q-hypergeometric equations - q-Riemann-Papperitz system and its multivariable extension -

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Abstract

We study various extensions of the Heine's equation. First, we give Riemann-Papperitz type integral solutions and series solutions in terms of the very-well-poised q-hypergeometric series ${}_8W_7$ for the variant of q-hypergeometric equation of degree three defined by Hatano-Matsunawa-Sato-Takemura. Second, we introduce an Euler type integral transformation for q-difference equations, and give representations for affine Weyl groups as an application. Third, we solve a connection problem for a multivariable q-hypergeometric system introduced by Park. Finally, we give a q-difference system associated with an extension of Riemann-Papperitz type Jackson integral and its solutions in terms of the Kajihara's q-hypergeometric series.

Contents

1 Introduction

The Gauss hypergeometric equation

$$\left[x(1-x)\frac{d^2}{dx^2}+(\gamma-(\alpha+\beta+1)x)\frac{d}{dx}-\alpha\beta\right]f(x)=0, \tag{1.1}$$

is a standard form of second order Fuchsian differential equations with three singularities on $\mathbb{P}^1=\mathbb{C}\cup\{\infty\}$. The equation (1.1) is characterized by the following Riemann scheme:

$$\left\{\begin{matrix} x=0 & x=1 & x=\infty \\ 0 & 0 & \alpha \\ 1-\gamma & \gamma-\alpha-\beta & \beta \end{matrix}\right\}. \tag{1.2}$$

The equation (1.1) has the integral solution

$$\int_C t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-xt)^{-\beta}dt, \tag{1.3}$$

where C is a suitable path. Also, the equation (1.1) has the series solution

$${}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};x\right)=1+\frac{\alpha\cdot\beta}{\gamma\cdot 1}x+\frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{\gamma(\gamma+1)\cdot 1\cdot 2}x^2+\cdots. \tag{1.4}$$

The Riemann scheme (1.2), the integral solution (1.3) and the series solution (1.4) are basic properties of the Gauss equation (1.1).

In this paper we fix $q\in\mathbb{C}$ with $0<|q|<1$. A q-difference analog of the Gauss hypergeometric equation (1.1) was introduced by Heine [22]. We consider the Heine's equation in the following form [27]

$$[x(1-aT_x)(1-bT_x)-(1-T_x)(1-cq^{-1}T_x)]f(x)=0, \tag{1.5}$$

where T_x is the q-shift operator of x, that is, $T_xf(x)=f(qx)$. The equation (1.5) is called the Heine's q-hypergeometric equation. This has the integral solution and the series solution

$$\int_C t^{\alpha-1}\frac{(qt)_\infty}{(ct/a)_\infty}\frac{(bxt)_\infty}{(xt)_\infty}d_qt, \tag{1.6}$$

$${}_2\varphi_1\left(\begin{matrix}a,b\\ c\end{matrix}\ x\right), \tag{1.7}$$

where $a=q^\alpha$. The notations in (1.6), (1.7) are defined below. These are used throughout this paper.

Definition 1.1. We define the Jackson integral of $f(t)$, the q-shifted factorial and the q-hypergeometric series as

$$\int_0^\tau f(t)d_qt=(1-q)\sum_{n=0}^\infty f(\tau q^n)\tau q^n,\quad \int_0^{\tau\infty} f(t)d_qt=(1-q)\sum_{n=-\infty}^\infty f(\tau q^n)\tau q^n, \tag{1.8}$$

$$\int_{\tau_1}^{\tau_2} f(t)d_qt=\int_0^{\tau_2} f(t)d_qt-\int_0^{\tau_1} f(t)d_qt, \tag{1.9}$$

$$(a)_\infty=\prod_{n=0}^\infty(1-aq^n),\quad (a)_m=\frac{(a)_\infty}{(aq^m)_\infty},\quad (a_1,\dots,a_r)_m=(a_1)_m\cdots(a_r)_m, \tag{1.10}$$

$$
{}_r\varphi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix};x\right) = \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_r)_n}{(b_1,\dots,b_s,q)_n}(-1)^{(r-s-1)n} q^{(r-s-1)\binom{n}{2}} x^n. \tag{1.11}
$$

Using the q-shifted factorial, the theta function is defined as

$$
\theta(x) = (x, q/x)_\infty. \tag{1.12}
$$

For details of these, see Gasper-Rahman [16] and references therein.

In this paper, we will study various extensions of the Heine's equation (1.5). The contents of this paper are as follows. In section 2, we study integral solutions, series solutions, connection formula and degenerations for the equation $\mathcal{H}_3$ defined in [21]. In section 3, we introduce an Euler-type integral transformation and construct representations for affine Weyl groups of types $D_5^{(1)}$, $E_n^{(1)}$ $(n = 6, 7, 8)$. In section 4, we give fundamental solutions for the system $E_{N,M}$ defined in [51, 52], and connection matrices among them. In section 5, we provide a q-difference system associated with an extension of the Jackson integral of Riemann-Papperitz type discussed in section 2, and its solutions in terms of Kajihara's q-hypergeometric series.

In the following, we summarize the contents of each section in more detail.

1.1 On section 2: q-Riemann-Papperitz system

The Riemann-Papperitz differential equation

$$
\begin{aligned}
&\left[\frac{d^2}{dx^2} + \left(\frac{1-\alpha_1-\beta_1}{x-t_1} + \frac{1-\alpha_2-\beta_2}{x-t_2} + \frac{1-\alpha_3-\beta_3}{x-t_3}\right)\frac{d}{dx} + \frac{1}{(x-t_1)(x-t_2)(x-t_3)}\right.\\
&\left.\times\left(\frac{\alpha_1\beta_1(t_1-t_2)(t_1-t_3)}{x-t_1} + \frac{\alpha_2\beta_2(t_2-t_1)(t_2-t_3)}{x-t_2} + \frac{\alpha_3\beta_3(t_3-t_1)(t_3-t_2)}{x-t_3}\right)\right] g(x) = 0,
\end{aligned} \tag{1.13}
$$

where $\alpha_1+\alpha_2+\alpha_3+\beta_1+\beta_2+\beta_3 = 1$, is another standard form of second order Fuchsian differential equations with three singularities. The equation (1.13) is characterized by the Riemann scheme

$$
\left\{\begin{matrix} x = t_1 & x = t_2 & x = t_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{matrix}\right\}. \tag{1.14}
$$

So the Riemann-Papperitz equation (1.13) and the Gauss equation (1.1) are transformed to each other by applying some Möbius transformation and gauge transformation. This makes some properties of the Gauss equation (1.1) clear (for example, see Whittaker-Watson [68]).

In the theory of q-difference equations, the points $x = 0$ and $x = \infty$ are the fixed points of the q-shift operator T_x. A notion of singularities that are useful for the analysis of q-difference equations is established only for the points $x = 0, \infty$. The Gauss equation (1.1) has singularities at $x = 0$ and $x = \infty$ (and also at $x = 1$), and then it is easy to obtain a q-analog of (1.1). However, the points $x = 0$ and $x = \infty$ are regular points for the Riemann-Papperitz equation (1.13). So it is difficult to construct a q-analog of (1.13).

In [21], two variants $\mathcal{H}_2 y = 0$, $\mathcal{H}_3 y = 0$ of q-hypergeometric equation were introduced. For background of these equations, see Remark 2.4. The equations $\mathcal{H}_2 y = 0$ and $\mathcal{H}_3 y = 0$, given in (2.4) and (2.1), are called the variant of q-difference equation of degree two and of degree three, respectively. It was found in [21] that the equation $\mathcal{H}_3$ degenerates to the equation $\mathcal{H}_2$ in some limit, and $\mathcal{H}_2$ degenerates to the Heine's equation (1.5). Several solutions for the equation

$\mathcal{H}_2$ are given in [21] in terms of ${}_3\varphi_2$ and the q-Appell hypergeometric series φ_1. However, only conjectural solutions were discussed for the equation $\mathcal{H}_3$ in [21].

Main results of section 2 are Theorem 2.20 and Theorem 2.27. These results are based on [46]. Theorem 2.20 gives integral solutions for the equation $\mathcal{H}_3$, and Theorem 2.27 gives series solutions for the equation $\mathcal{H}_3$ in terms of the very-well-poised q-hypergeometric series ${}_8W_7$. The key point to get these results is to regard the equation $\mathcal{H}_3$ as a q-Riemann-Papperitz system. The solutions are q-analogs of integral solutions and series solutions in terms of ${}_2F_1$ for the Riemann-Papperitz equation (1.13) (see Remark 2.23, 2.29). Connection formulas and degenerations for solutions of $\mathcal{H}_3$ will be also discussed in sections 2.4 and 2.5. These degenerations include Hatano-Matsunawa-Sato-Takemura's solutions [21], and q-analogs of Kummer's 24 solutions given by Hahn [20].

1.2 On section 3: Euler-type integral transformation for the q-Heun equation and affine Weyl group

The equations $\mathcal{H}_2 y = 0$, $\mathcal{H}_3 y = 0$ were introduced in [21] by reducing the q-Heun equation and its variants (see Remark 2.4). The q-Heun equation and its variants are not rigid (see also Remark 3.5, 3.17). So it is difficult to give an explicit solution for these equations, different from the equations $\mathcal{H}_2$, $\mathcal{H}_3$.

An integral transformation for Heun's differential equation was discussed by Takemura [59, 60]. Similarly, an integral transformation for the q-Heun equation was studied by Sasaki-Takagi-Takemura [56]. Recently, integral transformations for variants of the q-Heun equation were obtained by Takemura [63] from the viewpoint of the kernel functions.

In section 3, we introduce an Euler-type integral transformation

$$E^{\lambda}(f)(x;\sigma) = x^{\lambda-1}\int_0^{\sigma\infty} f(t)\frac{(qt/x)_\infty}{(q^\lambda t/x)_\infty}d_qt, \tag{1.15}$$

and study properties of it (see Proposition 3.3). Applying E^λ to solutions for suitable q-difference equations, including the q-Heun equation and its variants, we give representations for affine Weyl groups of types $D_5^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$. In particular, we provide integral transformations for the q-Heun equation and its variants. Main result of section 3 is Theorem 3.16, which gives a representation of affine Weyl group of type $E_8^{(1)}$ by means of the Euler-type integral transformation of the suitable q-difference equation. These results are based on [47].

1.3 On section 4: connection problem for a multivariable q-hypergeometric system

A connection problem, i.e. to get linear relations for the solutions, is an important and fundamental problem for the theory of hypergeometric equations. For example, the formula [64, 67]

$${}_2\varphi_1\left(\begin{matrix}a,b\\c\end{matrix};x\right) = \frac{(b,c/a)_\infty}{(b/a,c)_\infty}\frac{\theta(ax)}{\theta(x)}{}_2\varphi_1\left(\begin{matrix}a,aq/c\\aq/b\end{matrix};\frac{cq}{abx}\right) + \frac{(a,c/b)_\infty}{(a/b,c)_\infty}\frac{\theta(ax)}{\theta(x)}{}_2\varphi_1\left(\begin{matrix}b,bq/c\\bq/a\end{matrix};\frac{cq}{abx}\right), \tag{1.16}$$

is a connection formula for the Heine's equation.

In [51, 52], the q-hypergeometric series

$$\mathcal{F}_{N,M}\left(\begin{matrix}\{a_j\}_{1\le j\le N},\{b_i\}_{1\le i\le M}\\\{c_j\}_{1\le j\le N}\end{matrix};\{x_i\}_{1\le i\le M}\right) = \sum_{m_1,\dots,m_M\ge0}\prod_{j=1}^N\frac{(a_j)_{|m|}}{(c_j)_{|m|}}\prod_{i=1}^M\frac{(b_i)_{m_i}}{(q)_{m_i}}\prod_{i=1}^M x_i^{m_i}, \tag{1.17}$$

where $|m| = m_1 + \cdots + m_M$, was defined in order to get a certain isomonodromic deformation. This isomonodromic deformation is a q-analog of Tsuda's Hamiltonian system [65]. The function $\mathcal{F}_{N,M}$ satisfies the q-difference system $E_{N,M}$ [51, 52]

$$\left\{ x_s \prod_{j=1}^{N}(1-a_jT)\cdot(1-b_sT_s) - \prod_{j=1}^{N}(1-c_jq^{-1}T)\cdot(1-T_s) \right\}\mathcal{F} = 0 \quad (1 \le s \le M), \tag{1.18}$$

$$\{x_r(1-b_rT_r)(1-T_s) - x_s(1-b_sT_s)(1-T_r)\}\mathcal{F} = 0 \quad (1 \le r < s \le M), \tag{1.19}$$

where $T_s = T_{x_s}$ and $T = T_1 \cdots T_M$. In section 4, we will give fundamental solutions for the system $E_{N,M}$ and connection matrices among them. Main result of section 4 is Theorem 4.9. This provides connection matrices for fundamental solutions of the system $E_{N,M}$. Fundamental solutions for $E_{N,M}$ are given in Proposition 4.7. We also characterize such fundamental solutions by asymptotic behaviors in Proposition 4.6. These results are based on [45].

1.4 On section 5: a multivariable extension of the q-Riemann-Papperitz type integral

To get solutions for the equation $\mathcal{H}_3 y = 0$, we study the Jackson integral of Riemann-Papperitz type

$$\int_C \prod_{i=1}^{4} \frac{(a_it)_\infty}{(b_it)_\infty} d_qt, \;\; a_1 \cdots a_4 = q^2 b_1 \cdots b_4. \tag{1.20}$$

It is natural to consider an extension of this integral as

$$\int_C \prod_{i=1}^{M+3} \frac{(a_it)_\infty}{(b_it)_\infty} d_qt, \;\; a_1 \cdots a_{M+3} = q^2 b_1 \cdots b_{M+3}. \tag{1.21}$$

In section 5, we discuss some properties for this Riemann-Papperitz type integral. First we give a q-difference system of rank $M+1$, that the integral (1.21) satisfies in section 5.1. Second we show a transformation formula between the integral (1.21) and the Kajihara's q-hypergeometric series [29] in section 5.2, in particular we give solutions for the above system in terms of Kajihara's series. In section 5.3, we show that the above system degenerates the q-Appell-Lauricella system. Since it is very difficult to degenerate the Kajihara's q-hypergeometric series $W^{M,2}$ to the q-Appell-Lauricella series φ_D only by the definition of $W^{M,2}$, we will not discuss Kummer type solutions for the q-Appell-Lauricella system in this paper, unlike the one variable case.

Main results in section 5 are Theorem 5.4 and Theorem 5.12. Theorem 5.4 gives a q-difference system that the integral (1.21) satisfies. Theorem 5.12 provides a series solution for the q-difference system in terms of Kajihara's q-hypergeometric series $W^{M,2}$ [29]. These results are based on [48].

2 q-Riemann-Papperitz system

In this section, we will study a q-Riemann-Papperitz system. First, we will find that the variant of q-hypergeometric equation of degree three $\mathcal{H}_3$ [21] can be regarded as a q-Riemann-Papperitz system. Second, we will construct integral solutions and series solutions for the equation. Third, we will give linear relations for the solutions. As an application, we will obtain q-analogs of Kummer's 24 solutions.

The contents of section 2 are as follows. We show that the equation $\mathcal{H}_3$ can be regarded as a q-analog of the Riemann-Papperitz equation (1.13) in section 2.1. Integral solutions and series solutions for the equation $\mathcal{H}_3$ are given in section 2.2 and section 2.3, respectively. In section 2.4, we discuss connection formulas of the solutions. Finally we study degenerations for the q-Riemann-Papperitz system. This section is based on [46]. Some of the results in this section are also obtained in [15].

2.1 q-Riemann-Papperitz system: the variant of q-hypergeometric equation of degree three

In this subsection, we will show that the variant of q-hypergeometric equation of degree three introduced by [21], is a q-Riemann-Papperitz system.

Definition 2.1 ([21])**.** The variant of q-hypergeometric equation of degree three is defined as follows:

$$\mathcal{H}_3 y = 0, \tag{2.1}$$

$$\begin{aligned}\mathcal{H}_3 = &\prod_{i=1}^{3}(x - q^{h_i+1/2}t_i)\cdot T_x^{-1} + q^{2\alpha+1}\prod_{i=1}^{3}(x - q^{l_i-1/2}t_i)\cdot T_x \\ &+ q^{\alpha}\Bigg[-(q+1)x^3 + q^{1/2}\sum_{i=1}^{3}(q^{h_i}+q^{l_i})t_i x^2 \\ &- q^{(h_1+h_2+h_3+l_1+l_2+l_3+1)/2}t_1t_2t_3\sum_{i=1}^{3}((q^{-h_i}+q^{-l_i})/t_i)x \\ &+ q^{(h_1+h_2+h_3+l_1+l_2+l_3)/2}(q+1)t_1t_2t_3\Bigg].\end{aligned} \tag{2.2}$$

We call this equation as the q-Riemann-Papperitz system. For the reason why we call so, see Remark 2.8.

Remark 2.2. The equation $\mathcal{H}_3 y = 0$ has 10 parameters t_i, l_i, h_i $(i = 1, 2, 3)$ and α. We can delete the parameter α by the gauge transformation $y \to x^{-\alpha}y$. Also by putting the parameters $t'_i = q^{-h_i}t_i$ and $l'_i = l_i - h_i$, then $\mathcal{H}_3$ is rewritten by using t'_i and l'_i as

$$\begin{aligned}\mathcal{H}_3 = &\prod_{i=1}^{3}(x - q^{1/2}t'_i)\cdot T_x^{-1} + q^{2\alpha+1}\prod_{i=1}^{3}(x - q^{l'_i-1/2}t'_i)\cdot T_x \\ &+ q^{\alpha}\Bigg[-(q+1)x^3 + q^{1/2}\sum_{i=1}^{3}(1+q^{l'_i})t'_i x^2 \\ &- q^{(l'_1+l'_2+l'_3+1)/2}t'_1t'_2t'_3\sum_{i=1}^{3}((1+q^{-l'_i})/t'_i)x\end{aligned}$$

$$+ q^{(l'_1+l'_2+l'_3)/2}(q+1)t'_1t'_2t'_3\Big]. \tag{2.3}$$

In short, the equation (2.1) essentially depends on 6 parameters t'_i and l'_i $(i = 1, 2, 3)$.

Remark 2.3. In [21], another variant

$$\mathcal{H}_2 f(x) = 0, \tag{2.4}$$

of q-hypergeometric equation was introduced, where

$$\begin{aligned}\mathcal{H}_2 =& \prod_{i=1}^{2}(x - q^{h_i+1/2}t_i)\cdot T_x^{-1} + q^{\alpha_1+\alpha_2}\prod_{i=1}^{2}(x - q^{l_i-1/2}t_i)\cdot T_x \\ &- (q^{\alpha_1} + q^{\alpha_2})x^2 + Ex + p(q^{1/2} + q^{-1/2})t_1t_2, \qquad (2.5)\\ p =& q^{(h_1+h_2+l_1+l_2+\alpha_1+\alpha_2)/2}, \quad E = -p\{(q^{-h_2} + q^{-l_2})t_1 + (q^{-h_1} + q^{-l_1})t_2\}. \qquad (2.6)\end{aligned}$$

In this paper, we will discuss the equation $\mathcal{H}_3$ mainly. By taking the limit $t_3 \to \infty$ with some changes of parameters, the equation $\mathcal{H}_3$ degenerates to $\mathcal{H}_2$. Also the equation $\mathcal{H}_2$ becomes the Heine's q-hypergeometric equation by taking the limit $t_2 \to 0$. In this sense the equations $\mathcal{H}_3$ and $\mathcal{H}_2$ are called variants of q-hypergeometric equation. We can get some formulas about the equation $\mathcal{H}_2$ and Heine's equation by taking the same limits for the results of the equation $\mathcal{H}_3$. These degenerations are discussed in section 2.5.

Remark 2.4. We summarize background of the equations $\mathcal{H}_3$ and $\mathcal{H}_2$ here. The correspondences between some discrete Painlevé equations [54] and degenerations of the eigenvalue problem for the Ruijsenaars-van Diejen operator [13, 53] in 1 variable were discovered by [61] for q-difference case and [49] for elliptic case. Also Takemura [62] found that the fourth degeneration of the eigenvalue problem is equivalent to the q-Heun equation which was introduced by Hahn [19]. Heun's differential equation is a second order Fuchsian differential equation with 4 regular singularities. In addition, it was found in [62] that the second and third degenerations of the eigenvalue problem are also q-analogs of Heun's differential equation. These equations are called variants of the q-Heun equation of degree three and degree four. Heun's differential equation is not rigid, so it is difficult to get explicit solutions for the equation. If one of 4 singularities in Heun's equation is a regular point, then this equation has only 3 singularities, so it is equivalent to the Gauss equation. We can also take a similar specialization for the q-Heun equation and its variants. That is why variants $\mathcal{H}_2$, $\mathcal{H}_3$ of q-hypergeometric equation were introduced by Hatano-Matsunawa-Sato-Takemura [21]. The way [21] to derive the equations $\mathcal{H}_2 y = 0$ and $\mathcal{H}_3 y = 0$ from the q-Heun equation and its variant of degree three is to specialize so that the point $x = 0$ is essentially non-singular. Note that the points $x = 0$ and $x = \infty$ are essentially non-singular for the variant of the q-Heun equation of degree four, and then we can not apply the similar specialization for this case.

The following condition characterizes the equation $\mathcal{H}_3$.

Proposition 2.5 ([46])**.** *If the q-difference operator $H = \sum_{i=0}^{3}\sum_{j=-1}^{1} a_{i,j}x^iT_x^j$ has the following conditions:*

(1) We rewrite the operator H as $H = \sum_{i=0}^{3} x^i L_i(T_x)$. Then the Laurent polynomials L_0, L_1, L_2, L_3 are factorized in the following forms:

(i) $L_0(y) \propto (y-a)(y-aq)$, $L_1(y) \propto (y-a)$.

(ii) $L_3(y) \propto (y-b)(y-bq^{-1})$, $L_2(y) \propto (y-b)$.

(2) We rewrite the operator H *as* $H = \sum_{j=-1}^{1} P_j(x) T_x^j$. *Then the polynomials* P_{-1}, P_1 *are factorized as* $P_{-1}(x) \propto (x-c_1)(x-c_2)(x-c_3)$, $P_1(x) \propto (x-d_1)(x-d_2)(x-d_3)$.

Then the operator is equivalent to $\mathcal{H}_3$. *More precisely, we put* $a = q^{\nu-\alpha}$, $b = q^{\alpha}$, $c_i = q^{h_i+1/2}t_i$, $d_i = q^{l_i-1/2}t_i$ *where* $\nu = (h_1+h_2+h_3-l_1-l_2-l_3)/2$, *then we have* $H = a_{3,-1}\mathcal{H}_3$.

Proof. By the conditions of L_0, L_3, P_{-1}, P_1, the ratios of $\{a_{i,j} \mid (i,j) \neq (1,0),(2,0)\}$ are determined. Using the condition of L_1, $a_{1,0}$ can be written by $a_{1,1}$ and $a_{1,-1}$. More precisely, we have $L_1(y) = a_{1,1}y + a_{1,0} + a_{1,-1}y^{-1} = a_{1,1}y^{-1}(y-a)(y-a_{1,-1}/(a_{1,1}a))$, so we get $a_{1,0} = -(a_{1,1}a + a_{1,-1}/a)$. Similarly we find $a_{2,0} = -(a_{2,1}b + a_{2,-1}/b)$ by the condition of L_2. Therefore the operator H is determined by conditions (1) and (2).

On the other hand, we can check directly that the operator $\mathcal{H}_3$ satisfies conditions (1) and (2) by putting the parameters $a = q^{\nu-\alpha}$, $b = q^{\alpha}$, $c_i = q^{h_i+1/2}t_i$, $d_i = q^{l_i-1/2}t_i$. □

Remark 2.6. In [15], a "configuration" for q-difference equations was introduced. Also characterizations by means of it for the following equations were obtained:

- Variants of q-hypergeometric equation $\mathcal{H}_2$, $\mathcal{H}_3$.
- The Heine's q-hypergeometric equation.
- The q-Heun equation and its variant of degree three.

Remark 2.7. The condition (1.i) in Proposition 2.5 is the non-logarithmic condition at $x = 0$. That is the equation $Hy = 0$ has local solutions of the form

$$x^{\rho}(1+O(x)), \quad x^{\rho+1}(1+O(x)), \tag{2.7}$$

where $q^{\rho} = a$. The condition (1.ii) is also the non-logarithmic condition at $x = \infty$. For more general cases, see Proposition 3.1 in [41].

