies the following properties:

$$(1)\quad \mu(u,v;\alpha)=e^{-\pi i\alpha}\mu(u+1,v;\alpha)=e^{\pi i\alpha}\mu(u,v+1;\alpha), \tag{2.15}$$

$$(2)\quad \mu(u+\tau,v;\alpha)=-e^{2\pi i(u-v)}q^{\frac{\alpha}{2}}\mu(u,v;\alpha)+e^{\pi i(u-v)}q^{\frac{\alpha}{2}}\mu(u,v;\alpha-1), \tag{2.16}$$

$$\begin{aligned}(3)\quad \mu(u+z,v+z;\alpha+1)&=\Phi_{\alpha+1}(u+z,v+z)\mu(v,u;\alpha+1)\\ &+\frac{iq^{\frac{1}{8}}(q)_\infty^3\vartheta(z)\vartheta(u+v+z-\alpha\tau)\vartheta(u-v-\alpha\tau)}{\vartheta(u)\vartheta(v-\alpha\tau)\vartheta(u+z-\alpha\tau)\vartheta(v+z)\vartheta(u-v)}e^{2\pi i\alpha(u-v)}F_{\alpha+1}(u-v;\tau),\end{aligned} \tag{2.17}$$

$$(4)\quad \mu(u,v;\alpha)=\mu(u+\tau,v+\tau;\alpha), \tag{2.18}$$

$$(5)\quad \mu(u,v;\alpha)=\Phi_\alpha\mu(-u+\alpha\tau,-v+\alpha\tau;\alpha), \tag{2.19}$$

$$(6)\quad \mu(u,v;\alpha)=\Phi_\alpha\mu(v,u;\alpha). \tag{2.20}$$

Proof.
(1) is trivial.
(2) We have

$$\begin{aligned}&\mu(u+\tau,v;\alpha)\\ &=\frac{e^{\pi i\alpha(u-v+\tau)}}{\vartheta(v)}\sum_{n\in\mathbb{Z}}(-1)^n e^{2\pi i(n+\frac{1}{2})v}q^{\frac{n(n+1)}{2}}\frac{(e^{2\pi iu}q^{n+2})_\infty}{(e^{2\pi iu}q^{n-\alpha+2})_\infty}(e^{2\pi iu}q^{n+1}+1-e^{2\pi iu}q^{n+1})\\ &=\frac{e^{\pi i\alpha(u-v+\tau)}}{\vartheta(v)}\sum_{n\in\mathbb{Z}}e^{2\pi i(u-v)}(-1)^{n-1}e^{2\pi i(n+1\frac{1}{2})v}q^{\frac{n(n+1)}{2}}\frac{(e^{2\pi iu}q^{n+1})_\infty}{(e^{2\pi iu}q^{n-\alpha+1})_\infty}\\ &\quad+\frac{e^{\pi i\alpha(u-v+\tau)}}{\vartheta(v)}\sum_{n\in\mathbb{Z}}(-1)^n e^{2\pi i(n+\frac{1}{2})v}q^{\frac{n(n+1)}{2}}\frac{(e^{2\pi iu}q^{n+1})_\infty}{(e^{2\pi iu}q^{n-\alpha+2})_\infty}\\ &=e^{2\pi i(u-v)}q^{\frac{\alpha}{2}}\mu(u,v;\alpha)+e^{\pi i(u-v)}q^{\frac{\alpha}{2}}\mu(u,v;\alpha-1).\end{aligned}$$

(3) Putting $x=e^{2\pi i(u-v)},\lambda=-e^{2\pi iv},\lambda'=-e^{2\pi i(u+z)}$ in the equation (2.12), we have the equation (2.17) from the equation (2.14).
(4)-(6) Putting $z=\tau,z=0,z=-u-v+\alpha\tau$ in the equation (2.17), we have equations (2.18)-(2.20), respectively. ☐

Since the function $\mu(u,v;\alpha)$ is introduced as a solution of the q-Hermite-Weber equation (1.41), the function $\mu(u,v;\alpha)$ satisfies the following relation:

$$\mu(u+\tau,v;\alpha)=(1-e^{2\pi i(u-v)})q^{\frac{\alpha}{2}}\mu(u,v;\alpha)+e^{2\pi i(u-v)}\mu(u-\tau,v;\alpha). \tag{2.21}$$

In addition, using the equation (2.16), we obtain the following three terms relations of the function $\mu(u,v;\alpha)$.

$$(1)\quad \mu(u,v;\alpha)=-e^{2\pi i(u-v)}q^{-\frac{\alpha}{2}}\mu(u-\tau,v;\alpha)+e^{\pi i(u-v)}\mu(u,v;\alpha-1) \tag{2.22}$$

$$(2)\quad q^{\frac{\alpha}{2}}\mu(u,v;\alpha)=\mu(u+\tau,v;\alpha)+e^{\pi i(u-v)}(q^{\frac{\alpha}{2}}-q^{-\frac{\alpha}{2}})\mu(u,v;\alpha+1) \tag{2.23}$$

$$(3)\quad e^{\pi i(u-v)}q^{\frac{\alpha}{2}}\mu(u,v;\alpha)=-(q^{\frac{\alpha}{2}}-q^{-\frac{\alpha}{2}})\mu(u,v;\alpha+1)+e^{\pi i(u-v)}\mu(u-\tau,v;\alpha) \tag{2.24}$$

$$(4)\quad 2\cos\pi(u-v)\mu(u,v;\alpha)=(1-q^{-\alpha})\mu(u,v;\alpha+1)+\mu(u,v;\alpha-1) \tag{2.25}$$

In particular, the relation (2.25) is one of the specific properties of the function $\mu(u,v;\alpha)$, which coincides essentially with the q-Bessel equation:

$$\left[T_x-(q^{\frac{\nu}{2}}+q^{-\frac{\nu}{2}})T_x^{\frac{1}{2}}+\left(1+\frac{x^2}{4}\right)\right]f(x)=0,\quad T_x^{\frac{1}{2}}f(x):=f(xq^{\frac{1}{2}}). \tag{2.26}$$

Therefore, the function $\mu(u,v;\alpha)$ satisfies the q-Hermite-Weber equation with respect to the variables u and v, and satisfies the q-Bessel equation with respect to the variable α (Section 5). This fact plays important roles in Section 5.

2.4 Some properties of the function $\mu(u, v; k)$

For $k \in \mathbb{Z}$, we provide properties of the function

$$\mu(u, v; k) = \frac{e^{\pi i k(u-v)}}{\vartheta(v)} \sum_{n\in\mathbb{Z}} \frac{(-1)^k e^{2\pi i(n+\frac{1}{2})v} q^{\frac{n(n+1)}{2}}}{(e^{2\pi i u} q^{n-k+1})_k}. \tag{2.27}$$

First, we state the following corollary when $\alpha = k$ in Theorem 2.3.1.

Corollary 2.4.1. *For $k \in \mathbb{Z}$, the function $\mu(u, v; k)$ satisfies the following properties:*

(1) $\mu(u + 1, v; k) = \mu(u, v + 1; k) = (-1)^k \mu(u, v; k),$ (2.28)

(2) $\mu(u + \tau, v; k) = -e^{2\pi i(u-v)} q^{\frac{k}{2}} \mu(u, v; k) + e^{\pi i(u-v)} q^{\frac{k}{2}} \mu(u, v; k - 1),$ (2.29)

(3) $\mu(u + z, v + z; k + 1) = \mu(u, v; k + 1) + \dfrac{i q^{\frac{1}{8}} (q)_\infty^3 \vartheta(z)\vartheta(u + v + z)}{\vartheta(u)\vartheta(v)\vartheta(u + z)\vartheta(v + z)} F_{k+1}(u - v; \tau),$ (2.30)

(4) $\mu(u, v; k) = \mu(u + \tau, v + \tau; k),$ (2.31)

(5) $\mu(u, v; k) = \mu(-u, -v; k),$ (2.32)

(6) $\mu(u, v; k) = \mu(v, u; k).$ (2.33)

Noting that, since $\Phi_k(u, v) = 1$, the function $\mu(u, v; k)$ is truly symmetric with respect to variables u and v, unlike the function $\mu(u, v; \alpha)$. In particular, for $k > 0$, the function $\mu(u, v; k)$ is expressed using the original μ-function.

Theorem 2.4.2. *For $k \in \mathbb{Z}_{\geq 0}$, we have*

$$\begin{aligned} \mu(u, v; k + 1) &= F_{k+1}(u - v; \tau)\mu(u, v) \\ &\qquad - i q^{-\frac{1}{8}} \sum_{l=1}^{k} \frac{q^k}{(q)_l} F_{k-l+1}(u - v; \tau) H_{k-1}(\cos \pi(u - v); q), \end{aligned} \tag{2.34}$$

and

$$\sum_{m=0}^{k} \mu(u, v; m + 1) \frac{H_{k-m}(\cos \pi(u - v); q)}{(q)_{k-m}} q^{k-m} = -\frac{i q^{\frac{7}{8}} H_{k-1}(\cos \pi(u - v); q)}{(q)_k}, \tag{2.35}$$

where the function $H_k(z; q)$ is the q-Hermite polynomial in the equation (1.45).

We prove this theorem on page 23.

From the equation (2.34), the function $\mu(u, v; k)$ for $k \in \mathbb{Z}_{>0}$ is essentially the original μ-function, and we obtain the relations between the function $\mu(u, v; k)$ and modular transformations.

Theorem 2.4.3. *For $k \in \mathbb{Z}_{\geq 0}$, we define*

$$R_{k+1}(u; \tau) := R(u; \tau) + 2q^{-\frac{1}{8}} \sum_{l=1}^{k} \frac{q^l}{(q)_l} \frac{F_{k-l+1}(u; \tau)}{F_{k+1}(u; \tau)} H_{l-1}(\cos \pi u, q),$$

$$\tilde{\mu}(u, v; k + 1; \tau) := \frac{\mu(u, v; k + 1; \tau)}{F_{k+1}(u - v; \tau)} + \frac{i}{2} R_{k+1}(u - v; \tau).$$

Then we have

$$\tilde{\mu}(u,v;\tau)=\tilde{\mu}(u,v;k+1;\tau).$$

That is the function $\tilde{\mu}(u,v;k+1;\tau)$ *satisfies the following equations:*

$$\begin{aligned}\tilde{\mu}(u,v;k+1;\tau+1)&=e^{-\frac{\pi i}{4}}\tilde{\mu}(u,v;k+1;\tau),\\ \tilde{\mu}\left(\frac{u}{\tau},\frac{v}{\tau};k+1;-\frac{1}{\tau}\right)&=-i\sqrt{-i\tau}e^{\pi i\frac{(u-v)^2}{\tau}}\tilde{\mu}(u,v;k+1;\tau).\end{aligned}$$

Proof. From the equation (2.34), we have

$$\begin{aligned}\tilde{\mu}(u,v;\tau)&=\mu(u,v;\tau)+\frac{i}{2}R(u-v;\tau)\\ &=\frac{\mu(u,v;k+1,\tau)}{F_{k+1}(\cos\pi(u-v)\mid q)}+\frac{i}{2}R(u-v;\tau)\\ &\quad+iq^{-\frac{1}{8}}\sum_{l=1}^{k}\frac{q^l}{(q)_l}\frac{F_{k-l+1}(\cos\pi(u-v)\mid q)}{F_{k+1}(\cos\pi(u-v)\mid q)}H_{l-1}(\cos\pi(u-v)\mid q)\\ &=\frac{\mu(u,v;k+1,\tau)}{F_{k+1}(\cos\pi(u-v)\mid q)}+\frac{i}{2}R_{k+1}(u-v;\tau)\\ &=\tilde{\mu}(u,v;k+1;\tau)\end{aligned}$$

□

Remark . *T. Matsusaka has also obtained this result [Ma].*

It is striking that the error to modularity of the μ-function only depends on $u-v$. In an unpublished note, Zagier explained this phenomenon by splitting the μ-function (non-uniquely) as a sum of a Jacobi form in u, v and a mock Jacobi form which depends on $u-v$ and τ only [BFOR, p.142]. Here, we call the expression "Zagier expression". We provide that an expression analogous to the Zagier expression for the function $\mu(u,v;k)$ is derived in parallel with the case of the μ-function. First, we derive the Laurent series of the function $\vartheta(u)\vartheta(v)\mu(u,v;k)$ with respect to the variables $e^{2\pi iu}$ and $e^{2\pi iv}$. For this purpose, we present the following lemma.

Lemma 2.4.4 ([BFOR]p.143)**.** *For* $s\in\mathbb{Z}$*, we call the following series "truncated theta series":*

$$\gamma_s(q):=\sum_{n\in\mathbb{Z}+\frac{1}{2},n>s}(-1)^{n-\frac{1}{2}}q^{\frac{n^2}{2}}.$$

Then we have

$$(-1)^s i\frac{e^{\pi iu}q^{\frac{s(s+1)}{2}}}{1-e^{2\pi iu}q^s}\vartheta(u)=\sum_{r\in\mathbb{Z}}\gamma_{r-s}(q)e^{2\pi iru}q^{rs}.$$

Theorem 2.4.5. *For* $k\in\mathbb{Z}_{\geq 0}$*, we have*

$$\begin{aligned}&(-1)^k i\vartheta(u)\vartheta(v)\mu(u,v;k+1)\\ &=e^{\pi ik(u+v)}q^{\frac{k(k+1)}{2}}\sum_{m,n\in\mathbb{Z}}\left(\sum_{j=0}^{k}\frac{q^{mj+n(k-j)}}{(q)_j(q)_{k-j}}\gamma_{m-n-j}(q)\right)q^{mn}e^{2\pi imu}e^{2\pi inv}.\end{aligned}\tag{2.36}$$

Proof. From the definition (2.27) of the function $\mu(u,v;k)$, we have

$$\vartheta(u)\vartheta(v)\mu(u,v;k+1) = e^{\pi i(k+1)(u-v)}\vartheta(u)\sum_{n\in\mathbb{Z}}\frac{(-1)^n e^{2\pi i(n+\frac{1}{2})v}q^{\frac{n(n+1)}{2}}}{\prod_{j=0}^{k}(1-e^{2\pi iu}q^{n-j})}. \tag{2.37}$$

Note that the product of the right hand side in the equation (2.37) is

$$\prod_{j=0}^{k}\frac{1}{1-e^{2\pi iu}q^{n-j}} = \sum_{j=0}^{k}\frac{(-1)^{k-j}}{(q)_j(q)_{k-j}}\frac{q^{\frac{(k-j)(k-j+1)}{2}}}{1-e^{2\pi iu}q^{n-j}}, \tag{2.38}$$

then we have

$$\begin{aligned}&\vartheta(u)\vartheta(v)\mu(u,v;k+1)\\&= e^{\pi i(k+1)(u-v)}\vartheta(u)\sum_{n\in\mathbb{Z}}(-1)^n e^{2\pi i(n+\frac{1}{2})v}q^{\frac{n(n+1)}{2}}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{(q)_j(q)_{k-j}}\frac{q^{\frac{(k-j)(k-j+1)}{2}}}{1-e^{2\pi iu}q^{n-j}}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&(-1)^k i\vartheta(u)\vartheta(v)\mu(u,v;k+1)\\&= e^{\pi ik(u-v)}q^{\frac{k(k+1)}{2}}\sum_{n\in\mathbb{Z}}e^{2\pi inv}\sum_{j=0}^{k}(-1)^{n-j}i\frac{e^{\pi iu}q^{\frac{(n-j)(n-j+1)}{2}}}{1-e^{2\pi iu}q^{n-j}}\vartheta(u)\frac{q^{(n-k)j}}{(q)_j(q)_{k-j}}\\&= e^{\pi ik(u-v)}q^{\frac{k(k+1)}{2}}\sum_{m,n\in\mathbb{Z}}\left(\sum_{j=0}^{k}\frac{q^{(n-k)j+m(n-j)}}{(q)_j(q)_{k-j}}\gamma_{m-n+j}(q)\right)e^{2\pi imu}e^{2\pi inv}\end{aligned}$$

from Lemma 2.4.4, and we obtain the relatation (2.36). □

We derive Zagier expression of the function $\mu(u,v;k)$ using the Laurent series of the function $\mu(u,v;k)$.

Theorem 2.4.6. *For $k\in\mathbb{Z}$, we put*

$$M(u;k+1;\tau) = M(u;k+1) := e^{-\pi iku}\sum_{n\in\mathbb{Z}}\frac{e^{2\pi inu}q^{\frac{n(n+1)}{2}}}{(-q^{n-k})_{k+1}}.$$

Then we have

$$-i\mu(u,v;k+1) = \frac{M(u-v;k+1)}{\vartheta\left(u-v+\frac{1}{2}\right)} + \frac{(q)_\infty^2}{2(-q)_\infty^2}\frac{\vartheta\left(u+\frac{1}{2}\right)\vartheta\left(v+\frac{1}{2}\right)}{\vartheta\left(u-v+\frac{1}{2}\right)\vartheta(u)\vartheta(v)}F_{k+1}(u-v;\tau). \tag{2.39}$$

First, we provide the following lemma.

Lemma 2.4.7. *For $\varepsilon,\delta\in\{0,1\}$, we suppose*

$$\begin{aligned}\nu_\varepsilon(u,v;k+1,\tau) &= \nu_\varepsilon(u,v;k+1)\\&:= \vartheta(u)\vartheta(v)\mu(u,v;k+1)\\&\quad + (-1)^{\varepsilon+k}\vartheta\left(u+\frac{1}{2}\right)\vartheta\left(v+\frac{1}{2}\right)\mu\left(u+\frac{1}{2},v+\frac{1}{2};k+1\right),\end{aligned}$$

$$G_\varepsilon(u;k+1;\tau) = G_\varepsilon(u;k+1) := \sum_{n\equiv\varepsilon \mod 2}\left(\sum_{j=0}^{k}\frac{q^{nj}\gamma_{n-j}(q)}{(q)_j(q)_{k-j}}\right)q^{-\frac{n^2}{4}}e^{\pi inu},$$

$$\vartheta_{\varepsilon,\delta}(u,\tau) = \vartheta_{\varepsilon,\delta}(u) := \sum_{\nu\in\mathbb{Z}+\frac{\varepsilon}{2}}e^{\pi i\nu^2\tau+2\pi i\nu\left(u+\frac{\delta}{2}\right)}.$$

Then we have

$$(1)\quad \frac{i}{2}\nu_\varepsilon(u,v;k+1)=(-1)^k e^{\pi ik(u+v)}q^{\frac{k(k+1)}{2}}G_\varepsilon(u-v-k\tau;k+1)\vartheta_{\varepsilon,0}(u+v+k\tau,2\tau) \tag{2.40}$$

$$(2)\quad (-1)^k e^{\pi iku}q^{\frac{k(k+1)}{2}}\sum_{\varepsilon=0}^{1}G_\varepsilon(u-k\tau;k+1)\vartheta_{\varepsilon,0}(u+k\tau,2\tau)=q^{\frac{1}{8}}(q)_\infty^3F_{k+1}(u;\tau) \tag{2.41}$$

$$(3)\quad (-1)^{k+1} e^{\pi iku}q^{\frac{k(k+1)}{2}}\sum_{\varepsilon=0}^{1}(-1)^\varepsilon G_\varepsilon(u-k\tau;k+1)\vartheta_{\varepsilon,0}(u+k\tau,2\tau)=\vartheta\left(\frac{1}{2}\right)M(u;k+1) \tag{2.42}$$

Proof. (1) Putting $m=(r+s)/2$ and $n=(r-s)/2$ in the equation (2.36), we have

$$\begin{aligned}
&(-1)^k ie^{-\pi ik(u+v)}q^{-\frac{k(k+1)}{2}}\vartheta(u)\vartheta(v)\mu(u,v;k+1)\\
&=\sum_{r\equiv s \mod 2}\left(\sum_{j=0}^{k}\frac{q^{sj}\gamma_{s-j}(q)}{(q)_j(q)_{k-j}}\right)q^{\frac{k r}{2}+\frac{r^2-s^2}{4}}e^{\pi ir(u+v)}e^{\pi is(u-v)}\\
&=\sum_{\varepsilon=0}^{1}G_\varepsilon(u-v-k\tau;k+1)\vartheta_{\varepsilon,0}(u+v+k\tau,2\tau).
\end{aligned}\tag{2.43}$$

Then, putting $u\mapsto u+\frac{1}{2}, v\mapsto v+\frac{1}{2}$, we have

$$\begin{aligned}
&ie^{-\pi ik(u+v)}q^{-\frac{k(k+1)}{2}}\vartheta\left(u+\frac{1}{2}\right)\vartheta\left(v+\frac{1}{2}\right)\mu\left(u+\frac{1}{2},v+\frac{1}{2};k+1\right)\\
&=G_0(u-v-k\tau;k+1)\vartheta_{0,0}(u+v+k\tau,2\tau)-G_1(u-v-k\tau;k+1)\vartheta_{1,0}(u+v+k\tau,2\tau).
\end{aligned}$$

Therefore, we have the equation (2.40).
(2) From the equations (2.40) and (2.30), we have

$$\begin{aligned}
&(-1)^k e^{\pi ik(u+v)}q^{\frac{k(k+1)}{2}}\sum_{\varepsilon=0}^{1}G_\varepsilon(u-v-k\tau;k+1)\vartheta_{\varepsilon,0}(u+v+k\tau,2\tau)\\
&=\sum_{\varepsilon=0}^{1}\frac{i}{2}\nu_\varepsilon(u,v;k+1)=i\vartheta(u)\vartheta(v)\mu(u,v;k+1)\\
&=i\vartheta(u)\vartheta(v)\mu(u+z,v+z;k+1)\\
&\qquad+\frac{q^{\frac{1}{8}}(q)_\infty^3\vartheta(z)\vartheta(u+v+z)}{\vartheta(u+z)\vartheta(v+z)}F_{k+1}(u-v;\tau)
\end{aligned}$$

Then, putting $v=0$, we obtain the equation (2.41) since $\vartheta(0)=0$.
(3) From the equation (2.40), we have

$$(-1)^k e^{\pi ik(u+v)}q^{\frac{k(k+1)}{2}}\sum_{\varepsilon=0}^{1}(-1)^\varepsilon G_\varepsilon(u-v-k\tau;k+1)\vartheta_{\varepsilon,0}(u+v+k\tau,2\tau) \tag{2.44}$$

$$=\sum_{\varepsilon=0}^{1}\frac{i}{2}(-q)^\varepsilon\nu_\varepsilon(u,v;k+1)=i\vartheta\left(u+\frac{1}{2}\right)\vartheta\left(v+\frac{1}{2}\right)\mu\left(u+\frac{1}{2},v+\frac{1}{2};k+1\right). \tag{2.45}$$

Then, putting $v=0$ in the right hand side of the equation (2.44), we have

$$\begin{aligned}
i\vartheta\left(u+\frac{1}{2}\right)\vartheta\left(\frac{1}{2}\right)\mu\left(u+\frac{1}{2},\frac{1}{2};k+1\right) &= i\vartheta\left(u+\frac{1}{2}\right)\vartheta\left(\frac{1}{2}\right)\mu\left(\frac{1}{2},u+\frac{1}{2};k+1\right)\\
&= -\vartheta\left(\frac{1}{2}\right)e^{-\pi iku}\sum_{n\in\mathbb{Z}}\frac{e^{2\pi inu}q^{\frac{n(n+1)}{2}}}{(-q^{n-k})_{k+1}}\\
&= -\vartheta\left(\frac{1}{2}\right)M(u;k+1)
\end{aligned}$$

Therefore, we obtain the equation (2.42). □

Proof of Theorem 2.4.6. From the equations (2.41) and (2.42), we have

$$\begin{aligned}
&e^{\pi iku}q^{\frac{k(k+1)}{2}}\begin{pmatrix}G_0(u-k\tau;k+1)\vartheta_{0,0}(u+k\tau,2\tau)\\ G_1(u-k\tau;k+1)\vartheta_{1,0}(u+k\tau,2\tau)\end{pmatrix}\\
&\quad=\frac{(-1)^k}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}\eta(\tau)^3F_{k+1}(u;\tau)\\ -\vartheta(1/2)M(u;k+1)\end{pmatrix}.
\end{aligned}$$

Then, we rewrite the equation (2.43) as

$$\begin{aligned}
&ie^{-2\pi ikv}\vartheta(u)\vartheta(v)\mu(u,v;k+1)\\
&=\sum_{\varepsilon=0}^{1}\frac{\vartheta_{\varepsilon,0}(u+v+k\tau,2\tau)}{2\vartheta_{\varepsilon,0}(u-v+k\tau,2\tau)}\left(\eta(\tau)^3F_{k+1}(u-v;\tau)+(-1)^{\varepsilon}\vartheta\left(\frac{1}{2}\right)M(u-v;k+1)\right)\\
&=\frac{(q)_\infty^2}{2(-q)_\infty^2}\frac{\vartheta_{1,0}(u+k\tau)\vartheta_{1,0}(v)}{\vartheta_{1,0}(u-v+k\tau)}F_{k+1}(u-v;\tau)-\frac{\vartheta(u+k\tau)\vartheta(v)}{\vartheta_{1,0}(u-v+k\tau)}M(u-v;k+1).
\end{aligned}$$

The second equality follows from the addition formula for the Jacobi theta functions (for example, [Mu]):

$$\frac{\vartheta_{0,0}(u+v+k\tau,2\tau)}{2\vartheta_{0,0}(u-v+k\tau,2\tau)}+(-1)^{\delta}\frac{\vartheta_{1,0}(u+v+k\tau,2\tau)}{2\vartheta_{1,0}(u-v+k\tau,2\tau)}=\frac{\vartheta_{1,\delta}(u+k\tau)\vartheta_{1,\delta}(v)}{\vartheta_{1,0}(u-v+k\tau)\vartheta_{1,0}(0)}.$$

Finally, using the quasi-periodicity of the theta functions

$$\vartheta_{1,\delta}(u+l\tau)=(-1)^{l\delta}e^{-\pi il^2\tau-2\pi ilu}\vartheta_{1,\delta}(u),\quad (l\in\mathbb{Z}),$$

we have the equation (2.39). □

3 Relations between the function $\mu(u, v; \alpha)$ and q-hypergeometric series

In this section, we put $x = e^{2\pi iu}$, $y = e^{2\pi iv}$ and $q = e^{2\pi i\tau}$.

3.1 Relations to the continuous q-Hermite polynomial

Note that for $k \geq 0$, the function $\mu(u, v; -k)$ defined by an infinite series is a polynomial with respect to the variable $e^{\pi i(u-v)}$. In fact, from Theorem 2.2.5, we have

$$\begin{aligned}\mu(u, v; -k) &= -iq^{-\frac{1}{8}} f_0(q^{-k}, e^{2\pi i(u-v)}, -e^{2\pi iu}) \\ &= -iq^{-\frac{1}{8}} e^{-\pi ik(u-v)} \mathcal{L}^+ \circ \mathcal{B}^+ \left({}_2\phi_0 \left(\begin{matrix} q^{-k}, 0 \\ - \end{matrix}; q; xq^k\right)\right) (e^{2\pi i(u-v)}, -e^{2\pi iu}) \\ &= -iq^{-\frac{1}{8}} e^{-\pi ik(u-v)} {}_2\phi_0 \left(\begin{matrix} q^{-k}, 0 \\ - \end{matrix}; q; e^{2\pi i(u-v)} q^k\right)\end{aligned}$$

From the equation (1.46), we obtain the following theorem.

Theorem 3.1.1. *For $k \in \mathbb{Z}_{\geq 0}$, we have*

$$\mu(u, v; -k) = -iq^{-\frac{1}{8}} H_k(\cos \pi(u - v); q). \tag{3.1}$$

From this theorem, the function $\mu(u, v; k)$ is regarded as a "$-k$ degree" continuous q-Hermite polynomial.

A generating function of the continuous q-Hermite polynomial $H_k(z; q)$ has the following expression (for example [KLS, p.542]):

$$\sum_{k=0}^{\infty} \frac{H_k(\cos \theta; q)}{(q)_n} r^n = \frac{1}{(re^{i\theta}, re^{-i\theta})_\infty}. \tag{3.2}$$

Next, we provide q-difference equations and formulas with respect to a generating function of the function $\{\mu(u, v; k + 1)\}_{k\geq 0}$

$$S(u, v; r) = S(r) := \sum_{k=0}^{\infty} \mu(u, v; k + 1) r^k$$

which is an analogue of the generating function of the q-Hermite polynomial (3.2).

Theorem 3.1.2. (1) *The generating function $S(r)$ satisfies the following q-difference equations:*

$$S(r) = (1 - re^{\pi i(u-v)} q)(1 - re^{-\pi i(u-v)} q) S(rq) - irq^{\frac{7}{8}}, \tag{3.3}$$

$$\begin{aligned}qS(r) = &(1 + q(1 - re^{\pi i(u-v)} q)(1 - re^{-\pi i(u-v)} q)) S(rq) \\ &- (1 - re^{\pi i(u-v)} q^2)(1 - re^{-\pi i(u-v)} q^2) S(rq^2).\end{aligned} \tag{3.4}$$

(2) *The generating function* $S(r)$ *has the following expressions:*

$$\frac{S(r)}{(re^{\pi i(u-v)}q, re^{-\pi i(u-v)}q)_\infty} = \mu(u,v) - \frac{irq^{\frac{7}{8}}{}_3\phi_2\left(\begin{matrix} q, re^{\pi i(u-v)}q, re^{-\pi i(u-v)}q \\ 0, 0 \end{matrix}; q; q\right)}{(re^{\pi i(u-v)}q, re^{-\pi i(u-v)}q)_\infty}, \quad (3.5)$$

$$= \mu(u,v) - \frac{irq^{\frac{7}{8}}}{1-q}\Phi^{(1)}\left(\begin{matrix} q; 0, 0 \\ q^2 \end{matrix}; q; re^{\pi i(u-v)}q, re^{-\pi i(u-v)}q\right), \quad (3.6)$$

$$= \sum_{m\geq 0} \frac{\mu(u,v;1-m)}{(q)_m} q^m r^m, \quad (3.7)$$

where the function $\Phi^{(1)}$ *is the q-Appell function (for example, see [GR]):*

$$\Phi^{(1)}\left(\begin{matrix} a; b_1, b_2 \\ c \end{matrix}; q; x, y\right) := \sum_{m,n\geq 0} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}(q)_m(q)_n} x^m y^n. \quad (3.8)$$

Proof. (1) From the equation (2.25), we have

$$\begin{aligned} S(r) - S(r/q) &= \sum_{k=0}^{\infty}(1-q^{-k})\mu(u,v;k+1)r^k \\ &= \sum_{k=0}^{\infty}(2\cos\pi(u-v)\mu(u,v;k) - \mu(u,v;k+1))r^k \\ &= (2\cos\pi(u-v) - r^2)S(r) + (2\cos\pi(u-v) - r)\mu(u,v;0) - \mu(u,v;-1). \end{aligned}$$

Therefore, we have the equation (3.3) from the equation (1.44). Furthermore, by acting $(Tr - q)$ to both sides of the equation (3.3), we have the equation (3.4).
(2) For $N = 1, 2, \ldots$, we have

$$S(r) = (re^{\pi i(u-v)}q, re^{-\pi i(u-v)}q)_N S(rq^N) - irq^{\frac{7}{8}} \sum_{n=0}^{N-1} q^n (re^{\pi i(u-v)}q, re^{-\pi i(u-v)}q)_n \quad (3.9)$$

from the equation (3.3). Taking the limit $N \to \infty$ in the equation (3.9), we have the equation (3.5). Next, from a specialization of Andrews' formula (for example [GR, p.298])

$${}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q; x\right) = \frac{(ax, b, c)_\infty}{(x, d, e)_\infty}\Phi^{(1)}\left(\begin{matrix} x; d/b, e/c \\ ax \end{matrix}; q; b, c\right),$$

we immediately obtain the equation (3.6). Finally, we prove the equation (3.7). From equations (1.45) and (3.8), we have

$$\begin{aligned} \Phi^{(1)}\left(\begin{matrix} q; 0; 0 \\ q^2 \end{matrix}; q; re^{\pi i(u-v)}q, re^{-\pi i(u-v)}q\right) &= \sum_{m,n=0}^{\infty} \frac{(q)_{m+n} r^{m+n} q^{m+n}}{(q)_m (q)_n (q)_{m+n}} e^{\pi i(m-n)u} e^{\pi i(n-m)v} \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{(q)_m r^m q^m}{(q)_n (q)_{m-n} (q)_m} e^{\pi i(2n-m)(u-v)} \\ &= \sum_{m=0}^{\infty} \frac{H_m(\cos\pi(u-v);q)}{(q)_n} r^n q^n. \end{aligned}$$

Therefore, we obtain the equation (3.7) from the equation (3.1). □

Proof of Theorem 2.4.2. From the Taylor expansion of the q-exponential function [GR, (II.2)]

$$(x)_\infty = \sum_{n\geq 0} \frac{1}{(q)_n}(-1)^n q^{\frac{n(n-1)}{2}} x^n,$$

we have

$$\begin{aligned}
(e^{i\theta}rq, e^{-i\theta}rq)_\infty &= \sum_{n\geq 0}(-qr)^n \sum_{k=0}^{n} \frac{q^{\binom{k}{2}}q^{\binom{n-k}{2}}}{(q)_k(q)_{n-k}} e^{i(n-2k)\theta} \\
&= \sum_{n\geq 0} {}_1\phi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix}; q; e^{2i\theta}q\right) \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{(q)_n}(e^{-i\theta}rq)^n \\
&= \sum_{n\geq 0} F_{n+1}(\cos\pi(u-v) \mid q) r^n.
\end{aligned}$$

By expanding (3.7) and comparing the coefficient in (3.7), we obtain (2.34). To prove (2.35), we divide by both sides of (3.7) by $(e^{i\theta}rq, e^{-i\theta}rq)_\infty$;

$$\frac{1}{(re^{\pi i(u-v)}q, re^{-\pi i(u-v)}q)_\infty} S(r) = \sum_{m\geq 0} \frac{\mu(u,v;1-m)}{(q)_m} q^m r^m. \tag{3.10}$$

By the generating function of continuous q-Hermite polynomial (3.2), the left hand side of (3.10) is equal to

$$\sum_{m\geq 0}\sum_{k=0}^{m} \mu(u,v;k+1) \frac{H_{m-k}(\cos\pi(u-v) \mid q)}{(q)_{m-k}} q^{m-k} r^m.$$

Then we obtain the conclusion (2.35). □

3.2 Relations to the bilateral q-hypergeometric series

Since the function $\mu(u,v;\alpha)$ has the expression by using the bilateral q-hypergeometric series (1.43), we provide feedback to the function $\mu(u,v;\alpha)$ from the rich transformation formulas of the bilateral q-hypergeometric series. First, we prove the following proposition:

Proposition 3.2.1. *We have the following expressions:*

$$\begin{aligned}
iq^{\frac{1}{8}}\mu(u,v;\alpha) &= \frac{(x/y)^{\frac{\alpha}{2}}}{\theta_q(-x/a)} \frac{(aq/y)_\infty}{(q/y)_\infty} {}_1\psi_2\left(\begin{matrix} y/a \\ 0, y \end{matrix}; q; x\right) &(3.11)\\
&= \left(\frac{x}{y}\right)^{\frac{\alpha}{2}} \frac{(a,q,aq/x,aq/y)_\infty}{\theta_q(-y)\theta_q(-x/a)} {}_2\psi_2\left(\begin{matrix} x/a, y/a \\ 0, 0 \end{matrix}; q; a\right) &(3.12)\\
&= \left(\frac{x}{y}\right)^{\frac{\alpha}{2}} \frac{(a,q,x,y)_\infty}{\theta_q(-y)\theta_q(-x/a)} {}_0\psi_2\left(\begin{matrix} - \\ x, y \end{matrix}; q; \frac{xy}{a}\right) &(3.13)
\end{aligned}$$

These equations (3.11)-(3.13) are proved by using the degenerated Bailey's transformations (1.37)-(1.39).

Further, by some Bailey's transformation formula [AB1, (3.2.4)], [B, (3.2)], [GR, p. 147, Exercise 5.11]

$$\begin{aligned}
{}_2\psi_2\left(\begin{matrix} a, b \\ c, d \end{matrix}; q, x\right) &= \frac{(ax, bx, qc/abx, qd/abx)_\infty}{(x, abx, q^2/abx, cd/abx)_\infty} \\
&\cdot {}_6\psi_8\left(\begin{matrix} \sqrt{qabx}, -\sqrt{qabx}, abx/c, abx/d, a, b \\ \sqrt{abx/q}, -\sqrt{abx/q}, ax, bx, c, d, 0, 0 \end{matrix}; q, \frac{cdx}{q}\right),
\end{aligned} \tag{3.14}$$

we also obtain the following expression for the function $\mu(u,v;\alpha)$ by the (degenerate) very-well-poised bilateral q-hypergeometric series.