Remark 2.8 ([15, 21])**.** By taking the classical limit $q \to 1$, the equation (2.1) becomes a Fuchsian differential equation with the following Riemann scheme:

$$\left\{ \begin{array}{ccccc} x=t_1 & x=t_2 & x=t_3 & x=0 & x=\infty \\ 0 & 0 & 0 & \nu-\alpha & \alpha \\ l_1-h_1 & l_2-h_2 & l_3-h_3 & \nu-\alpha+1 & \alpha+1 \end{array} \right\}, \tag{2.8}$$

where $\nu = (h_1+h_2+h_3-l_1-l_2-l_3)/2$. Here, $x = 0$ and $x = \infty$ are non-logarithmic singularities. By some gauge transformation, we can delete two singularities $x = 0$ and $x = \infty$ from that equation. Therefore the equation (2.1) can be regarded as a q-analog of the Riemann-Papperitz system. Note that the classical limit of the equation (2.1) was calculated by Hatano-Matsunawa-Sato-Takemura [21]. However the solutions of it were not fully studied.

The Riemann-Papperitz system has an integral solution and a series solution as

$$\prod_{i=1}^{3}(x-t_i)^{\alpha_i}\int_C (t-t_1)^{\alpha_2+\alpha_3+\beta_1-1}(t-t_2)^{\alpha_1+\alpha_3+\beta_2-1}(t-t_1)^{\alpha_1+\alpha_2+\beta_3-1}(t-x)^{-\alpha_1-\alpha_2-\alpha_3}dt, \tag{2.9}$$

$$\left(\frac{x-t_1}{x-t_3}\right)^{\alpha_1}\left(\frac{x-t_2}{x-t_3}\right)^{\alpha_2} {}_2F_1\left(\begin{array}{c}\alpha_1+\alpha_2+\alpha_3, \alpha_1+\alpha_2+\beta_3 \\ \alpha_1-\beta_1+1\end{array}; \frac{x-t_1}{x-t_3}\frac{t_2-t_3}{t_2-t_1}\right). \tag{2.10}$$

We will construct similar solutions for the q-Riemann-Papperitz system (2.1) in the following subsections.

2.2 Integral solutions for the q-Riemann-Papperitz system

In this subsection, we will give Jackson integral solutions for the equation (2.1). These solutions are q-analogs of the integral (2.9). First, we derive a q-difference equation for the Jackson integral of Jordan-Pochhammer type:

$$\varphi = \varphi(x,\tau) = \int_0^{\tau\infty} \psi \frac{d_q t}{t}, \tag{2.11}$$

$$\psi = \psi(x,t) = t^\alpha \prod_{i=1}^{M} \frac{(a_i t)_\infty}{(b_i t)_\infty} \frac{(Axt)_\infty}{(Bxt)_\infty}. \tag{2.12}$$

Next, we show integral solutions for the q-Riemann-Papperitz system (2.1) by considering a special case of the Jordan-Pochhammer integral.

Remark 2.9. A linear q-difference system associated with the Jackson integral of Jordan-Pochhammer type was obtained by Mimachi [38] and Matsuo [35]. A q-difference system for the Jackson integral of Selberg type, which contains the system for the Jordan-Pochhammer type, was obtained by Mimachi [39]. In [24], a q-difference system for the Jackson integral of Selbarg type was also discussed, and the above results were summarized. Hence, for more details on the Jackson integral of Jordan-Pochhammer type, see [24, 35, 38, 39]. In this paper, we derive a q-difference equation for the Jackson integral of Jordan-Pochhammer type by integrating the equation that the integrand satisfies.

Lemma 2.10 plays an essential role in deriving the q-difference equation which the Jackson integral of Jordan-Pochhammer type satisfies.

Lemma 2.10 ([15])**.** *The integrand (2.12) satisfies the following equation*

$$\sum_{k=0}^{M} (-1)^k x^{M-k} [e_k(a) T_t T_x^{-1} - q^\alpha e_k(b)] \prod_{n=0}^{M-k-1} (B - Aq^{M-k-1} T_x) \prod_{n=0}^{k-1} (1 - q^{-n} T_x) \psi = 0. \tag{2.13}$$

Here, e_i is the elementary symmetric polynomial of degree i.

Proof. The equations

$$xt(B - AT_x)\psi = (1 - T_x)\psi, \tag{2.14}$$

$$\prod_{i=1}^{M} (1 - a_i t) \cdot T_t T_x^{-1} \psi = q^\alpha \prod_{i=1}^{M} (1 - b_i t) \cdot \psi, \tag{2.15}$$

can be verified by direct calculations. Using the elementary symmetric functions, the equation (2.15) can be rewritten as

$$\sum_{k=0}^{M} (-t)^k [e_k(a) T_t T_x^{-1} - q^\alpha e_k(b)] \psi = 0. \tag{2.16}$$

Since $[(xt)(B - AT_x)]^k = (xt)^k \prod_{n=0}^{k-1} (B - Aq^{k-1} T_x)$ holds, we have

$$\begin{aligned} &x^M \prod_{n=0}^{M-1} (B - Aq^n T_x) t^k (T_t T_x^{-1})^\varepsilon \psi \\ =&x^{M-k} (xt)^k (T_t T_x^{-1})^\varepsilon \prod_{n=k}^{M-1} (B - Aq^n T_x) \prod_{n=0}^{k-1} (B - Aq^n T_x) \psi \end{aligned}$$

$$
\begin{aligned}
&=x^{M-k}(T_tT_x^{-1})^{\varepsilon}\prod_{n=k}^{M-1}(B-Aq^{n-k}T_x)(xt)^k\prod_{n=0}^{k-1}(B-Aq^nT_x)\psi \\
&=x^{M-k}(T_tT_x^{-1})^{\varepsilon}\prod_{n=0}^{M-k-1}(B-Aq^nT_x)[xt(B-AT_x)]^k\psi, \qquad (2.17)
\end{aligned}
$$

for $\varepsilon = 0, 1$. By using the equation (2.14), we get

$$
[(xt)(B-AT_x)]^k\psi = \prod_{n=0}^{k-1}(1-q^{-n}T_x)\psi. \qquad (2.18)
$$

Therefore, multiplying the equation (2.16) by $x^M(B-AT_x)(B-AqT_x)\cdots(B-Aq^{M-1}T_x)$ on the left yields the desired equation (2.13). □

Remark 2.11. By the integral transformation

$$
f(x) \mapsto \int_0^{\tau\infty} f(t)\frac{(Axt)_\infty}{(Bxt)_\infty}d_qt, \qquad (2.19)
$$

we can transform a linear q-difference equation. For more properties, see section 3. In section 3, we give an integral transformation for the q-Heun equation and its variants [62].

If the integral path (the cycle of integration) is chosen appropriately, then we can derive the q-difference equation that the Jackson integral of Jordan-Pochhammer type satisfies.

Proposition 2.12 ([15])**.** *We assume $T_x\tau = q^l\tau$ for $l \in \mathbb{Z}$. The integral (2.11) satisfies the q-difference equation*

$$
\sum_{k=0}^{M}(-1)^kx^{M-k}[e_k(a)T_x^{-1}-q^\alpha e_k(b)]\prod_{n=0}^{M-k-1}(B-Aq^nT_x)\prod_{n=0}^{k-1}(1-q^{-n}T_x)\varphi = 0. \qquad (2.20)
$$

Proof. Since $T_x\tau = q^l\tau$, we get

$$
\begin{aligned}
\int_0^{\tau\infty}(T_t^iT_x^j\psi(x,t))\frac{d_qt}{t} &= (1-q)\sum_{n\in\mathbb{Z}}\psi(T_x^ix, q^{n+j}\tau) \\
&= (1-q)\sum_{n\in\mathbb{Z}}\psi(T_x^ix, q^{n+j-il}T_x^i\tau) \\
&= (1-q)\sum_{n\in\mathbb{Z}}\psi(T_x^ix, q^nT_x^i\tau) \\
&= (1-q)\sum_{n\in\mathbb{Z}}T_x^i\psi(x, q^n\tau) \\
&= T_x^i\int_0^{\tau\infty}\psi(x,t)\frac{d_qt}{t}. \qquad (2.21)
\end{aligned}
$$

Therefore we have the desired equation (2.20) by integrating the equation (2.13). □

Remark 2.13. We put $M = 1$, and then the equation (2.20) is the Heine's q-hypergeometric equation. With the changes in the variable and parameters

$$
z = \frac{qBx}{a_1}, \quad a = q^\alpha, \quad b = \frac{A}{B}, \quad c = q^{\alpha+1}\frac{b_1}{a_1}, \qquad (2.22)
$$

we find

$$
\begin{aligned}
&[x(T_x^{-1}-q^\alpha)(B-AT_x)-(a_1T_x^{-1}-q^\alpha b_1)(1-T_x)]\\
&=T_z^{-1}[z(1-aT_z)(1-bT_z)-(1-cq^{-1}T_z)(1-T_z)].
\end{aligned}
\tag{2.23}
$$

Therefore we have integral solutions $\displaystyle\int_0^{\tau\infty} t^\alpha \frac{(Axt,a_1t)_\infty}{(Bxt,b_1t)_\infty}\frac{d_qt}{t}$ for Heine's equation.

We will reduce the special case of (2.20) to the variant of the q-hypergeometric equation of degree three. First we put $\alpha=1$ and

$$
Aa_1\cdots a_M=q^2Bb_1\cdots b_M. \tag{2.24}
$$

Then the equation (2.20) is reducible:

$$
\begin{aligned}
&\sum_{k=0}^{M}(-1)^k x^{M-k}[e_k(a)T_x^{-1}-qe_k(b)]\prod_{n=0}^{M-k-1}(B-Aq^nT_x)\prod_{n=0}^{k-1}(1-q^{-n}T_x)\\
=&(B-Aq^{-1}T_x)(1-q^{1-M})E_M,
\end{aligned}
\tag{2.25}
$$

where the rank $M-1$ operator E_M is defined as

$$
\begin{aligned}
E_M&=E_M(A,B,a=\{a_i\}_{1\le i\le M},b=\{b_i\}_{1\le i\le M})\\
&=x^MT_x^{-1}\prod_{n=0}^{M-2}(B-Aq^nT_x)\\
&+\sum_{k=1}^{M-1}(-1)^k x^{M-k}[e_k(a)T_x^{-1}-qe_k(b)]\prod_{n=0}^{M-k-2}(B-Aq^nT_x)\prod_{n=0}^{k-2}(1-q^{-n}T_x)\\
&+(-1)^M\frac{a_1\cdots a_M}{B}T_x^{-1}\prod_{n=0}^{M-2}(1-q^{-n}T_x).
\end{aligned}
\tag{2.26}
$$

Remark 2.14. We remark on the characterization of the operator E_M.

- If $H=\sum_{i=0}^M x^iL_i(T_x)$, then

$$
L_0(y)\propto\prod_{j=0}^{M-2}(y-aq^j),\ L_1(y)\propto\prod_{j=0}^{M-3}(y-aq^j),\ \ldots,\ L_{M-2}(y)\propto(y-a).,\tag{2.27}
$$

$$
L_M(y)\propto\prod_{j=0}^{M-2}(y-bq^{-j}),\ L_{M-1}(y)\propto\prod_{j=0}^{M-3}(y-bq^{-j}),\ \ldots,\ L_2(y)\propto(y-b).\tag{2.28}
$$

- If $H=\sum_{j=-1}^{M-2}P_j(x)T_x^j$, then

$$
P_{-1}(x)\propto(x-c_1)\cdots(x-c_M),\ P_{M-2}(x)\propto(x-d_1)\cdots(x-d_M).\tag{2.29}
$$

Then we have $H=E_M$ with suitable parameters a, b, c_i, d_i. This characterization can be proved in a similar manner to Proposition 2.5.

When $M=3$, the q-difference operator E_3 satisfies the conditions in Proposition 2.5. So the equation $E_3 y=0$ is equivalent to the q-Riemann-Papperitz system (2.1). More precisely, by putting $q^{-\nu}=B/A$, $q^{l_i-1/2}t_i=b_i/A$, $q^{h_i+1/2}t_i=a_i/B$, and by the gauge transformation $Y=x^{\nu-\alpha}y$, we have $\mathcal{H}_3 Y=0$.

Now the integral (2.11) satisfies the equation $(B-Aq^{-1}T_x)(1-q^{1-M}T_x)E_M\varphi=0$. Thus we have $E_M\varphi=C_1x^{M-1}+C_2x^{\lambda}$, where $q^{\lambda}=Bq/A$ and C_1 C_2 are pseudo constants, i.e. functions in x that satisfy $T_xC_i=C_i$. We want to delete pseudo constants C_i in order to obtain solutions for the equation $E_My=0$. The following lemma is useful for calculating the pseudo constants C_1, C_2.

Lemma 2.15. *We have*

$$\lim_{z\to 1}(1-z)_M\psi_M\left(\begin{array}{c}a_1,\dots,a_M\\ b_1,\dots,b_M\end{array};z\right)=\frac{(a_1,\dots,a_M)_\infty}{(b_1,\dots,b_M)_\infty},\tag{2.30}$$

where ${}_M\psi_M$ is the bilateral q-hypergeometric function:

$${}_M\psi_M\left(\begin{array}{c}a_1,\dots,a_M\\ b_1,\dots,b_M\end{array};z\right)=\sum_{n\in\mathbb{Z}}\frac{(a_1,\dots,a_M)_n}{(b_1,\dots,b_M)_n}z^n.\tag{2.31}$$

Proof. The negative power part of the function ${}_M\psi_M$ is holomorphic at $z=1$. Thus we have

$$\lim_{z\to 1}(1-z)_M\psi_M\left(\begin{array}{c}a_1,\dots,a_M\\ b_1,\dots,b_M\end{array};z\right)=\lim_{z\to 1}(1-z)\sum_{n\geq 0}\frac{(a_1,\dots,a_M)_n}{(b_1,\dots,b_M)_n}z^n.\tag{2.32}$$

Using the q-binomial theorem, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(a_1,\dots,a_M)_n}{(b_1,\dots,b_M)_n}z^n\\
=&\frac{(a_1,\dots,a_M)_\infty}{(b_1,\dots,b_M)_\infty}\sum_{n=0}^{\infty}\frac{(b_1q^n,\dots,b_Mq^n)_\infty}{(a_1q^n,\dots,a_Mq^n)_\infty}z^n\\
=&\frac{(a_1,\dots,a_M)_\infty}{(b_1,\dots,b_M)_\infty}\sum_{n=0}^{\infty}\sum_{k_1=0}^{\infty}\cdots\sum_{k_M=0}^{\infty}\frac{(b_1/a_1)_{k_1}\cdots(b_M/a_M)_{k_M}}{(q)_{k_1}\cdots(q)_{k_M}}a_1^{k_1}\cdots a_M^{k_M}(zq^{k_1+\cdots+k_M})^n\\
=&\frac{(a_1,\dots,a_M)_\infty}{(b_1,\dots,b_M)_\infty}\sum_{k_1=0}^{\infty}\cdots\sum_{k_M=0}^{\infty}\frac{(b_1/a_1)_{k_1}\cdots(b_M/a_M)_{k_M}}{(q)_{k_1}\cdots(q)_{k_M}}a_1^{k_1}\cdots a_M^{k_M}\frac{1}{1-zq^{k_1+\cdots+k_M}}\\
=&\frac{(a_1,\dots,a_M)_\infty}{(b_1,\dots,b_M)_\infty}\left(\frac{1}{1-z}+\sum_{k_1+\cdots+k_M>0}\frac{(b_1/a_1)_{k_1}\cdots(b_M/a_M)_{k_M}}{(q)_{k_1}\cdots(q)_{k_M}}a_1^{k_1}\cdots a_M^{k_M}\frac{1}{1-zq^{k_1+\cdots+k_M}}\right).
\end{aligned}\tag{2.33}$$

Therefore we get the desired equation (2.30). □

Lemma 2.16. *Suppose $\tau\in\{q/a_1,\dots,q/a_M,q/(Ax)\}$ and $B/A\notin q^{\mathbb{Z}}$. Then the Jackson integral of Jordan-Pochhammer type (2.11) satisfies the equation*

$$E_M\varphi=-\prod_{n=0}^{M-3}(B-Aq^n)\cdot q(1-q)x^{M-1},\tag{2.34}$$

where E_M is given in equation (2.26). In particular, the integral

$$\varphi_{i,j} = \int_{\tau_i}^{\tau_j} \prod_{k=1}^{M} \frac{(a_k t)_\infty}{(b_k t)_\infty} \frac{(Axt)_\infty}{(Bxt)_\infty} d_q t, \tag{2.35}$$

satisfies the equation $E_M \varphi_{i,j} = 0$. Here, $\tau_i = q/a_i$ $(i = 1, \ldots, M)$ and $\tau_{M+1} = q/(Ax)$.

Proof. First, if $\tau = q/a_k$, then the integral

$$\varphi(x, \tau) = \int_0^{\tau} \prod_{i=1}^{M} \frac{(a_i t)_\infty}{(b_i t)_\infty} \frac{(Axt)_\infty}{(Bxt)_\infty} d_q t, \tag{2.36}$$

is holomorphic at $x = 0$. Also if $\tau = q/(Ax)$, then this integral is holomorphic at $x = \infty$. Thus we have $E_M \varphi = C x^{M-1}$ where C is a constant. For $\tau = q/a_k$, we have

$$\varphi(x, \tau) = \int_0^{\tau} \prod_{i=1}^{M} \frac{(a_i t)_\infty}{(b_i t)_\infty} \sum_{n=0}^{\infty} \frac{(A/B)_n}{(q)_n} (Bxt)^n d_q t = \sum_{n=0}^{M-1} \frac{(A/B)_n}{(q)_n} (Bx)^n \int_0^{\tau} t^n \prod_{i=1}^{M} \frac{(a_i t)_\infty}{(b_i t)_\infty} d_q t + O(x^M), \tag{2.37}$$

due to the q-binomial theorem. So we have

$$\begin{aligned}
C = &\sum_{k=1}^{M-1} (-1)^k [e_k(a) q^{1-k} - q e_k(b)] \prod_{n=0}^{M-k-2} (B - A q^n q^{k-1}) \prod_{n=0}^{k-2} (1 - q^{-n} q^{k-1}) \\
&\times \frac{(A/B)_{k-1}}{(q)_{k-1}} B^{k-1} \int_0^{\tau} t^{k-1} \prod_{i=1}^{M} \frac{(a_i t)_\infty}{(b_i t)_\infty} d_q t \\
&+ (-1)^M \frac{a_1 \cdots a_M}{B} q^{1-M} \prod_{n=0}^{M-2} (1 - q^{-n} q^{M-1}) \frac{(A/B)_{M-1}}{(q)_{M-1}} B^{M-1} \int_0^{\tau} t^{M-1} \prod_{i=1}^{M} \frac{(a_i t)_\infty}{(b_i t)_\infty} d_q t \\
= &\sum_{k=1}^{M-1} (-1)^k [e_k(a) q^{1-k} - q e_k(b)] \prod_{n=0}^{M-3} (B - A q^n) \int_0^{\tau} t^{k-1} \prod_{i=1}^{M} \frac{(a_i t)_\infty}{(b_i t)_\infty} d_q t \\
&+ (-1)^M q^{1-M} \frac{a_1 \cdots a_M}{B} \prod_{n=0}^{M-3} (B - A q^n) \cdot (B - A q^{M-2}) \int_0^{\tau} t^{M-1} \prod_{i=1}^{M} \frac{(a_i t)_\infty}{(b_i t)_\infty} d_q t \\
= &\prod_{n=0}^{M-3} (B - A q^n) \sum_{k=1}^{M} [e_k(a) q^{1-k} - q e_k(b)] \int_0^{\tau} t^{k-1} \prod_{i=1}^{M} \frac{(a_i t)_\infty}{(b_i t)_\infty} d_q t \\
= &\prod_{n=0}^{M-3} (B - A q^n)(1 - q) q \prod_{i=1}^{M} \frac{(a_i \tau)_\infty}{(b_i \tau)_\infty} \sum_{k=1}^{M} [e_k(a) q^{-k} - e_k(b)] \tau^k {}_M\psi_M \left(\begin{matrix} b_1\tau, \ldots, b_M\tau \\ a_1\tau, \ldots, a_M\tau \end{matrix} ; q^k \right).
\end{aligned} \tag{2.38}$$

Here, we used the condition $a_1 \cdots a_M A = q^2 b_1 \cdots b_M B$. The function ${}_M\psi_M \left(\begin{matrix} b_1\tau, \ldots, b_M\tau \\ a_1\tau, \ldots, a_M\tau \end{matrix} ; z \right)$ satisfies the following equation:

$$[(1 - a_1 \tau q^{-1} T_z) \cdots (1 - a_M \tau q^{-1} T_z) - z(1 - b_1 \tau T_z) \cdots (1 - b_M \tau T_z)] {}_M\psi_M = 0. \tag{2.39}$$

Using the elementary symmetric polynomial e_k, we obtain

$$\sum_{k=0}^{M} [e_k(a) q^{-k} - z e_k(b)] \tau^k {}_M\psi_M \left(\begin{matrix} b_1\tau, \ldots, b_M\tau \\ a_1\tau, \ldots, a_M\tau \end{matrix} ; z q^k \right) = 0. \tag{2.40}$$

So we have

$$\sum_{k=1}^{M}[e_k(a)q^{-k}-e_k(b)]\tau^k{}_M\psi_M\left(\begin{array}{c}b_1\tau,\dots,b_M\tau\\ a_1\tau,\dots,a_M\tau\end{array};q^k\right)$$
$$=-\lim_{z\to 1}(1-z)_M\psi_M\left(\begin{array}{c}b_1\tau,\dots,b_M\tau\\ a_1\tau,\dots,a_M\tau\end{array};z\right)=-\frac{(b_1\tau,\dots,b_M\tau)_\infty}{(a_1\tau,\dots,a_M\tau)_\infty}. \tag{2.41}$$

Therefore we have

$$C=-\prod_{n=0}^{M-3}(B-Aq^n)q(1-q). \tag{2.42}$$

For $\tau=q/(Ax)$, the integral can be expanded at $x=\infty$ as follows:

$$\int_0^{q/(Ax)}\frac{(Axt)_\infty}{(Bxt)_\infty}\prod_{i=1}^{M}\frac{(a_it)_\infty}{(b_it)_\infty}d_qt$$
$$=\frac{1}{x}\int_0^{q/A}\frac{(At)_\infty}{(Bt)_\infty}\prod_{i=1}^{M}\frac{(a_it/x)_\infty}{(b_it/x)_\infty}d_qt$$
$$=\frac{1}{x}\left(\int_0^{q/A}\frac{(At)_\infty}{(Bt)_\infty}d_qt+O\left(\frac{1}{x}\right)\right). \tag{2.43}$$

The last integral is factorized by the q-binomial theorem as follows:

$$\int_0^{q/A}\frac{(At)_\infty}{(Bt)_\infty}d_qt=(1-q)\frac{q}{A}\frac{(q)_\infty}{(qB/A)_\infty}\frac{(q^2B/A)_\infty}{(q)_\infty}=\frac{(1-q)q}{A-Bq}. \tag{2.44}$$

The degree of the operator E_M in x is M, so we have

$$E_M\varphi(x,q/(Ax))=\frac{(1-q)q}{A-Bq}q(B-Aq^{-1})\prod_{n=0}^{M-3}(B-Aq^n)x^{M-1}+O(x^{M-2}). \tag{2.45}$$

Hence, the integral $\varphi(x,\tau)$ satisfies the equation (2.34)

$$E_M\varphi(x,\tau)=Cx^{M-1}. \tag{2.46}$$

This completes the proof. $\square$

Proposition 2.17 ([46])**.** *Suppose $a_i\neq a_j$ and $b_i\neq b_j$ for $1\le i<j\le M$. The integrals $\varphi_{1,2},\dots,\varphi_{1,M}$ are linearly independent over the field K_x of pseudo constants. Here*

$$K_x=\{C(x)\mid T_xC(x)=C(x)\}. \tag{2.47}$$

In particular, $\varphi_{1,2},\dots,\varphi_{1,M}$ are basis of the space of solutions for $E_My=0$.