Theorem 3.2.2. *We define*

$$\mathcal{W}(a;b,c;q) := \frac{1-a}{(1-a/b)(1-a/c)}\, {}_4\psi_8 \left(\begin{matrix} \sqrt{a}q, -\sqrt{a}q, b, c \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, 0,0,0,0 \end{matrix}; q; \frac{a^3q^2}{bc} \right) \tag{3.15}$$

$$= \sum_{n\in\mathbb{Z}} \frac{(1-aq^{2n})(b,c;q)_n}{(a/b,a/c)_{n+1}} q^{2n^2} \left(\frac{a^3}{bc}\right)^n. \tag{3.16}$$

Then, we have

$$iq^{\frac{1}{8}}\mu(u,v;\alpha) = \left(1-\frac{x}{q}\right)\left(1-\frac{y}{q}\right) \frac{(x/y)^{\frac{\alpha}{2}}(x,aq/y)_\infty}{(q/y,x/a)_\infty \theta_q(-xy/aq)} \mathcal{W}\left(\frac{xy}{aq}; \frac{x}{a}\frac{y}{a}; q\right) \tag{3.17}$$

$$= \frac{(x/y)^{\frac{\alpha}{2}}(x,aq/y)_\infty}{(q/y,x/a)_\infty \theta_q(-xy/aq)} \sum_{n\in\mathbb{Z}} \left(1-\frac{xy}{a}q^{2n-1}\right) \frac{(x/a,y/a)_n}{(x,y)_n} q^{2n^2-3n} \left(\frac{x^2y^2}{a}\right)^n. \tag{3.18}$$

In particular, if $\alpha=1$, we have

$$-iq^{\frac{1}{8}}\mu(u,v) = \frac{\sqrt{xy}}{\theta_q(-xy)} \mathcal{W}(xy;,x,y;q) \tag{3.19}$$

$$= \frac{\sqrt{xy}}{\theta_q(-xy)} \sum_{n\in\mathbb{Z}} \frac{1-xyq^{2n}}{(1-xq^n)(1-yq^n)} x^{2n}y^{2n}q^{2n^2}. \tag{3.20}$$

Proof. By taking the limit $c\to 0, d\to 0$ in the equation (3.14), we have

$${}_2\psi_2\left(\begin{matrix} a,b \\ 0,0 \end{matrix}; q; x\right) = \frac{(ax,bx)_\infty}{(x,abx,q^2/abx)_\infty}\, {}_4\psi_8 \left(\begin{matrix} \sqrt{abxq}, -\sqrt{abxq}, a, b \\ \sqrt{abx/q}, -\sqrt{abx/q}, ax, bx, 0,0,0,0 \end{matrix}; q; \frac{a^2b^2x^3}{q} \right).$$

Therefore, from the equation (3.12), we obtain the equation (3.17). ∎

We recall two kind of Slater's transformation formulas.

Lemma 3.2.3 ([GR, Chapter 5 (5.4.3), (5.5.2)], [Sl, (4)])**.** *For appropriate complex parameters $a, a_1, \ldots, a_r, b_1, \ldots, b_{2r+2}, c_1 \ldots, c_r, z$, we have*

$$\theta_q(-dzq)\left(\prod_{j=1}^r \frac{(b_j,q/a_j)_\infty}{\theta_q(-c_j)}\right) {}_r\psi_r\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_r \end{matrix}; q; z\right)$$
$$= \sum_{m=1}^r \frac{\theta_q(-c_m dz)}{\theta_q(-c_m)} (c_m/a_m, b_mq/c_m)_\infty \tag{3.21}$$
$$\cdot \left(\prod_{j\neq m} \frac{(c_m/a_j, b_jq/c_m)_\infty}{\theta_q(-c_m/c_j)}\right) {}_r\psi_r\left(\begin{matrix} a_1q/c_m,\ldots,a_rq/c_m \\ b_1q/c_m,\ldots,b_rq/c_m \end{matrix}; q; z\right),$$

and

$$\left(\frac{\prod_{1\le j\le 2r}(q/b_j,aq/b_j)_\infty}{\prod_{1\le k\le r}\theta_q(-a_k)\theta_q(-a_k/a)}\right){}_{2r}\psi_{2r}\left(\begin{matrix}b_1,\ldots,b_{2r}\\ aq/b_1,\ldots,aq/b_{2r}\end{matrix};q;-\frac{a^rq^r}{b_1\cdots b_{2r}}\right)$$
$$=\frac{\theta_q(-a)}{\theta_q(-\sqrt{a})\theta_q(\sqrt{a})}\sum_{m=1}^{r}\frac{a_m\theta_q(\sqrt{a}/a_m)\theta_q(-\sqrt{a}/a_m)}{\theta_q(-a_m)\theta_q(-a/a_m)\theta_q(-a_m^2/a)}$$
$$\left(\frac{\prod_{1\le j\le 2r}(a_mq/b_j,aq/a_mb_j)_\infty}{\prod_{k\neq m}\theta_q(-a_k/a_m)\theta_q(-a_ma_k/a)}\right){}_{2r}\psi_{2r}\left(\begin{matrix}a_mb_1/a,\ldots,a_mb_{2r}/a\\ a_mq/b_1,\ldots,a_mq/b_{2r}\end{matrix};q;-\frac{a^rq^r}{b_1\cdots b_{2r}}\right),\tag{3.22}$$

where

$$d=\prod_{j=1}^{r}\frac{a_j}{c_j}.$$

In particular, if $a_1=b_1=\sqrt{a}q, a_2=b_2=-\sqrt{a}q$ *in the equation* (3.22)*, we have*

$$\left(\frac{\prod_{3\le j\le 2r}(q/b_j,aq/b_j)_\infty}{\prod_{3\le k\le r}\theta_q(-a_k)\theta_q(-a_k/a)}\right){}_{2r}\psi_{2r}\left(\begin{matrix}\sqrt{a}q,-\sqrt{a}q,b_3,\ldots,b_{2r}\\ \sqrt{a},-\sqrt{a},aq/b_3,\ldots,aq/b_{2r}\end{matrix};q;\frac{a^{r-1}q^{r-2}}{b_3\cdots b_{2r}}\right)$$
$$=\frac{\theta_q(-a)}{1-a}\sum_{m=3}^{r}\frac{1-a_m^2/a}{\theta_q(-a_m)\theta_q(-a_m/a)\theta_q(-a_m^2/a)}\left(\frac{\prod_{3\le j\le 2r}(a_mq/b_j,aq/a_mb_j)_\infty}{\prod_{k\neq m}\theta_q(-a_k/a_m)\theta_q(-a_ka_m/a)}\right)$$
$$\cdot{}_{2r}\psi_{2r}\left(\begin{matrix}a_mq/\sqrt{a},-a_mq/\sqrt{a},a_mb_3/a,\ldots,a_mb_{2r}/a\\ a_m/\sqrt{a},-a_m/\sqrt{a},a_mq/b_3,\ldots,a_mq/b_{2r}\end{matrix};q;\frac{a^{r-1}q^{r-2}}{b_3\cdots b_{2r}}\right).\tag{3.23}$$

Using Slater's transformation formulas (3.21) and (3.23), we obtain the following transformation formulas for the function $\mu(u,v;\alpha)$. In particular, the formula (3.25) coinsides with the property (2.17) of the function $\mu(u,v;\alpha)$.

Theorem 3.2.4. *We have the following expressions, where* $\xi_1=e^{2\pi iz_1},\xi_2=e^{2\pi iz_2}$.

(1) $$\theta_q\left(-\frac{xy\xi_1\xi_2}{aq}\right)\frac{\theta_q(-y)\theta_q(-x/a)}{\theta_q(-\xi_1\theta_q(-\xi_2)}\mu(u,v;\alpha)$$
$$=\theta_q\left(-\frac{xy\xi_2}{aq}\right)\frac{\theta_q(-y\xi_1)\theta_q(-x\xi_1/a)}{\theta_q(-\xi_1)\theta_q(-\xi_2/\xi_1)}\mu(u+z_1,v+z_1;\alpha)\tag{3.24}$$
$$+\theta_q\left(-\frac{xy\xi_1}{aq}\right)\frac{\theta_q(-y\xi_2)\theta_q(-x\xi_2/a)}{\theta_q(-\xi_2)\theta_q(-\xi_1/\xi_2)}\mu(u+z_2,v+z_2;\alpha).$$

(2) $$\mu(u,v;\alpha)-\mu(u+z_1,v+z_1;\alpha)$$
$$=iq^{-\frac{1}{8}}\left(\frac{x}{y}\right)^{\frac{\alpha}{2}}\frac{(a,q,q)_\infty\theta_q(-xy\xi_1/a)\theta_q(-\xi_1)}{\theta_q(-a/x)\theta_q(-y\xi_1)\theta_q(-y)\theta_q(-x\xi_1/a)}{}_1\phi_1\left(\begin{matrix}q/a\\0\end{matrix};q;\frac{xq}{y}\right).\tag{3.25}$$

(3) $$iq^{\frac{1}{8}}\left(\frac{x}{y}\right)^{-\frac{\alpha}{2}}\frac{\theta_q(-y)\theta_q(-x/a)}{(a,q,q)_\infty}\mu(u,v;\alpha)$$
$$=\frac{\theta_q(-y)\theta_q(-x/a)}{\theta_q(-x/y)\theta_q(-a)}{}_1\phi_1\left(\begin{matrix}q/a\\0\end{matrix};q;\frac{yq}{x}\right)+\frac{\theta_q(-x)\theta_q(-y/a)}{\theta_q(-y/x)\theta_q(-a)}{}_1\phi_1\left(\begin{matrix}q/a\\0\end{matrix};q;\frac{xq}{y}\right).\tag{3.26}$$

Proof of Theorem 3.2.4 *using the equation* (3.21). Putting $r = 2, a_1 = x/a, a_2 = y/a, b_1 = b_2 = 0, c_1 = q/\xi_1, c_2 = q/\xi_2$ in the equation(3.21), we have

$$\begin{aligned}
&\theta_q\left(-\frac{xy\xi_1\xi_2}{aq}\right)\frac{(aq/x,aq/y)_\infty}{\theta_q(-\xi_1)\theta_q(-\xi_2)}{}_2\psi_2\left(\begin{matrix}x/a,y/a\\0,0\end{matrix};q;a\right)\\
&\qquad=\theta_q\left(-\frac{xy\xi_2}{aq}\right)\frac{(aq/x\xi_1,aq/y\xi_1)_\infty}{\theta_q(-\xi_1)\theta_q(-\xi_2/\xi_1)}{}_2\psi_2\left(\begin{matrix}x\xi_1/a,y\xi_1/a\\0,0\end{matrix};q;a\right)\\
&\qquad+\theta_q\left(-\frac{xy\xi_1}{aq}\right)\frac{(aq/x\xi_2,aq/y\xi_2)_\infty}{\theta_q(-\xi_2)\theta_q(-\xi_1/\xi_2)}{}_2\psi_2\left(\begin{matrix}x\xi_2/a,y\xi_2/a\\0,0\end{matrix};q;a\right).
\end{aligned}\tag{3.27}$$

Note that

$${}_2\psi_2\left(\begin{matrix}x,1\\0,0\end{matrix};q;a\right)={}_0\phi_1\left(\begin{matrix}-\\q/x\end{matrix};q;\frac{q^2}{ax}\right)=\frac{1}{(q/x)_\infty}{}_1\phi_1\left(\begin{matrix}q/a\\0\end{matrix};q;\frac{q}{x}\right),$$

putting $\xi_2 = a/x$ in the equation (3.27), we have

$$\begin{aligned}
&\frac{(aq/x.aq/y)_\infty}{\theta_q(-x/a)\theta_q(-y)}{}_2\psi_2\left(\begin{matrix}x/a,y/a\\0,0\end{matrix};q;a\right)-\frac{(aq/x\xi_1,aq/y\xi_1)_\infty}{\theta_q(-x\xi_1/a)\theta_q(-y\xi_1)}{}_2\psi_2\left(\begin{matrix}x\xi_1/a,y\xi_1/a\\0,0\end{matrix};q;a\right)\\
&\qquad=-\frac{\theta_q(-xy\xi_1/a)\theta_q(-\xi_1)(q)_\infty}{\theta_q(-a/x)\theta_q(-y\xi_1)\theta_q(-y)\theta_q(-x\xi_1/a)}{}_1\phi_1\left(\begin{matrix}q/a\\0\end{matrix};q;\frac{xq}{y}\right).
\end{aligned}\tag{3.28}$$

Furthermore, putting $\xi_1 = a/y$ in the equation (3.28), we have

$$\begin{aligned}
\left(\frac{aq}{x},\frac{aq}{y}\right)_\infty{}_2\psi_2\left(\begin{matrix}x/a,y/a\\0,0\end{matrix};q;a\right)&=\frac{\theta_q(-y)\theta_q(-x/a)(q)_\infty}{\theta_q(-x/y)\theta_q(-a)}{}_1\phi_1\left(\begin{matrix}q/a\\0\end{matrix};q;\frac{yq}{x}\right)\\
&+\frac{\theta_q(-x)\theta_q(-y/a)(q)_\infty}{\theta_q(-y/x)\theta_q(-a)}{}_1\phi_1\left(\begin{matrix}q/a\\0\end{matrix};q;\frac{xq}{y}\right).
\end{aligned}\tag{3.29}$$

Therefore, we obtain the equations (3.24)-(3.26) from equations (3.27)-(3.29) respectively using the formula (3.12). □

Proof of Theorem 3.2.4 *using the equation* (3.23). Putting $r = 2$ and taking the limit $b_5 \to \infty, \ldots, b_8 \to \infty$ in the equation (3.23), we have

$$\begin{aligned}
&\frac{\theta_q(-a_3a_4/a)}{\theta_q(-a)}(1-a){}_4\psi_8\left(\begin{matrix}\sqrt{a}q,-\sqrt{a}q,b_3,b_4\\\sqrt{a},-\sqrt{a},aq/b_3,aq/b_4,0,0,0,0\end{matrix};q;\frac{a^3q^2}{b_3b_4}\right)\\
&=\frac{\theta_q(-a_4)\theta_q(-a_4/a)(a_3q/b_3,a_3q/b_4,aq/a_3b_3,aq/a_3b_4)_\infty}{\theta_q(-a_3^2/a)\theta_q(-a_4/a_3)(q/b_3,q/b_4,aq/b_3,aq/b_4)_\infty}\\
&\cdot\left(1-\frac{a_3^2}{a}\right){}_4\psi_8\left(\begin{matrix}a_3q/\sqrt{a},-a_3q/\sqrt{a},a_3b_3/a,a_3b_4/a\\a_3/\sqrt{a},-a_3/\sqrt{a},a_3q/b_3,a_3q/b_4,0,0,0,0\end{matrix};q;\frac{a_3^4q^2}{ab_3b_4}\right)\\
&+\frac{\theta_q(-a_3)\theta_q(-a_3/a)(a_4q/b_3,a_4q/b_4,aq/a_4b_3,aq/a_4b_4)_\infty}{\theta_q(-a_4^2/a)\theta_q(-a_3/a_4)(q/b_3,q/b_4,aq/b_3,aq/b_4)_\infty}\\
&\cdot\left(1-\frac{a_4^2}{a}\right){}_4\psi_8\left(\begin{matrix}a_4q/\sqrt{a},-a_4q/\sqrt{a},a_4b_3/a,a_4b_4/a\\a_4/\sqrt{a},-a_4/\sqrt{a},a_4q/b_3,a_4q/b_4,0,0,0,0\end{matrix};q;\frac{a_4^4q^2}{ab_3b_4}\right).
\end{aligned}\tag{3.30}$$

Putting $a \mapsto \frac{xy}{aq}$ and $b_3 = x/a, b_4 = y/a, a_3 = xy\xi_1/aq, a_4 = xy\xi_2/aq$ in the equation

(3.30), we have

$$\begin{aligned}
&\frac{\theta_q(-xy\xi_1\xi_2/aq)}{\theta_q(-xy/aq)}\left(1-\frac{xy}{aq}\right){}_4\psi_8\left(\begin{matrix}\sqrt{xyq/a},-\sqrt{xyq/a},x/a,y/a\\ \sqrt{xy/aq},-\sqrt{xy/aq},x,y,0,0,0,0\end{matrix};q;\frac{x^2y^2}{aq}\right)\\
&=\frac{\theta_q(-xy\xi_2/aq)\theta_q(-\xi_2)(x\xi_1,y\xi_1,aq/x\xi_1,aq/y\xi_1)_\infty}{\theta_q(-xy\xi_1^2/aq)\theta_q(-\xi_2/\xi_1)(aq/x,aq/y,x,y)_\infty}\\
&\quad\cdot\left(1-\frac{xy\xi_1^2}{aq}\right){}_4\psi_8\left(\begin{matrix}\xi_1\sqrt{xyq/a},-\xi_1\sqrt{xyq/a},x\xi_1/a,y\xi_1/a\\ \xi_1\sqrt{xy/aq},-\xi_1\sqrt{xy/aq},x\xi_1,y\xi_1,0,0,0,0\end{matrix};q;\frac{x^2y^2\xi_1^4}{aq}\right)\\
&+\frac{\theta_q(-xy\xi_1/aq)\theta_q(-\xi_1)(x\xi_2,y\xi_2,aq/x\xi_2,aq/y\xi_2)_\infty}{\theta_q(-xy\xi_2^2/aq)\theta_q(-\xi_1/\xi_2)(aq/x,aq/y,x,y)_\infty}\\
&\quad\cdot\left(1-\frac{xy\xi_2^2}{aq}\right){}_4\psi_8\left(\begin{matrix}\xi_2\sqrt{xyq/a},-\xi_2\sqrt{xyq/a},x\xi_2/a,y\xi_2/a\\ \xi_2\sqrt{xy/aq},-\xi_2\sqrt{xy/aq},x\xi_2,y\xi_1,0,0,0,0\end{matrix};q;\frac{x^2y^2\xi_2^4}{aq}\right).
\end{aligned}\tag{3.31}$$

Note that

$$ {}_4\psi_8\left(\begin{matrix}\sqrt{axq},-\sqrt{axq},x,1\\ \sqrt{ax/q},-\sqrt{ax/q},a,ax,0,0,0,0\end{matrix};q;\frac{a^3x^2}{q}\right)=\frac{(q^2/ax)_\infty}{(q/x)_\infty}{}_1\phi_1\left(\begin{matrix}q/a\\0\end{matrix};q;\frac{q}{x}\right)$$

by using the equation (3.14) to take the limit $c\to 0, d\to 0, b\to 1$. Putting $\xi_2=a/x$ in the equation (3.31), we have

$$\begin{aligned}
&\frac{(aq/x,aq/y,x,y)_\infty}{\theta_q(-xy/aq)}\left(1-\frac{xy}{aq}\right){}_4\psi_8\left(\begin{matrix}\sqrt{xyq/a},-\sqrt{xyq/a},x/a,y/a\\ \sqrt{xy/aq},-\sqrt{xy/aq},x,y,0,0,0,0\end{matrix};q;\frac{x^2y^2}{aq}\right)\\
&=\frac{\theta_q(-y)\theta_q(-x/a)(x\xi_1,y\xi_1,aq/x\xi_1,aq/y\xi_1)_\infty}{\theta_q(-y\xi_1)\theta_q(-x\xi_1/a)\theta_q(-xy\xi_1^2/aq)}\\
&\quad\cdot\left(1-\frac{xy\xi_1^2}{aq}\right){}_4\psi_8\left(\begin{matrix}\xi_1\sqrt{xyq/a},-\xi_1\sqrt{xyq/a},x\xi_1/a,y\xi_1/a\\ \xi_1\sqrt{xy/aq},-\xi_1\sqrt{xy/aq},x\xi_1,y\xi_1,0,0,0,0\end{matrix};q;\frac{x^2y^2\xi_1^4}{aq}\right)\\
&+\frac{\theta_q(-xy\xi_1/aq)\theta_q(-\xi_1)(a)_\infty}{\theta_q(-x\xi_1/a)\theta_q(-y\xi_1/q)}{}_1\phi_1\left(\begin{matrix}q/a\\0\end{matrix};q;\frac{xq}{y}\right).
\end{aligned}\tag{3.32}$$

Furthermore, putting $\xi_1=a/x$ in the equation (3.32), we have

$$\begin{aligned}
&\frac{(aq/x,aq/y,x,y)_\infty}{\theta_q(-xy/aq)}\left(1-\frac{xy}{aq}\right){}_4\psi_8\left(\begin{matrix}\sqrt{xyq/a},-\sqrt{xyq/a},x/a,y/a\\ \sqrt{xy/aq},-\sqrt{xy/aq},x,y,0,0,0,0\end{matrix};q;\frac{x^2y^2}{aq}\right)\\
&=\frac{(a)_\infty\theta_q(-y)\theta_q(-x/a)}{\theta_q(-a)\theta_q(-x/y)}{}_1\phi_1\left(\begin{matrix}q/a\\0\end{matrix};q;\frac{yq}{x}\right)+\frac{(a)_\infty\theta_q(-x)\theta_q(-y/a)}{\theta_q(-a)\theta_q(-y/x)}{}_1\phi_1\left(\begin{matrix}q/a\\0\end{matrix};q;\frac{xq}{y}\right)
\end{aligned}\tag{3.33}$$

Therefore, we obtain the equations (3.24)-(3.26) from equations (3.31)-(3.33) using the formula (3.17),respectively. □

From the expression (3.19), we have expressions for Ramanujan's mock theta functions using the degenerated very-well-poised bilateral q-hypergeometric series (see Appendix).