Proof. It is sufficient to prove it in a special case. We can put $a_i=b_i$ $(i=1,\dots,M)$ by the assumptions. Also we put $B=1$, $A=q^2$. By a simple calculation, we have

$$\varphi_{1,i}=\int_{q/a_1}^{q/a_i}\frac{1}{(1-xt)(1-qxt)}d_qt=\frac{(1-q)q(a_1-a_i)}{(a_1-xq)(a_i-xq)}. \tag{2.48}$$

We assume

$$C_2\varphi_{1,2} + \cdots + C_M\varphi_{1,M} = 0, \tag{2.49}$$

where $C_i \in K_x$. We have

$$(C_2, \ldots, C_M)((p_2, \ldots, p_M)^{\mathrm{T}}) = 0, \tag{2.50}$$

where v^{T} is the transpose of v and

$$p_i = (a_1 - a_i) \prod_{\substack{2\le j\le M \\ j\neq i}} (a_i - xq). \tag{2.51}$$

By the assumption $a_i \neq a_j$, the polynomials p_i of degree $M-2$ are basis of $\{f(x) \in \mathbb{C}[x] \mid \deg f(x) \le M-2\}$. So there exists a constant matrix $G \in GL(M-1, \mathbb{C})$ such that

$$(p_2, \ldots, p_M)^{\mathrm{T}} = M(1, x, \ldots, x^{M-2})^{\mathrm{T}}. \tag{2.52}$$

We put $(\tilde{C}_0, \ldots, \tilde{C}_{M-2}) = (C_2, \ldots, C_M)G$, then $\tilde{C}_0, \ldots, \tilde{C}_{M-2}$ are pseudo constants and

$$(\tilde{C}_0, \ldots, \tilde{C}_{M-2})((1, x, \ldots, x^{M-2})^{\mathrm{T}}) = 0. \tag{2.53}$$

Applying T_x^k $(k = 1, \ldots, M-2)$, we have

$$(\tilde{C}_0, \ldots, \tilde{C}_{M-2}) \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x & qx & \cdots & q^{M-2}x \\ \vdots & \vdots & & \vdots \\ x^{M-2} & (qx)^{M-2} & \cdots & (q^{M-2}x)^{M-2} \end{pmatrix} = (0, \ldots, 0). \tag{2.54}$$

By the Vandermonde determinant, we have $(\tilde{C}_0, \ldots, \tilde{C}_{M-2}) = (0, \ldots, 0)$. Since G is invertible, we finally obtain $(C_2, \ldots, C_M) = (0, \ldots, 0)$. □

Remark 2.18. In [15], the linearly independence of integrals for the case of $M = 3$ is proved in another way. Their proof is based on that a non-constant rational function is not a pseudo constant.

Remark 2.19. In some cases of q-difference equations, linearly independence of solutions can be easily checked by considering characteristic exponents at $x = 0$ or $x = \infty$. However, the points $x = 0$ and $x = \infty$ are essentially non-singular for the equation $E_M y = 0$. Therefore it is difficult to prove the linearly independence by this way here. Our method in the proof of Proposition 2.17 does not focus on the characteristic exponents. We proved the linearly independence by using the Vandermonde determinant.

We consider the case $M = 3$ here. As mentioned above, the equation E_3 is equivalent to the q-Riemann-Papperitz system (2.1). By putting the parameters suitably, we finally get integral solutions for the q-Riemann-Papperitz system (2.1).

Theorem 2.20 ([46])**.** *The integral*

$$\varphi_{i,j} = x^{\nu-\alpha} \int_{\tau_i}^{\tau_j} \frac{(q^{\nu}xt, q^{h_1+\frac{1}{2}}t_1t, q^{h_2+\frac{1}{2}}t_2t, q^{h_3+\frac{1}{2}}t_3t)_\infty}{(xt, q^{\nu+l_1-\frac{1}{2}}t_1t, q^{\nu+l_2-\frac{1}{2}}t_2t, q^{\nu+l_3-\frac{1}{2}}t_3t)_\infty} d_qt \quad (1 \le i, j \le 4), \tag{2.55}$$

satisfies the q-Riemann-Papperitz system (2.1). Here, $\nu = \frac{1}{2}(h_1 + h_2 + h_3 - l_1 - l_2 - l_3 + 1)$, $\tau_i = q^{\frac{1}{2}-h_i}/t_i$ $(i = 1, 2, 3)$, $\tau_4 = q^{1-\nu}/x$.

Proof. By putting $q^{-\nu} = B/A$, $q^{l_i-1/2}t_i = b_i/A$, $q^{h_i+1/2}t_i = a_i/B$, and by the gauge transformation $Y = x^{\nu-\alpha}y$, the equation $E_3y = 0$ is transformed to the q-Riemann-Papperitz system $\mathcal{H}_3Y = 0$. Using Lemma 2.16, we have the desired result. □

Remark 2.21. In [15], other integral solutions are given by considering the following integral:

$$\tilde{\varphi} = \int_0^{\tau\infty} \psi \times C(t) d_qt, \tag{2.56}$$

where $C(t)$ is a suitable pseudo constant.

Remark 2.22. In [9], the integral in Theorem 2.20 is discussed by using the q-middle convolution defined by Sakai-Yamaguchi [55].

Remark 2.23. The Riemann-Papperitz equation (1.13) has an integral solution of the form

$$(\text{gauge factor}) \times \int_C (t-t_1)^{\nu_1}(t-t_2)^{\nu_2}(t-t_3)^{\nu_3}(t-x)^{\nu_4}dt, \tag{2.57}$$

where $\nu_1+\nu_2+\nu_3+\nu_4 = -2$. The integral (2.55) is a q-analog of this integral. So we call (2.55) the Jackson integral of Riemann-Papperitz type. Note that the condition (2.24) corresponds to the relation $\nu_1+\nu_2+\nu_3+\nu_4 = -2$.

2.3 Series solutions for the q-Riemann-Papperitz system

In this section, we will present series solutions for the q-Riemann-Papperitz system (2.1) in terms of the very-well-poised q-hypergeometric function ${}_8W_7$. These solutions are q-analogs of a series solution (2.62) for the Riemann-Papperitz system (1.13).

First we show some formulas for the the very-well-poised q-hypergeometric function ${}_8W_7$ without proofs. For details of such formulas, see Gasper-Rahman's book [16].

Definition 2.24 ([16], (2.1.11))**.** The very-well-poised q-hypergeometric series is defined as follows:

$${}_{r+1}W_r(a_1;a_4,\ldots,a_{r+1};z) = \sum_{n=0}^{\infty} \frac{1-a_1q^{2n}}{1-a_1}\frac{(a_1,a_4,\ldots,a_{r+1})_n}{(q,qa_1/a_4,\ldots,qa_1/a_{r+1})_n}z^n. \tag{2.58}$$

This series converges for $|z|<1$, and is called balanced if $z = \dfrac{(\pm(a_1q)^{1/2})^{r-3}}{a_4a_5\cdots a_{r+1}}$.

Lemma 2.25 ([16], (2.10.19))**.** *If $cd = abefgh$ and $|ah|<1$, then we have*

$$\begin{aligned}&\int_a^b \frac{(qt/a,qt/b,ct,dt)_\infty}{(et,ft,gt,ht)_\infty}d_qt\\ =&b(1-q)\frac{(q,bq/a,a/b,cd/(eh),cd/(fh),cd/(gh),bc,bd)_\infty}{(ae,af,ag,be,bf,bg,bh,bcd/h)_\infty} \times {}_8W_7\left(\frac{bcd}{hq};be,bf,bg,\frac{c}{h},\frac{d}{h};ah\right).\end{aligned} \tag{2.59}$$

This formula is known as the Bailey's transformation formula.

Lemma 2.26 ([16], (2.10.1))**.** *Let $\mu = qa^2/(bcd)$. We have*

$${}_8W_7\left(a;b,c,d,e,f;\frac{a^2q^2}{bcdef}\right) = \frac{(aq,aq/(ef),\mu q/e,\mu q/f)_\infty}{(aq/e,aq/f,\mu q,\mu q/(ef))_\infty}{}_8W_7\left(\mu;\frac{\mu b}{a},\frac{\mu c}{a},\frac{\mu d}{a},e,f;\frac{aq}{ef}\right), \tag{2.60}$$

if $\max(|aq/(ef)|,|\mu q/(ef)|)<1$.

Lemma 2.26 is mainly used in section 2.5. By applying Lemma 2.25 to the integral (2.55), we obtain series solutions for the q-Riemann-Papperitz system (2.1).

Theorem 2.27 ([46]). *Let $\{i,j,k\} = \{1,2,3\}$. The function*

$$x^{\nu-\alpha}\frac{(q^{\nu-h_k+\frac{1}{2}}x/t_k)_\infty}{(q^{\frac{1}{2}-h_k}x/t_k)_\infty}{}_8W_7\left(\frac{t_iq^{h_i-h_k+\nu}}{t_k};\frac{t_iq^{-h_k+l_i+\nu}}{t_k},\frac{t_jq^{-h_k+l_j+\nu}}{t_k},q^{-h_k+l_k+\nu},\frac{t_iq^{h_i+\frac{1}{2}}}{x},q^{\nu};\frac{xq^{\frac{1}{2}-h_j}}{t_j}\right), \tag{2.61}$$

satisfies the q-Riemann-Papperitz system (2.1), where $\nu = \frac{1}{2}(h_1+h_2+h_3-l_1-l_2-l_3+1)$.

Remark 2.28. In [15], a configuration of q-difference equations was introduced and the symmetry of the q-Riemann-Papperitz system (2.1) was studied by using this idea. Also the symmetry was realized by some gauge transformations, so we get many solutions in terms of ${}_8W_7$ by using the symmetry. For more details, see [15].

Remark 2.29. The Riemann-Papperitz equation (1.13) has a series solution as

$$\left(\frac{z-t_1}{z-t_3}\right)^{\alpha_1}\left(\frac{z-t_2}{z-t_3}\right)^{\alpha_2}{}_2F_1\left(\begin{matrix}\alpha_1+\alpha_2+\alpha_3,\alpha_1+\alpha_2+\beta_3\\ \alpha_1-\beta_1+1\end{matrix};\frac{z-t_1}{z-t_3}\frac{t_2-t_3}{t_2-t_1}\right). \tag{2.62}$$

The series ${}_8W_7$ in the above theorem is a q-analog of the series (2.62), namely we have

$$\begin{aligned}
&{}_8W_7\left(\frac{t_iq^{h_i-h_k+\nu}}{t_k};\frac{t_iq^{-h_k+l_i+\nu}}{t_k},\frac{t_jq^{-h_k+l_j+\nu}}{t_k},q^{-h_k+l_k+\nu},\frac{t_iq^{h_i+\frac{1}{2}}}{x},q^{\nu};\frac{xq^{\frac{1}{2}-h_j}}{t_j}\right)\\
&=\sum_{n=0}^{\infty}\frac{1-t_iq^{\bullet}/t_k}{1-t_iq^{\bullet}/t_k}\frac{(t_iq^{\bullet}/t_k,t_iq^{\bullet}/t_k,t_jq^{\bullet}/t_k,q^{-h_k+l_k+\nu},t_iq^{\bullet}/x,q^{\nu})_n}{(q,q^{1+h_i-l_i},t_iq^{\bullet}/t_j,t_iq^{\bullet}/t_k,xq^{\bullet}/t_k,t_iq^{\bullet}/t_k)_n}\left(\frac{xq^{\bullet}}{t_j}\right)\\
&\xrightarrow{q\to1}\sum_{n=0}^{\infty}\frac{(-h_k+l_k+\nu,n)(\nu,n)}{(1,n)(1+h_i-l_i,n)}\left(\frac{(1-t_i/t_k)(1-t_i/t_k)(1-t_j/t_k)(1-t_i/x)}{(1-t_i/t_j)(1-t_i/t_k)(1-x/t_k)(1-t_i/t_k)}\frac{x}{t_j}\right)^n\\
&={}_2F_1\left(\begin{matrix}-h_k+l_k+\nu,\nu\\ 1+h_i-l_i\end{matrix};\frac{x-t_i}{x-t_k}\frac{t_j-t_k}{t_j-t_i}\right).
\end{aligned} \tag{2.63}$$

Note that explicit forms of $\bullet$ are not essential for taking the limit $q\to1$.

Remark 2.30. The solution (2.61) can be obtained by applying the formula (2.59) to the integral $\varphi_{i,j}$ $(1\le i,j\le 3)$. Applying it to $\varphi_{i,4}$, we have the following solution:

$$\begin{aligned}
&x^{\nu-\alpha-1}\frac{(t_iq^{h_i-\nu+3/2}/x,xq^{-h_i+\nu-1/2}/t_i,t_jq^{h_j+h_k-l_k-\nu+3/2}/x,t_jq^{h_j-\nu+3/2}/x,t_kq^{h_k-\nu+3/2}/x)_\infty}{(xq^{1/2-h_i}/t_i,t_iq^{l_i+1/2}/x,t_jq^{l_j+1/2}/x,t_kq^{l_k+1/2}/x,t_jq^{h_j+h_k-l_k-2\nu+5/2}/x)_\infty}\\
&\times{}_8W_7\left(\frac{t_jq^{h_j+h_k-l_k-2\nu+\frac{3}{2}}}{x};q^{1-\nu},\frac{t_iq^{l_i+\frac{1}{2}}}{x},\frac{t_jq^{l_j+\frac{1}{2}}}{x},\frac{t_jq^{h_j-l_k-\nu+1}}{t_k},q^{h_k-l_k-\nu+1};\frac{t_kq^{-h_i+l_k+\nu}}{t_i}\right).
\end{aligned} \tag{2.64}$$

Also we get several solutions for the q-Riemann-Papperitz system by Lemma 2.26. For example,

$$\begin{aligned}
&x^{\nu-\alpha}\frac{(q^{\nu-h_k+\frac{1}{2}}x/t_k,xq^{h_i-l_j-l_i-\nu+3/2}/t_j,xq^{h_k+h_i-l_j-l_k-l_i-\nu+3/2}/t_j)_\infty}{(q^{\frac{1}{2}-h_k}x/t_k,xq^{h_i-l_j-l_i+3/2}/t_j,xq^{-h_k+h_i-l_j+l_k-l_i+2\nu+3/2}/t_j)_\infty}\\
&\times{}_8W_7\left(\frac{xq^{h_i-l_j-l_i+\frac{1}{2}}}{t_j};\frac{xq^{\frac{1}{2}-l_j}}{t_j},\frac{t_kq^{h_k+h_i-l_j-l_i-\nu+1}}{t_j},\frac{xq^{\frac{1}{2}-l_i}}{t_i},q^{\nu},q^{-h_k+l_k+\nu};\frac{t_iq^{h_i-l_k-\nu+1}}{t_k}\right),
\end{aligned} \tag{2.65}$$

$$x^{\nu-\alpha-1}\frac{(t_jq^{h_j-\nu+3/2}/x, t_kq^{h_k-\nu+3/2}/x, t_iq^{h_i-\nu+3/2}/x, xq^{-h_j+\nu-1/2}/t_j, t_kt_iq^{h_k+h_i-l_j-\nu+3/2}/t_jx)_\infty}{(t_jq^{l_j+1/2}/x, t_kq^{l_k+1/2}/x, t_iq^{l_i+1/2}/x, t_kt_iq^{h_k+h_i-\nu+2}/x^2)_\infty}$$
$$\times (t_iq^{h_k+h_i-l_k-\nu+3/2}/x, t_kq^{h_k+h_i-l_i-\nu+3/2}/x)_\infty$$
$$\times {}_8W_7\left(\frac{t_kt_iq^{h_k+h_i-\nu+1}}{x^2}; \frac{t_jq^{l_j+\frac{1}{2}}}{x}, \frac{t_kq^{l_k+\frac{1}{2}}}{x}, \frac{t_iq^{l_i+\frac{1}{2}}}{x}, \frac{t_kq^{h_k+\frac{1}{2}}}{x}, \frac{t_iq^{h_i+\frac{1}{2}}}{x}; \frac{xq^{\frac{1}{2}-h_j}}{t_j}\right). \tag{2.66}$$

These solutions are useful for considering degenerations of the q-Riemann-Papperitz system (2.1). Degenerations are discussed in section 2.5.

Remark 2.31. The very-well-poised-balanced q-hypergeometric series ${}_8W_7$ is also known as the Askey-Wilson function [33]. Well known linear q-difference equations satisfied by the very-well-poised-balanced ${}_8W_7$ are the ones considered by Ismail and Rahman [23]. We put $\phi = {}_8W_7\left(a; b, c, d, e, f; \frac{(aq)^2}{bcdef}\right)$, and $T_1 = T_bT_c^{-1}$, $T_2 = T_a^2T_cT_dT_eT_f$. In [23], linear relations for $T_1\phi$, ϕ, $T_1^{-1}\phi$, and for $T_2\phi$, ϕ, $T_2^{-1}\phi$ were obtained. These equations are as follows:

$$\left[(b-c)\left(1-\frac{aq}{bc}\right)\left(1-\frac{a^2q^2}{bcdef}\right)+L+M\right]\phi = \frac{(1-c)(1-a/c)}{(1-b/q)(1-aq/b)}T_1^{-1}\phi + \frac{(1-b)(1-a/b)}{(1-c/q)(1-aq/c)}T_1\phi, \tag{2.67}$$

$$\begin{aligned}&\left[\frac{b(1-a/(bc))(1-a/(bd))(1-a/(be))(1-a/(bf))}{(1-a/b)(1-a/(bq))}\right.\\ &\left.+\frac{a^2(1-c)(1-d)(1-e)(1-f)q}{(1-a/b)(1-aq/b)bcdef} + (1-b)\left(1-\frac{a^2q}{bcdef}\right)\right]\phi\\ &= \frac{(1-a/c)(1-a/d)(1-a/e)(1-a/f)}{(1-a)(1-aq)}T_2^{-1}\phi\\ &+\frac{a^2q}{cdef}\frac{(1-c)(1-d)(1-e)(1-f)(aq)_2(1-aq/(bc))(1-aq/(bd))(1-aq/(be))(1-aq/(bf))}{(a/b)_2(aq/b)_2(1-aq/c)(1-aq/d)(1-aq/e)(1-aq/f)}T_2\phi,\end{aligned} \tag{2.68}$$

where

$$L = \frac{c(1-c/q)(1-aq/(cd))(1-aq/(ce))(1-aq/(cf))}{b-c/q}, \tag{2.69}$$
$$M = \frac{b(1-b/q)(1-aq/(bd))(1-aq/(be))(1-aq/(bf))}{c-b/q}. \tag{2.70}$$

On the other hand, we find that the function ${}_8W_7$ satisfies the q-Riemann-Papperitz system (2.1) under some gauge transformation. Our result gives linear relations for $T_3\phi$, ϕ, $T_3^{-1}\phi$, and for $T_4\phi$, ϕ, $T_4^{-1}\phi$, and $T_5\phi$, ϕ, $T_5^{-1}\phi$. Here, $T_3 = T_b$, $T_4 = T_aT_bT_c$, and $T_5 = T_a^2T_bT_cT_dT_eT_f$. Therefore the q-Riemann-Papperitz system (2.1) can be regarded as a contiguity relation for the Askey-Wilson function ${}_8W_7$. In [21], it was mentioned that some solution for the equation $\mathcal{H}_2y = 0$ is related to the big q-Jacobi polynomial. The big q-Jacobi polynomial can be obtained from the Askey-Wilson polynomial by taking some limit. In addition, the little q-Jacobi polynomial, which is related to the Heine's q-hypergeometric equation, can be obtained from the big q-Jacobi polynomial. Degenerations of orthogonal polynomials are summarized in the Askey scheme [32]. In conclusion the degeneration "$\mathcal{H}_3 \to \mathcal{H}_2 \to$ Heine" can be regarded as (a part of) the Askey scheme.

2.4 Connection problem

In this subsection, we will solve the connection problem associated with the q-Riemann-Papperitz system (2.1). In Theorem 2.20, we get 6 integral solutions for the equation (2.1). So there are 4 linear relations among the integrals. In this subsection, we will give 4 relations. Three of 4 linear relations are trivially obtained, and another one can be obtained by reducing Mimachi's formula [38]. However this reduction is not straightforward as discussed below.

A connection formula for the Jackson integral of Jordan-Pochhammer type was obtained by Mimachi [38] by calculating a certain contour integral. First, we review Mimachi's connection formula [38] for the Jackson integral of Jordan-Pochhammer type.

Proposition 2.32 ([38])**.** *We have*

$$\theta(q^\rho)\int_0^{qb_0} t^{\rho-1}\prod_{i=0}^{M}\frac{(t/b_i)_\infty}{(t/a_i)_\infty}d_qt=\sum_{k=0}^{M}C_k\int_0^{q/a_k}t^{\alpha-1}\prod_{i=0}^{M}\frac{(a_it)_\infty}{(b_it)_\infty}d_qt, \tag{2.71}$$

where $q^\rho=a_1\cdots a_{N+1}/(b_1\cdots b_N q^\alpha)$ *and*

$$C_k=(qb_0)^\rho\left(\frac{a_k}{q}\right)^\alpha\theta(q^{\rho+1}b_0/a_k)\prod_{j=1}^{M}\theta\left(\frac{a_k}{b_j}\right)\prod_{\substack{0\le j\le M\\ j\ne k}}\theta\left(\frac{a_k}{a_j}\right)^{-1}. \tag{2.72}$$

Proof. We consider the following contour integral:

$$\int_L\frac{(c_1s,\ldots,c_Ms,qs/x,x/s)_\infty}{(d_0s,\ldots,d_Ms,1/s)_\infty}\frac{ds}{s}. \tag{2.73}$$

Here the contour L is a deformation of the positively oriented unit circle so that the poles of $1/(d_1s,\ldots,d_{M+1}s)_\infty$ lie outside L, and the poles of $1/(1/s)_\infty$ and 0 lie inside L. By the Cauchy's residue formula to the inside and outside, we get the following formula:

$$\begin{aligned}&\theta(x)\sum_{n=0}^{\infty}\frac{(qq^n,c_1q^n,\ldots,c_Mq^n)_\infty}{(d_0q^n,\ldots,d_Mq^n)_\infty}x^n\\ &=\sum_{k=0}^{M}\theta(xd_k)\prod_{i=1}^{M}\theta(c_i/d_k)\prod_{\substack{0\le i\le M\\ i\ne k}}\theta(d_i/d_k)^{-1}\sum_{n=0}^{\infty}\frac{(qd_kq^n/d_0,\ldots,qd_kq^n/d_M)_\infty}{(qd_kq^n/c_1,\ldots,qd_kq^n/c_M,d_kq^n)_\infty}\left(\frac{c_1\cdots c_Mq}{d_0\cdots d_Mx}\right)^n.\end{aligned} \tag{2.74}$$

Putting c_i, d_i and x suitably, we obtain the desired formula (2.71). □

Remark 2.33. The formula (2.71) is equivalent to the connection formula of the generalized q-hypergeometric series ${}_{M+1}\varphi_M$ [64, 67]. The connection formula of ${}_{M+1}\varphi_M$ is also written in section 4, (4.36).

Remark 2.34. A connection formula for the Jackson integral of type A was studied by Ito-Noumi [26]. Their formula includes Mimachi's one (2.71) as a special case.

We put $\alpha=1$ and $a_0\cdots a_M=q^2b_0\cdots b_M$, then $\theta(q^\rho)=\theta(q)=0$. So we get

$$0=\sum_{k=0}^{M}C_k\int_0^{q/a_k}t^{\alpha-1}\prod_{i=0}^{M}\frac{(a_it)_\infty}{(b_it)_\infty}d_qt. \tag{2.75}$$

By a simple calculation, we have

$$\sum_{k=1}^{M}\frac{C_k}{C_0}\int_{q/a_0}^{q/a_k}\prod_{i=0}^{M}\frac{(a_it)_\infty}{(b_it)_\infty}d_qt=-\left(1+\frac{C_1}{C_0}+\cdots+\frac{C_M}{C_0}\right)\int_0^{q/a_0}\prod_{i=0}^{M}\frac{(a_it)_\infty}{(b_it)_\infty}d_qt. \tag{2.76}$$

We put $a_0 = Ax$ and $b_0 = Bx$. Due to Lemma 2.16, the left-hand-side of the above formula satisfies the equation $E_My = 0$ because coefficients C_k/C_0 $(k = 1,\dots,M)$ are pseudo constants of x. On the other hand, the integral

$$\int_0^{q/a_0}\prod_{i=0}^{M}\frac{(a_it)_\infty}{(b_it)_\infty}d_qt, \tag{2.77}$$

of the right-hand-side is not a solution for $E_My = 0$. Thus the coefficient $1+C_1/C_0+\cdots+C_M/C_0$ vanishes. Therefore we have the following formula:

Theorem 2.35. *Suppose $a_0\cdots a_M = q^2b_0\cdots b_M$. We have*

$$\sum_{k=1}^{M}\tilde{C}_k\int_{q/a_0}^{q/a_k}\prod_{i=0}^{M}\frac{(a_it)_\infty}{(b_it)_\infty}d_qt=0. \tag{2.78}$$

Here

$$\tilde{C}_k=\frac{C_k}{C_0}=\left(\frac{a_k}{a_0}\right)^2\prod_{i=0}^{M}\frac{\theta(a_k/b_i)}{\theta(a_0/b_i)}\prod_{\substack{0\le i\le M\\ i\neq 0}}\theta(a_0/a_i)\prod_{\substack{0\le i\le M\\ i\neq k}}\theta(a_k/a_i)^{-1}. \tag{2.79}$$

Remark 2.36. We obtained the linear relation (2.78) by reducing Mimachi's formula (2.71). However this reduction is not obtained straightforward. The key point in order to prove $1 + C_1/C_0 + \cdots + C_M/C_0 = 0$ is that the integral (2.77) is not a solution of $E_My = 0$. The direct proof of the relation $1 + C_1/C_0 + \cdots + C_M/C_0 = 0$ seems to be difficult.