4 Higher level Appell functions and q-difference equations

In this Section, we put $x = e^{2\pi i u}$, $y = e^{2\pi i v}$, $q = e^{2\pi i \tau}$.

4.1 Definition of higher level Appell functions

Kac and Wakimoto introduced the Multivariable generalization of Appell function by explicitly computing characters related to affine Lie superalgebras $\mathfrak{gl}(l \mid 1)^\wedge$ and $\mathfrak{sl}(l \mid 1)^\wedge$ [KW2]:

$$A_{B,s}(x; z_1, \ldots, z_l; q) := \sum_{k \in \mathbb{Z}^l} \frac{q^{\frac{1}{2}{}^t kBk} z_1^{k_1} \cdots z_l^{k_l}}{1 + xq^{s(k)}},$$

where B be a positive definite symmetric matrix and s be a linear function on $\mathbb{C}^l$. In particular, if $l = 1$, $B = (m)$ and $s(k) = k$, the function

$$A_m(x, y; q) = A_m(x, y) := x^{\frac{m}{2}} A_{B,s}(-x; (-1)^m y q^{\frac{m}{2}}; q) = x^{\frac{m}{2}} \sum_{k \in \mathbb{Z}} \frac{(-1)^{mk} y^k q^{\frac{mk(k+1)}{2}}}{1 - xq^k}$$

is called higher level Appell function. Zwegers provided an expression of the higher level Appell function $A_m(x, y)$ using the μ-function from a simple algebraic identity:

$$A_m(x, (-1)^{m-1} y) = \sum_{k=0}^{m-1} -iq^{-\frac{m}{8}} x^k \frac{\theta_{q^m}(-yq^k)}{\sqrt{yq^k}} \mu(mu, v + k\tau; m\tau). \tag{4.1}$$

Furthermore, he introduced the completed level m Appell function

$$\begin{aligned}&\widehat{A}_m(u, v; \tau)\\ &:= A_m(x, y; q) + \frac{i}{2} \sum_{k=0}^{m-1} x^k \vartheta\left(v + k\tau + \frac{m-1}{2}, m\tau\right) R\left(mu - v - k\tau - \frac{m-1}{2}; m\tau\right),\end{aligned}$$

where the function $R(u; \tau)$ is defined by the equation (1.23), and showed that the completed level m Appell function satisfies the following modular transformation formula with respect to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ [Zw2]:

$$\widehat{A}_m\left(\frac{u}{c\tau + d}, \frac{v}{c\tau + d}; \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d) e^{\frac{\pi i c}{c\tau + d}(-mu^2 + 2uv)} \widehat{A}_m(u, v; \tau).$$

In this section, we provide q-difference equations satisfied by the higher level Appell function $A_m(x, y)$ and give another proof of the equation (4.1) as an application.

4.2 q-difference equations satisfied by the higher level Appell functions

We define

$$G_m(x, y) = G_m(x, y; q) := x^{-\frac{m}{2}} A_m(x, (-1)^{m-1} y) = \sum_{n \in \mathbb{Z}} \frac{(-1)^n y^n q^{\frac{mn(n+1)}{2}}}{1 - xq^n}.$$

First, we provide a q-difference equation satisfied by the function $G_m(x, y)$.

Theorem 4.2.1. *The function $G_m(x, y)$ satisfies the following q-difference equation:*

$$yG_m(xq, y) + x^m G_m(x, y) + \sum_{k=0}^{m-1} x^k \theta_{q^m}(-yq^k) = 0, \tag{4.2}$$

$$\left[\prod_{k=0}^{m-1}(T_x - q^k)\right](yT_x + x^m)G_m(x, y) = 0. \tag{4.3}$$

Proof. From definitions of the function $G_m(x, y)$ and theta function, we have

$$\begin{aligned}
G_m(xq, y) &= \sum_{n\in\mathbb{Z}} \frac{(-1)^n y^n q^{\frac{mn(n+1)}{2}}}{1 - xq^{n+1}}(x^m q^{m(n+1)} + 1 - x^m q^{m(n+1)}) \\
&= x^m \sum_{n\in\mathbb{Z}} \frac{(-1)^n y^n q^{\frac{m(n+1)(n+2)}{2}}}{1 - xq^{n+1}} + \sum_{n\in\mathbb{Z}} (-1)^n y^n q^{\frac{mn(n+1)}{2}} \sum_{k=0}^{m-1} x^k q^{k(n+1)} \\
&= -\frac{x^m}{y} G_m(x, y) - \frac{1}{y}\sum_{k=0}^{m-1} x^k \theta_{q^m}(-yq^k).
\end{aligned}$$

Therefore, the equation (4.2) holds. Furthermore, by applying $\prod_{k=0}^{m-1}(T_x - q^k)$ to both sides of the equation (4.2), we obtain the equation (4.3) from

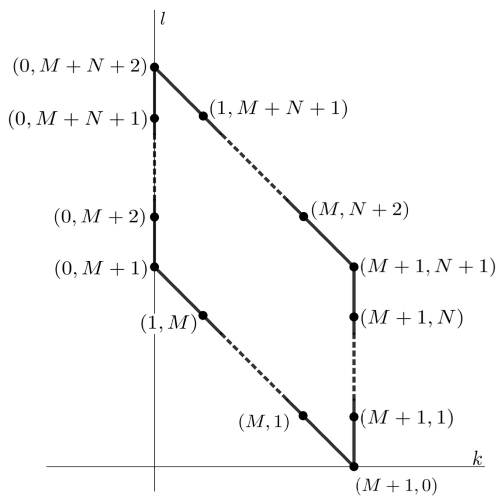
$$\prod_{k=0}^{m-1}(T_x - q^k)\sum_{k=0}^{m-1} x^k \theta_{q^m}(-yq^k) = 0.$$

□

Generalizing the equation (4.3), we consider the following factorized q-difference equation:

$$\begin{aligned}
&\left[\prod_{l=0}^{N}(T_x - q^{\beta_l})\right]\left[\prod_{k=0}^{M}(T_x + xq^{\alpha_k + k})\right] f(x) \\
&:= \left[\prod_{l=0}^{N}(T_x - q^{\beta_l})\right](T_x + xq^{\alpha_M + M}) \cdots (T_x + xq^{\alpha_0}) f(x) = 0.
\end{aligned} \tag{4.4}$$

The Newton-Puiseux diagram for the q-difference equation (4.4) is as follows.



Theorem 4.2.2. *For distinct complex numbers $\alpha_0, \ldots, \alpha_M, \beta_0, \ldots, \beta_N$, the q-difference equation* (4.4) *has the following fundamental solutions:*

$$x^{\beta_n - M - \frac{1}{2}} \sum_{k=0}^{M} q^{-\frac{\alpha_k}{2}} \left(\prod_{\substack{1 \le j \le M \\ j \ne k}} \frac{1}{1 - q^{\alpha_k - \alpha_j}} \right) \mu(u + v + M\tau, v + (\beta_n - \alpha_k)\tau), \tag{4.5}$$

$$\frac{1}{\theta_q(-xq^{\alpha_m})}, \tag{4.6}$$

for $m = 0, \ldots, M$ and $n = 0, \ldots, N$

Proof. Since the factors $(T_x + ax)$, $(T_x + bx)$ are noncommutative and satisfy the following commutative relation:

$$(T_x + ax)(T_x + bx) = (T_x + bxq)(T_x + ax/q),$$

the q-difference equation (4.4) has the following solutions

$$\frac{1}{\theta_q(-xq^{\alpha_m})} \quad (m = 0, \ldots, M).$$

Next, let the formal series $x^\rho \sum_{n \ge 0} a_n x^n$ be a solution of the following q-difference equation:

$$\begin{aligned}(T_x - q^\alpha) &\left[\prod_{k=0}^{M} (T_x + xq^{\alpha_k + k}) \right] f(x) \\ &:= (T_x - q^\alpha)(T_x + xq^{\alpha_M + M}) \cdots (T_x + xq^{\alpha_0}) f(x) = 0.\end{aligned} \tag{4.7}$$

Putting

$$\sum_{k=0}^{M+1} e_k z^k := \prod_{k=0}^{M} (1 + zq^{\alpha_k}),$$

by an induction on M, we have

$$\left[\prod_{k=0}^{M} (T_x + xq^{\alpha_k + k}) \right] \sum_{n=0}^{\infty} a_n x^{n+\rho} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{M+1} a_{n-k} q^{(M+1-k)(n+\rho) + \frac{k(k-1)}{2}} e_k \right) x^{n+\rho},$$

where $a_0 = 1, a_{-1} = \ldots + a_{-M-1} = 0$. Therefore, putting $a_n = A_n q^{-\frac{n(n+1)}{2} - \alpha n}$, we obtain

$$\begin{aligned}&(T_x - q^\alpha) \left[\prod_{k=0}^{M} (T_x + xq^{\alpha_k + k}) \right] x^\rho \sum_{n \ge 0} A_n q^{-\frac{n(n+1)}{2}} (x/q^\alpha)^n \\ &= x^\rho q^{(M+1)\rho} \sum_{n=0}^{\infty} (q^{n+\rho} - q^\alpha) \left(\sum_{k=0}^{M+1} A_{n-k} q^{(\alpha - \rho)k} e_k \right) q^{-\frac{n(n+1)}{2}} (xq^{M+1-\alpha})^n = 0,\end{aligned}$$

namely the coefficients A_n satisfy the following relations:

$$\sum_{k=0}^{M+1} A_{n-k} e_k = 0 \quad (n \ge 1),$$
$$A_0 = 1, A_{-1} = \ldots = A_{-M-1} = 0,$$

Therefore, we have

$$\left[\prod_{k=0}^{M}(1+zq^{\alpha_k})\right]\sum_{n\geq 0}A_n z^n = \left(\sum_{k=0}^{M+1} e_k z^k\right)\left(\sum_{n=0}^{\infty} A_n z^n\right)$$
$$= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{M+1} e_k A_{m-k}\right) z^m = 1. \tag{4.8}$$

Using the relation (4.8), we can compute the q-Borel-Laplace image of the formal solution $\sum_{n\geq 0} a_n x^n$ as follows:

$$\begin{aligned}
&x^\alpha \mathcal{L}^+ \circ \mathcal{B}^+ \left(\sum_{n=0}^{\infty} a_n x^n\right)(x, -\lambda^{-1})\\
&= x^\alpha \mathcal{L}^+ \circ \mathcal{B}^+ \left(\sum_{n=0}^{\infty} A_n q^{-\frac{n(n+1)}{2}} (xq^{-\alpha})^n\right)(x, -\lambda^{-1})\\
&= x^\alpha \mathcal{L}^+ \left(\prod_{k=0}^{M} \frac{1}{1+xq^{\alpha_k-\alpha-1}}\right)(x, -\lambda^{-1})\\
&= \frac{x^\alpha}{\theta_q(-x\lambda)} \sum_{n\in\mathbb{Z}} (-x\lambda q^M)^n q^{\frac{n(n+1)}{2}} \prod_{k=0}^{M} \frac{-\lambda q^{\alpha-\alpha_k}}{1-\lambda q^{n+\alpha-\alpha_k}}\\
&= -\frac{x^{\alpha-M}\lambda e_{M+1}^{-1}}{\theta_q(-x\lambda q^M)} q^{(M+1)\alpha-\frac{M(M-1)}{2}} \sum_{k=0}^{M}\left(\prod_{j\neq k}\frac{1}{1-q^{\alpha_k-\alpha_j}}\right)\sum_{n\in\mathbb{Z}}\frac{(-x\lambda q^M)^n q^{\frac{n(n+1)}{2}}}{1-\lambda q^{n+\alpha-\alpha_k}}.
\end{aligned} \tag{4.9}$$

The final equality follows form the equation (1.1) and a partial fraction expansion

$$\prod_{k=0}^{M}\frac{1}{1-\lambda q^{-\alpha_k}} = \sum_{k=0}^{M}\left(\prod_{j\neq k}\frac{1}{1-q^{\alpha_k-\alpha_j}}\right)\frac{1}{1-\lambda q^{-\alpha_k}}.$$

Thus the function

$$x^{\alpha-M-\frac{1}{2}}\sum_{k=0}^{M} q^{-\frac{\alpha_k}{2}}\left(\prod_{j\neq k}\frac{1}{1-q^{\alpha_k-\alpha_j}}\right)\mu(u+v+M\tau, v+(\alpha-\alpha_k)\tau) \tag{4.10}$$

is a solution of the q-difference equation (4.7) by putting $\lambda = e^{2\pi i v}$ in the equation (4.9). The desired solution (4.5) is given by putting $\alpha = \beta_n$. □

Remark . *We provide another derivation of the solution* (4.10) (*or* (4.9)) *of the q-difference equation* (4.7) *without using the q-Borel-Laplace transformation. We put*

$$\left[\prod_{k=0}^{M}(T_x + xq^{\alpha_k+k})\right] f(x) = \tilde{f}(x),$$

then the function $\tilde{f}(x)$ has the pseudo-periodicity $\tilde{f}(xq) = q^\alpha \tilde{f}(x)$. That is, $\tilde{f}(x) = x^\alpha$ is a solution of the q-difference equation (4.7)*, and it sufficient to solve the following non-homogeneous q-difference equation:*

$$\left[\prod_{k=0}^{M}(T_x + xq^{\alpha_k+k})\right] f(x) = x^\alpha. \tag{4.11}$$

By putting $f(x)=\frac{g(x)}{\theta_q(-x\lambda)}$*, we rewrite the q-difference equation* (4.11) *as*

$$\left[\prod_{k=0}^{M}(T_x-\lambda^{-1}q^{\alpha_k})\right]g(x)=x^{\alpha}\theta_q(-x\lambda q^{M+1}).$$

If $g(x)=\sum_{n\in\mathbb{Z}}A_n x^{n+\rho}$*, coefficients* A_n *and characteristic exponent* ρ *satisfy the following relation:*

$$\sum_{n\in\mathbb{Z}}A_n\left(\prod_{k=0}^{M}(q^{n+\rho}-\lambda^{-1}q^{\alpha_k})\right)x^{n+\rho}=x^{\alpha}\sum_{n\in\mathbb{Z}}(-x\lambda q^{M})^n q^{\frac{n(n+1)}{2}}.$$

Therefore, we have $\rho=\alpha$, $A_n\prod_{k=0}^{M}(q^{n+\alpha}-\lambda^{-1}q^{\alpha_k})=(-\lambda q^M)^n q^{\frac{n(n+1)}{2}}$*. Hence, the function*

$$f(x)=\frac{x^{\alpha}(-\lambda)^{M+1}}{\theta_q(-x\lambda)}\sum_{n\in\mathbb{Z}}(-x\lambda q^M)^n q^{\frac{n(n+1)}{2}}\prod_{k=0}^{M}\frac{q^{-\alpha_k}}{1-\lambda q^{n+\alpha-\alpha_k}}$$

is a solution of the q-difference (4.11)*. This solution is the same as that given in* (4.9) *up to a normalization factor* $q^{(M+1)\alpha}$*.*

4.3 An application of the Theorem 4.2.2

By Theorem 4.2.2, the q-difference equation (4.3) satisfied by the function $G_m(x,y)$ has the following fundamental solutions

$$\frac{1}{\theta_{q^m}(-x^m/y)},\quad x^{j-\frac{m}{2}}\mu(mu+z,v+z+j\tau;m\tau).$$

Hence, the function $G_m(x,y)$ is expressed as a linear combination of these functions. We provide another proof of the Zwegers' result (4.1) by determining coefficients of the linear combination.

Proof of the equation (4.1)*.* Putting

$$G_m(x,y)=\frac{C(x)}{\theta_{q^m}(-x^m/y)}+\sum_{k=0}^{m-1}C_k(x)x^{k-\frac{m}{2}}\mu(mu+z,v+z+k\tau;m\tau).$$

From the pseudo-periodicity (4.2) and (1.26), we have

$$yG_m(xq,y)+x^mG_m(x,y)=-i\sqrt{y}q^{-\frac{m}{8}}\sum_{k=0}^{m-1}C_k(x)x^kq^{\frac{k}{2}}=-\sum_{k=0}^{m-1}x^k\theta_{q^m}(-yq^k).$$

That is,

$$C_k(x)=-iq^{\frac{m}{8}}\frac{\theta_{q^m}(-yq^k)}{\sqrt{yq^k}}.$$

Therefore, we have

$$C(x)=\theta_{q^m}(-x^m/y)\left\{G_m(x,y)+iq^{\frac{m}{8}}\sum_{k=0}^{m-1}\frac{\theta_{q^m}(-yq^k)}{\sqrt{yq^k}}x^{k-\frac{m}{2}}\mu(mu+z,v+z+k\tau;m\tau)\right\}. \tag{4.12}$$

Putting $z=0$, since the right hand side of the equation (4.12) is an elliptic function with a simple pole at $u=0$, the function $c(x)$ is a constant function. Therefore, we have

$$\begin{aligned}
&C(x)\\
&= \lim_{x\to^m\sqrt{y}} \theta_{q^m}(-x^m/y)\left\{G_m(x,y) + iq^{\frac{m}{8}}\sum_{k=0}^{m-1}\frac{\theta_{q^m}(-yq^k)}{\sqrt{yq^k}}x^{k-\frac{m}{2}}\mu(mu, v+k\tau; m\tau)\right\}\\
&= 0.
\end{aligned}$$

Thus the function $G_m(x,y)$ is expressed as follows

$$G_m(x,y) = -iq^{\frac{m}{8}}\sum_{k=0}^{m-1}\frac{\theta_{q^m}(-yq^k)}{\sqrt{yq^k}}x^{k-\frac{m}{2}}\mu(mu, v+k\tau; m\tau),$$

hence we have the equation (4.1). □

The q-difference equation (4.3) degenerates at $v=0$. Since the calculations are similar, we only state as the following corollary.

Corollary 4.3.1.
(1) *The function $G_m(x,1)$ satisfies the following q-difference equation.*

$$G_m(xq,1) + x^m G_m(x,1) = -\sum_{k=1}^{m-1} x^k \theta_{q^m}(-q^k)$$

(2) *The function $G_m(x,1)$ satisfies the following m-th order linear q-difference equation:*

$$\left[\prod_{k=1}^{m-1}(T_x - q^k)\right](T_x + x^m)G_m(x,1) = 0. \tag{4.13}$$

Furthermore, the q-difference equation has the following fundamental solutions

$$\frac{1}{\theta_{q^m}(-x^m)}, \quad x^{j-\frac{m}{2}}\mu(mu+z, z+j\tau \mid m\tau) \quad (j=1,\ldots,m-1).$$

(3) *The expression of the function $G_m(x,1)$ using the μ-function is as follows:*

$$G_m(x,1) = \frac{(q^m;q^m)_\infty^3}{\theta_{q^m}(-x^m)} - iq^{\frac{m}{8}}\sum_{j=1}^{m-1}\theta_{q^m}(-q^j)x^{j-\frac{m}{2}}q^{-\frac{j}{2}}\mu(mu, j\tau \mid m\tau). \tag{4.14}$$

Since the universal mock theta functions g_2 and g_3 are expressed as

$$g_2(x,q) = \frac{(-q)_\infty}{(q)_\infty}G_2(x,1), \quad g_3(x,q) = \frac{1}{(q)_\infty}G_3(x,1). \tag{4.15}$$

we have q-difference equations satisfied by the functions g_2 and g_3 (1.48)-(1.51) by Corollary 4.3.1. These relations may be known, but we cannot find suitable the literature. Furthermore, from the equation (4.14) with $m=2$ and $m=3$, we obtain another proof of the equations (1.31) and (1.32), respectively.

5 Application to discrete Painlevé equations

In this section, we put $x = e^{\pi i u}$, $q = e^{\pi i \tau}$.

5.1 Definition of the type $(A_2 + A_1)^{(1)}$ q-Painlevé equation

Discrete Painlevé equations are introduced from various fields such as physics and discrete integrable systems (for examples [BK], [FIK], [RGH]). They were classified by H. Sakai in 2001 in a geometric framework based on the type of rational surfaces [Sa]. Among the classified equations, multiplicative discrete Painlevé equations are sometimes called the q-Painlevé equations. For example, let W be the extended affine Weyl group $W := \mathrm{Aut}(A_5^{(1)}) \ltimes W((A_2 + A_1)^{(1)})$. That is, the group W has generators $s_0, s_1, s_2, r_0, r_1, \iota, \pi$ satisfying the following relations:

$$s_j^2 = (s_j s_{j+1})^3 = r_k^2 = \iota^2 = \pi^6 = 1, \quad \pi s_j = s_{j+1}\pi, \quad r_k \pi = \pi r_{k+1}, \quad r_0 \iota = \iota r_1,$$
$$(j \in \mathbb{Z}/3\mathbb{Z}, k \in \mathbb{Z}/2\mathbb{Z}).$$

The group W is constructed a representation on the field $K := \mathbb{C}\,(q, x, a_0, a_1, a_2, f_0, f_1, f_2)$ by the following actions (for example, see [T]) (and for recent study from the quiver theory, for example, see [BGM], [SO]):

$$\begin{aligned}
s_j(a_j) &= \frac{1}{a_j}, \quad s_j(a_{j+1}) = a_j a_{j+1}, \quad s_j(a_{j+2}) = a_j a_{j+2}, \\
s_j(f_{j+1}) &= f_{j+1}\frac{a_j + f_j}{1 + a_j f_j}, \quad s_j(f_{j+2}) = f_{j+2}\frac{1 + a_j f_j}{a_j + f_j}, \\
r_0(x) &= \frac{q^2}{x}, \quad r_0(f_j) = \frac{a_j a_{j+1}}{f_{j+2}} \frac{a_j a_{j+2} + a_{j+2} f_j + f_j f_{j+2}}{a_j a_{j+1} + a_j f_{j+1} + f_j f_{j+1}}, \\
r_1(x) &= \frac{1}{x}, \quad r_1(f_j) = \frac{1}{a_j a_{j+1} f_{j+1}} \frac{1 + a_j f_j + a_j a_{j+1} f_j f_{j+1}}{1 + a_{j+2} f_{j+2} + a_j a_{j+2} f_j f_{j+2}}, \\
\iota(q) &= \frac{1}{q}, \quad \iota(x) = xq, \quad \iota(a_j) = \frac{1}{a_{2j}}, \quad \iota(f_j) = f_{2j}, \\
\pi(x) &= \frac{q}{x}, \quad \pi(a_j) = a_{j+1}, \quad \pi(f_j) = \frac{1}{f_{j+1}}, \quad (j \in \mathbb{Z}/3\mathbb{Z}).
\end{aligned}$$

The group W has a subgroup isomorphic to $\mathbb{Z}^3$ whose generators are $T := \pi^3 r_0, T_1 := \pi^2 s_2 s_0, T_2 := \pi^4 s_1 s_0$. If $a_0 a_1 a_2 = q, f_0 f_1 f_2 = x^2 q$, these generators satisfy the following actions:

$$T(x, a_1, a_2) = (xq, a_1, a_2), \quad T_1(x, a_1, a_2) = (x, a_1 q, a_2)\,, \quad T_2(x, a_1, a_2) = (x, a_1, a_2 q)\,, \tag{5.1}$$

$$T(f_j) = a_j a_{j+1} f_{j+1} \frac{1 + a_{j+2} f_{j+2} + a_{j+2} a_j f_{j+2} f_j}{1 + a_j f_j + a_j a_{j+1} f_j f_{j+1}}, \quad (j \in \mathbb{Z}/3\mathbb{Z}), \tag{5.2}$$

$$\frac{T_1(f_2)}{f_1} = \frac{1 + a_1 f_1 + a_1^2 a_2 f_2 + a_1 a_2 f_1 f_2}{a_1 a_2 + a_1 f_1 + a_1^2 a_2 f_1 + f_1 f_2}, \quad \frac{T_1^{-1}(f_1) q}{f_2} = \frac{x^2 q^2 + a_1 a_2 f_1 f_2}{x^2 a_1 a_2 + f_1 f_2}, \tag{5.3}$$

$$\frac{T_2(f_1)}{f_2} = \frac{a_! a_2^2 + f_1 + a_1 a_2 f_2 + a_2 f_1 f_2}{a_2 + a_1 a_2 f_1 + f_2 + a_1 a_2^2 f_1 f_2}, \quad \frac{T_2^{-1}(f_2)}{f_1 q} = \frac{a_1 a_2 x^2 + f_1 f_2}{x^2 q^2 + a_1 a_2 f_1 f_2}. \tag{5.4}$$

The equations (5.1)-(5.4) are called the type $(A_2 + A_1)^{(1)}$ q-Painlevé equations. It is known that the type $(A_2 + A_1)^{(1)}$ q-Painlevé equation has q-hypergeometric solutions satisfying the q-Bessel equation (2.26) and q-Hermite-Weber equation (2.2), when $a_0 = 1, f_0 = -1$ (for details, see [KK], [KMNOY1], [KMNOY2], [KNY2], [RGTT], etc).

In the next subsection, we prove the following theorem.

Theorem 5.1.1. *Putting $k, m \in \mathbb{C}, n \in \mathbb{Z}_{\geq 0}$ and*

$$\xi_{m,n,k} := \det\left[\mu(u+v+k\tau, v; m-j-j')\right]_{j,j'=0}^{n}. \tag{5.5}$$

Then,

$$(x, a_1, a_2; f_1, f_2) = \left(x, -q^m, q^{-n}; \frac{\xi_{m,n,1}}{\xi_{m,n,0}}\frac{\xi_{m,n-1,0}}{\xi_{m,n-1,1}}, -\frac{\xi_{m,n-1,1}}{\xi_{m,n-1,0}}\frac{\xi_{m-1,n-1,0}}{\xi_{m-1,n-1,1}}\right) \tag{5.6}$$

are solutions of the type $(A_2 + A_1)^{(1)}$ q-Painlevé equation, where $x := e^{\pi i u}$.

Note that the parameter v is an independent parameter of the equation, so the solution (5.6) is a one parameter family of solutions of the type $(A_2 + A_1)^{(1)}$ q-Painlevé equation. Furthermore, we obtain the following remark by applying the action s_2 to the solution (5.6).

Remark . *Putting $k, m \in \mathbb{C}, n \in \mathbb{Z}_{\geq 0}$, then*

$$(x, a_0, a_2; f_0, f_2) = \left(x, -q^{-m}, q^{n+1}; \frac{\xi_{m,n,1}\xi_{m,n-1,0}}{\xi_{m,n,0}\xi_{m,n-1,1}}, -\frac{\xi_{m+1,n,1}\xi_{m,n,0}}{\xi_{m+1,n,0}\xi_{m,n,1}}\right)$$

are solutions of the type $(A_2 + A_1)^{(1)}$ q-Painlevé equation.

The q-hypergeometric solutions of the q-Painlevé equations are obtained from rewriting the equations as a kind of Riccati type q-difference equations

$$f(xq) = \frac{af(x) + b}{cf(x) + d}$$

by specializing the parameters, and linearizing them. Since the function $\mu(u, v; \alpha)$ satisfies the q-Hermite-Weber equation and q-Bessel equation, it is expected that the function $\mu(u, v; \alpha)$ is related to a solution of the type $(A_2 + A_1)^{(1)}$ q-Painlevé equation. The fact stated in Theorem 5.1.1 is a novel claim and an important result that connects discrete integrable systems and the theory of mock modular forms.

5.2 Bilinear relations satisfied by a τ-function of the type $(A_2 + A_1)^{(1)}$ q-Painlevé equation

First, we present bilinear relations satisfied by the functions $\xi_{m,n,k}$.

Theorem 5.2.1. *The functions $\xi_{m,n,k}$ defined by the equation* (5.5) *satisfy the following* *bilinear equations:*

$$x^2q^{m+2k}\xi_{m-1,n-1,k+1}\xi_{m,n,k} - xq^{m-n+k}\xi_{m,n-1,k+1}\xi_{m-1,n,k} + q^n\xi_{m,n,k+1}\xi_{m-1,n-1,k} = 0, \tag{5.7}$$

$$x^2q^{2n+2k-m}\xi_{m-1,n-1,k}\xi_{m,n,k-1} - xq^k\xi_{m-1,n,k}\xi_{m,n-1,k-1} + q^n\xi_{m,n,k}\xi_{m-1,n-1,k-1} = 0, \tag{5.8}$$

$$xq^{m+k-n}\xi_{m-1,n-2,k}\xi_{m,n,k-1} - q^n\xi_{m-1,n-1,k}\xi_{m,n-1,k-1} + \xi_{m,n-1,k}\xi_{m-1,n-1,k-1} = 0, \tag{5.9}$$

$$x\xi_{m-1,n-1,k+1}\xi_{m,n-1,k} - xq^n\xi_{m,n-1,k+1}\xi_{m-1,n-1,k} + q^{m-n-k}\xi_{m-1,n-2,k}\xi_{m,n,k+1} = 0, \tag{5.10}$$

$$x^2q^{2k+m}\xi_{m-2,n-1,k+1}\xi_{m,n,k} - xq^{k+m-n}\xi_{m-1,n-1,k+1}\xi_{m-1,n,k} + \xi_{m,n,k+1}\xi_{m-2,n-1,k} = 0, \tag{5.11}$$

$$x^2q^{2k-m}\xi_{m-2,n-1,k}\xi_{m,n,k-1} - xq^{k-n}\xi_{m-1,n,k}\xi_{m-1,n-1,k-1} + \xi_{m,n,k}\xi_{m-2,n-1,k-1} = 0, \tag{5.12}$$

$$xq^{k+2m-n}\xi_{m,n,k}\xi_{m-1,n-2,k} - q^n\xi_{m,n-1,k}\xi_{m-1,n-1,k} + \xi_{m,n-1,k+1}\xi_{m-1,n-1,k-1} = 0, \tag{5.13}$$

$$xq^k\xi_{m,n-1,k-1}\xi_{m-1,n-1,k+1} - xq^{n+k}\xi_{m-1,n-1,k}\xi_{m,n-1,k} + q^{2m-n}\xi_{m,n,k}\xi_{m-1,n-2,k} = 0. \tag{5.14}$$

Proof. We first show the case $k = 0$ (or $k = \pm 1$) and then prove to put $x \mapsto xq^k$. Putting $\nu_j := \mu(u+v, v; m-j)$. Since the function $\mu(u, v; \alpha)$ satisfies the relations (2.16) and (2.22), we obtain

$$\xi_{m-1,n-1,\pm 1} = x^{\pm n}q^{n(m-n)} \det \begin{bmatrix} 1 & x^{\pm 1} & \dots & x^{\pm n} \\ \nu_1 & \nu_2 & \dots & \nu_{n+1} \\ \nu_2 & \nu_3 & \dots & \nu_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_n & \nu_{n+1} & \dots & \nu_{2n} \end{bmatrix}. \tag{5.15}$$

Hence we have

$$\begin{aligned}
\text{LHS of } (5.7)_{k=0} =& x^{n+1}q^{mn-n^2+m} \det \begin{bmatrix} x & x^2 & \dots & x^{n+1} \\ \nu_1 & \nu_2 & \dots & \nu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_n & \nu_{n+1} & \dots & \nu_{2n} \end{bmatrix} \det[\nu_{j+j'}]_{j,j'=0}^{n} \\
&- x^{n+1}q^{mn-n^2+m} \det \begin{bmatrix} 1 & x & \dots & x^n \\ \nu_0 & \nu_1 & \dots & \nu_n \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{n-1} & \nu_n & \dots & \nu_{2n-1} \end{bmatrix} \det[\nu_{j+j'+1}]_{j,j'=0}^{n} \\
&+ x^{n+1}q^{mn-n^2+m} \det \begin{bmatrix} 1 & x & \dots & x^{n+1} \\ \nu_0 & \nu_1 & \dots & \nu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_n & \nu_{n+1} & \dots & \nu_{2n+1} \end{bmatrix} \det[\nu_{j+j'+1}]_{j,j'=0}^{n-1}.
\end{aligned}$$

In general, the coefficients of the following polynomial with respect to the variable x

$$
\begin{aligned}
&\det\begin{bmatrix} x & x^2 & \dots & x^{n+1} \\ a_1 & a_2 & \dots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \dots & a_{2n} \end{bmatrix} \det[a_{j+j'}]_{j,j'=0}^{n} \\
&- \det\begin{bmatrix} 1 & x & \dots & x^{n} \\ a_0 & a_1 & \dots & a_{n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n} & \dots & a_{2n-1} \end{bmatrix} \det[a_{j+j'+1}]_{j,j'=0}^{n} \\
&+ \det\begin{bmatrix} 1 & x & \dots & x^{n+1} \\ a_0 & a_1 & \dots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n} & a_{n+1} & \dots & a_{2n+1} \end{bmatrix} \det[a_{j+j'+1}]_{j,j'=0}^{n-1}
\end{aligned}
$$

are zero by the Plücker relation. Therefore, we obtain LHS of $(5.7)_{k=0} = 0$. From similar calculations, we also obtain equations (5.8)–(5.12).