We consider the case $M = 3$. As mentioned above, the equation $E_3y = 0$ is equivalent to the q-Riemann-Papperitz system. So the formula (2.78) gives a connection formula for integral solutions of the q-Riemann-Papperitz system (2.1).

Theorem 2.37. *We put*

$$C_1=\frac{\theta\left(q^{h_1-l_1-\nu+1}\right)\theta\left(\frac{t_1q^{h_1-\frac12}}{x}\right)\theta\left(\frac{t_1q^{h_1-l_2-\nu+1}}{t_2}\right)\theta\left(\frac{t_1q^{h_1-l_3-\nu+1}}{t_3}\right)\theta\left(\frac{xq^{-h_2+\nu-\frac12}}{t_2}\right)\theta\left(\frac{xq^{-h_3+\nu-\frac12}}{t_3}\right)}{\theta\left(q^{\nu-1}\right)\theta\left(\frac{t_1q^{h_1-h_2}}{t_2}\right)\theta\left(\frac{t_1q^{h_1-h_3}}{t_3}\right)\theta\left(\frac{xq^{\frac12-l_1}}{t_1}\right)\theta\left(\frac{xq^{\frac12-l_2}}{t_2}\right)\theta\left(\frac{xq^{\frac12-l_3}}{t_3}\right)}, \tag{2.80}$$

$$C_2=\frac{\theta\left(q^{h_2-l_2-\nu+1}\right)\theta\left(\frac{t_2q^{h_2-\frac12}}{x}\right)\theta\left(\frac{t_2q^{h_2-l_1-\nu+1}}{t_1}\right)\theta\left(\frac{t_2q^{h_2-l_3-\nu+1}}{t_3}\right)\theta\left(\frac{xq^{-h_1+\nu-\frac12}}{t_1}\right)\theta\left(\frac{xq^{-h_3+\nu-\frac12}}{t_3}\right)}{\theta\left(q^{\nu-1}\right)\theta\left(\frac{t_2q^{h_2-h_1}}{t_1}\right)\theta\left(\frac{t_2q^{h_2-h_3}}{t_3}\right)\theta\left(\frac{xq^{\frac12-l_1}}{t_1}\right)\theta\left(\frac{xq^{\frac12-l_2}}{t_2}\right)\theta\left(\frac{xq^{\frac12-l_3}}{t_3}\right)}, \tag{2.81}$$

$$C_3=\frac{\theta\left(q^{h_3-l_3-\nu+1}\right)\theta\left(\frac{t_3q^{h_3-\frac12}}{x}\right)\theta\left(\frac{t_3q^{h_3-l_1-\nu+1}}{t_1}\right)\theta\left(\frac{t_3q^{h_3-l_2-\nu+1}}{t_2}\right)\theta\left(\frac{xq^{-h_1+\nu-\frac12}}{t_1}\right)\theta\left(\frac{xq^{-h_2+\nu-\frac12}}{t_2}\right)}{\theta\left(q^{\nu-1}\right)\theta\left(\frac{t_3q^{h_3-h_1}}{t_1}\right)\theta\left(\frac{t_3q^{h_3-h_2}}{t_2}\right)\theta\left(\frac{xq^{\frac12-l_1}}{t_1}\right)\theta\left(\frac{xq^{\frac12-l_2}}{t_2}\right)\theta\left(\frac{xq^{\frac12-l_3}}{t_3}\right)}. \tag{2.82}$$

Then the integrals $\varphi_{1,4}$, $\varphi_{2,4}$, $\varphi_{3,4}$ in Theorem 2.20 satisfy the following linear relation:

$$C_1\varphi_{1,4} + C_2\varphi_{2,4} + C_3\varphi_{3,4} = 0. \tag{2.83}$$

Remark 2.38. The integrals $\varphi_{i,j}$ satisfies the trivial relation:

$$\varphi_{i,j} + \varphi_{j,k} + \varphi_{k,i} = 0, \tag{2.84}$$

due to the definition of the Jackson integral. Therefore we get

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & C_1 & 0 & C_2 & C_3 \end{pmatrix} \begin{pmatrix} \varphi_{1,2} \\ \varphi_{1,3} \\ \varphi_{1,4} \\ \varphi_{2,3} \\ \varphi_{2,4} \\ \varphi_{3,4} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.85}$$

The rank of this 6×4 matrix is 4 in general.

Remark 2.39. We obtained a three term relation (2.83) for the integral $\varphi_{i,j}$. On the other hand, the integral can be transformed to the very-well-poised q-hypergeometric function ${}_8W_7$. So we get a new linear relation for ${}_8W_7$:

$$\begin{aligned} &{}_8W_7(a; b, c, d, e, f) \\ &= \frac{(aq/(be), f, f/a, aq, bq/a, aq/(bc), bq/c, aq/(bd), bq/d, bq/e, be/a)_\infty}{(q/e, be/a, f/b, bf/a, aq/b, b^2q/a, q/c, aq/c, q/d, aq/d, aq/e)_\infty} {}_8W_7\left(\frac{b^2}{a}; b, \frac{be}{a}, \frac{bc}{a}, \frac{bd}{a}, \frac{bf}{a}\right) \\ &+ \mathrm{idem}(b, f). \end{aligned}$$

Here, the symbol $\mathrm{idem}(b, c)$ after an expression means that the preceding expression is repeated with b and c interchanged. Note that a similar formula for the function ${}_8W_7$ is written in Gasper-Rahman [16], equation (2.11.1), however our relation is not the same as it.

2.5 Degenerations

In this section, we consider some degenerations for the q-Riemann-Papperitz system (2.1). By taking the limit $t_3 \to \infty$ with suitable parameters, the q-Riemann-Papperitz system (2.1) becomes the equation (2.4), and taking the limit $t_2 \to 0$ in (2.4), we get the Heine's equation (1.5). More precisely, we put $(\alpha_1, \alpha_2) = (\alpha, \alpha - h_3 + l_3)$, then we have

$$\lim_{t_3 \to \infty} \frac{1}{-q^{h_3+1/2}t_3} \mathcal{H}_3 = \mathcal{H}_2. \tag{2.86}$$

Here, as mentioned in Remark 2.3, $\mathcal{H}_2$ is given by:

$$\begin{aligned} \mathcal{H}_2 = &\prod_{i=1}^{2}(x - q^{h_i+1/2}t_i) \cdot T_x^{-1} + q^{\alpha_1+\alpha_2} \prod_{i=1}^{2}(x - q^{l_i-1/2}t_i) \cdot T_x \\ &- (q^{\alpha_1} + q^{\alpha_2})x^2 + Ex + p(q^{1/2} + q^{-1/2})t_1t_2, \end{aligned} \tag{2.87}$$

$$p = q^{(h_1+h_2+l_1+l_2+\alpha_1+\alpha_2)/2}, \quad E = -p\{(q^{-h_2} + q^{-l_2})t_1 + (q^{-h_1} + q^{-l_1})t_2\}. \tag{2.88}$$

Also we put $t_1 = 1$, $h_1 = 1/2$, $h_2 - l_2 = \alpha_1 + \alpha_2 + l_1 - 3/2$, $a = q^{\alpha_1}$, $b = q^{\alpha_2}$, $c = q^{\alpha_1+\alpha_2+l_1-1/2}$ and take the limit $t_2 \to 0$, then the equation $\mathcal{H}_2 y = 0$ becomes the Heine's equation. For the details of these degenerations, see [21]. Taking the same limits for the integral solution (2.55) and the series solution (2.61), we get solutions for the equations (2.4) and (1.5).

First, we consider degenerations for the integral solutions of the q-Riemann-Papperitz system (2.1). If the function $C(t)$ is a pseudo constant, i.e. $C(qt) = C(t)$, then we have

$$\int_0^\tau f(t)C(t)d_qt = C(\tau)\int_0^\tau f(t)d_qt. \tag{2.89}$$

So we have the following:

$$\begin{aligned}&\frac{1}{-q^{h_3+1/2}t_3}\mathcal{H}_3 x^{\nu-\alpha}\int_0^\tau \frac{(q^\nu xt, q^{h_1+\frac12}t_1t, q^{h_2+\frac12}t_2t, q^{h_3+\frac12}t_3t)_\infty}{(xt, q^{\nu+l_1-\frac12}t_1t, q^{\nu+l_2-\frac12}t_2t, q^{\nu+l_3-\frac12}t_3t)_\infty}C(t)d_qt\\ &=\frac{1}{q^{h_3+1/2}t_3}(1-q^\nu)q(1-q)x^2C(\tau),\end{aligned} \tag{2.90}$$

where $\tau \in \{q^{1/2-h_1}/t_1, q^{1/2-h_2}/t_2, q^{1-\nu}/x\}$. We put $C(t) = t^{\nu+l_3-h_3-1}\dfrac{\theta(q^{\nu+l_3-\frac12}t_3t)}{\theta(q^{h_3+\frac12}t_3t)}$, then we have

$$\lim_{t_3\to\infty}\frac{(q^\nu xt, q^{h_1+\frac12}t_1t, q^{h_2+\frac12}t_2t, q^{h_3+\frac12}t_3t)_\infty}{(xt, q^{\nu+l_1-\frac12}t_1t, q^{\nu+l_2-\frac12}t_2t, q^{\nu+l_3-\frac12}t_3t)_\infty}C(t) = t^{\nu+l_3-h_3-1}\frac{(q^\nu xt, q^{h_1+\frac12}t_1t, q^{h_2+\frac12}t_2t)_\infty}{(xt, q^{\nu+l_1-\frac12}t_1t, q^{\nu+l_2-\frac12}t_2t)_\infty}. \tag{2.91}$$

Therefore we have integral solutions for the equation $\mathcal{H}_2y = 0$, namely:

Theorem 2.40 ([46])**.** *Suppose $|q^{\nu+\alpha_2-\alpha_1}| < 1$. The integral*

$$x^{\nu-\alpha_1}\int_0^\tau t^{\nu+\alpha_2-\alpha_1-1}\frac{(q^\nu xt, q^{h_1+\frac12}t_1t, q^{h_2+\frac12}t_2t)_\infty}{(xt, q^{\nu+l_1-\frac12}t_1t, q^{\nu+l_2-\frac12}t_2t)_\infty}d_qt, \tag{2.92}$$

satisfies $\mathcal{H}_2y = 0$, where $\nu = \frac12(h_1+h_2-l_1-l_2+1-\alpha_1+\alpha_2)$, $\tau \in \{q^{1/2-h_1}/t_1, q^{1/2-h_2}/t_2, q^{1-\nu}/x\}$.

Remark 2.41. The condition $|q^{\nu+\alpha_1-\alpha_2}| < 1$ is necessary for the integral to converge.

Remark 2.42. In [15], other integral solutions were obtained. See also Remark 2.21.

Remark 2.43. In [21], the following solution was given:

$$x^{-\alpha_2}\varphi_1\left(\begin{matrix} q^{\nu-\alpha_1+\alpha_2}; q^{\nu-\alpha_1+\alpha_2-h_2+l_2}, q^{\nu-\alpha_1+\alpha_2-h_1+l_1}\\ q^{\alpha_2-\alpha_1+1}\end{matrix}; q^{l_1+1/2}\frac{t_1}{x}, q^{l_2+1/2}\frac{t_2}{x}\right), \tag{2.93}$$

where, φ_1 is the q-Appell series defined as

$$\varphi_1\left(\begin{matrix} a; b_1, b_2\\ c\end{matrix}; x_1, x_2\right) = \sum_{n_1=0}^\infty\sum_{n_2=0}^\infty \frac{(a)_{n_1+n_2}(b_1)_{n_1}(b_2)_{n_2}}{(c)_{n_1+n_2}(q)_{n_1}(q)_{n_2}}x_1^{n_1}x_2^{n_2}. \tag{2.94}$$

The q-Appell-Lauricella series has an integral representation [3]:

$$\varphi_1\left(\begin{matrix} a; b_1, b_2\\ c\end{matrix}; x_1, x_2\right) = \frac{(a, c/a)_\infty}{(q, c)_\infty(1-q)}\int_0^1 t^{\alpha-1}\frac{(qt, b_1x_1t, b_2x_2t)_\infty}{(ct/a, x_1t, x_2t)_\infty}d_qt, \tag{2.95}$$

where $q^\alpha = a$. The solution using the q-Appell series is also obtained by applying this formula to our integral solution:

$$x^{-\alpha_2}\varphi_1\left(\begin{matrix} q^{\nu-\alpha_1+\alpha_2}; q^{\nu-\alpha_1+\alpha_2-h_2+l_2}, q^{\nu-\alpha_1+\alpha_2-h_1+l_1}\\ q^{\alpha_2-\alpha_1+1}\end{matrix}; q^{l_1+1/2}\frac{t_1}{x}, q^{l_2+1/2}\frac{t_2}{x}\right)$$

$$= \frac{(q^{\nu-\alpha_1+\alpha_2}, q^{1-\nu})_\infty}{(q, q^{\alpha_2-\alpha_1+1})_\infty(1-q)} q^{(\nu-1)(\nu-\alpha_1+\alpha_2)} x^{\nu-\alpha_1} \int_0^{q^{1-\nu}/x} t^{\nu+\alpha_2-\alpha_1-1} \frac{(q^\nu xt, q^{h_1+\frac{1}{2}}t_1 t, q^{h_2+\frac{1}{2}}t_2 t)_\infty}{(xt, q^{\nu+l_1-\frac{1}{2}}t_1 t, q^{\nu+l_2-\frac{1}{2}}t_2 t)_\infty} d_q t. \tag{2.96}$$

See also [21].

Next, we put $t_1 = 1$, $h_1 = 1/2$, $h_2 - l_2 = \alpha_1 + \alpha_2 + l_1 - 3/2$, $a = q^{\alpha_1}$, $b = q^{\alpha_2}$, $c = q^{\alpha_1+\alpha_2+l_1-1/2}$ and take the limit $t_2 \to 0$ in the above theorem. Then we get the following known integral solution for the Heine's equation.

Corollary 2.44. *Suppose $|a| < 1$. The integral*

$$\int_0^\tau t^{\alpha-1} \frac{(bxt, qt)_\infty}{(xt, ct/a)_\infty} d_q t, \tag{2.97}$$

satisfies the Heine's equation

$$[(1-T_x)(1-cq^{-1}T_x) - x(1-aT_x)(1-bT_x)]y = 0, \tag{2.98}$$

where $q^\alpha = a$, $\tau \in \{1, q/(bx), d\infty\}$ and $d \in \mathbb{C}$ is an arbitrary point.

In the following, we consider the degenerations for the series solutions (2.61) in terms of the very-well-poised q-hypergeometric series. Several limits of the very-well-poised-balanced q-hypergeometric series ${}_8W_7$ are expressed by the q-hypergeometric series ${}_3\varphi_2$ as follows:

$$\lim_{l\to\infty} {}_8W_7\left(al; bl, cl, d, e, f; \frac{((al)q)^2}{(bl)(cl)def}\right) = {}_3\varphi_2\left(\begin{matrix} d, e, f \\ aq/b, , aq/c \end{matrix}; q\right), \tag{2.99}$$

$$\lim_{l\to 0} {}_8W_7\left(al; bl, cl, d, e, f; \frac{((al)q)^2}{(bl)(cl)def}\right) = {}_3\varphi_2\left(\begin{matrix} d, e, f \\ aq/b, aq/c \end{matrix}; \frac{(aq)^2}{bcdef}\right). \tag{2.100}$$

These limits can be calculated by the definition of ${}_8W_7$. Using the transformation (2.60), we have

$$\begin{aligned}
&\lim_{b\to\infty} {}_8W_7\left(a; b, c, d, e, f; \frac{(aq)^2}{bcdef}\right) = \lim_{b\to\infty} \frac{(aq, aq/(ef), \mu q/e, \mu q/f)_\infty}{(aq/e, aq/f, \mu q, \mu q/(ef))_\infty} {}_8W_7\left(\mu; \frac{\mu b}{a}, \frac{\mu c}{a}, \frac{\mu d}{a}, e, f; \frac{aq}{ef}\right) \\
&= \frac{(aq, aq/(ef))_\infty}{(aq/e, aq/f)_\infty} {}_3\varphi_2\left(\begin{matrix} aq/(cd), e, f \\ aq/c, aq/d \end{matrix}; \frac{aq}{ef}\right),
\end{aligned} \tag{2.101}$$

$$\begin{aligned}
&\lim_{b\to 0} \frac{(\mu q, \mu q/(ef))_\infty}{(\mu q/e, \mu q/f)_\infty} {}_8W_7\left(a; b, c, d, e, f; \frac{(aq)^2}{bcdef}\right) = \lim_{b\to 0} \frac{(aq, aq/(ef))_\infty}{(aq/e, aq/f)_\infty} {}_8W_7\left(\mu; \frac{\mu b}{a}, \frac{\mu c}{a}, \frac{\mu d}{a}, e, f; \frac{aq}{ef}\right) \\
&= \frac{(aq, aq/(ef))_\infty}{(aq/e, aq/f)_\infty} {}_3\varphi_2\left(\begin{matrix} aq/(cd), e, f \\ aq/c, aq/d \end{matrix}; q\right),
\end{aligned} \tag{2.102}$$

$$\begin{aligned}
&\lim_{l\to\infty} \frac{((aq)^2 l/(bcd), aql/e, aql/f)_\infty}{(aql^2)_\infty} {}_8W_7\left(al^2; bl, cl, dl, el, fl; \frac{(aq)^2}{bcdefl}\right) \\
&= \lim_{l\to\infty} \frac{(aq/ef, (aq)^2/(bcde), (aq)^2/(bcdf))_\infty}{((aq)^2/(bcdefl))_\infty} {}_8W_7\left(\mu l; \frac{\mu b}{a}, \frac{\mu c}{a}, \frac{\mu d}{a}, el, fl; \frac{aq}{ef}\right) \\
&= (aq/ef, (aq)^2/(bcde), (aq)^2/(bcdf))_\infty {}_3\varphi_2\left(\begin{matrix} aq/bc, aq/cd, aq/bd \\ (aq)^2/(bcde), (aq)^2/(bcdf) \end{matrix}; q\right),
\end{aligned} \tag{2.103}$$

$$\begin{aligned}
&\lim_{l\to 0} ((aq)^2/(bcdefl))_\infty {}_8W_7\left(al^2; bl, cl, dl, el, fl; \frac{(aq)^2}{bcdefl}\right) \\
&= \lim_{l\to 0} \frac{(aql^2, aq/ef, (aq)^2/(bcde), (aq)^2/(bcdf))_\infty}{((aq)^2 l/(bcd), aql/e, aql/f)_\infty} {}_8W_7\left(\mu l; \frac{\mu b}{a}, \frac{\mu c}{a}, \frac{\mu d}{a}, el, fl; \frac{aq}{ef}\right)
\end{aligned}$$

$$=(aq/ef,(aq)^2/(bcde),(aq)^2/(bcdf))_\infty {}_3\varphi_2\left(\begin{matrix} aq/bc, aq/cd, aq/bd \\ (aq)^2/(bcde),(aq)^2/(bcdf)\end{matrix};\frac{aq}{ef}\right). \tag{2.104}$$

By using the formula (2.100), we have

$$\lim_{t_3\to\infty} x^{\nu-\alpha}\frac{(q^{\nu-h_3+\frac12}x/t_3)_\infty}{(q^{\frac12-h_3}x/t_3)_\infty}{}_8W_7\left(\frac{t_iq^{h_i-h_3+\nu}}{t_3};\frac{t_iq^{-h_3+l_i+\nu}}{t_3},\frac{t_jq^{-h_3+l_j+\nu}}{t_3},q^{-h_3+l_3+\nu},\frac{t_iq^{h_i+\frac12}}{x},q^\nu;\frac{xq^{\frac12-h_j}}{t_j}\right)$$

$$=x^{\nu-\alpha}{}_3\varphi_2\left(\begin{matrix} q^{-h_3+l_3+\nu},q^\nu,t_iq^{h_i+1/2}/x \\ q^{1+h_i-l_i},q^{1+h_i-l_j}t_i/t_j\end{matrix};\frac{xq^{h_i+h_3-l_i-l_j-l_3-2\nu+\frac32}}{t_j}\right). \tag{2.105}$$

Similar to this calculation, we have the following theorem:

Theorem 2.45 ([15]). *The functions*

$$x^{\nu-\alpha_1}{}_3\varphi_2\left(\begin{matrix} q^{\nu+\alpha_1-\alpha_2},q^\nu,t_iq^{h_i+1/2}/x \\ q^{1+h_i-l_i},q^{1+h_i-l_j}t_i/t_j\end{matrix};\frac{xq^{-h_j+1/2}}{t_j}\right), \tag{2.106}$$

$$x^{\nu-\alpha_1}\frac{(q^{\nu-h_i+\frac12}x/t_i)_\infty}{(q^{\frac12-h_i}x/t_i)_\infty}{}_3\varphi_2\left(\begin{matrix} t_jq^{-h_i+l_j+\nu}/t_i,q^{-h_i+l_i+\nu},q^\nu \\ q^{-\alpha_1+\alpha_2+1},xq^{-h_i+\nu+\frac12}/t_i\end{matrix};q\right), \tag{2.107}$$

$$x^{\nu-\alpha_1}\frac{(q^{\nu-h_j+\frac12}x/t_j)_\infty}{(q^{\frac12-h_j}x/t_j)_\infty}{}_3\varphi_2\left(\begin{matrix} xq^{1/2-l_i}/t_i,q^\nu,q^{-h_j+l_j+\nu}, \\ xq^{-h_j+\nu+1/2}/t_j,q^{h_i-l_i+1}\end{matrix};\frac{t_iq^{h_i-l_j-\nu+1}}{t_j}\right), \tag{2.108}$$

$$x^{\nu-\alpha_1}\frac{(xq^{1/2-l_i}/t_i,xq^{1/2-l_j}/t_j)_\infty}{(xq^{-h_j+\nu-1/2}/t_j,xq^{1/2-h_i}/t_i)_\infty}{}_3\varphi_2\left(\begin{matrix} q^{1-\nu},t_iq^{l_i+1/2}/x,q^{h_j-l_j-\nu+1} \\ q^{h_j+h_2-l_j-l_2-2\nu+2},t_jq^{h_j-\nu+3/2}/x\end{matrix};q\right), \tag{2.109}$$

$$x^{\nu-\alpha_1}\frac{(q^{1/2-l_j})x/t_i}{(q^{l_i+\nu-h_i-h_j-3/2}x/t_i)}{}_3\varphi_2\left(\begin{matrix} q^{\nu+\alpha_1-\alpha_2},q^{h_i-l_i+1},q^{h_j-l_i-\nu+1}t_j/t_i \\ q^{h_i+h_j-l_i-l_j-2\nu+2},q^{h_i+h_j-l_i-\nu+3/2}t_j/x\end{matrix};\frac{q^{l_i+\nu+1/2}t_i}{x}\right), \tag{2.110}$$

$$x^{\nu-\alpha_1}{}_3\varphi_2\left(\begin{matrix} q^\nu,q^{\nu+\alpha_1-\alpha_2},q^{-l_i+1/2}x/t_i \\ q^{h_i-l_i+1},q^{h_j-l_i+1}t_j/t_i\end{matrix};q\right), \tag{2.111}$$

satisfy the equation $\mathcal{H}_2y=0$*. Here,* $\nu=\frac12(h_1+h_2-l_1-l_2-\alpha_1+\alpha_2+1)$.

Remark 2.46. The solution (2.111) was obtained in [21].

Remark 2.47. Solutions in Theorem 2.45 are q-analogs of the series (2.62) with $t_3=\infty$, i.e. q-analogs of the following series:

$$(x-t_1)^{\mu_1}(x-t_2)^{\mu_2}{}_2F_1\left(\begin{matrix}\nu_1,\nu_2\\ \nu_3\end{matrix};\frac{t_1-t_2}{x-t_2}\right), \tag{2.112}$$

$$(x-t_1)^{\mu_1}(x-t_2)^{\mu_2}{}_2F_1\left(\begin{matrix}\nu_1,\nu_2\\ \nu_3\end{matrix};\frac{x-t_2}{t_1-t_2}\right), \tag{2.113}$$

$$(x-t_1)^{\mu_1}(x-t_2)^{\mu_2}{}_2F_1\left(\begin{matrix}\nu_1,\nu_2\\ \nu_3\end{matrix};\frac{x-t_1}{x-t_2}\right). \tag{2.114}$$

Remark 2.48. A symmetry of the equation $\mathcal{H}_2$ was found in [15] by using a configuration (see also Remark 2.28). Using the symmetry, we can get many solutions in principle.