Next, note that

$$
(x^{-1} - x)\xi_{m-1,n-1,0} = q^{2n(m-n-1)} \det\begin{bmatrix} 1 & x & \dots & x^{n+1} \\ 1 & x^{-1} & \dots & x^{-n-1} \\ \nu_1 & \nu_2 & \dots & \nu_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_n & \nu_{n+1} & \dots & \nu_{2n+1} \end{bmatrix}. \tag{5.16}
$$

In fact, since

$$
\begin{aligned}
&\xi_{m-1,n-1,0} \\
&= \left(\frac{x}{q}\right)^n q^{n(m-n)} \det\begin{bmatrix} 1 & \dots & (x/q)^n \\ \mu(u+v-\tau, v; m-1) & \dots & \mu(u+v-\tau, v; m-n-1) \\ \vdots & \ddots & \vdots \\ \mu(u+v-\tau, v; m-n) & \dots & \mu(u+v-\tau, v; m-2n) \end{bmatrix} \\
&= q^{2n(m-n-1)} \det\begin{bmatrix} 1 & \dots & x^n \\ \nu_2 - \frac{\nu_1}{x} & \dots & \nu_{n+2} - \frac{\nu_n+1}{x} \\ & \dots & \\ \nu_{n+1} - \frac{\nu_n}{x} & \dots & \nu_{2n+1} - \frac{\nu_{2n}}{x} \end{bmatrix},
\end{aligned}
$$

we put $X_j := \sum_{k=0}^{j} x^{2k-j}$ which implies $X_j - x^{-1}X_{j-1} = x^j$, so we have

$$
\begin{aligned}
&\xi_{m-1,n-1,0}\\
&= q^{2n(m-n-1)} \det \begin{bmatrix} X_0 - 0 & X_1 - \frac{X_0}{x} & \dots & X_n - \frac{X_{n-1}}{x} \\ \nu_2 - \frac{\nu_1}{x} & \nu_3 - \frac{\nu_2}{x} & \dots & \nu_{n+2} - \frac{\nu_{n+1}}{x} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{n+1} - \frac{\nu_n}{x} & \nu_{n+2} - \frac{\nu_{n+1}}{x} & \dots & \nu_{2n+1} - \frac{\nu_{2n}}{x} \end{bmatrix}\\
&= q^{2n(m-n-1)} \sum_{j=0}^{n+1} \det \begin{bmatrix} 0 & -X_0/x & \dots & -X_{j-2}/x & X_j & \dots & X_n \\ -\nu_1/x & -\nu_2/x & \dots & -\nu_j/x & \nu_{j+2} & \dots & \nu_{n+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\nu_n/x & -\nu_{n+1}/x & \dots & -\nu_{n+j-1}/x & \nu_{n+j+1} & \dots & \nu_{2n+1} \end{bmatrix}\\
&= q^{2n(m-n-1)} \det \begin{bmatrix} 1 & x^{-1} & \dots & x^{-n-1} \\ 0 & X_0 & \dots & X_n \\ \nu_1 & \nu_2 & \dots & \nu_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_n & \nu_{n+1} & \dots & \nu_{2n+1} \end{bmatrix}\\
&= \frac{q^{2n(m-n-1)}}{x - x^{-1}} \det \begin{bmatrix} 1 & x^{-1} & \dots & x^{-n-1} \\ 0 & x - x^{-1} & \dots & x^{n+1} - x^{-n-1} \\ \nu_1 & \nu_2 & \dots & \nu_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_n & \nu_{n+1} & \dots & \nu_{2n+1} \end{bmatrix}.
\end{aligned}
$$

Therefore, the equation (5.16) holds. Using the equation (5.16), we have

$$
\begin{aligned}
&\xi_{m,n-1,1}\xi_{m-1,n-1,-1} - \xi_{m-1,n-1,1}\xi_{m,n-1,-1} + (x - x^{-1})q^{2m-n}\xi_{m,n,0}\xi_{m-1,n-2,0}\\
&= q^{n(2m-2n+1)} \det \begin{bmatrix} 1 & \dots & x^n \\ \nu_0 & \dots & \nu_n \\ & \dots & \\ \nu_{n-1} & \dots & \nu_{2n-1} \end{bmatrix} \det \begin{bmatrix} 1 & \dots & x^{-n} \\ \nu_1 & \dots & \nu_{n+1} \\ & \dots & \\ \nu_n & \dots & \nu_{2n} \end{bmatrix}\\
&\quad - q^{n(2m-2n+1)} \det \begin{bmatrix} 1 & \dots & x^n \\ \nu_1 \dots & \nu_{n+1} & \\ & \dots & \\ \nu_n & \dots & \nu_{2n} \end{bmatrix} \det \begin{bmatrix} 1 & \dots & x^{-n} \\ \nu_0 & \dots & \nu_n \\ & \dots & \\ \nu_{n-1} & \dots & \nu_{2n-1} \end{bmatrix}\\
&\quad + (x - x^{-1})q^{2m-n}\xi_{m,n,0}\xi_{m-1,n-2,0}\\
&= q^{n(2m-2n+1)} \det \begin{bmatrix} \nu_0 & \dots & \nu_n \\ & \dots & \\ \nu_n & \dots & \nu_{2n} \end{bmatrix} \det \begin{bmatrix} 1 & \dots & x^n \\ 1 & \dots & x^{-n} \\ \nu_1 & \dots & \nu_{n+1} \\ & \dots & \\ \nu_{n-1} & \dots & \nu_{2n-1} \end{bmatrix}\\
&\quad - (x^{-1} - x)q^{2m-n}\xi_{m,n,0}\xi_{m-1,n-2,0}\\
&= 0,
\end{aligned}
$$

and similar calculations, we have

$$
x^{-1}\xi_{m,n-1,1}\xi_{m-1,n-1,-1} - x\xi_{m-1,n-1,1}\xi_{m,n-1,-1} + (x - x^{-1})q^n\xi_{m,n-1,0}\xi_{m-1,n-1,0} = 0.
$$

Therefore, From

$$\begin{bmatrix} 1 & -1 \\ x & -x^{-1} \end{bmatrix} \begin{bmatrix} \xi_{m-1,n-1,1}\xi_{m,n-1,-1} \\ \xi_{m,n-1,1}\xi_{m-1,n-1,-1} \end{bmatrix} = (x - x^{-1}) \begin{bmatrix} \xi_{m,n,0}\xi_{m-1,n-2,0} \\ q^n \xi_{m,n-1,0}\xi_{m-1,n-1,0} \end{bmatrix},$$

we have equations (5.13) and (5.14). □

Next, we present the following fact that solutions of the type $(A_2 + A_1)^{(1)}$ q-Painlevé equation can be constructed by using functions satisfying certain bilinear forms. The following theorem have been implicitly known since a long time ago (for examples, [KNY1], [T], [N]), however there does not seem to be any paper that gives full details of this.

Theorem 5.2.2. *Suppose that a function $\xi = \xi(a_1, a_2, x)$ satisfies the following bilinear relations:*

$$0 = \frac{1}{a_1 a_2}\tilde{\xi}_{1,1,0}\tilde{\xi}_{0,0,1} - x^2\tilde{\xi}_{1,1,1}\tilde{\xi}_{0,0,0} + a_2 x\tilde{\xi}_{1,0,0}\tilde{\xi}_{0,1,1}, \tag{5.17}$$

$$0 = \frac{1}{a_1 a_2}\tilde{\xi}_{1,1,0}\tilde{\xi}_{0,0,-1} - \frac{1}{x^2}\tilde{\xi}_{1,1,-1}\tilde{\xi}_{0,0,0} + \frac{a_2}{x}\tilde{\xi}_{1,0,0}\tilde{\xi}_{0,1,-1}, \tag{5.18}$$

$$0 = a_1 a_2 x\tilde{\xi}_{1,2,0}\tilde{\xi}_{0,0,-1} - \tilde{\xi}_{1,1,-1}\tilde{\xi}_{0,1,0} + \frac{1}{a_2}\tilde{\xi}_{1,1,0}\tilde{\xi}_{0,1,-1}, \tag{5.19}$$

$$0 = \frac{a_1 a_2}{x}\tilde{\xi}_{1,2,0}\tilde{\xi}_{0,0,1} - \tilde{\xi}_{1,1,1}\tilde{\xi}_{0,1,0} + \frac{1}{a_2}\tilde{\xi}_{1,1,0}\tilde{\xi}_{0,1,1}, \tag{5.20}$$

$$0 = \tilde{\xi}_{2,1,0}\tilde{\xi}_{0,0,1} - a_1 x^2\tilde{\xi}_{2,1,1}\tilde{\xi}_{0,0,0} + a_1 a_2 x\tilde{\xi}_{1,1,1}\tilde{\xi}_{1,0,0}, \tag{5.21}$$

$$0 = \tilde{\xi}_{2,1,0}\tilde{\xi}_{0,0,-1} - \frac{a_1}{x^2}\tilde{\xi}_{2,1,-1}\tilde{\xi}_{0,0,0} + \frac{a_1 a_2}{x}\tilde{\xi}_{1,1,-1}\tilde{\xi}_{1,0,0}, \tag{5.22}$$

$$0 = a_1 a_2 x\tilde{\xi}_{0,0,0}\tilde{\xi}_{1,2,0} - \frac{1}{a_1 a_2}\tilde{\xi}_{1,1,0}\tilde{\xi}_{0,1,0} + \frac{1}{a_1}\tilde{\xi}_{0,1,1}\tilde{\xi}_{1,1,-1}, \tag{5.23}$$

$$0 = \frac{a_1 a_2}{x}\tilde{\xi}_{0,0,0}\tilde{\xi}_{1,2,0} - \frac{1}{a_1 a_2}\tilde{\xi}_{1,1,0}\tilde{\xi}_{0,1,0} + \frac{1}{a_1}\tilde{\xi}_{0,1,-1}\tilde{\xi}_{1,1,1}, \tag{5.24}$$

where $\tilde{\xi}_{a,b,c} := T_1^{-a}T_2^b T^c \xi$. Then, the pair

$$(x, a_1, a_2; f_1, f_2) = \left(x, a_1, a_2; \frac{\tilde{\xi}_{0,0,1}\tilde{\xi}_{0,1,0}}{\tilde{\xi}_{0,0,0}\tilde{\xi}_{0,1,1}}, -\frac{\tilde{\xi}_{0,1,1}\tilde{\xi}_{1,1,0}}{\tilde{\xi}_{0,1,0}\tilde{\xi}_{1,1,1}} \right)$$

gives a solution of the type $(A_2 + A_1)^{(1)}$ q-Painlevé equation (5.1)-(5.4).

Remark . *If a function $\xi(a_1, a_2, x)$ is a solution of the bilinear relations* (5.17)-(5.24)*, functions $a_1^\alpha a_2^\beta x^\gamma q^\delta \xi(a_1, a_2, x)$ and $\xi(a_1, a_2, x^{-1})$ are also solutions.*

Proof of Theorem 5.2.2. Putting

$$(x, a_1, a_2; f_1, f_2) = \left(x, a_1, a_2; \frac{\tilde{\xi}_{0,0,1}\tilde{\xi}_{0,1,0}}{\tilde{\xi}_{0,0,0}\tilde{\xi}_{0,1,1}}, -\frac{\tilde{\xi}_{0,1,1}\tilde{\xi}_{1,1,0}}{\tilde{\xi}_{0,1,0}\tilde{\xi}_{1,1,1}} \right),$$

we will show equations (5.2)-(5.4) or equavalently

$$
\begin{aligned}
f_1T(f_1) &= \frac{a_1a_2f_1f_2+a_1x^2q^2f_1+x^2q^2}{a_1a_2f_1f_2+a_1f_1+1}, && (5.25)\\
\frac{f_2T(f_2)}{x^2q^2} &= \frac{a_1a_2f_1f_2+a_1f_1+1}{a_1a_2f_1f_2+a_1f_1+x^2q^2}, && (5.26)\\
\frac{T_1^{-1}(f_1)q}{f_2} &= \frac{a_1a_2f_1f_2+x^2q^2}{f_1f_2+a_1a_2x^2}, && (5.27)\\
\frac{f_1f_2T_1^{-1}(f_2)}{x^2} &= \frac{a_1^2a_2f_1f_2^2+q^2f_1f_2+a_1x^2q^2f_2+a_1a_2x^2q^2}{a_1a_2f_1f_2^2+a_1f_1f_2+x^2q^2f_2+a_1^2a_2x^2}, && (5.28)\\
\frac{f_1f_2T_2^{-1}(f_1)}{x^2} &= \frac{q^2f_1^2f_2+a_1a_2^2f_1f_2+a_1a_2x^2q^2f_1+a_2x^2q^2}{a_2f_1^2f_2+a_1a_2f_1f_2+a_1a_2^2x^2f_1+x^2q^2}, && (5.29)\\
\frac{T_2^{-1}(f_2)}{qf_1} &= \frac{f_1f_2+a_1a_2x^2}{a_1a_2f_1f_2+x^2q^2}. && (5.30)
\end{aligned}
$$

From equations (5.17), (5.20) and (5.23), note that

$$
\begin{aligned}
0 &= a_1a_2x\tilde{\xi}_{0,0,0}\tilde{\xi}_{1,2,0}-\frac{1}{a_1a_2}\tilde{\xi}_{1,1,0}\tilde{\xi}_{0,1,0}+\frac{1}{a_1}\tilde{\xi}_{0,1,1}\tilde{\xi}_{1,1,-1}\\
&= x^2\frac{\tilde{\xi}_{0,0,0}}{\tilde{\xi}_{0,0,1}}\left(\tilde{\xi}_{0,1,0}\tilde{\xi}_{1,1,1}-\frac{1}{a_2}\tilde{\xi}_{0,1,1}\tilde{\xi}_{1,1,0}\right)+\frac{1}{a_1}\tilde{\xi}_{0,1,1}\tilde{\xi}_{1,1,-1}-\frac{1}{a_1a_2}\tilde{\xi}_{0,1,0}\tilde{\xi}_{1,1,0}\\
&= \frac{a_1a_2x^2\tilde{\xi}_{0,0,0}\tilde{\xi}_{1,1,1}-\tilde{\xi}_{0,0,1}\tilde{\xi}_{1,1,0}}{a_1^2a_2^2x\tilde{\xi}_{1,0,0}\tilde{\xi}_{0,0,1}}\left(a_1a_2x\tilde{\xi}_{0,1,0}\tilde{\xi}_{1,0,0}-\frac{a_1x^2}{a_2}\tilde{\xi}_{0,0,0}\tilde{\xi}_{1,1,0}+\tilde{\xi}_{0,0,1}\tilde{\xi}_{1,1,-1}\right)\\
&= \frac{\tilde{\xi}_{0,1,1}}{a_1a_2\tilde{\xi}_{0,0,1}}\left(a_1a_2^2x\tilde{\xi}_{0,1,0}\tilde{\xi}_{1,0,0}-\tilde{\xi}_{0,0,0}\tilde{\xi}_{1,1,0}+a_2\tilde{\xi}_{0,0,1}\tilde{\xi}_{1,1,-1}\right). \qquad (5.31)
\end{aligned}
$$

Using equations (5.18), (5.19), (5.23) and (5.31), we have

$$
\begin{aligned}
\text{RHS of (5.25)} &= \frac{a_1x^2q^2\tilde{\xi}_{0,0,1}\tilde{\xi}_{0,1,0}\tilde{\xi}_{1,1,1}+\tilde{\xi}_{0,1,1}(x^2q^2\tilde{\xi}_{1,1,1}\tilde{\xi}_{0,0,0}-a_1a_2\tilde{\xi}_{0,0,1}\tilde{\xi}_{1,1,0})}{\tilde{\xi}_{0,0,1}(a_1\tilde{\xi}_{0,1,0}\tilde{\xi}_{1,1,1}-a_1a_2\tilde{\xi}_{1,1,0}\tilde{\xi}_{0,1,1})+\tilde{\xi}_{0,0,0}\tilde{\xi}_{1,1,1}\tilde{\xi}_{0,1,1}}\\
&= \frac{\tilde{\xi}_{0,1,0}}{\tilde{\xi}_{0,0,0}}\frac{-a_1a_2^2xq\tilde{\xi}_{0,1,1}\tilde{\xi}_{1,0,1}+a_1x^2q^2\tilde{\xi}_{0,0,1}\tilde{\xi}_{1,1,1}}{\tilde{\xi}_{1,1,1}\tilde{\xi}_{0,1,1}-a_1^2a_2^2xq\tilde{\xi}_{0,0,1}\tilde{\xi}_{1,2,1}} = \frac{\tilde{\xi}_{0,1,0}\tilde{\xi}_{0,0,2}}{\tilde{\xi}_{0,0,0}\tilde{\xi}_{0,1,2}}\\
&= \text{LHS of (5.25)}.
\end{aligned}
$$

Using equations (5.19), (5.23) and (5.24), we have

$$
\begin{aligned}
\text{RHS of (5.26)} &= \frac{\tilde{\xi}_{1,1,1}\tilde{\xi}_{0,0,0}\tilde{\xi}_{0,1,1}+\tilde{\xi}_{0,0,1}(a_1\tilde{\xi}_{0,1,0}\tilde{\xi}_{1,1,1}-a_1a_2\tilde{\xi}_{1,1,0}\tilde{\xi}_{0,1,1})}{x^2q^2\tilde{\xi}_{1,1,1}\tilde{\xi}_{0,0,0}\tilde{\xi}_{0,1,1}+\tilde{\xi}_{0,0,1}(a_1\tilde{\xi}_{1,1,0}\tilde{\xi}_{0,1,1}-a_1a_2\tilde{\xi}_{1,1,0}\tilde{\xi}_{0,1,1})}\\
&= \frac{\tilde{\xi}_{1,1,1}\tilde{\xi}_{0,1,1}-a_1^2a_2^2xq\tilde{\xi}_{0,0,1}\tilde{\xi}_{1,2,1}}{x^2q^2\tilde{\xi}_{1,1,1}\tilde{\xi}_{0,1,1}-a_1^2a_2^2xq\tilde{\xi}_{0,0,1}\tilde{\xi}_{1,2,1}} = \frac{1}{x^2q^2}\frac{\tilde{\xi}_{1,1,0}\tilde{\xi}_{0,1,2}}{\tilde{\xi}_{0,1,0}\tilde{\xi}_{1,1,2}} = \text{LHS of (5.26)}.
\end{aligned}
$$

Using equations (5.17) and (5.18)), we have

$$
\text{RHS of (5.27)} = \frac{x^2q^2\tilde{\xi}_{1,1,1}\tilde{\xi}_{0,0,0}-a_1a_2\tilde{\xi}_{0,0,1}\tilde{\xi}_{1,1,0}}{a_1a_2x^2\tilde{\xi}_{0,0,0}\tilde{\xi}_{1,1,1}-\tilde{\xi}_{0,0,1}\tilde{\xi}_{1,1,0}} = -q\frac{\tilde{\xi}_{1,0,1}\tilde{\xi}_{0,1,0}}{\tilde{\xi}_{0,1,1}\tilde{\xi}_{1,0,0}} = \text{LHS of (5.27)}.
$$

Using equations $(5.17), (5.18), (5.21)$ and (5.22), we have

$$\text{RHS of (5.28)} = q\frac{a_1\tilde{\xi}_{1,1,0}\tilde{\xi}_{1,0,1} + q\tilde{\xi}_{1,1,1}\tilde{\xi}_{1,0,0}}{a_1\tilde{\xi}_{1,1,1}\tilde{\xi}_{1,0,0} + q\tilde{\xi}_{1,1,0}\tilde{\xi}_{1,0,1}} = \frac{1}{x^2}\frac{\tilde{\xi}_{0,0,1}\tilde{\xi}_{2,1,0}}{\tilde{\xi}_{2,1,1}\tilde{\xi}_{0,0,0}} = \text{LHS of (5.28)}.$$

Using equations $(5.17), (5.18), (5.19)$ and (5.20), we have

$$\text{RHS of (5.29)} = q\frac{q\tilde{\xi}_{0,0,1}\tilde{\xi}_{1,0,0} - a_2\tilde{\xi}_{0,0,0}\tilde{\xi}_{1,0,1}}{a_2\tilde{\xi}_{0,0,1}\tilde{\xi}_{1,0,0} - q\tilde{\xi}_{0,0,0}\tilde{\xi}_{1,0,1}} = -\frac{1}{x^2}\frac{\tilde{\xi}_{1,1,0}\tilde{\xi}_{0,-1,1}}{\tilde{\xi}_{1,1,1}\tilde{\xi}_{0,-1,0}} = \text{LHS of (5.29)}.$$

Using equations (5.17) and (5.18), we have

$$\text{RHS of (5.30))} = \frac{a_1a_2x^2\tilde{\xi}_{0,0,0}\tilde{\xi}_{1,1,1} - \tilde{\xi}_{0,0,1}\tilde{\xi}_{1,1,0}}{x^2q^2\tilde{\xi}_{1,1,1}\tilde{\xi}_{0,0,0} - a_1a_2\tilde{\xi}_{0,0,1}\tilde{\xi}_{1,1,0}} = -\frac{1}{q}\frac{\tilde{\xi}_{0,1,1}\tilde{\xi}_{1,0,0}}{\tilde{\xi}_{1,0,1}\tilde{\xi}_{0,1,0}} = \text{LHS of (5.30)}.$$

□

Proof of Theorem 5.1.1. Putting $(a_1, a_2, x) = (-q^m, q^{-n}, xq^k)$ in the bilinear relations (5.17)-(5.24). Then the function $\tilde{\xi}_{0,0,0} = \det\,[\mu(u+v+k\tau, v; m-j-j')]_{j,j'=0}^{n}$ satisfies the bilinear equations (5.17)–(5.24) from Theorem 5.2.1. Hence a solution of the type $(A_2+A_1)^{(1)}$ q-Painlevé equation is expressed using the function $\tilde{\xi}_{m,n,k}$, and Theorem 5.1.1 holds. □

The bilinear relations (5.17) and (5.18) are the relations for three pairs of opposite vertices of a cube, respectively. From these relations, we obtain bilinear relations for any

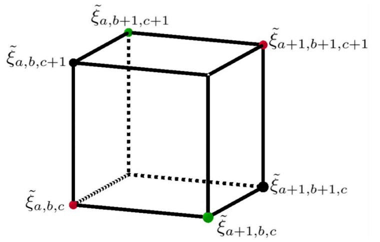


Figure 1: Equation (5.17)

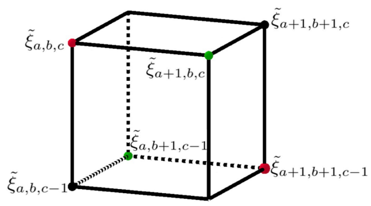


Figure 2: Equation (5.18)

three pairs of opposite vertices of a cube. That is, if we give a set of initial values for the six vertices which include all the vertices of a certain face of a cube, we obtain the value of any vertices of the cube. From the bilinear relations (5.19)-(5.24), we obtain the values of two vertices of a cube whose faces are adjacent to the original cube. Therefore, if we give the initial values, the values of any $\xi_{m,n,k}$ is uniquely determined. In particular, the solution (5.6) corresponds to the initial values $\tilde{\xi}_{0,n+1,0} = \tilde{\xi}_{0,n+1,1} = \tilde{\xi}_{1,n+1,0} = \tilde{\xi}_{1,n+1,1} = 1, \tilde{\xi}_{0,n,0} = \mu(u+v+k\tau, v; m)$ and $\tilde{\xi}_{0,n,1} = \mu(u+v+(k+1)\tau, v; m)$.

Appendix

In this appendix, as a corollary of Theorem 3.2.2, we obtain a list of new expressions for all of Ramanujan's mock theta functions by the degenerated very-well-poised bilateral q-hypergeometric series $\mathcal{W}$. The function $\mathcal{W}$ is defined by the equation (3.16)

Order 2 mock theta functions

$$\begin{aligned}
A(q) :&= \sum_{n=0}^{\infty} \frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}} \\
&= \frac{q^2}{\theta_{q^4}(-q^5)} \mathcal{W}(q^5;q^3,q^2;q^4) \\
&= \frac{1}{\theta_{q^4}(-q^5)} \sum_{n\in\mathbb{Z}} \frac{(1-q^{8n+5})q^{8n^2+10n+2}}{(1-q^{4n+3})(1-q^{4n+2})} \\
B(q) :&= \sum_{n=0}^{\infty} \frac{q^{n}(-q;q^2)_n}{(q;q^2)_{n+1}} \\
&= \frac{q^2}{\theta_{q^4}(-q^6)} \mathcal{W}(q^6;q^3,q^3;q^4) \\
&= \frac{1}{\theta_{q^4}(-q^6)} \sum_{n\in\mathbb{Z}} \frac{1+q^{4n+3}}{1-q^{4n+3}} q^{8n^2+12n+2} \\
\mu(q) :&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q^2,q^2)_n^2} \\
&= \frac{4}{\theta_{q^4}(q)} \mathcal{W}(-q;q,-1;q^4) - \frac{(q^2;q^2)_\infty^8}{(q;q)_\infty^3(q^4;q^4)_\infty^4} \\
&= \frac{4}{\theta_{q^4}(q)} \sum_{n\in\mathbb{Z}} \frac{(1+q^{8n+1})q^{8n^2+2n}}{(1-q^{4n+1})(1+q^{4n})} - \frac{(q^2;q^2)_\infty^8}{(q;q)_\infty^3(q^4;q^4)_\infty^4}
\end{aligned}$$

Order 3 mock theta functions

$$\begin{aligned}
f(q) :&= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} \\
&= -\frac{4q}{\theta_{q^3}(q^3)} \mathcal{W}(-q^3;-q^2,q;q^3) + \frac{(q^3;q^3)_\infty^4}{(q;q)_\infty(q^6;q^6)_\infty^2} \\
&= -\frac{4}{\theta_{q^3}(q^3)} \sum_{n\in\mathbb{Z}} \frac{(1+q^{6n+3})q^{6n^2+6n+1}}{(1-q^{3n+1})(1+q^{3n+2})} + \frac{(q^3;q^3)_\infty^4}{(q;q)_\infty(q^6;q^6)_\infty^2} \\
\phi(q) :&= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n} \\
&= -\frac{2q^2}{\theta_{-q^3}(-q^5)} \mathcal{W}(q^5;q^3,q^2;-q^3) + 2q\frac{(q^6;q^6)_\infty(q^{12};q^{12})_\infty^2}{(q^3;q^3)_\infty(q^4;q^4)_\infty} \\
&= -\frac{2}{\theta_{-q^3}(-q^5)} \sum_{n\in\mathbb{Z}} \frac{(1-q^{6n+5})q^{6n^2+10n+2}}{(1-(-1)^n q^{3n+2})(1-(-1)^n q^{3n+3})} + 2q\frac{(q^6;q^6)_\infty(q^{12};q^{12})_\infty^2}{(q^3;q^3)_\infty(q^4;q^4)_\infty}
\end{aligned}$$

$$
\begin{aligned}
\psi(q) :&= \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n} \\
&= \frac{q}{\theta_{-q^3}(-q^3)} \mathcal{W}(q^3;q^2,q;-q^3) + q\frac{(q^6;q^6)_\infty(q^{12};q^{12})_\infty^2}{(q^3;q^3)_\infty(q^4;q^4)_\infty} \\
&= \frac{1}{\theta_{-q^3}(-q^3)} \sum_{n\in\mathbb{Z}} \frac{(1-q^{6n+3})q^{6n^2+6n+1}}{(1-(-1)^n q^{3n+1})(1-(-1)^n q^{3n+2})} + q\frac{(q^6;q^6)_\infty(q^{12};q^{12})_\infty^2}{(q^3;q^3)_\infty(q^4;q^4)_\infty} \\
\chi(q) :&= \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q)_n}{(-q^3;q^3)_n} \\
&= -\frac{q}{\theta_{q^3}(q^3)} \mathcal{W}(-q^3;-q^2,q;q^3) + \frac{(q^3;q^3)_\infty^4}{(q;q)_\infty(q^6;q^6)_\infty^2} \\
&= -\frac{1}{\theta_{q^3}(q^3)} \sum_{n\in\mathbb{Z}} \frac{(1+q^{6n+3})q^{6n^2+6n+1}}{(1-q^{3n+1})(1+q^{3n+2})} + \frac{(q^3;q^3)_\infty^4}{(q;q)_\infty(q^6;q^6)_\infty^2} \\
\omega(q) :&= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2} \\
&= \frac{2q}{\theta_{q^6}(-q^5)} \mathcal{W}(q^5;q^3,q^2;q^6) + \frac{(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty(q^3;q^3)_\infty^2} \\
&= \frac{2}{\theta_{q^6}(-q^5)} \sum_{n\in\mathbb{Z}} \frac{(1-q^{12n+5})q^{12n^2+10n+1}}{(1-q^{6n+2})(1-q^{6n+3})} + \frac{(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty(q^3;q^3)_\infty^2} \\
\nu(q) :&= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}} \\
&= \frac{2q^2}{\theta_{q^{12}}(-q^8)} \mathcal{W}(q^8;-q^5,-q^3;q^{12}) + \frac{(q;q)_\infty(q^3;q^3)_\infty(q^{12};q^{12})_\infty}{(q^2;q^2)_\infty(q^6;q^6)_\infty} \\
&= \frac{2}{\theta_{q^{12}}(-q^8)} \sum_{n\in\mathbb{Z}} \frac{(1-q^{24n+8})q^{24n^2+16n+2}}{(1+q^{12n+5})(1+q^{12n+3})} + \frac{(q;q)_\infty(q^3;q^3)_\infty(q^{12};q^{12})_\infty}{(q^2;q^2)_\infty(q^6;q^6)_\infty} \\
\rho(q) :&= \sum_{n=1}^{\infty} \frac{q^{2n(n-1)}(q;q^2)_n}{(q^3;q^6)_n} \\
&= \frac{1}{\theta_{q^6}(-q^3)} \mathcal{W}(q^3;-q^2,-q;q^6) \\
&= \frac{1}{\theta_{q^6}(-q^3)} \sum_{n\in\mathbb{Z}} \frac{(1-q^{12n+3})q^{12n^2+6n}}{(1+q^{6n+1})(1+q^{6n+2})}
\end{aligned}
$$

Order 5 mock theta functions

$$
\begin{aligned}
f_0(q) :&= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n} \\
&= -\frac{2q^4}{\theta_{q^{30}}(-q^{22})} \mathcal{W}(q^{22};q^{18},q^4;q^{30}) \\
&\quad - \frac{2q^2}{\theta_{q^{30}}(-q^{12})} \mathcal{W}(q^{12};q^8,q^4;q^{30}) \frac{(q^5;q^5)_\infty^3}{\theta_{q^5}(-q)(q^{10};q^{10})_\infty}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{2}{\theta_{q^{30}}(-q^{22})}\sum_{n\in\mathbb{Z}}\frac{(1-q^{60n+22})q^{60n^2+44n+4}}{(1-q^{30n+18})(1-q^{30n+4})}\\
&\quad -\frac{2}{\theta_{q^{30}}(-q^{12})}\sum_{n\in\mathbb{Z}}\frac{(1-q^{60n+12})q^{60n^2+24n+2}}{(1-q^{30n+8})(1-q^{30n+4})}+\frac{(q^5;q^5)_\infty^3}{\theta_{q^5}(-q)(q^{10};q^{10})_\infty}\\
f_1(q) :&= \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(-q;q)_n}\\
&= -\frac{2q^7}{\theta_{q^{30}}(-q^{24})}\mathcal{W}(q^{24};q^{16},q^8;q^{30})\\
&\quad -\frac{2q^3}{\theta_{q^{30}}(-q^{14})}\mathcal{W}(q^{14};q^8,q^6;q^{30})+\frac{(q^5;q^5)_\infty^3}{\theta_{q^5}(-q^2)(q^{10};q^{10})_\infty}\\
&= -\frac{2}{\theta_{q^{30}}(-q^{24})}\sum_{n\in\mathbb{Z}}\frac{(1-q^{60n+24})q^{60n^2+48n+7}}{(1-q^{30n+16})(1-q^{30n+8})}\\
&\quad -\frac{2q^3}{\theta_{q^{30}}(-q^{14})}\sum_{n\in\mathbb{Z}}\frac{(1-q^{60n+14})q^{60n^2+28n+3}}{(1-q^{30n+8})(1-q^{30n+6})}+\frac{(q^5;q^5)_\infty^3}{\theta_{q^5}(-q^2)(q^{10};q^{10})_\infty}\\
F_0(q) :&= \sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q^2)_n}\\
&= \frac{1}{\theta_{q^{15}}(-q^4)}\mathcal{W}(q^4;q^3,q;q^{15})\\
&\quad -\frac{q^4}{\theta_{q^{15}}(-q^{16})}\mathcal{W}(q^{16};q^{12},q^4;q^{15})-\frac{q(q^{10};q^{10})_\infty^3}{(q^5;q^5)_\infty\theta_{q^{10}}(-q^4)}\\
&= \frac{1}{\theta_{q^{15}}(-q^4)}\sum_{n\in\mathbb{Z}}\frac{(1-q^{30n+4})q^{30n^2+8n}}{(1-q^{15n+3})(1-q^{15n+1})}\\
&\quad -\frac{1}{\theta_{q^{15}}(-q^{16})}\sum_{n\in\mathbb{Z}}\frac{(1-q^{30n+16})q^{30n^2+32n+4}}{(1-q^{15n+12})(1-q^{15n+4})}-\frac{q(q^{10};q^{10})_\infty^3}{(q^5;q^5)_\infty\theta_{q^{10}}(-q^4)}\\
F_1(q) :&= \sum_{n=1}^{\infty}\frac{q^{2n(n-1)}}{(q;q^2)_n}\\
&= \frac{q}{\theta_{q^{15}}(-q^7)}\mathcal{W}(q^7;q^4,q^3;q^{15})\\
&\quad +\frac{q^3}{\theta_{q^{15}}(-q^{12})}\mathcal{W}(q^{12};q^8,q^4;q^{15})+\frac{(q^{10};q^{10})_\infty^3}{(q^5;q^5)_\infty\theta_{q^{10}}(-q^2)}\\
&= \frac{1}{\theta_{q^{15}}(-q^7)}\sum_{n\in\mathbb{Z}}\frac{(1-q^{30n+7})q^{30n^2+14n+1}}{(1-q^{15n+4})(1-q^{15n+3})}\\
&\quad +\frac{1}{\theta_{q^{15}}(-q^{12})}\sum_{n\in\mathbb{Z}}\frac{(1-q^{30n+12})q^{30n^2+24n+3}}{(1-q^{15n+8})(1-q^{15n+4})}+\frac{(q^{10};q^{10})_\infty^3}{(q^5;q^5)_\infty\theta_{q^{10}}(-q^2)}\\
\phi_0(q) :&= \sum_{n=0}^{\infty}q^{n^2}(-q;q^2)_n\\
&= \frac{q^8}{\theta_{-q^{15}}(-q^{20})}\mathcal{W}(q^{20};q^{11},q^9;-q^{15})-\frac{q^9}{\theta_{-q^{15}}(q^{25})}\mathcal{W}(-q^{25};-q^{16},q^9;-q^{15})
\end{aligned}
$$

$$= \frac{1}{\theta_{-q^{15}}(-q^{20})} \sum_{n\in\mathbb{Z}} \frac{(1-q^{30n+20})q^{30n^2+40n+8}}{(1-(-1)^n q^{15n+11})(1-(-1)^n q^{15n+9})}$$

$$- \frac{1}{\theta_{-q^{15}}(q^{25})} \sum_{n\in\mathbb{Z}} \frac{(1+q^{30n+25})q^{30n^2+50n+9}}{(1+(-1)^n q^{15n+16})(1-(-1)^n q^{15n+9})}$$

$$\phi_1(q) := \sum_{n=0}^{\infty} q^{(n+1)^2}(-q;q^2)_n$$

$$= \frac{q^3}{\theta_{-q^{15}}(-q^{10})}\mathcal{W}(q^{10};q^7,q^3;-q^{15}) + \frac{q}{\theta_{-q^{15}}(q^5)}\mathcal{W}(-q^5;q^3,-q^2;-q^{15})$$

$$= \frac{1}{\theta_{-q^{15}}(-q^{10})} \sum_{n\in\mathbb{Z}} \frac{(1-q^{30n+10})q^{30n^2+20n+3}}{(1-(-1)^n q^{15n+7})(1-(-1)^n q^{15n+3})}$$

$$+ \frac{1}{\theta_{-q^{15}}(q^5)} \sum_{n\in\mathbb{Z}} \frac{(1+q^{30n+5})q^{30n^2+10n+1}}{(1-(-1)^n q^{15n+3})(1+(-1)^n q^{15n+2})}$$

$$\psi_0(q) := \sum_{n=0}^{\infty} q^{\frac{(n+1)(n+2)}{2}}(-q;q)_n$$

$$= \frac{q^3}{\theta_{q^{30}}(-q^{20})}\mathcal{W}(q^{20};q^{17},q^3;q^{30}) + \frac{q}{\theta_{q^{30}}(-q^{10})}\mathcal{W}(q^{10};q^7,q^3;q^{30})$$

$$= \frac{1}{\theta_{q^{30}}(-q^{20})} \sum_{n\in\mathbb{Z}} \frac{(1-q^{60n+20})q^{60n^2+40n+3}}{(1-q^{30n+17})(1-q^{30n+3})}$$

$$+ \frac{1}{\theta_{q^{30}}(-q^{10})} \sum_{n\in\mathbb{Z}} \frac{(1-q^{60n+10})q^{60n^2+20n+1}}{(1-q^{30n+7})(1-q^{30n+3})}$$

$$\psi_1(q) := \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}(-q;q)_n$$

$$= \frac{1}{\theta_{q^{30}}(-q^{10})}\mathcal{W}(q^{10};q^9,q;q^{30}) + \frac{q^6}{\theta_{q^{30}}(-q^{20})}\mathcal{W}(q^{20};q^{11},q^9;q^{30})$$

$$= \frac{1}{\theta_{q^{30}}(-q^{10})} \sum_{n\in\mathbb{Z}} \frac{(1-q^{60n+10})q^{60n^2+20n}}{(1-q^{30n+9})(1-q^{30n+1})}$$

$$+ \frac{1}{\theta_{q^{30}}(-q^{20})} \sum_{n\in\mathbb{Z}} \frac{(1-q^{60n+20})q^{60n^2+40n+6}}{(1-q^{30n+11})(1-q^{30n+9})}$$

$$\chi_0(q) := \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_n}$$

$$= 2 + \frac{3q^{-6}}{\theta_{q^{15}}(-q^{-5})}\mathcal{W}(q^{-5};q,q^{-6};q^{15}) + \frac{3q^3}{\theta_{q^{15}}(-q^{10})}\mathcal{W}(q^{10};q^6,q^4;q^{15}) + 2\frac{\theta_{q^5}(-q^2)^3}{(q;q)_\infty^2}$$

$$= 2 + \frac{3}{\theta_{q^{15}}(-q^{-5})} \sum_{n\in\mathbb{Z}} \frac{(1-q^{30n-5})q^{30n^2-10n-6}}{(1-q^{30n+1})(1-q^{30n-6})}$$

$$+ \frac{3}{\theta_{q^{15}}(-q^{10})} \sum_{n\in\mathbb{Z}} \frac{(1-q^{30n+10})q^{30n^2+20n+3}}{(1-q^{15n+6})(1-q^{15n+4})} + 2\frac{\theta_{q^5}(-q^2)^3}{(q;q)_\infty^2}$$

$$
\begin{aligned}
\chi_1(q) :&= \sum_{n=1}^{\infty} \frac{q^{n-1}}{(q^n;q)_n} \\
&= \frac{3q^2}{\theta_{q^{15}}(-q^{10})}\mathcal{W}(q^{10};q^7,q^3;q^{15}) + \frac{3}{\theta_{q^{15}}(-q^5)}\mathcal{W}(q^5;q^3,q^2;q^{15}) - 2\frac{\theta_{q^5}(-q)^3}{(q;q)_\infty^2} \\
&= \frac{3}{\theta_{q^{15}}(-q^{10})}\sum_{n\in\mathbb{Z}} \frac{(1-q^{30n+10})q^{30n^2+20n+2}}{(1-q^{15n+7})(1-q^{15n+3})} \\
&\quad + \frac{3}{\theta_{q^{15}}(-q^5)}\sum_{n\in\mathbb{Z}} \frac{(1-q^{30n+5})q^{30n^2+10n}}{(1-q^{15n+3})(1-q^{15n+2})} - 2\frac{\theta_{q^5}(-q)^3}{(q;q)_\infty^2} \\
\Psi_0(q) :&= -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n} \\
&= \frac{q}{\theta_{q^{15}}(-q^6)}\mathcal{W}(q^6;q^4,q^2;q^{15}) + \frac{q^2}{\theta_{q^{15}}(-q^{11})}\mathcal{W}(q^{11};q^9,q^2;q^{15}) \\
&= \frac{1}{\theta_{q^{15}}(-q^6)}\sum_{n\in\mathbb{Z}} \frac{(1-q^{30n+6})q^{30n^2+12n+1}}{(1-q^{15n+4})(1-q^{15n+2})} \\
&\quad + \frac{1}{\theta_{q^{15}}(-q^{11})}\sum_{n\in\mathbb{Z}} \frac{(1-q^{30n+11})q^{30n^2+22n+2}}{(1-q^{15n+9})(1-q^{15n+2})} \\
\Psi_1(q) :&= -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q^2;q^5)_{n+1}(q^3;q^5)_n} \\
&= \frac{q^2}{\theta_{q^{15}}(-q^7)}\mathcal{W}(q^7;q^4,q^3;q^{15}) + \frac{q^4}{\theta_{q^{15}}(-q^{12})}\mathcal{W}(q^{12};q^8,q^4;q^{15}) \\
&= \frac{1}{\theta_{q^{15}}(-q^7)}\sum_{n\in\mathbb{Z}} \frac{(1-q^{30n+7})q^{30n^2+14n+2}}{(1-q^{15n+4})(1-q^{15n+3})} \\
&\quad + \frac{1}{\theta_{q^{15}}(-q^{12})}\sum_{n\in\mathbb{Z}} \frac{(1-q^{30n+12})q^{30n^2+24n+4}}{(1-q^{15n+8})(1-q^{15n+4})}
\end{aligned}
$$