Next, we consider the degeneration to the Heine's equation. Taking the limit $t_2\to 0$ with suitable parameters for ${}_3\varphi_2$, we obtain series solutions for the Heine's equation. The limit can be calculated easily, so we only show the result. In the following, we list series solutions for the Heine's equation, which are q-analogs of Kummer's 24 solutions. More precisely, each series becomes one of Kummer's 24 solutions by taking the limit $q\to 1$ with suitable parameters.

Theorem 2.49 ([15])**.** *The following 32 series satisfy the Heine's equation:*

$$\left\{\begin{aligned}
&{}_2\varphi_1\left(\begin{matrix} a,b \\ c\end{matrix};x\right), \frac{(abx/c)_\infty}{(x)_\infty}{}_2\varphi_1\left(\begin{matrix} c/a,c/b \\ c\end{matrix};\frac{abx}{c}\right),\\
&x^{1-\gamma}{}_2\varphi_1\left(\begin{matrix} aq/c,bq/c \\ q^2/c\end{matrix};x\right), x^{1-\gamma}\frac{(abx/c)_\infty}{(x)_\infty}{}_2\varphi_1\left(\begin{matrix} q/a,q/b \\ q^2/c\end{matrix};\frac{abx}{c}\right),
\end{aligned}\right. \tag{2.115}$$

$$\left\{\begin{aligned}
&x^{-\alpha}{}_2\varphi_1\left(\begin{matrix} a,aq/c \\ aq/b\end{matrix};\frac{cq}{abx}\right), x^{-\alpha}\frac{(q/x)_\infty}{(cq/(abx))_\infty}{}_2\varphi_1\left(\begin{matrix} q/b,c/b \\ aq/b\end{matrix};\frac{q}{x}\right),\\
&x^{-\beta}{}_2\varphi_1\left(\begin{matrix} b,bq/c \\ bq/a\end{matrix};\frac{cq}{abx}\right), x^{-\beta}\frac{(q/x)_\infty}{(cq/(abx))_\infty}{}_2\varphi_1\left(\begin{matrix} q/a,c/a \\ bq/a\end{matrix};\frac{q}{x}\right),
\end{aligned}\right. \tag{2.116}$$

$$\left\{\begin{aligned}
&{}_3\varphi_2\left(\begin{matrix} a,b,abx/c \\ abq/c,0\end{matrix};q\right), \frac{(abx/c)_\infty}{(x)_\infty}{}_3\varphi_2\left(\begin{matrix} c/a,c/b,x \\ cq/(ab),0\end{matrix};q\right),\\
&x^{1-\gamma}{}_3\varphi_2\left(\begin{matrix} aq/c,bq/c,abx/c \\ abq/c,0\end{matrix};q\right), x^{1-\gamma}\frac{(abx/c)_\infty}{(x)_\infty}{}_3\varphi_2\left(\begin{matrix} q/a,q/b,x \\ cq/(ab),0\end{matrix};q\right),
\end{aligned}\right. \tag{2.117}$$

$$\left\{\begin{aligned}
&x^{-\alpha}{}_3\varphi_2\left(\begin{matrix} a,aq/c,q/x, \\ abq/c,0\end{matrix};q\right), x^{-\alpha}\frac{(q/x)_\infty}{(cq/(abx))_\infty}{}_3\varphi_2\left(\begin{matrix} q/b,c/b,cq/(abx), \\ cq/(ab),0\end{matrix};q\right),\\
&x^{-\beta}{}_3\varphi_2\left(\begin{matrix} a,aq/c,q/x, \\ abq/c,0\end{matrix};q\right), x^{-\beta}\frac{(q/x)_\infty}{(cq/(abx))_\infty}{}_3\varphi_2\left(\begin{matrix} q/a,c/b,q/x, \\ cq/ab,0\end{matrix};q\right),
\end{aligned}\right. \tag{2.118}$$

$$\left\{\begin{aligned}
&\frac{(ax)_\infty}{(x)_\infty}{}_2\varphi_2\left(\begin{matrix} c/b,a \\ c,ax\end{matrix};bx\right), \frac{(bx)_\infty}{(x)_\infty}{}_2\varphi_2\left(\begin{matrix} b,c/a \\ c,bx\end{matrix};ax\right),\\
&x^{1-\gamma}\frac{(aqx/c)_\infty}{(x)_\infty}{}_2\varphi_2\left(\begin{matrix} aq/c,q/b \\ q^2/c,aqx/c\end{matrix};\frac{bqx}{c}\right), x^{1-\gamma}\frac{(bqx/c)_\infty}{(x)_\infty}{}_2\varphi_2\left(\begin{matrix} bq/c,q/a \\ q^2/c,bqx/c\end{matrix};\frac{aqx}{c}\right),\\
&x^{-\alpha}\frac{(cq/(bx))_\infty}{(cq/(abx))_\infty}{}_3\varphi_2\left(\begin{matrix} c/b,a,0 \\ c,cq/(bx)\end{matrix};q\right), x^{-\alpha}\frac{(q^2/(bx))_\infty}{(cq/(abx))_\infty}{}_3\varphi_2\left(\begin{matrix} q/b,aq/c,0 \\ q^2/c,q^2/(bx)\end{matrix};q\right),\\
&x^{-\beta}\frac{(cq/(ax))_\infty}{(cq/(abx))_\infty}{}_3\varphi_2\left(\begin{matrix} c/a,b,0 \\ c,cq/(ax)\end{matrix};q\right), x^{-\beta}\frac{(q^2/(ax))_\infty}{(cq/(abx))_\infty}{}_3\varphi_2\left(\begin{matrix} q/a,bq/c,0 \\ q^2/c,q^2/(ax)\end{matrix};q\right),
\end{aligned}\right. \tag{2.119}$$

$$\left\{\begin{aligned}
&\frac{(abx/c)_\infty}{(bx/c)_\infty}{}_2\varphi_2\left(\begin{matrix} c/b,a \\ aq/b,cq/(bx)\end{matrix};\frac{q^2}{bx}\right), \frac{(abx/c)_\infty}{(bx/c)_\infty}{}_2\varphi_2\left(\begin{matrix} c/a,b \\ bq/a,cq/(ax)\end{matrix};\frac{q^2}{ax}\right),\\
&x^{1-\gamma}\frac{(abx/c)_\infty}{(bx/q)_\infty}{}_2\varphi_2\left(\begin{matrix} aq/c,q/b \\ aq/b,q^2/(bx)\end{matrix};\frac{cq}{bx}\right), x^{1-\gamma}\frac{(abx/c)_\infty}{(bx/c)_\infty}{}_2\varphi_2\left(\begin{matrix} bq/c,q/a \\ bq/a,q^2/(ax)\end{matrix};\frac{cq}{ax}\right),\\
&\frac{(ax)_\infty}{(x)_\infty}{}_3\varphi_2\left(\begin{matrix} c/b,a,0 \\ aq/b,ax\end{matrix};q\right), \frac{(bx)_\infty}{(x)_\infty}{}_3\varphi_2\left(\begin{matrix} c/a,b,0 \\ bq/a,bx\end{matrix};q\right),\\
&x^{1-\gamma}\frac{(aqx/c)_\infty}{(x)_\infty}{}_3\varphi_2\left(\begin{matrix} q/b,aq/c,0 \\ bq/a,aqx/c\end{matrix};q\right), x^{1-\gamma}\frac{(bqx/c)_\infty}{(x)_\infty}{}_3\varphi_2\left(\begin{matrix} q/a,bq/c,0 \\ aq/b,bqx/c\end{matrix};q\right).
\end{aligned}\right. \tag{2.120}$$

Here, $q^\alpha = a$, $q^\beta = b$, $q^\gamma = c$. *The above series can be considered as q-analogs of Kummer's 24 solutions as follows (see also Remark 2.51).*

solutions of Heine's equation	the limit $q \to 1$ (Kummer's solutions)
(2.115)	(gauge factor) $\times\, {}_2F_1\left(\begin{matrix} *, * \\ * \end{matrix}; x\right)$
(2.116)	(gauge factor) $\times\, {}_2F_1\left(\begin{matrix} *, * \\ * \end{matrix}; \frac{1}{x}\right)$
(2.117)	(gauge factor) $\times\, {}_2F_1\left(\begin{matrix} *, * \\ * \end{matrix}; 1-x\right)$
(2.118)	(gauge factor) $\times\, {}_2F_1\left(\begin{matrix} *, * \\ * \end{matrix}; \frac{x-1}{x}\right)$
(2.119)	(gauge factor) $\times\, {}_2F_1\left(\begin{matrix} *, * \\ * \end{matrix}; \frac{x}{x-1}\right)$
(2.120)	(gauge factor) $\times\, {}_2F_1\left(\begin{matrix} *, * \\ * \end{matrix}; \frac{1}{1-x}\right)$

Remark 2.50. Suppose the function $f(a,b,c;x)$ satisfies the Heine's equation:

$$[(1-T_x)(1-cq^{-1}T_x) - x(1-aT_x)(1-bT_x)]f(a,b,c;x) = 0. \tag{2.121}$$

By simple calculations, we find that the following functions satisfy the Heine's equation:

$$x^{1-\gamma} f\left(\frac{aq}{c}, \frac{bq}{c}, \frac{q^2}{c}; x\right), \ \frac{(abx/c)_\infty}{(x)_\infty} f\left(\frac{c}{b}, \frac{c}{a}, c; \frac{abx}{c}\right), \ x^{-\alpha} f\left(a, \frac{aq}{c}, \frac{aq}{b}; \frac{cq}{abx}\right), \ f(b,a,c;x). \tag{2.122}$$

For detailed calculations, see appendix A. Also the function

$$x^\rho \frac{\theta(q^\rho x)}{\theta(x)} f(a,b,c;x), \tag{2.123}$$

satisfies the Heine's equation because $x^\rho \dfrac{\theta(q^\rho x)}{\theta(x)}$ is a pseudo constant. The following 4 functions are solutions in the list of Theorem 2.49, which are not transformed to each other by (2.122) and (2.123):

$${}_2\varphi_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right), \tag{2.124}$$

$${}_3\varphi_2\left(\begin{matrix} a, b, abx/c \\ abq/c, 0 \end{matrix}; q\right), \tag{2.125}$$

$$\frac{(ax)_\infty}{(x)_\infty} {}_2\varphi_2\left(\begin{matrix} c/b, a \\ c, ax \end{matrix}; bx\right), \tag{2.126}$$

$$\frac{(ax)_\infty}{(x)_\infty} {}_3\varphi_2\left(\begin{matrix} c/b, a, 0 \\ aq/b, ax \end{matrix}; q\right). \tag{2.127}$$

Conversely, we can get series solutions by the transformations (2.122). We put

$$s_1 : (a,b,c;x) \mapsto \left(\frac{aq}{c}, \frac{bq}{c}, \frac{q^2}{c}; x\right), \ s_2 : (a,b,c;x) \mapsto \left(\frac{c}{b}, \frac{c}{a}, c; \frac{abx}{c}\right), \tag{2.128}$$

$$s_3 : (a,b,c;x) \mapsto \left(a, \frac{aq}{c}, \frac{aq}{b}; \frac{cq}{abx}\right), \ s_4 :\mapsto (b,a,c;x). \tag{2.129}$$

Then we can check directly that

$$s_i^2 = \mathrm{id}\ (1 \le i \le 4), \ s_i s_j = s_j s_i\ (2 \le i \ne j \le 4), \tag{2.130}$$

$$(s_i s_1)^4 = \mathrm{id}\ (2 \le i \le 4),\ (s_i s_j s_1)^2 = \mathrm{id}\ (2 \le i \ne j \le 4). \tag{2.131}$$

We put $G = \langle s_1, s_2, s_3, s_4 \rangle$, then we have

$$G = \langle s_1 s_2 \rangle \rtimes \langle s_2, s_3, s_4 \rangle \cong (\mathbb{Z}/4\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}). \tag{2.132}$$

This equality is checked by the computer algebra system GAP. Here the actions s_1, s_2, s_3, s_4 were also found in [15] by using the configuration of Heine's q-hypergeometric equation. The order $|G|$ is 32, therefore we formally obtain 128 solutions for the Heine's equation, by applying transformations (2.122) to solutions (2.124), (2.125), (2.126) and (2.127). Note that there are many duplications in 128 solutions up to (2.123), for example

$${}_2\varphi_1\left(\begin{matrix} a, b \\ c \end{matrix} ; x \right) = {}_2\varphi_1\left(\begin{matrix} b, a \\ c \end{matrix} ; x \right). \tag{2.133}$$

When we delete duplications from the 128 solutions, we get 32 solutions which is the same as the list in Theorem 2.49.

Remark 2.51. For the Gauss hypergeometric differential equation (1.1), there are solutions which are written by power series of x, $1/x$, $1-x$, $1/(1-x)$, $(x-1)/x$, $x/(x-1)$. These solutions are called Kummer's 24 solutions (see appendix A). The solutions (2.115), ..., (2.120) are q-analogs of Kummer's 24 solutions. In [20], q-analogs of Kummer's 24 solutions were obtained. More precisely, the ${}_3\varphi_2$ solution (2.125) and the ${}_2\varphi_2$ solution (2.126) in Remark 2.50 were obtained. Note that the solution (2.127) was not discussed in [20]. In [11], relations for Kummer type solutions, including (2.127), were studied by using a transformation formula for ${}_3\varphi_2$.

Remark 2.52. The dimension of the space of solutions for the Heine's equation, which are regular at $x = 0$, is 1, and then we have

$${}_2\varphi_1\left(\begin{matrix} a, b \\ c \end{matrix} ; x \right) = \frac{(abx/c)_\infty}{(x)_\infty} {}_2\varphi_1\left(\begin{matrix} c/a, c/b \\ c \end{matrix} ; \frac{abx}{c} \right). \tag{2.134}$$

Also if $a = q^{-n}$ $(n \in \mathbb{Z}_{\ge 0})$, the solutions (2.125), (2.126), (2.127) are regular at $x = 0$. Moreover if $a = q^{-n}$, then the function

$$x^\alpha \frac{\theta(abx/c)}{\theta(bx/c)} \cdot x^{-\alpha} \frac{(cq/(bx))_\infty}{(cq/(abx))_\infty} {}_3\varphi_2\left(\begin{matrix} c/b, a, 0 \\ c, cq/(bx) \end{matrix} ; q \right) = (q^{-n}bx/c)_n \cdot {}_3\varphi_2\left(\begin{matrix} c/b, q^{-n}, 0 \\ c, cq/(bx) \end{matrix} ; q \right), \tag{2.135}$$

is regular at $x = 0$, too. Therefore we get

$$\begin{aligned} &{}_2\varphi_1\left(\begin{matrix} q^{-n}, b \\ c \end{matrix} ; x \right) = (q^{-n}x)_n \cdot {}_2\varphi_2\left(\begin{matrix} q^{-n}, c/b \\ c, q^{-n}x \end{matrix} ; bx \right) \\ =&\frac{(c/b)_n}{(c)_n} {}_3\varphi_2\left(\begin{matrix} q^{-n}, b, q^{-n}bx/c \\ q^{1-n}b/c, 0 \end{matrix} ; q \right) = \frac{(b)_n}{(c)_n} (q^{-n}x)_n \cdot {}_3\varphi_2\left(\begin{matrix} q^{-n}, c/b, 0 \\ q^{1-n}/b, q^{-n}x \end{matrix} ; q \right) \\ =&(q^{-n}bx/c)_n \cdot {}_3\varphi_2\left(\begin{matrix} c/b, q^{-n}, 0 \\ c, cq/(bx) \end{matrix} ; q \right). \end{aligned} \tag{2.136}$$

We used the q-Vandermonde formula

$${}_2\varphi_1\left(\begin{matrix} a, q^{-n} \\ c \end{matrix} ; q \right) = \frac{(c/a)_n}{(c)_n} a^n \quad (n \in \mathbb{Z}_{\ge 0}), \tag{2.137}$$

to lead the formulas (2.136). The q-Vandermonde formula can be obtained by the Heine's transformation formula (2.143), given below. Heine's transformation formula also can be found from the viewpoint of solutions for the Heine's equation (see Remark 2.53 below). Note that a part of the equation (2.136) holds with the general parameter a:

$$
{}_2\varphi_1\left(\begin{matrix} a,b \\ c \end{matrix};x\right) = \frac{(ax)_\infty}{(x)_\infty}{}_2\varphi_2\left(\begin{matrix} a,c/b \\ c,ax \end{matrix};bx\right), \tag{2.138}
$$

The formulas (2.134), (2.136), (2.137), (2.138) are summarized in [16].

Remark 2.53. We note that the functions

$$
{}_3\varphi_1\left(\begin{matrix} a,b,q/x \\ abq/c \end{matrix};\frac{x}{c}\right), \tag{2.139}
$$

$$
\frac{(bx)_\infty}{(x)_\infty}{}_2\varphi_1\left(\begin{matrix} c/a,x \\ bx \end{matrix};a\right), \tag{2.140}
$$

also satisfy the Heine's equation. These can be found by the same degeneration $t_2 \to 0$ to obtain the list in Theorem 2.49. However, we did not put them in the list. The reason is as follows.

(1) The series ${}_3\varphi_1$ is a divergent series. Thus this solution is formal if the series does not terminate. If $a = q^{-n}$ $(n \in \mathbb{Z}_{\geq 0})$, we have

$$
{}_2\varphi_1\left(\begin{matrix} q^{-n},b \\ c \end{matrix};x\right) = \frac{(c/b)_n}{(c)_n}b^n{}_3\varphi_1\left(\begin{matrix} q^{-n},b,q/x \\ q^{1-n}b/c \end{matrix};\frac{x}{c}\right), \tag{2.141}
$$

in the same way as Remark 2.52. Note that we put $a = q^\alpha$, $b = q^\beta$, $c = q^\gamma$, then we have

$$
{}_3\varphi_1\left(\begin{matrix} a,b,q/x \\ abq/c \end{matrix};\frac{x}{c}\right) \xrightarrow{q\to 1} {}_2F_1\left(\begin{matrix} \alpha,\beta \\ \alpha+\beta-\gamma+1 \end{matrix};1-x\right), \tag{2.142}
$$

formally.

(2) The series (2.140) is the integral solution (1.6) with the integral path $C = [0,1]$ up to the normalization. This series is regular at $x = 0$, and then we get

$$
{}_2\varphi_1\left(\begin{matrix} a,b \\ c \end{matrix};x\right) = D\frac{(bx)_\infty}{(x)_\infty}{}_2\varphi_1\left(\begin{matrix} c/a,x \\ bx \end{matrix};a\right), \tag{2.143}
$$

and by using the q-binomial theorem, we have $D = \dfrac{(a)_\infty}{(c)_\infty}$. The transformation (2.143) is well known as Heine's transformation formula [16].

3 Euler-type integral transformation for the q-Heun equation and affine Weyl group

In this section, we introduce the q-Euler transformation E^{λ}, and we give integral transformations for the q-Heun equation and its variants [62]. As an application, we give representations of affine Weyl groups of types $D_5^{(1)}$, $E_n^{(1)}$ $(n = 6, 7, 8)$. As mentioned in introduction, variants of q-hypergeometric equation [21] are introduced by considering special cases of the q-Heun equation and its variant of degree three. In section 2.2, we give integral solutions for the variants of q-hypergeometric equation. For the q-Heun equation and its variants in general parameters, it is difficult to give an integral solution, but we can construct integral transformations for the equations. This section is based on [47].

The Heun's differential equation is a standard form of second order Fuchsian differential equations with four singularities on $\mathbb{P}^1$. An integral transformation is studied by Takemura [59, 60]. A q-analog of Heun's differential equation was discovered by Hahn [22], and studied by Takemura [61, 62] from the viewpoint of the Ruijsenaars-van Diejen system [13, 53]. And in [62], variants of the q-Heun equation were introduced. In [56], an integral transformation of the q-Heun equation was obtained by using Sakai-Yamaguchi's q-middle convolution [55]. More precisely, they construct an integral transformation for the Lax linear equation of Jimbo-Sakai's q-Painlevé equation [28], and by specializing the linear equation, they give an integral transformation for the q-Heun equation. However integral transformations for variants of the q-Heun equation were not studied by the q-middle convolution. Sakai-Yamaguchi's q-middle convolution is an integral transformation for first order q-difference systems, but the q-Heun equation and its variants are second order q-difference equations. So rewriting them as systems of first order q-difference equations is needed for applying the q-middle convolution. It is difficult to rewrite variants as a first order system in simple forms, that is why integral transformations for the variants by using the q-middle convolution were not studied.

Our integral transformation can be applied to higher order q-difference equations, not to systems of first order q-difference equations. So no rewriting is needed for the q-Heun equation and its variants. Therefore we can get integral transformations for the q-Heun equation and its variants by easy calculations compared to [56].

3.1 Euler-type integral transformation

In this subsection, we introduce an Euler-type integral transformation E^{λ}.

Definition 3.1. Using the Jackson integral, we define the q-Euler transformation of the function $f(x)$ as follows:

$$E^{\lambda}(f)(x;\sigma) = x^{\lambda-1}\int_0^{\sigma\infty} f(t)\frac{(qt/x)_\infty}{(q^{\lambda}t/x)_\infty}d_qt. \tag{3.1}$$

Remark 3.2. The q-Euler transformation $E^{\lambda}(f)$ is a q-analog of the Riemann-Liouville integral

$$\int_C f(t)(x-t)^{\lambda-1}dt. \tag{3.2}$$

The q-Euler transformation was first considered by Agarwal [1] and Al-Salam [2] as a transformation of functions. The transformation E^{λ} can be regarded as a transformation of some q-difference equations, by applying it to solutions for the q-difference equations.

Proposition 3.3. *Suppose $T_x\sigma = q^l\sigma$ $(l \in \mathbb{Z})$. We have*

$$E^{\lambda}(T_xf)(x;\sigma) = q^{-\lambda}T_xE^{\lambda}(f)(x;\sigma), \tag{3.3}$$

$$(1-T_x)E^\lambda(x\cdot f)(x;\sigma)=x(1-q^{1-\lambda}T_x)E^\lambda(f)(x;\sigma). \tag{3.4}$$

Proof. First we prove the equation (3.3). By the definition of the Jackson integral, we find

$$\int_0^{\sigma\infty}F(qt)d_qt=q^{-1}\int_0^{\sigma\infty}F(t)d_qt. \tag{3.5}$$

Thus we have

$$\begin{aligned}E^\lambda(T_xf)(x;\sigma)&=x^{\lambda-1}\int_0^{\sigma\infty}f(qt)\frac{(qt/x)_\infty}{(q^\lambda t/x)_\infty}d_qt\\&=q^{-1}x^{\lambda-1}\int_0^{\sigma\infty}f(t)T_t^{-1}\left(\frac{(qt/x)_\infty}{(q^\lambda t/x)_\infty}\right)d_qt\\&=q^{-1}x^{\lambda-1}\int_0^{\sigma\infty}f(t)T_x\left(\frac{(qt/x)_\infty}{(q^\lambda t/x)_\infty}\right)d_qt\\&=q^{-1}x^{\lambda-1}\int_0^{\sigma\infty}T_x\left(f(t)\frac{(qt/x)_\infty}{(q^\lambda t/x)_\infty}\right)d_qt.\end{aligned} \tag{3.6}$$

By the condition $T_x\sigma=q^l\sigma$ $(l\in\mathbb{Z})$, we have

$$\int_0^{\sigma\infty}T_xF(t;x)d_qt=T_x\int_0^{\infty}F(t;x)d_qt. \tag{3.7}$$

Therefore we have

$$\begin{aligned}q^{-1}x^{\lambda-1}\int_0^{\sigma\infty}T_x\left(f(t)\frac{(qt/x)_\infty}{(q^\lambda t/x)_\infty}\right)d_qt&=q^{-1}x^{\lambda-1}T_x\int_0^{\sigma\infty}f(t)\frac{(qt/x)_\infty}{(q^\lambda t/x)_\infty}d_qt\\&=q^{-\lambda}T_x(E^\lambda(f)(x;\sigma)).\end{aligned} \tag{3.8}$$

Second, we prove the equation (3.4). From a direct calculation, we get

$$(1-T_x)\left(x^{\lambda-1}tf(t)\frac{(qt/x)_\infty}{(q^\lambda t/x)_\infty}\right)=x(1-q^{1-\lambda}T_x)\left(x^{\lambda-1}f(t)\frac{(qt/x)_\infty}{(q^\lambda t/x)_\infty}\right). \tag{3.9}$$

By integrating both sides, we finally find

$$(1-T_x)\left(x^{\lambda-1}\int_0^{\sigma\infty}tf(t)\frac{(qt/x)_\infty}{(q^\lambda t/x)_\infty}d_qt\right)=x(1-q^{1-\lambda}T_x)\left(x^{\lambda-1}\int_0^{\infty}f(t)\frac{(qt/x)_\infty}{(q^\lambda t/x)_\infty}d_qt\right). \tag{3.10}$$

This completes the proof of (3.4). □

3.2 $D_5^{(1)}$ case: q-Heun equation

In this subsection, we construct an integral transformation for the q-Heun equation, and we give a representation of affine Weyl group of type $D_5^{(1)}$ as an application.