Order 6 mock theta functions

$$
\begin{aligned}
\phi(q) :&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q)_{2n}} \\
&= \frac{2}{\theta_{q^3}(-q)}\mathcal{W}(q;-q,-1;q^3) \\
&= \frac{2}{\theta_{q^3}(-q)}\sum_{n\in\mathbb{Z}} \frac{(1-q^{6n+1})q^{6n^2+2n}}{(1+q^{3n+1})(1+q^{3n})} \\
\psi(q) :&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q)_{2n+1}} \\
&= \frac{q}{\theta_{q^3}(-q^2)}\mathcal{W}(q^2;-q,-q;q^3) \\
&= \frac{1}{\theta_{q^3}(-q^2)}\sum_{n\in\mathbb{Z}} \frac{1-q^{3n+1}}{1+q^{3n+1}}q^{6n^2+4n+1}
\end{aligned}
$$

$$
\begin{aligned}
\rho(q) :&= \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}(-q;q)_n}{(q;q^2)_{n+1}} \\
&= \frac{1}{\theta_{q^6}(-q^2)} \mathcal{W}(q^2;q,q;q^6) \\
&= \frac{1}{\theta_{q^6}(-q^2)} \sum_{n\in\mathbb{Z}} \frac{1+q^{6n+1}}{1-q^{6n+1}} q^{12n^2+4n} \\
\sigma(q) :&= \sum_{n=0}^{\infty} \frac{q^{\frac{(n+1)(n+2)}{2}}(-q;q)_n}{(q;q^2)_{n+1}} \\
&= \frac{q}{\theta_{q^6}(-q^4)} \mathcal{W}(q^4;q^3,q;q^6) \\
&= \frac{1}{\theta_{q^6}(-q^4)} \sum_{n\in\mathbb{Z}} \frac{(1-q^{12n+4})q^{12n^2+8n+1}}{(1-q^{6n+3})(1-q^{6n+1})} \\
\lambda(q) :&= \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q^2)_n}{(-q;q)_n} \\
&= \frac{2q}{\theta_{q^6}(-q^4)} \mathcal{W}(q^4;-q^2,-q^2;q^6) + \frac{(q;q)_\infty^3 (q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^3 (q^3;q^3)_\infty} \\
&= \frac{2}{\theta_{q^6}(-q^4)} \sum_{n\in\mathbb{Z}} \frac{1-q^{6n+2}}{1+q^{6n+2}} q^{12n^2+8n+1} + \frac{(q;q)_\infty^3 (q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^3 (q^3;q^3)_\infty} \\
\mu(q) :&= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+1}(1+q^n)(q;q^2)_n}{(-q;q)_{n+1}} \\
&= \frac{2}{\theta_{q^6}(-q^2)} \mathcal{W}(q^2;-q^2,-1;q^6) - \frac{1}{2} \frac{(q;q)_\infty^2 (q^3;q^3)_\infty^2}{(q^2;q^2)_\infty^2 (q^6;q^6)_\infty} \\
&= \frac{2}{\theta_{q^6}(-q^2)} \sum_{n\in\mathbb{Z}} \frac{(1-q^{12n+2})q^{12n^2+4n}}{(1+q^{6n+2})(1+q^{6n})} - \frac{1}{2} \frac{(q;q)_\infty^2 (q^3;q^3)_\infty^2}{(q^2;q^2)_\infty^2 (q^6;q^6)_\infty} \\
\gamma(q) :&= \sum_{n=0}^{\infty} \frac{q^{n^2}(q;q)_n}{(q^3;q^3)_n} \\
&= \frac{3}{\theta_{q^3}(-q)} \mathcal{W}(q,;-q,-1;q^3) - \frac{\theta_{q^2}(-q)^2}{2\theta_{q^3}(q)} \\
&= \frac{3}{\theta_{q^3}(-q)} \sum_{n\in\mathbb{Z}} \frac{(1-q^{6n+1})q^{6n^2+2n}}{(1+q^{3n+1})(1+q^{3n})} - \frac{\theta_{q^2}(-q)^2}{2\theta_{q^3}(q)} \\
\phi_-(q) :&= \sum_{n=1}^{\infty} \frac{q^n(-q;q)_{2n-1}}{(q;q^2)_n} \\
&= -\frac{q^{\frac{1}{2}}}{\theta_{q^3}(-q^2)} \mathcal{W}(q^2;-q^{\frac{3}{2}},-q^{\frac{1}{2}};q^3) + \frac{q^{\frac{1}{2}}(q^2;2^2)_\infty^2 (q^6;q^6)_\infty^2}{(q;q)_\infty^2 (q^3;q^3)_\infty} \\
&= -\frac{1}{\theta_{q^3}(-q^2)} \sum_{n\in\mathbb{Z}} \frac{(1-q^{6n+2})q^{6n^2+4n+\frac{1}{2}}}{(1+q^{3n+\frac{3}{2}})(1+q^{3n+\frac{1}{2}})} + \frac{q^{\frac{1}{2}}(q^2;2^2)_\infty^2 (q^6;q^6)_\infty^2}{(q;q)_\infty^2 (q^3;q^3)_\infty} \\
\psi_-(q) :&= \sum_{n=1}^{\infty} \frac{q^n(-q;q)_{2n-2}}{(q;q^2)_n} \\
&= \frac{1}{2} \frac{q}{\theta_{q^3}(-q^2)} \mathcal{W}(q^2;q,q;q^3) + \frac{q}{2} \frac{(q^6;q^6)_\infty^3}{(q;q)_\infty (q^2;q^2)_\infty}
\end{aligned}
$$

$$= \frac{1}{2\theta_{q^3}(-q^2)} \sum_{n\in\mathbb{Z}} \frac{1+q^{3n+1}}{1-q^{3n+1}} q^{6n^2+4n+1} + \frac{q}{2} \frac{(q^6;q^6)_\infty^3}{(q;q)_\infty (q^2;q^2)_\infty}$$

Order 7 mock theta functions

$$\begin{aligned}
\mathcal{F}_0(q) :&= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1};q)_n} \\
&= \frac{2q^4}{\theta_{q^{21}}(-q^{14})} \mathcal{W}(q^{14}; q^9, q^5; q^{21}) \\
&\quad - \frac{2q^9}{\theta_{q^{21}}(-q^{28})} \mathcal{W}(q^{28}; q^{19}, q^9; q^{21}) + \frac{(q^3, q^4, q^7; q^7)_\infty}{(q, q^2, q^5, q^6; q^7)_\infty} \\
&= \frac{2}{\theta_{q^{21}}(-q^{14})} \sum_{n\in\mathbb{Z}} \frac{(1-q^{42n+14}) q^{42n^2+28n+4}}{(1-q^{21n+9})(1-q^{21n+5})} \\
&\quad - \frac{2}{\theta_{q^{21}}(-q^{28})} \sum_{n\in\mathbb{Z}} \frac{(1-q^{42n+28}) q^{42n^2+56n+9}}{(1-q^{21n+19})(1-q^{21n+9})} + \frac{(q^3, q^4, q^7; q^7)_\infty}{(q, q^2, q^5, q^6; q^7)_\infty} \\
\mathcal{F}_1(q) :&= \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^{n};q)_n} \\
&= \frac{2q^3}{\theta_{q^{21}}(-q^{14})} \mathcal{W}(q^{14}; q^{11}, q^3; q^{21}) \\
&\quad + \frac{2q}{\theta_{q^{21}}(-q^{7})} \mathcal{W}(q^{7}; q^{4}, q^3; q^{21}) - \frac{q(q, q^6, q^7; q^7)_\infty}{(q^2, q^3, q^4, q^5; q^7)_\infty} \\
&= \frac{2}{\theta_{q^{21}}(-q^{14})} \sum_{n\in\mathbb{Z}} \frac{(1-q^{42n+14}) q^{42n^2+28n+3}}{(1-q^{21n+11})(1-q^{21n+3})} \\
&\quad + \frac{2}{\theta_{q^{21}}(-q^{7})} \sum_{n\in\mathbb{Z}} \frac{(1-q^{42n+7}) q^{42n^2+14n+1}}{(1-q^{21n+4})(1-q^{21n+3})} - \frac{q(q, q^6, q^7; q^7)_\infty}{(q^2, q^3, q^4, q^5; q^7)_\infty} \\
\mathcal{F}_2(q) :&= \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(q^{n};q)_n} \\
&= \frac{2q^5}{\theta_{q^{21}}(-q^{17})} \mathcal{W}(q^{17}; q^{11}, q^6; q^{21}) \\
&\quad + \frac{2q^2}{\theta_{q^{21}}(-q^{10})} \mathcal{W}(q^{10}; q^{6}, q^4; q^{21}) + \frac{(q^2, q^5, q^7; q^7)_\infty}{(q, q^3, q^4, q^6; q^7)_\infty} \\
&= \frac{2}{\theta_{q^{21}}(-q^{17})} \sum_{n\in\mathbb{Z}} \frac{(1-q^{42n+17}) q^{42n^2+34n+5}}{(1-q^{21n+11})(1-q^{21n+6})} \\
&\quad + \frac{2}{\theta_{q^{21}}(-q^{10})} \sum_{n\in\mathbb{Z}} \frac{(1-q^{21n+10}) q^{21n^2+20n+2}}{(1-q^{21n+6})(1-q^{21n+4})} + \frac{(q^2, q^5, q^7; q^7)_\infty}{(q, q^3, q^4, q^6; q^7)_\infty}
\end{aligned}$$

Order 8 mock theta functions

$$
\begin{aligned}
S_0(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n} \\
&= \frac{2}{\theta_{q^8}(q^3)}\mathcal{W}(-q^3;q^3,-1;q^8) + q\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2\theta_{q^8}(q)}{(q^4;q^4)_\infty^2\theta_{q^8}(-q^3)^2} \\
&= \frac{2}{\theta_{q^8}(q^3)}\sum_{n\in\mathbb{Z}} \frac{(1+q^{16n+3})q^{16n^2+6n}}{(1-q^{8n+3})(1+q^{8n})} + q\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2\theta_{q^8}(q)}{(q^4;q^4)_\infty^2\theta_{q^8}(-q^3)^2} \\
S_1(q) &:= \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n} \\
&= -\frac{2q^{-1}}{\theta_{q^8}(q)}\mathcal{W}(-q;q,-1;q^8) + q^{-1}\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2\theta_{q^8}(q^3)}{(q^4;q^4)_\infty^2\theta_{q^8}(-q)^2} \\
&= -\frac{2}{\theta_{q^8}(q)}\sum_{n\in\mathbb{Z}} \frac{(1+q^{16n+1})q^{16n^2+2n-1}}{(1+q^{8n})(1-q^{8n+1})} + q^{-1}\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2\theta_{q^8}(q^3)}{(q^4;q^4)_\infty^2\theta_{q^8}(-q)^2} \\
T_0(q) &:= \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}} \\
&= \frac{q^2}{\theta_{q^8}(q^7)}\mathcal{W}(-q^7;-q^5,q^2;q^8) \\
&= \frac{1}{\theta_{q^8}(q^7)}\sum_{n\in\mathbb{Z}} \frac{(1+q^{16n+7})q^{16n^2+14n+2}}{(1-q^{8n+2})(1+q^{8n+5})} \\
T_1(q) &:= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}} \\
&= -\frac{q^5}{\theta_{q^8}(q^{13})}\mathcal{W}(-q^{13};-q^7,q^6;q^8) \\
&= \frac{-1}{\theta_{q^8}(q^{13})}\sum_{n\in\mathbb{Z}} \frac{(1+q^{16n+13})q^{16n^2+26n+5}}{(1-q^{8n+6})(1+q^{8n+7})} \\
U_0(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n} \\
&= \frac{2}{\theta_{q^4}(q)}\mathcal{W}(-q;q,-1;q^4) \\
&= \frac{2}{\theta_{q^4}(q)}\sum_{n\in\mathbb{Z}} \frac{(1+q^{8n+1})q^{8n^2+2n}}{(1+q^{4n})(1-q^{4n+1})} \\
U_1(q) &:= \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}} \\
&= -\frac{q^2}{\theta_{q^4}(q^5)}\mathcal{W}(-q^5;q^3,-q^2;q^4) \\
&= \frac{-1}{\theta_{q^4}(q^5)}\sum_{n\in\mathbb{Z}} \frac{(1+q^{8n+5})q^{8n^2+10n+2}}{(1+q^{4n+2})(1-q^{4n+3})}
\end{aligned}
$$

$$
\begin{aligned}
V_0(q) :&= -1 + 2\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n} \\
&= \frac{2}{\theta_{q^8}(-q^2)}\mathcal{W}(q^2;q,q;q^8) - \frac{(q^2;q^2)_\infty^3(q^4;q^4)_\infty}{(q;q)_\infty^2(q^8;q^8)_\infty} \\
&= \frac{2}{\theta_{q^8}(-q^2)}\sum_{n\in\mathbb{Z}} \frac{1+q^{8n+1}}{1-q^{8n+1}}q^{16n^2+4n} - \frac{(q^2;q^2)_\infty^3(q^4;q^4)_\infty}{(q;q)_\infty^2(q^8;q^8)_\infty} \\
V_1(q) :&= \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}} \\
&= \frac{q}{\theta_{q^8}(-q^4)}\mathcal{W}(q^4;q^3,q;q^8) \\
&= \frac{1}{\theta_{q^8}(-q^4)}\sum_{n\in\mathbb{Z}} \frac{(1-q^{16n+4})q^{16n^2+8n+1}}{(1-q^{8n+1})(1-q^{8n+3})}
\end{aligned}
$$

Order 10 mock theta functions

$$
\begin{aligned}
\phi(q) :&= \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(q;q^2)_{n+1}} \\
&= \frac{2q}{\theta_{q^{10}}(-q^5)}\mathcal{W}(q^5;q^3,q^2;q^{10}) + \frac{(q^2;q^2)_\infty(q^5;q^5)_\infty(q^{10};q^{10})_\infty^2}{\theta_{q^5}(-q^2)\theta_{q^{10}}(-q^2)^2} \\
&= \frac{2}{\theta_{q^{10}}(-q^5)}\sum_{n\in\mathbb{Z}} \frac{(1-q^{20n+5})q^{20n^2+10n+1}}{(1-q^{10n+2})(1-q^{10n+3})} + \frac{(q^2;q^2)_\infty(q^5;q^5)_\infty(q^{10};q^{10})_\infty^2}{\theta_{q^5}(-q^2)\theta_{q^{10}}(-q^2)^2} \\
\psi(q) :&= \sum_{n=0}^{\infty} \frac{q^{\frac{(n+1)(n+2)}{2}}}{(q;q^2)_{n+1}} \\
&= \frac{2q}{\theta_{q^{10}}(-q^5)}\mathcal{W}(q^5;q^4,q;q^{10}) - \frac{(q^2;q^2)_\infty(q^5;q^5)_\infty(q^{10};q^{10})_\infty^2}{\theta_{q^5}(-q)\theta_{q^{10}}(-q^4)^2} \\
&= \frac{2}{\theta_{q^{10}}(-q^5)}\sum_{n\in\mathbb{Z}} \frac{(1-q^{20n+5})q^{20n^2+10n+1}}{(1-q^{10n+1})(1-q^{10n+4})} - q\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty(q^{10};q^{10})_\infty^2}{\theta_{q^5}(-q)\theta_{q^{10}}(-q^4)^2} \\
X(q) :&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}} \\
&= -\frac{2q^{-1}}{\theta_{q^5}(1)}\mathcal{W}(-1;-q,q^{-1};q^5) - \frac{(q^5;q^5)_\infty^2\theta_{q^{10}}(-q^3)}{(q^{10};q^{10})_\infty\theta_{q^5}(-q)} \\
&= \frac{-2}{\theta_{q^5}(1)}\sum_{n\in\mathbb{Z}} \frac{(1+q^{10n})q^{10n^2-1}}{(1-q^{5n-1})(1+q^{5n+1})} - \frac{(q^5;q^5)_\infty^2\theta_{q^{10}}(-q^3)}{(q^{10};q^{10})_\infty\theta_{q^5}(-q)} \\
\chi(q) :&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}} \\
&= -\frac{2q^2}{\theta_{q^5}(q^5)}\mathcal{W}(-q^5;-q^3,q^2;q^5) + q\frac{(q^5;q^5)_\infty^2\theta_{q^{10}}(-q)}{(q^{10};q^{10})_\infty\theta_{q^5}(-q^2)} \\
&= \frac{-2}{\theta_{q^5}(q^5)}\sum_{n\in\mathbb{Z}} \frac{(1+q^{10n+5})q^{10n^2+10n+2}}{(1-q^{5n+2})(1+q^{5n+3})} + q\frac{(q^5;q^5)_\infty^2\theta_{q^{10}}(-q)}{(q^{10};q^{10})_\infty\theta_{q^5}(-q^2)}
\end{aligned}
$$

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