Definition 3.4 (q-Heun equation [19])**.** Suppose $a_1a_2d_1d_2=b_1b_2c_1c_2$. We define the q-difference operator H as follows:

$$\begin{aligned}H=&x^{-1}(1-a_1^{-1}T_x)(1-a_2^{-1}T_x)T_x^{-1}+(c_1c_2)^{-1}x(1-b_1^{-1}T_x)(1-b_2^{-1}T_x)T_x^{-1}\\&+(-c_1^{-1}-c_2^{-1})T_x^{-1}+(a_1a_2)^{-1}(-d_1^{-1}-d_2^{-1})T_x.\end{aligned} \tag{3.11}$$

For $E\in\mathbb{C}$, the equation $(H-E)f(x)=0$ is called the q-Heun equation.

Remark 3.5. The eigenvalue E plays an accessory parameter. See also [61, 62] for the q-Heun equation.

Proposition 3.6. *The operator H is characterized by the following conditions: The operator H is rewritten as*

$$H=\sum_{i=-1}^{1}\sum_{j=-1}^{1}a_{i,j}x^iT_x^j=\sum_{i=-1}^{1}x^iL_i(T_x)=\sum_{j=-1}^{1}P_j(x)T_x^j\quad(a_{0,0}=0). \tag{3.12}$$

Then we have the followings:

(1) The Laurent polynomials $L_{-1}(y)$, $L_1(y)$ satisfy

$$L_{-1}(y)\propto(y-a_1)(y-a_2),\ L_1(y)\propto(y-b_1)(y-b_2). \tag{3.13}$$

(2) The Laurent polynomials $P_{-1}(x)$ $P_1(x)$ satisfy

$$P_{-1}(x)\propto(x-c_1)(x-c_2),\ P_1(x)\propto(x-d_1)(x-d_2). \tag{3.14}$$

Proof. We can prove this proposition in a similar way to the proof of Proposition 2.5. □

Remark 3.7. If parameters do not satisfy the condition $a_1a_2d_1d_2=b_1b_2c_1c_2$, then there is no operator that satisfies the conditions (1) and (2) in Proposition 3.6. This is easily checked by the compatibility of the two representations of H:

$$H=\sum_{i=-1}^{1}x^iL_i(T_x)=\sum_{j=-1}^{1}P_j(x)T_x^j. \tag{3.15}$$

Proposition 3.8. *Let the function $f(x)$ satisfy the q-Heun equation $(H-E)f(x)=0$. We put $q^{\rho_1}=a_1^{-1}$, $q^{\lambda}=a_1b_1^{-1}q^{-1}$, $q^{\rho_2}=q^{-\lambda-\rho_1}=b_1q$. Then the function*

$$F(x)=x^{\rho_2}E^{\lambda}(x^{\rho_1}\cdot f)(x;\sigma)=x^{\rho_2+\lambda-1}\int_0^{\sigma\infty}t^{\rho_1}f(t)\frac{(qt/x)_\infty}{(q^\lambda t/x)_\infty}d_qt, \tag{3.16}$$

satisfies the equation $(\tilde{H}-E)F(x)=0$ if $T_x\sigma=q^l\sigma$ $(l\in\mathbb{Z})$. Here,

$$\begin{aligned}\tilde{H}=&x^{-1}(1-(b_1q)^{-1}T_x)(1-a_2^{-1}T_x)T_x^{-1}+(c_1c_2)^{-1}x(1-a_1^{-1}qT_x)(1-b_2^{-1}T_x)T_x^{-1}\\&+(-c_1^{-1}-c_2^{-1})T_x^{-1}+(a_1a_2)^{-1}(-d_1^{-1}-d_2^{-1})T_x,\end{aligned} \tag{3.17}$$

i.e.

$$\tilde{H}=H\Big|_{a_1\to b_1q,\ b_1\to a_1q^{-1},\ d_i\to d_i\frac{a_1}{b_1q}\ (i=1,2)}. \tag{3.18}$$

Proof. First, by the gauge transformation $f(x)\mapsto f_1(x)=x^{\rho_1}f(x)$, the equation $(H-E)f(x)=0$ becomes $(H_1-E)f_1(x)=0$, where

$$\begin{aligned}H_1=&x^{-1}(1-T_x)(1-a_1a_2^{-1}T_x)a_1^{-1}T_x^{-1}+(c_1c_2)^{-1}x(1-a_1b_1^{-1}T_x)(1-a_1b_2^{-1}T_x)a_1^{-1}T_x^{-1}\\&+(-c_1^{-1}-c_2^{-1})a_1^{-1}T_x^{-1}+(a_1a_2)^{-1}(-d_1^{-1}-d_2^{-1})a_1T_x.\end{aligned} \tag{3.19}$$

Using the relation (3.3) in Proposition 3.3, we get

$$E^{\lambda}\left(x^{-1}(1-T_x)(1-a_1a_2^{-1}T_x)a_1^{-1}T_x^{-1}f_1(x)\right)(x;\sigma)$$

$$
\begin{aligned}
&=E^{\lambda}\left((1-qT_x)(1-a_1a_2^{-1}qT_x)a_1^{-1}q^{-1}T_x^{-1}x^{-1}f_1(x)\right)(x;\sigma)\\
&=(1-q^{1-\lambda}T_x)(1-a_1a_2^{-1}q^{1-\lambda}T_x)a_1^{-1}q^{-1+\lambda}T_x^{-1}E^{\lambda}\left(x^{-1}f_1(x)\right)(x;\sigma). \qquad (3.20)
\end{aligned}
$$

By using (3.4) in Proposition 3.3, we find

$$
\begin{aligned}
&(1-q^{1-\lambda}T_x)(1-a_1a_2^{-1}q^{1-\lambda}T_x)a_1^{-1}q^{-1+\lambda}T_x^{-1}E^{\lambda}\left(x^{-1}f_1(x)\right)(x;\sigma)\\
&=(1-a_1a_2^{-1}q^{1-\lambda}T_x)a_1^{-1}q^{-1+\lambda}T_x^{-1}(1-q^{-1-\lambda}T_x)E^{\lambda}(x^{-1}f_1(x))(x;\sigma)\\
&=(1-a_1a_2^{-1}q^{1-\lambda}T_x)a_1^{-1}q^{-1+\lambda}T_x^{-1}x^{-1}(1-T_x)E^{\lambda}(f_1)(x;\sigma)\\
&=x^{-1}(1-T_x)(1-a_1a_2^{-1}q^{-\lambda}T_x)a_1^{-1}q^{\lambda}T_x^{-1}E^{\lambda}(f_1)(x;\sigma). \qquad (3.21)
\end{aligned}
$$

We similarly get

$$
\begin{aligned}
&E^{\lambda}\left(x(1-a_1b_1^{-1}T_x)(1-a_1b_2^{-1}T_x)a_1^{-1}T_x^{-1}f_1(x)\right)(x;\sigma)\\
&=(1-a_1b_1^{-1}q^{-1-\lambda}T_x)(1-a_1b_2^{-1}q^{-1-\lambda}T_x)a_1^{-1}q^{1+\lambda}T_x^{-1}E^{\lambda}(xf_1(x))(x;\sigma)\\
&=(1-a_1b_2^{-1}q^{-1-\lambda}T_x)a_1^{-1}q^{1+\lambda}T_x^{-1}x(1-q^{1-\lambda}T_x)E^{\lambda}(f_1)(x;\sigma)\\
&=x(1-q^{1-\lambda}T_x)(1-a_1b_2^{-1}q^{-\lambda}T_x)a_1^{-1}q^{\lambda}T_x^{-1}E^{\lambda}(f_1)(x;\sigma). \qquad (3.22)
\end{aligned}
$$

Here we use $q^{\lambda}=a_1b_1^{-1}q^{-1}$. Therefore we have $(H_2-E)E^{\lambda}(f_1)(x;\sigma)=0$ by considering $E^{\lambda}((H_1-E)f_1)(x;\sigma)$. Here,

$$
\begin{aligned}
H_2=&x^{-1}(1-T_x)(1-a_1a_2^{-1}q^{-\lambda}T_x)a_1^{-1}q^{\lambda}T_x^{-1}+(c_1c_2)^{-1}x(1-q^{1-\lambda}T_x)(1-a_1b_2^{-1}q^{-\lambda}T_x)a_1^{-1}q^{-\lambda}T_x^{-1}\\
&+(-c_1^{-1}-c_2^{-1})a_1^{-1}q^{\lambda}T_x^{-1}+(a_1a_2)^{-1}(-d_1^{-1}-d_2^{-1})a_1q^{-\lambda}T_x. \qquad (3.23)
\end{aligned}
$$

By the gauge transformation $f_2(x)\mapsto x^{\rho_2}f_2(x)=F(x)$, we get the desired equation $(\tilde{H}-E)F(x)=0$. □

Remark 3.9. Proposition 3.8 is also obtained by Sasaki-Takagi-Takemura [56]. They apply the q-middle convolution [55] to a linear q-difference equation which is associated with Jimbo-Sakai's q-Painlevé equation [28]. As a special case, they give an integral transformation for the q-Heun equation. Also an integral transformation of Heun's differential equation was obtained in [59] by using the middle convolution, which was introduced by Katz [31]. See also Dettweiler-Reiter [12] for the middle convolution.

Clearly, the q-Heun equation has the symmetry $a_1\leftrightarrow a_2$, $b_1\leftrightarrow b_2$, $c_1\leftrightarrow c_2$, $d_1\leftrightarrow d_2$. In addition, by the gauge transformation $f(x)\mapsto f^g(x)=f(x)\dfrac{(x/d_1)_\infty}{(qx/c_1)_\infty}$, the q-Heun equation $(H-E)f(x)=0$ is transformed to the equation $(H^g-E)f^g(x)=0$, where

$$
\begin{aligned}
H^g=&x^{-1}(1-a_1^{-1}T_x)(1-a_2^{-1}T_x)T_x^{-1}+((d_1q)c_2)^{-1}x(1-b_1^{-1}c_1^{-1}d_1qT_x)(1-b_2^{-1}c_1^{-1}d_1qT_x)T_x^{-1}\\
&+(-d_1^{-1}q^{-1}-c_2^{-1})T_x^{-1}+(a_1a_2)^{-1}(-c_1^{-1}q-d_2^{-1})T_x, \qquad (3.24)
\end{aligned}
$$

i.e.

$$
H^g=H\Big|_{c_1\to d_1q,\ d_1\to c_1q^{-1},\ b_i\to b_i\frac{c_1}{d_1q}\ (i=1,2)}. \qquad (3.25)
$$

In conclusion we get a representation of the affine Weyl group of type $D_5^{(1)}$ by using the transformations of the q-Heun equation as follows.

Theorem 3.10. *We define the actions of $s_0, \dots, s_5$ on parameters as follows:*

$$s_0 : a_1 \leftrightarrow a_2, \ s_1 : b_1 \leftrightarrow b_2, \ s_4 : c_1 \leftrightarrow c_2, \ s_5 : d_1 \leftrightarrow d_2, \tag{3.26}$$

$$s_2 : a_1 \to b_1 q, \ b_1 \to a_1 q^{-1}, \ d_i \to d_i \frac{a_1}{b_1 q} \ (i = 1, 2), \tag{3.27}$$

$$s_3 : c_1 \to d_1 q, \ d_1 \to c_1 q^{-1}, \ b_i \to b_i \frac{c_1}{d_1 q} \ (i = 1, 2). \tag{3.28}$$

These transformations $\{s_i\}$ satisfy the Coxeter relations for the following Dynkin diagram:

$$\begin{tikzcd}
s_0 \arrow[dr, dash] & & & s_4 \arrow[dl, dash] \\
& s_2 \arrow[r, dash] & s_3 & \\
s_1 \arrow[ur, dash] & & & s_5 \arrow[ul, dash]
\end{tikzcd}$$

We also define the actions of $\tilde{s}_0, \dots, \tilde{s}_5$ on a function $f(x)$ as follows:

$$\tilde{s}_2 : f(x) \mapsto x^{\rho_2} E^{\lambda} (x^{\rho_1} \cdot f)(x; \sigma), \tag{3.29}$$

$$\tilde{s}_3 : f(x) \mapsto f(x) \frac{(x/d_1)_\infty}{(qx/c_1)_\infty}, \tag{3.30}$$

$$\tilde{s}_i : \text{trivial} \ \ (i = 0, 1, 4, 5). \tag{3.31}$$

Here $q^{\rho_1} = a_1^{-1}$, $q^{\lambda} = a_1 b_1^{-1} q^{-1}$, $q^{\rho_2} = q^{-\lambda - \rho_1} = b_1 q$. If the function $f(x)$ satisfies the q-Heun equation $(H - E) f(x) = 0$, then we have $((s_i H) - E)(\tilde{s}_i f(x)) = 0$ $(0 \le i \le 5)$.

Proof. We already proved $((s_i H) - E)(\tilde{s}_i f(x)) = 0$. We only have to check the braid relations for the transformations (3.26), (3.27), (3.28). These can be found directly. For example,

$$s_0 s_2 s_0 : (a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2) \to \left(a_1, b_1 q, \frac{a_2}{q}, b_2, c_1, c_2, \frac{a_2 d_1}{b_1 q}, \frac{a_2 d_2}{b_1 q} \right), \tag{3.32}$$

$$s_2 s_0 s_2 : (a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2) \to \left(a_1, b_1 q, \frac{a_2}{q}, b_2, c_1, c_2, \frac{a_2 d_1}{b_1 q}, \frac{a_2 d_2}{b_1 q} \right). \tag{3.33}$$

□

Remark 3.11. The transformations (3.29), (3.30), (3.31) do not satisfy the Coxeter relations in general.

3.3 $E_n^{(1)}$ case

By similar method to obtain Theorem 3.10, we get the representations of affine Weyl group of type $E_n^{(1)}$ in terms of transformations of suitable q-difference equations. Following theorems can be proved in similar ways of the $D_5^{(1)}$ case.

<u>Case $E_6^{(1)}$:</u>

We consider the q-difference equation $\mathcal{L}_{E_6^{(1)}} f(x) = \sum_{i=-1}^{2} \sum_{j=-1}^{1} c_{i,j} x^i T_x^j f(x) = 0$ with the following conditions:

(1) We rewrite $\mathcal{L}_{E_6^{(1)}} = \sum_{i=-1}^{2} x^i L_i(T_x)$, then $L_{-1}(y) \propto (y - a_1)(y - a_2)$ and $L_2(y) \propto (y - b)(y - bq^{-1})$, $L_1(y) \propto (y - b)$.

(2) We rewrite $\mathcal{L}_{E_6^{(1)}} = \displaystyle\sum_{j=-1}^{1} P_j(x) T_x^j$, then $P_{-1}(x) \propto (x-c_1)(x-c_2)(x-c_3)$ and $P_1(x) \propto (x-d_1)(x-d_2)(x-d_3)$.

Here, parameters satisfy $a_1 a_2 d_1 d_2 d_3 = q^{-1} b^2 c_1 c_2 c_3$.

Remark 3.12. The equation $\mathcal{L}_{E_6^{(1)}} f(x) = 0$ is equivalent to the variant of the q-Heun equation of degree three [62].

We put $q^{\rho_1} = a_1^{-1}$, $q^{\lambda} = a_1 b^{-1} q^{-1}$, $q^{\rho_2} = q^{-\lambda-\rho_1} = bq$, and

$$F(x) = x^{\rho_2} E^{\lambda}(x^{\rho_1} \cdot f)(x;\sigma) = x^{\rho_2} \int_0^{\sigma\infty} t^{\rho_1} f(t) \frac{(qt/x)_\infty}{(q^\lambda t/x)_\infty} d_q t. \tag{3.34}$$

Similar to Proposition 3.8, we find easily $\tilde{\mathcal{L}}_{E_6^{(1)}} F(x) = 0$ if $\mathcal{L}_{E_6^{(1)}} f(x) = 0$, where

$$\tilde{\mathcal{L}}_{E_6^{(1)}} = \mathcal{L}_{E_6^{(1)}} \Big|_{a_1 \to bq,\ b \to a_1 q^{-1},\ d_i \to d_i \frac{a_1}{bq}\ (i=1,2,3)} \tag{3.35}$$

Also the equation $\mathcal{L}_{E_6^{(1)}} f(x) = 0$ has the symmetries $a_1 \leftrightarrow a_2$, $c_i \leftrightarrow c_{i+1}$ $(i=1,2)$, $d_i \leftrightarrow d_{i+1}$ $(i=1,2)$, and by the gauge transformation $f(x) \mapsto f(x)\dfrac{(x/d_1)_\infty}{(qx/c_1)_\infty}$, the equation $\mathcal{L}_{E_6^{(1)}} f(x) = 0$ is transformed to $\mathcal{L}^g_{E_6^{(1)}} f^g(x) = 0$, where

$$\mathcal{L}^g_{E_6^{(1)}} = \mathcal{L}_{E_6^{(1)}} \Big|_{c_1 \to d_1 q,\ d_1 \to c_1 q^{-1},\ b \to b\frac{c_1}{d_1 q}}. \tag{3.36}$$

Therefore we have the following result.

Theorem 3.13. *We define the actions of $s_0, \ldots, s_6$ on parameters as follows:*

$$s_0 : a_1 \leftrightarrow a_2,\ s_1 : c_3 \leftrightarrow c_2,\ s_2 : c_2 \leftrightarrow c_1,\ s_4 : d_1 \leftrightarrow d_2,\ s_5 : d_2 \leftrightarrow d_3, \tag{3.37}$$

$$s_3 : c_1 \to d_1 q,\ d_1 \to c_1 q^{-1},\ b \to b\frac{c_1}{d_1 q}, \tag{3.38}$$

$$s_6 : a_1 \to bq,\ b \to a_1 q^{-1},\ d_i \to d_i \frac{a_1}{bq}\ (i=1,2,3). \tag{3.39}$$

These transformations $\{s_i\}$ satisfy the Coxeter relations for the following Dynkin diagram:

$$\begin{array}{ccccccccc} & & & & s_0 & & & & \\ & & & & | & & & & \\ & & & & s_6 & & & & \\ & & & & | & & & & \\ s_1 & \text{---} & s_2 & \text{---} & s_3 & \text{---} & s_4 & \text{---} & s_5 \end{array}$$

We also define the actions of $\tilde{s}_0, \ldots, \tilde{s}_6$ on a function $f(x)$ as follows:

$$\tilde{s}_3 : f(x) \mapsto f(x)\frac{(x/d_1)_\infty}{(qx/c_1)_\infty}, \tag{3.40}$$

$$\tilde{s}_6 : f(x) \mapsto x^{\rho_2} E^{\lambda}(x^{\rho_1} \cdot f)(x;\sigma), \tag{3.41}$$

$$\tilde{s}_i : \textit{trivial}\ \ (i=0,1,2,4,5). \tag{3.42}$$

Here, $q^{\rho_1} = a_1^{-1}$, $q^{\lambda} = a_1 b^{-1} q^{-1}$, $q^{\rho_2} = q^{-\lambda-\rho_1} = bq$. If the function $f(x)$ satisfies $\mathcal{L}_{E_6^{(1)}} f(x) = 0$, then we have $(s_i \mathcal{L}_{E_6^{(1)}})(\tilde{s}_i f(x)) = 0$ $(0 \le i \le 6)$.

Case $E_7^{(1)}$:

We consider the equation $\mathcal{L}_{E_7^{(1)}} f(x) = \sum_{i=-2}^{2} \sum_{j=-1}^{1} c_{i,j} x^i T_x^j f(x) = 0$ with the following conditions:

(1) We rewrite $\mathcal{L}_{E_7^{(1)}} = \sum_{i=-2}^{2} x^i L_i(T_x)$, then $L_{-2}(y) \propto (y-a)(y-aq)$, $L_{-1}(y) \propto (y-a)$ and $L_2(y) \propto (y-b)(y-bq^{-1})$, $L_1(y) \propto (y-b)$.

(2) We rewrite $\mathcal{L}_{E_7^{(1)}} = \sum_{j=-1}^{1} P_j(x) T_x^j$, then $P_{-1}(x) \propto (x-c_1)(x-c_2)(x-c_3)$ and $P_1(x) \propto (x-d_1)(x-d_2)(x-d_3)$.

Here, parameters satisfy $qa^2 d_1 d_2 d_3 = q^{-1} b^2 c_1 c_2 c_3$.

Remark 3.14. The equation $\mathcal{L}_{E_7^{(1)}} f(x) = 0$ is equivalent to the variant of the q-Heun equation og degree four [62].

Similar calculation to the $D_5^{(1)}$ case and $E_6^{(1)}$ case, we find that the equation $\mathcal{L}_{E_7^{(1)}} f(x) = 0$ has the $E_7^{(1)}$-symmetry. Here we only give the result.

Theorem 3.15. *We define the actions of $s_0, \ldots, s_7$ on parameters as follows:*

$$s_0 : a_1 \leftrightarrow a_2,\ s_1 : c_4 \leftrightarrow c_3,\ s_2 : c_3 \leftrightarrow c_2,\ s_3 : c_2 \leftrightarrow c_1,\ s_5 : d_1 \leftrightarrow d_2,\ s_6 : d_2 \leftrightarrow d_3,\ s_7 : d_3 \leftrightarrow d_4, \tag{3.43}$$

$$s_4 : c_1 \to d_1 q,\ d_1 \to c_1 q^{-1},\ b \to b\frac{c_1}{d_1 q}, \tag{3.44}$$

$$s_0 : a_1 \to bq,\ b \to a_1 q^{-1},\ d_i \to d_i \frac{a_1}{bq}\ (i = 1,2,3,4). \tag{3.45}$$

These transformations $\{s_i\}$ satisfy the Coxeter relations for the following Dynkin diagram:

$$\begin{array}{ccccccccccccc} & & & & & & s_0 & & & & & & \\ & & & & & & | & & & & & & \\ s_1 & — & s_2 & — & s_3 & — & s_4 & — & s_5 & — & s_6 & — & s_7 \end{array}$$

We also define the actions of $\tilde{s}_0, \ldots, \tilde{s}_7$ on a function $f(x)$ as follows:

$$\tilde{s}_4 : f(x) \mapsto f(x)\frac{(x/d_1)_\infty}{(qx/c_1)_\infty}, \tag{3.46}$$

$$\tilde{s}_0 : f(x) \mapsto x^{\rho_2} E^{\lambda}(x^{\rho_1} \cdot f)(x;\sigma), \tag{3.47}$$

$$\tilde{s}_i : \textit{trivial}\ \ (i = 1,2,3,5,6,7). \tag{3.48}$$

Here, $q^{\rho_1} = a^{-1}$, $q^{\lambda} = ab^{-1}q^{-1}$, $q^{\rho_2} = q^{-\lambda-\rho_1} = bq$. If the function $f(x)$ satisfies the equation $\mathcal{L}_{E_7^{(1)}} f(x) = 0$, then we have $(s_i \mathcal{L}_{E_7^{(1)}})(\tilde{s}_i f(x)) = 0$ $(0 \le i \le 7)$.

Case $E_8^{(1)}$:

We consider the equation $\mathcal{L}_{E_8^{(1)}} f(x) = \sum_{i=-3}^{3} \sum_{j=-1}^{2} c_{i,j} x^i T_x^j f(x) = 0$ with the following conditions:

(1) We rewrite $\mathcal{L}_{E_8^{(1)}} = \sum_{i=-3}^{3} x^i L_i(T_x)$, then $L_{-3}(y) \propto (y-a)(y-aq)(y-aq^2)$, $L_{-2}(y) \propto (y-a)(y-aq)$, $L_{-1}(y) \propto (y-a)$ and $L_3(y) \propto (y-b)(y-bq^{-1})(y-bq{-}2)$, $L_2(y) \propto (y-b)(y-bq^{-1})$, $L_1(y) \propto (y-b)$.

(2) We rewrite $\mathcal{L}_{E_8^{(1)}} = \sum_{j=-1}^{2} P_j(x)T_x^j$, then $P_{-1}(x) \propto (x-c_1)(x-c_2)(x-c_3)(x-c_4)(x-c_5)(x-c_6)$ and $P_2(x) \propto (x-d_1)(x-d_1q^{-1})(x-d_2)(x-d_2q^{-1})(x-d_3)(x-d_3q^{-1})$, $P_1(x) \propto (x-d_1)(x-d_2)(x-d_3)$.

Here, parameters satisfy $a^3d_1^2d_2^2d_3^2 = q^{-3}b^3c_1c_2c_3c_4c_5c_6$. Note that the rank of the equation $\mathcal{L}_{E_8^{(1)}}f(x) = 0$ is 3, unlike in the cases $D_5^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$. Similar calculation to the above cases, we find that the equation $\mathcal{L}_{E_8^{(1)}}f(x) = 0$ has the $E_8^{(1)}$-symmetry. We only show the result since the proof is almost the same as the $D_5^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$ cases.

Theorem 3.16. *We define the actions of $s_0, \dots, s_8$ on parameters as follows:*

$$s_1 : d_3 \leftrightarrow d_2,\ s_2 : d_2 \leftrightarrow d_1,\ s_4 : c_1 \leftrightarrow c_2,\ s_5 : c_2 \leftrightarrow c_3,\ s_6 : c_3 \leftrightarrow c_4,\ s_7 : c_4 \leftrightarrow c_5,\ s_8 : c_5 \leftrightarrow c_6, \tag{3.49}$$

$$s_3 : c_1 \to d_1q,\ d_1 \to c_1q^{-1},\ b \to b\frac{c_1}{d_1q}, \tag{3.50}$$

$$s_0 : a_1 \to bq,\ b \to a_1q^{-1},\ d_i \to d_i\frac{a_1}{bq}\ (i = 1,2,3). \tag{3.51}$$

These transformations $\{s_i\}$ satisfy the Coxeter relations for the following Dynkin diagram:

$$\begin{array}{ccccccccccccccc} & & & & s_0 & & & & & & & & & & \\ & & & & | & & & & & & & & & & \\ s_1 & — & s_2 & — & s_3 & — & s_4 & — & s_5 & — & s_6 & — & s_7 & — & s_8 \end{array}$$

We also define the actions of $\tilde{s}_0, \dots, \tilde{s}_8$ on a function $f(x)$ as follows:

$$\tilde{s}_3 : f(x) \mapsto f(x)\frac{(x/d_1)_\infty}{(qx/c_1)_\infty}, \tag{3.52}$$

$$\tilde{s}_0 : f(x) \mapsto x^{\rho_2}E^{\lambda}(x^{\rho_1} \cdot f)(x;\sigma), \tag{3.53}$$

$$\tilde{s}_i : \textit{trivial}\ \ (i = 1,2,4,5,6,7,8). \tag{3.54}$$

Here, $q^{\rho_1} = a^{-1}$, $q^{\lambda} = ab^{-1}q^{-1}$, $q^{\rho_2} = q^{-\lambda-\rho_1} = bq$. If the function $f(x) = 0$ satisfies the equation $\mathcal{L}_{E_8^{(1)}}f(x) = 0$, then we have $(s_i\mathcal{L}_{E_8^{(1)}})(\tilde{s}_if(x)) = 0$ $(0 \le i \le 8)$.

Remark 3.17. The equations $\mathcal{L}_{E_n^{(1)}}f(x) = 0$ $(n = 6,7,8)$ are characterized by the corresponding conditions up to the constant term $c_{0,0}$. In other words, these equations have one accessory parameter.

Remark 3.18. The operator $\mathcal{L}_{E_8^{(1)}}$ is discussed by Moriyama-Yamada [41] in order to give quantum representations for affine Weyl groups. The operators $\mathcal{L}_{E_7}^{(1)}$, $\mathcal{L}_{E_6}^{(1)}$ and H are also studied in [41].

Remark 3.19. The integral transformations of the cases of $D_5^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$ are also obtained by Takemura [63] by using kernel functions of q-difference operators. The integral transformation of the case of $E_8^{(1)}$ is a new result [48].

4 Connection problem for a multivariable q-hypergeometric system

In this section, we will give local solutions and connection matrices for the q-hypergeometric system $E_{N,M}$ [51, 52], which is an extension of the q-Appell-Lauricella system. A characterization of these solutions will be also given. This section is based on [45].

Definition 4.1 ([51, 52]). We define the system $E_{N,M}$ as

$$\left\{ x_s \prod_{j=1}^{N} (1 - a_j T) \cdot (1 - b_s T_s) - \prod_{j=1}^{N} (1 - c_j q^{-1} T) \cdot (1 - T_s) \right\} \mathcal{F} = 0 \quad (1 \le s \le M), \tag{4.1}$$

$$\{ x_r (1 - b_r T_r)(1 - T_s) - x_s (1 - b_s T_s)(1 - T_r) \} \mathcal{F} = 0 \quad (1 \le r < s \le M), \tag{4.2}$$

where $T_s = T_{x_s}$ and $T = T_1 \cdots T_M$ in this section.

The function

$$\mathcal{F}_{N,M} \left(\begin{matrix} \{a_j\}_{1 \le j \le N}, \{b_i\}_{1 \le i \le M} \\ \{c_j\}_{1 \le j \le N} \end{matrix} ; \{x_i\}_{1 \le i \le M} \right) = \sum_{m_1, \dots, m_M \ge 0} \prod_{j=1}^{N} \frac{(a_j)_{|m|}}{(c_j)_{|m|}} \prod_{i=1}^{M} \frac{(b_i)_{m_i}}{(q)_{m_i}} \prod_{i=1}^{M} x_i^{m_i}, \tag{4.3}$$

satisfies the system $E_{N,M}$. This function and the corresponding system $E_{N,M}$ were introduced by Park [51, 52] in order to construct an isomonodromic deformation.

Theorem 4.2 ([52]). *The rank of the system $E_{N,M}$ is $NM + 1$.*

The function $\mathcal{F}_{N,M}$ is a generalization both of the q-Appell-Lauricella function φ_D and of the generalized q-hypergeometric function ${}_{N+1}\varphi_N$, i.e. $\mathcal{F}_{1,M} = \varphi_D$ and $\mathcal{F}_{N,1} = {}_{N+1}\varphi_N$. Thus our results contain solving the connection problem related with φ_D and also related with ${}_{N+1}\varphi_N$. The equality (4.36) given below can be considered as a part of the connection relations for ${}_{N+1}\varphi_N$, which was first studied by J. Thomae [64] and was proved by G. N. Watson [67]. Also the q-Appell-Lauricella function has a Jackson integral representation of Jordan-Pochhammer type, and the connection problem related with the Jackson integral of Jordan-Pochhammer type was solved by Mimachi [38]. Note that a connection formula for the Jackson integral of type A, including Mimachi's formula, was studied by Ito-Noumi [26]. In differential case, fundamental solutions of the equation related with the Appell-Lauricella function F_D were obtained by Gelfand-Kapranov-Zelevinsky [17] as an application of the theory of the GKZ hypergeometric function. Our solutions include q-analogs of the solutions of [17] for the Appell-Lauricella system.

4.1 Fundamental solutions

In the following, we give a fundamental solution for the system $E_{N,M}$.

Definition 4.3 ([45]). We define series $\mathcal{F}^{L}_{N,M}, \mathcal{F}^{L;k,l}_{N,M}$ and $\mathcal{G}^{L;k,l'}_{N,M}$ as

$$\mathcal{F}^{L}_{N,M} \left(\begin{matrix} \{a_j\}_{1 \le j \le N}, \{b_i\}_{1 \le i \le M} \\ \{c_j\}_{1 \le j \le N} \end{matrix} ; \{x_i\}_{1 \le i \le M} \right)$$
$$= \sum_{m_1, \dots, m_M \ge 0} \prod_{j=1}^{N} \frac{(a_j/(b_{L+1} \cdots b_M))_{m(L)}}{(c_j/(b_{L+1} \cdots b_M))_{m(L)}} \prod_{i=1}^{M} \frac{(b_i)_{m_i}}{(q)_{m_i}} \prod_{i=1}^{L} x_i^{m_i} \prod_{i=L+1}^{M} \left(\frac{q}{b_i x_i} \right)^{m_i}, \tag{4.4}$$

$$
\begin{aligned}
&\mathcal{F}^{L;k,l}_{N,M}\left(\begin{matrix}\{a_j\}_{1\le j\le N},\{b_i\}_{1\le i\le N}\\ \{c_j\}_{1\le j\le N}\end{matrix};\{x_i\}_{1\le i\le M}\right)\\
&=\sum_{m_1,\dots,m_M\ge 0}\left\{\prod_{j=1}^{N}\frac{(qa_k/c_j)_{m_{L+1}}}{(qa_k/a_j)_{m_{L+1}}}\prod_{i=1}^{L}\frac{(b_i)_{m_i}}{(q)_{m_i}}\prod_{i=L+1}^{l-1}\frac{(b_i)_{m_{i+1}}}{(q)_{m_{i+1}}}\prod_{i=l+1}^{M}\frac{(b_i)_{m_i}}{(q)_{m_i}}\right.\\
&\quad\left.\times\frac{(a_k/(b_{l+1}\cdots b_M))_{m(l)}}{(qa_k/(b_l\cdots b_M))_{m(l)}}\prod_{i=1}^{L}\left(\frac{qx_i}{b_lx_l}\right)^{m_i}\prod_{i=L+1}^{l-1}\left(\frac{qx_i}{b_lx_l}\right)^{m_{i+1}}\prod_{i=l+1}^{M}\left(\frac{b_lx_l}{b_ix_i}\right)^{m_i}\left(\prod_{j=1}^{N}\frac{c_j}{a_j}\cdot\frac{q}{b_lx_l}\right)^{m_{L+1}}\right\},
\end{aligned}
\tag{4.5}
$$

$$
\begin{aligned}
&\mathcal{G}^{L;k,l'}_{N,M}\left(\begin{matrix}\{a_j\}_{1\le j\le N},\{b_i\}_{1\le i\le M}\\ \{c_j\}_{1\le j\le N}\end{matrix};\{x_i\}_{1\le i\le M}\right)=\sum_{m_1,\dots,m_M\ge 0}\left\{\prod_{j=1}^{N}\frac{(qa_j/c_k)_{m_L}}{(qc_j/c_k)_{m_L}}\prod_{i=1}^{l'-1}\frac{(b_i)_{m_i}}{(q)_{m_i}}\prod_{i=l'+1}^{L}\frac{(b_i)_{m_{i-1}}}{(q)_{m_{i-1}}}\right.\\
&\left.\times\prod_{i=L+1}^{M}\frac{(b_i)_{m_i}}{(q)_{m_i}}\frac{(c_k/qb_{l'+1}\cdots b_M)_{m(l'-1)}}{(c_k/b_{l'}\cdots b_M)_{m(l'-1)}}\prod_{i=1}^{l'-1}\left(\frac{qx_i}{b_{l'}x_{l'}}\right)^{m_i}\prod_{i=l'+1}^{L}\left(\frac{b_{l'}x_{l'}}{b_ix_i}\right)^{m_{i-1}}\prod_{i=L+1}^{M}\left(\frac{b_{l'}x_{l'}}{b_ix_i}\right)^{m_i}\cdot\left(\frac{b_{l'}x_{l'}}{q}\right)^{m_L}\right\},
\end{aligned}
\tag{4.6}
$$

where $0\le L\le M$, $1\le k\le N$, $L+1\le l\le M$ and $1\le l'\le L$. Here, $m(l)=\sum_{i=1}^{l}m_i-\sum_{i=l+1}^{M}m_i$.

Proposition 4.4 ([45])**.** *For $0\le L\le M$, the functions*

$$
\prod_{i=L+1}^{M}x_i^{-\beta_i}\cdot\mathcal{F}^{L}_{N,M}\left(\begin{matrix}\{a_j\}_{1\le j\le N},\{b_i\}_{1\le i\le M}\\ \{c_j\}_{1\le j\le N}\end{matrix};\{x_i\}_{1\le i\le M}\right),\tag{4.7}
$$

$$
x_l^{1+\sum_{i=l+1}^{M}\beta_i-\gamma_k}\prod_{i=l+1}^{M}x_i^{-\beta_i}\cdot\mathcal{G}^{L;k,l}_{N,M}\left(\begin{matrix}\{a_j\}_{1\le j\le N},\{b_i\}_{1\le i\le M}\\ \{c_j\}_{1\le j\le N}\end{matrix};\{x_i\}_{1\le i\le M}\right)\quad(1\le k\le N,\ 1\le l\le L),\tag{4.8}
$$

$$
x_l^{-\alpha_k+\sum_{i=l+1}^{M}\beta_i}\prod_{i=l+1}^{M}x_i^{-\beta_i}\cdot\mathcal{F}^{L;k,l}_{N,M}\left(\begin{matrix}\{a_j\}_{1\le j\le N},\{b_i\}_{1\le i\le M}\\ \{c_j\}_{1\le j\le N}\end{matrix};\{x_i\}_{1\le i\le M}\right)\quad(1\le k\le N,\ L+1\le l\le M),\tag{4.9}
$$

satisfy the q-difference system $E_{N,M}$. Here $q^{\alpha_k}=a_k$, $q^{\beta_i}=b_i$, $q^{\gamma_k}=c_k$.

Proof. This proposition can be checked directly. □

Remark 4.5. When $N=1$, the system $E_{N,M}$ reduces the q-Appell-Lauricella system:

$$
\{x_s(1-aT)(1-b_sT_s)-(1-cq^{-1}T)(1-T_s)\}\mathcal{F}=0\quad(1\le s\le M),\tag{4.10}
$$
$$
\{x_r(1-b_rT_r)(1-T_s)-x_s(1-b_sT_s)(1-T_r)\}\mathcal{F}=0\quad(1\le r<s\le M).\tag{4.11}
$$

The series $\mathcal{F}^{L}_{1,M}$ (4.7), $\mathcal{F}^{L;1,l}_{1,M}$ (4.8), $\mathcal{G}^{L;1,l}_{1,M}$ (4.9) are rewritten as

$$
\begin{aligned}
&\mathcal{F}^{L}_{1,M}\left(\begin{matrix}a,\{b_i\}_{1\le i\le M}\\ c\end{matrix};\{x_i\}_{1\le i\le M}\right)\\
&=\varphi_{D,L}\left(\begin{matrix}a/(b_{L+1}\cdots b_M);\{b_i\}_{1\le i\le M}\\ c/(b_{L+1}\cdots b_M)\end{matrix};\left\{x_1,\dots,x_L,\frac{q}{b_{L+1}x_{L+1}},\dots,\frac{q}{b_Mx_M}\right\}\right),
\end{aligned}
\tag{4.12}
$$

$$
\mathcal{F}^{L;1,l}_{1,M}\left(\begin{matrix}a,\{b_i\}_{1\le i\le M}\\ c\end{matrix};\{x_i\}_{1\le i\le M}\right)
$$

$$= \varphi_{D,l}\left(\begin{matrix} a/(b_{l+1}\cdots b_M); \{b_1,\dots,b_L, aq/c, b_{L+1},\dots,b_{l-1},b_{l+1},\dots,b_M\} \\ aq/(b_l\cdots b_M) \end{matrix} ; \right.$$

$$\left. \left\{ \frac{qx_1}{b_l x_l},\dots,\frac{qx_L}{b_l x_l},\frac{cq}{ab_l x_l},\frac{qx_{L+1}}{b_l x_l},\dots,\frac{qx_{l-1}}{b_l x_l},\frac{b_l x_l}{b_{l+1}x_{l+1}},\dots,\frac{b_l x_l}{b_M x_M} \right\} \right), \tag{4.13}$$

$$\mathcal{G}_{1,M}^{L;1,l'}\left(\begin{matrix} a,\{b_i\}_{1\le i\le M} \\ c \end{matrix} ; \{x_i\}_{1\le i\le M} \right)$$

$$= \varphi_{D,l'-1}\left(\begin{matrix} c/(b_{l'+1}\cdots b_M q); \{b_1,\dots,b_{l'-1},b_{l'+1},\dots,b_L, aq/c, b_{L+1},\dots,b_M\} \\ c/(b_{l'}\cdots b_M) \end{matrix} ; \right.$$

$$\left. \left\{ \frac{qx_1}{b_{l'} x_{l'}},\dots,\frac{qx_{l'-1}}{b_{l'} x_{l'}},\frac{b_{l'}x_{l'}}{b_{l'+1}x_{l'+1}},\dots,\frac{b_{l'}x_{l'}}{b_L x_L},\frac{b_{l'}x_{l'}}{q},\frac{b_{l'}x_{l'}}{b_{L+1}x_{L+1}},\dots\frac{b_{l'}x_{l'}}{b_M x_M} \right\} \right), \tag{4.14}$$

where $0 \le L \le M$, $L+1 \le l \le M$, $1 \le l' \le L$ and

$$\varphi_{D,j}\left(\begin{matrix} a;\{b_i\}_{1\le i\le M} \\ c \end{matrix} ; \{x_i\}_{1\le i\le M} \right) = \sum_{m_1,\dots,m_M\ge 0} \frac{(a)_{m(j)}}{(c)_{m(j)}} \prod_{i=1}^{M} \frac{(b_i)_{m_i}}{(q)_{m_i}} \prod_{i=1}^{M} x_i^{m_i}. \tag{4.15}$$

These solutions are q-analogs of series solutions for the Appell-Lauricella system which were obtained by Gelfand-Kapranov-Zelevinsky [17].

In the general theory of q-difference systems in several variables, the basis of local solutions should be characterized by the asymptotic behavior near the singularity, i.e. origin or infinity, in some prescribed sector [4, 5]. The following proposition answers this requirement.

Proposition 4.6 ([45])**.** *We put* $z = (x_1/x_2,\dots,x_{L-1}/x_L, x_L, 1/x_{L+1}, x_{L+1}/x_{L+2},\dots,x_{M-1}/x_M)$*. The solutions (4.7), (4.8), (4.9) are characterized by the following asymptotic behavior at* $z = (0,\dots,0)$*:*

$$\prod_{i=L+1}^{M} x_i^{-\beta_i}(1+O(||z||)), \tag{4.16}$$

$$x_l^{1+\sum_{i=l+1}^{M}\beta_i-\gamma_k} \prod_{i=l+1}^{M} x_i^{-\beta_i}(1+O(||z||)), \tag{4.17}$$

$$x_l^{-\alpha_k+\sum_{i=l+1}^{M}\beta_i} \prod_{i=l+1}^{M} x_i^{-\beta_i}(1+O(||z||)), \tag{4.18}$$

where $||z|| = \sum_{i=1}^{M}|z_i|$*.*

Proof. The equations (4.1) and (4.2) are rewritten as

$$\begin{cases} \left\{ \frac{x_s}{x_{s+1}}\cdots\frac{x_{L-1}}{x_L}x_L \prod_{j=1}^{N}(1-a_jT)\cdot(1-b_sT_s) - \prod_{j=1}^{N}(1-c_jq^{-1}T)\cdot(1-T_s) \right\}\mathcal{F} = 0, & (1 \le s \le L), \\ \left\{ \prod_{j=1}^{N}(1-a_jT)\cdot(1-b_sT_s) - \frac{1}{x_{L+1}}\frac{x_{L+1}}{x_{L+2}}\cdots\frac{x_{s-1}}{x_s} \prod_{j=1}^{N}(1-c_jq^{-1}T)\cdot(1-T_s) \right\}\mathcal{F} = 0, & (L < s \le M), \end{cases} \tag{4.19}$$

and

$$\left\{ \frac{x_r}{x_s}(1-b_rT_r)(1-T_s) - (1-b_sT_s)(1-T_r) \right\}\mathcal{F} = 0 \quad (1 \le r < s \le M). \tag{4.20}$$

If a function $f(x)$ is a solution of $E_{N,M}$ and has the asymptotic behavior

$$f(x) = x_1^{\delta_1} \cdots x_M^{\delta_M}(1 + O(||z||)), \tag{4.21}$$

at $x = (0, \ldots, 0)$. Then, by checking the coefficient of the lowest power of z, we find that $\delta = (\delta_1, \ldots, \delta_M)$ must satisfy

$$\begin{cases} \prod_{j=1}^{N}(1 - c_j q^{-1} q^{\delta_1+\cdots+\delta_M})(1 - q^{\delta_s}) = 0, \quad (1 \leq s \leq L), \\ \prod_{j=1}^{N}(1 - a_j q^{\delta_1+\cdots+\delta_M})(1 - b_s q^{\delta_s}) = 0, \quad (L < s \leq M), \end{cases} \tag{4.22}$$

$$(1 - b_s q^{\delta_s})(1 - q^{\delta_r}) = 0 \quad (1 \leq r < s \leq M). \tag{4.23}$$

In this sense, the solution $(\delta_1, \ldots, \delta_M)$ of the above equations should be called "the characteristic exponent of the system $E_{N,M}$ at $z = (0, \ldots, 0)$", like a characteristic exponent in one variable case. Solving those equations, we have

$$\begin{aligned} &(\delta_1, \ldots, \delta_M) \\ =&(0, \ldots, 0, -\beta_{L+1}, \ldots, -\beta_M) : \text{ one solution,} \end{aligned} \tag{4.24}$$

$$\begin{cases} (1 + \sum_{i=2}^{M} \beta_i - \gamma_k, -\beta_2, \ldots, -\beta_M), \\ (0, 1 + \sum_{i=3}^{M} \beta_i - \gamma_k, -\beta_3, \ldots, -\beta_M), \\ \vdots \\ (0, \ldots, 0, 1 + \sum_{i=L+1}^{M} \beta_i - \gamma_k, -\beta_{L+1}, \ldots, -\beta_M), \end{cases} \quad : NL \text{ solutions,} \tag{4.25}$$

$$\begin{cases} (0, \ldots, 0, 0, -\alpha_k + \sum_{i=L+2}^{M} \beta_i, -\beta_{L+2}, \ldots, -\beta_M), \\ (0, \ldots, 0, 0, 0, -\alpha_k + \sum_{i=L+3}^{M} \beta_i, -\beta_{L+3}, \ldots, -\beta_M), \\ \vdots \\ (0, \ldots, 0, 0, \ldots, 0, -\alpha_k), \end{cases} \quad : N(M-L) \text{ solutions} \tag{4.26}$$

where $1 \leq k \leq N$. This completes the proof. $\square$

Proposition 4.7 ([45]). *For $0 \leq L \leq M$ and $\sigma \in \mathfrak{S}_M$, we set*

$$u_0^{L,\sigma} = \prod_{i=L+1}^{M} x_{\sigma_i}^{-\beta_{\sigma_i}} \cdot \mathcal{F}_{N,M}^{L}\left(\begin{matrix} \{a_j\}_{1\leq j\leq N}, \{b_{\sigma(i)}\}_{1\leq i\leq M} \\ \{c_j\}_{1\leq j\leq N} \end{matrix} ; \{x_{\sigma(i)}\}_{1\leq i\leq M} \right), \tag{4.27}$$

$$u_{k,l}^{L,\sigma} = \begin{cases} x_{\sigma_l}^{1+\sum_{i=l+1}^{M}\beta_{\sigma_i}-\gamma_k} \prod_{i=l+1}^{M} x_{\sigma_i}^{-\beta_{\sigma_i}} \cdot \mathcal{G}_{N,M}^{L;k,l}\left(\begin{matrix} \{a_j\}_{1\leq j\leq N}, \{b_{\sigma(i)}\}_{1\leq i\leq M} \\ \{c_j\}_{1\leq j\leq N} \end{matrix} ; \{x_{\sigma(i)}\}_{1\leq i\leq M} \right), \\ \qquad (1 \leq k \leq N,\ 1 \leq l \leq L), \\ x_{\sigma_l}^{-\alpha_k+\sum_{i=l+1}^{M}\beta_{\sigma_i}} \prod_{i=l+1}^{M} x_{\sigma_i}^{-\beta_{\sigma_i}} \cdot \mathcal{F}_{N,M}^{L;k,l}\left(\begin{matrix} \{a_j\}_{1\leq j\leq N}, \{b_{\sigma(i)}\}_{1\leq i\leq M} \\ \{c_j\}_{1\leq j\leq N} \end{matrix} ; \{x_{\sigma(i)}\}_{1\leq i\leq M} \right), \\ \qquad (1 \leq k \leq N,\ L+1 \leq l \leq M). \end{cases} \tag{4.28}$$

We set

$$\boldsymbol{u}^{L,\sigma} = (u_0^{L,\sigma}, u_{1,1}^{L,\sigma}, \ldots, u_{1,M}^{L,\sigma}, u_{2,1}^{L,\sigma}, \ldots, u_{N,M}^{L,\sigma})^{\mathrm{T}}. \tag{4.29}$$

Then $\boldsymbol{u}^{L,\sigma}$ is a fundamental solution of the q-difference system $E_{N,M}$ in the region

$$D^{L,\sigma} = \left\{ |x_{\sigma(i)}| < 1 \ (1 \le i \le L), \ \left| \prod_{j=1}^{N} \frac{c_j}{a_j} \cdot \frac{q}{b_{\sigma(i)} x_{\sigma(i)}} \right| < 1 \ (L+1 \le i \le M), \right.$$
$$\left. \left| \frac{q x_{\sigma(i)}}{b_{\sigma(j)} x_{\sigma(j)}} \right| < 1 \ (1 \le i < j \le M) \right\}. \tag{4.30}$$

Proof. We only have to prove the linearly independence of $u_0^{L,\mathrm{id}}$, $u_{k,l}^{L,\mathrm{id}}$ $(1 \le k \le N, \ 1 \le l \le M)$. Suppose

$$C_0 u_0^{L,\mathrm{id}} + \sum_{k=1}^{N} \sum_{l=1}^{M} C_{k,l} u_{k,l}^{l,\mathrm{id}} = 0, \tag{4.31}$$

where C_0, $C_{k,l}$ are pseudo constants in x. The solutions $u_0^{L,\mathrm{id}}$, $u_{k,l}^{L,\mathrm{id}}$ are characterized by the asymptotic behavior in Proposition 4.6. We rewrite these asymptotic behavior as

$$u_0^{L,\mathrm{id}} = x^{\delta_0}(1 + O(||z||)), \ u_{k,l}^{L,\mathrm{id}} = x^{\delta_{k,l}}(1 + O(||z||)), \tag{4.32}$$

where $x^\delta = x_1^{\delta_1} \cdots x_M^{\delta_M}$ for $\delta = (\delta_1, \ldots, \delta_M)$. We can take $m = (m_1, \ldots, m_M) \in \mathbb{Z}^M$ such that $(m, \delta_0) \neq (m, \delta_{k,l}) \neq (m, \delta_{k',l'})$ for any $(k,l) \neq (k',l')$, where $(m, \delta) = m_1\delta_1 + \cdots + m_M\delta_M$. We define the operator $R = T_1^{m_1} \cdots T_M^{m_M}$ for such m. We put

$$\boldsymbol{v} = (C_0 u_0^{l,\mathrm{id}}, C_{1,1} u_{1,1}^{L,\mathrm{id}}, C_{1,2} u_{1,2}^{L,\mathrm{id}}, \ldots, C_{1,M} u_{1,M}^{L,\mathrm{id}}, C_{2,1} u_{2,1}^{L,\mathrm{id}}, \ldots, C_{N,M} u_{N,M}^{L,\mathrm{id}})^{\mathrm{T}}, \tag{4.33}$$

and consider the following determinant

$$D = \left| \boldsymbol{v}, R\boldsymbol{v}, \ldots, R^{NM}\boldsymbol{v} \right|. \tag{4.34}$$

Using the asymptotic behaviors (4.32), we have

$$D = C_0 x^{\delta_0} \prod_{k,l} (C_{k,l} x^{\delta_{k,l}}) \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 \\ q^{(m,\delta_0)} & q^{(m,\delta_{1,1})} & \cdots & q^{(m,\delta_{N,M})} \\ (q^{(m,\delta_0)})^2 & (q^{(m,\delta_{1,1})})^2 & \cdots & (q^{(m,\delta_{N,M})})^2 \\ \vdots & \vdots & & \vdots \\ (q^{(m,\delta_0)})^{NM} & (q^{(m,\delta_{1,1})})^{NM} & \cdots & (q^{(m,\delta_{N,M})})^{NM} \end{vmatrix} (1 + O(||z||)). \tag{4.35}$$

By the assumption (4.31), we have $D = 0$. Also the Vandermonde determinant is not vanished by the condition of m. Therefore one of the pseudo constants C_0, $C_{k,l}$ is 0. Using the above method inductively, we finally obtain $C_0 = C_{k,l} = 0$ for any (k,l). □

4.2 Connection matrices

In the following, we consider the connection formula between $\boldsymbol{u}^{L_1,\sigma_1}$ and $\boldsymbol{u}^{L_2,\sigma_2}$, where $0 \le L_1, L_2 \le M$ and $\sigma_1, \sigma_2 \in \mathfrak{S}_M$. We can solve this problem in principle by calculating the following matrices:

(1) the matrix which connects $\boldsymbol{u}^{L,\sigma}$ with $\boldsymbol{u}^{L+1,\sigma}$ $(0 \le L \le M-1)$,

(2) the matrix which connects $\boldsymbol{u}^{L,\sigma}$ with $\boldsymbol{u}^{L-1,\sigma}$ $(1 \le L \le M)$,

(3) the matrix which connects $\boldsymbol{u}^{M,s_r\sigma}$ with $\boldsymbol{u}^{M,\sigma}$ $(0 \le r \le M-1)$,

where $s_r = (r, r+1) \in \mathfrak{S}_M$. These matrices can be calculated by the Thomae-Watson's formula [64, 67].

Lemma 4.8 ([64, 67])**.** The connection formula of ${}_{N+1}\varphi_N$ is given as follows:

$$
\begin{aligned}
&{}_{N+1}\varphi_N\left(\begin{matrix} a_1, \dots, a_{N+1} \\ b_1, \dots, b_N \end{matrix} ; x\right) \\
&= \sum_{k=1}^{N+1} \prod_{j=1}^{N} \frac{(b_j/a_k)_\infty}{(b_j)_\infty} \prod_{\substack{1 \le j \le N+1 \\ j \ne k}} \frac{(a_j)_\infty}{(a_j/a_k)_\infty} \cdot \frac{\theta(xa_k)}{\theta(x)} {}_{N+1}\varphi_N\left(\begin{matrix} \{qa_k/b_j\}_{1\le j\le N},\, a_k \\ \{qa_k/a_j\}_{1 \le j \le N+1,\, j\ne k} \end{matrix} ; \frac{b_1\cdots b_N q}{a_1 \cdots a_{N+1} x}\right).
\end{aligned}
\tag{4.36}
$$

Proof. This formula can be proved in a similar way to Proposition 2.32. □

Computation of case (1):

First, we consider the matrix which connects $\boldsymbol{u}^{L,\mathrm{id}}$ with $\boldsymbol{u}^{L+1,\mathrm{id}}$. If $l \ne L+1$, then we have

$$
u_{k,l}^{L,\mathrm{id}} = u_{k,l}^{L+1,\mathrm{id}}, \tag{4.37}
$$

easily. Hence, we should calculate the connection formula of $u_0^{L,\mathrm{id}}$, $u_{k,L+1}^{L,\mathrm{id}}$. By rewriting the definition (4.4), we have

$$
\begin{aligned}
&\mathcal{F}_{N,M}^{L}\left(\begin{matrix} \{a_j\}, \{b_i\} \\ \{c_j\} \end{matrix} ; \{x_i\}\right) = \sum_{m_1,\dots,m_M \ge 0} \prod_{j=1}^{N} \frac{(a_j/(b_{L+1}\cdots b_M))_{m(L)}}{(c_j/(b_{L+1}\cdots b_M))_{m(L)}} \prod_{i=1}^{M} \frac{(b_i)_{m_i}}{(q)_{m_i}} \prod_{i=1}^{L} x_i^{m_i} \prod_{i=L+1}^{M} \left(\frac{q}{b_i x_i}\right)^{m_i} \\
&= \sum_{m_1,\dots,m_L,m_{L+2},\dots,m_M \ge 0} \left\{ \prod_{j=1}^{N} \frac{(a_j/(b_{L+1}\cdots b_M))_{m(L)'}}{(c_j/(b_{L+1}\cdots b_M))_{m(L)'}} \prod_{\substack{1 \le i \le M \\ i \ne L+1}} \frac{(b_i)_{m_i}}{(q)_{m_i}} \prod_{i=1}^{L} x_i^{m_i} \prod_{i=L+2}^{M} \left(\frac{q}{b_i x_i}\right)^{m_i} \right. \\
&\left. \times\ {}_{N+1}\varphi_N\left(\begin{matrix} \{qb_{L+1}\cdots b_M/c_j q^{m(L)'}\}_{1\le j\le N},\, b_{L+1} \\ \{qb_{L+1}\cdots b_M/a_j q^{m(L)'}\}_{1\le j\le N} \end{matrix} ; \prod_{j=1}^{N} \frac{c_j}{a_j} \cdot \frac{q}{b_{L+1}x_{L+1}}\right)\right\}.
\end{aligned}
\tag{4.38}
$$

Here and in the following, we use the notation

$$
m(l)' = \sum_{i=1}^{l} m_i - \sum_{i=l+2}^{M} m_i, \tag{4.39}
$$

for $m = (m_1, \dots, m_M)$ and $0 \le l \le M$. By applying the formula (4.36) to ${}_{N+1}\varphi_N$, we obtain

$$
\begin{aligned}
&\mathcal{F}_{N,M}^{L}\left(\begin{matrix} \{a_j\}_{1\le j\le N}, \{b_i\}_{1\le i\le M} \\ \{c_j\}_{1\le j\le N} \end{matrix} ; \{x_i\}_{1\le i\le M}\right) \\
&= \prod_{j=1}^{N} \frac{(qb_{L+2}\cdots b_M/a_j, qb_{L+1}\cdots b_M/c_j)_\infty}{(qb_{L+1}\cdots b_M/a_j, qb_{L+2}\cdots b_M/c_j)_\infty} \cdot \frac{\theta(x_{L+1}a_1\cdots a_N/(c_1\cdots c_N))}{\theta(x_{L+1}b_{L+1}a_1\cdots a_N/(c_1\cdots c_N))} \\
&\times \mathcal{F}_{N,M}^{L+1}\left(\begin{matrix} \{a_j\}_{1\le j\le N}, \{b_i\}_{1\le i\le M} \\ \{c_j\}_{1\le j\le N} \end{matrix} ; \{x_i\}_{1\le i\le M}\right) \\
&+ \sum_{d=1}^{N} \left\{ \prod_{j=1}^{N} \frac{(c_d/a_j)_\infty}{(qb_{L+1}\cdots b_M/a_j)_\infty} \prod_{\substack{1\le j\le N \\ j\ne d}} \frac{(qb_{L+1}\cdots b_M/c_j)_\infty}{(c_d/c_j)_\infty} \cdot \frac{(b_{L+1})_\infty}{(c_d/(qb_{L+2}\cdots b_M))_\infty} \right.
\end{aligned}
$$

$$\times \frac{\theta(x_{L+1}a_1\cdots a_N c_d/(qb_{L+2}\cdots b_M c_1\cdots c_N))}{\theta(x_{L+1}b_{L+1}a_1\cdots a_N/(c_1\cdots c_N))}\mathcal{F}^{L+1;d,L+1}_{N,M}\left(\begin{matrix}\{a_j\}_{1\le j\le N},\{b_i\}_{1\le i\le M}\\ \{c_j\}_{1\le j\le N}\end{matrix};\{x_i\}_{1\le i\le M}\right)\Bigg\}. \tag{4.40}$$

By a similar calculation of (4.38), we have

$$\mathcal{F}^{L;k,L+1}_{N,M}\left(\begin{matrix}\{a_j\}_{1\le j\le N},\{b_i\}_{1\le i\le M}\\ \{c_j\}_{1\le j\le N}\end{matrix};\{x_i\}_{1\le i\le M}\right)=\prod_{\substack{1\le j\le N\\ j\ne k}}\frac{(qb_{L+2}\cdots b_M/a_j)_\infty}{(qa_k/a_j)_\infty}\cdot\frac{(q/b_{L+1})_\infty}{(qa_k/(b_{L+1}\cdots b_M))_\infty}$$
$$\times\prod_{j=1}^N\frac{(qa_k/c_j)_\infty}{(qb_{L+2}\cdots b_M/c_j)_\infty}\frac{\theta(x_{L+1}b_{L+1}\cdots b_M a_1\cdots a_N/(a_k c_1\cdots c_N))}{\theta(x_{L+1}b_{L+1}a_1\cdots a_N/(c_1\cdots c_N))}$$
$$\times\mathcal{F}^{L+1}_{N,M}\left(\begin{matrix}\{a_j\}_{1\le j\le N},\{b_i\}_{1\le i\le M}\\ \{c_j\}_{1\le j\le N}\end{matrix};\{x_i\}_{1\le i\le M}\right)$$
$$+\sum_{d=1}^N\Bigg\{\prod_{\substack{1\le j\le N\\ j\ne k}}\frac{(c_d/a_j)_\infty}{(qa_k/a_j)_\infty}\cdot\frac{(c_d/(b_{L+1}\cdots b_M))_\infty}{(qa_k/(b_{L+1}\cdots b_M))_\infty}\prod_{\substack{1\le j\le N\\ j\ne d}}\frac{(qa_k/c_j)_\infty}{(c_d/c_j)_\infty}\cdot\frac{(a_k/(b_{L+2}\cdots b_M))_\infty}{(c_d/(qb_{L+2}\cdots b_M))_\infty}$$
$$\times\frac{\theta(x_{L+1}b_{L+1}a_1\cdots a_N c_d/(qc_1\cdots c_N a_k))}{\theta(x_{L+1}b_{L+1}a_1\cdots a_N/(c_1\cdots c_N))}\mathcal{G}^{L+1;d,L+1}_{N,M}\left(\begin{matrix}\{a_j\}_{1\le j\le N},\{b_i\}_{1\le i\le M}\\ \{c_j\}_{1\le j\le N}\end{matrix};\{x_i\}_{1\le i\le M}\right)\Bigg\}. \tag{4.41}$$

Therefore, we set

$$A^{L,\mathrm{id}}=A^{L,\mathrm{id}}\left(\begin{matrix}\{a_j\}_{1\le j\le N},\{b_i\}_{1\le i\le M}\\ \{c_j\}_{1\le j\le N}\end{matrix};x_{L+1}\right)=\left(\begin{array}{c|ccc}A_{0,0}&A_{0,1}&\cdots&A_{0,N}\\ \hline A_{1,0}&A_{1,1}&\cdots&A_{1,N}\\ \vdots&\vdots&&\vdots\\ A_{N,0}&A_{N,1}&\cdots&A_{N,N}\end{array}\right), \tag{4.42}$$

where

$$A_{0,0}=\prod_{j=1}^N\frac{(qb_{L+2}\cdots b_M/a_j,qb_{L+1}\cdots b_M/c_j)_\infty}{(qb_{L+1}\cdots b_M/a_j,qb_{L+2}\cdots b_M/c_j)_\infty}\cdot\frac{\theta(x_{L+1}a_1\cdots a_N/(c_1\cdots c_N))}{\theta(x_{L+1}b_{L+1}a_1\cdots a_N/(c_1\cdots c_N))}x_{L+1}^{-\beta_{L+1}}, \tag{4.43}$$

and $A_{0,d}$, $A_{k,0}$ $(1\le d,k\le N)$ are M components vectors given by

$$A_{0,d}=(A_{0,(1,d)},A_{0,(2,d)},\dots,A_{0,(M,d)})=(0,\dots,0,A_{0,(L+1,d)},0,\dots,0), \tag{4.44}$$
$$A_{0,(L+1,d)}=\prod_{j=1}^N\frac{(c_d/a_j)_\infty}{(qb_{L+1}\cdots b_M/a_j)_\infty}\prod_{\substack{1\le j\le N\\ j\ne d}}\frac{(qb_{L+1}\cdots b_M/c_j)_\infty}{(c_d/c_j)_\infty}\cdot\frac{(b_{L+1})_\infty}{(c_d/(qb_{L+2}\cdots b_M))_\infty}$$
$$\times\frac{\theta(x_{L+1}a_1\cdots a_N c_d/(qb_{L+2}\cdots b_M c_1\cdots c_N))}{\theta(x_{L+1}b_{L+1}a_1\cdots a_N/(c_1\cdots c_N))}x_{L+1}^{-1-\sum_{i=L+1}^M\beta_i+\gamma_d}, \tag{4.45}$$
$$A_{k,0}=(A_{(1,k),0},A_{(2,k),0},\dots,A_{(N,k),0})^{\mathrm{T}}=(0,\dots,0,A_{(L+1,k),0},0,\dots,0)^{\mathrm{T}}, \tag{4.46}$$
$$A_{(L+1,k),0}=\prod_{\substack{1\le j\le N\\ j\ne k}}\frac{(qb_{L+2}\cdots b_M/a_j)_\infty}{(qa_k/a_j)_\infty}\cdot\frac{(q/b_{L+1})_\infty}{(qa_k/(b_{L+1}\cdots b_M))_\infty}\prod_{j=1}^N\frac{(qa_k/c_j)_\infty}{(qb_{L+2}\cdots b_M/c_j)_\infty}$$
$$\times\frac{\theta(x_{L+1}b_{L+1}\cdots b_M a_1\cdots a_N/(a_k c_1\cdots c_N))}{\theta(x_{L+1}b_{L+1}a_1\cdots a_N/(c_1\cdots c_N))}x_{L+1}^{-\alpha_k+\sum_{i=L+2}^M\beta_i}. \tag{4.47}$$

The other blocks $A_{k,d}$ $(1 \leq d, k \leq N)$ are $M \times M$ matrices given by

$$A_{k,d} = \begin{pmatrix} I_L & O & O \\ O & A_{(L+1,k),(L+1,d)} & O \\ O & O & I_{M-L-1} \end{pmatrix}, \tag{4.48}$$

where I_n is the unit matrix of degree n, O is the null matrix and

$$A_{(L+1,k),(L+1,d)} = \prod_{\substack{1\leq j\leq N \\ j\neq k}} \frac{(c_d/a_j)_\infty}{(qa_k/a_j)_\infty} \cdot \frac{(c_d/(b_{L+1}\cdots b_M))_\infty}{(qa_k/(b_{L+1}\cdots b_M))_\infty} \prod_{\substack{1\leq j\leq N \\ j\neq d}} \frac{(qa_k/c_j)_\infty}{(c_d/c_j)_\infty} \cdot \frac{(a_k/(b_{L+2}\cdots b_M))_\infty}{(c_d/(qb_{L+2}\cdots b_M))_\infty}$$

$$\times \frac{\theta(x_{L+1}b_{L+1}a_1\cdots a_N c_d/(qc_1\cdots c_N a_k))}{\theta(x_{L+1}b_{L+1}a_1\cdots a_N/(c_1\cdots c_N))} x_{L+1}^{-1-\alpha_k+\gamma_d}. \tag{4.49}$$

Then we have

$$\boldsymbol{u}^{L,\mathrm{id}} = A^{L,\mathrm{id}}\boldsymbol{u}^{L+1,\mathrm{id}}. \tag{4.50}$$

In addition, we put

$$A^{L,\sigma} = A^{L,\mathrm{id}}\left(\begin{matrix} \{a_j\}_{1\leq j\leq N}, \{b_{\sigma(i)}\}_{1\leq i\leq M} \\ \{c_j\}_{1\leq j\leq N} \end{matrix} ; x_{\sigma(L+1)} \right), \tag{4.51}$$

then we have

$$\boldsymbol{u}^{L,\sigma} = A^{L,\sigma}\boldsymbol{u}^{L+1,\sigma}, \tag{4.52}$$

for $\sigma \in \mathfrak{S}_M$ since $\boldsymbol{u}^{L,\sigma} = \boldsymbol{u}^{L,\mathrm{id}}\left(\begin{matrix} \{a_j\}, \{b_{\sigma(i)}\} \\ \{c_j\} \end{matrix} ; \{x_{\sigma(i)}\} \right)$.

Computation of case (2):

The matrix of the case (2) can be calculated by a similar method in the case (1). So we only show the result. We set

$$B^{L,\mathrm{id}} = B^{L,\mathrm{id}}\left(\begin{matrix} \{a_j\}_{1\leq j\leq N}, \{b_i\}_{1\leq i\leq M} \\ \{c_j\}_{1\leq j\leq N} \end{matrix} ; x_L \right) = \left(\begin{array}{c|cccc} B_{0,0} & B_{0,1} & \cdots & B_{0,N} \\ \hline B_{1,0} & B_{1,1} & \cdots & B_{1,N} \\ \vdots & \vdots & & \vdots \\ B_{N,0} & B_{N,1} & \cdots & B_{N,N} \end{array} \right), \tag{4.53}$$

where

$$B_{0,0} = \prod_{j=1}^{N} \frac{(a_j/(b_{L+1}\cdots b_M), c_j/(b_L\cdots b_M))_\infty}{(a_j/(b_L\cdots b_M), c_j/(b_{L+1}\cdots b_M))_\infty} \cdot \frac{\theta(x_L b_L)}{\theta(x_L)} x_L^{\beta_L}, \tag{4.54}$$

and $B_{0,d}$, $B_{k,0}$ $(1 \leq d, k \leq N)$ are M components vectors given by

$$B_{0,d} = (B_{0,(1,d)}, B_{0,(2,d)}, \ldots, B_{0,(M,d)}) = (0, \ldots, 0, B_{0,(L,d)}, 0, \ldots, 0), \tag{4.55}$$

$$B_{0,(L,d)} = \prod_{j=1}^{N} \frac{(c_j/a_d)_\infty}{(c_j/(b_{L+1}\cdots b_M))_\infty} \prod_{\substack{1\leq j\leq N \\ j\neq d}} \frac{(a_j/(b_{L+1}\cdots b_M))_\infty}{(a_j/a_d)_\infty} \cdot \frac{(b_L)_\infty}{(b_L\cdots b_M/a_d)_\infty} \tag{4.56}$$

$$\times \frac{\theta(x_L a_d/(b_{L+1}\cdots b_M))}{\theta(x_L)} x_L^{\alpha_d - \sum_{i=L+1}^{M}\beta_i}, \tag{4.57}$$

$$B_{k,0} = (B_{(1,k),0}, B_{(2,k),0}, \ldots, B_{(M,k),0})^{\mathrm{T}} = (0, \ldots, 0, B_{(L,k),0}, 0, \ldots, 0)^{\mathrm{T}}, \tag{4.58}$$

$$B_{(L,k),0} = \prod_{\substack{1\leq j\leq N \\ j\neq k}} \frac{(c_j/(b_L\cdots b_M))_\infty}{(qc_j/c_k)_\infty} \cdot \frac{(q/b_L)_\infty}{(q^2 b_{L+1}\cdots b_M/c_k)_\infty} \prod_{j=1}^{N} \frac{(qa_j/c_k)_\infty}{(a_j/(b_L\cdots b_M))_\infty}$$

$$\times \frac{\theta(x_L q b_L \cdots b_M/c_k)}{\theta(x_L)} x_L^{1+\sum_{i=L}^{M}\beta_i-\gamma_k}. \tag{4.59}$$

And the other blocks $B_{k,d}$ $(1 \le d, k \le N)$ are $M \times M$ matrices given by

$$B_{k,d} = \begin{pmatrix} I_{L-1} & O & O \\ O & B_{(L,k),(L,d)} & O \\ O & O & I_{M-L} \end{pmatrix}, \tag{4.60}$$

$$B_{(L,k),(L,d)} = \prod_{\substack{1\le j\le N \\ j\neq k}} \frac{(c_j/a_d)_\infty}{(qc_j/c_k)_\infty} \cdot \frac{(qb_{L+1}\cdots b_M/a_d)_\infty}{(q^2 b_{L+1}\cdots b_M/c_k)_\infty} \prod_{\substack{1\le j\le N \\ j\neq d}} \frac{(qa_j/c_k)_\infty}{(a_j/a_d)_\infty} \cdot \frac{(qb_L\cdots b_M/c_k)_\infty}{(b_L\cdots b_M/a_d)_\infty}$$
$$\times \frac{\theta(x_L q a_d/c_k)}{\theta(x_L)} x_L^{1+\alpha_d-\gamma_k}. \tag{4.61}$$

Then we have

$$\boldsymbol{u}^{L,\mathrm{id}} = B^{L,\mathrm{id}} \boldsymbol{u}^{L-1,\mathrm{id}}. \tag{4.62}$$

Similar to the argument from (4.50) to (4.52), we have

$$\boldsymbol{u}^{L,\sigma} = B^{L,\sigma} \boldsymbol{u}^{L-1,\sigma}, \tag{4.63}$$

where $B^{L,\sigma}$ is defined in the same as (4.51) for $\sigma \in \mathfrak{S}_M$.

Computation of case (3)

We first consider the matrix which connects $\boldsymbol{u}^{M,s_r}$ with $\boldsymbol{u}^{M,\mathrm{id}}$ like the computation of the case (1). We have

$$u_0^{M,s_r} = u_0^{M,\mathrm{id}}, \tag{4.64}$$
$$u_{k,l}^{M,s_r} = u_{k,l}^{M,\mathrm{id}}, \tag{4.65}$$

easily if $l \neq r, r+1$. So we should calculate the connection formula of $u_{k,r}^{M,s_r}$, $u_{k,r+1}^{M,s_r}$. We have

$$\mathcal{G}_{N,M}^{M;k,r}\left(\begin{matrix} \{a_j\}_{1\le j\le N}, \{b_{s_r(i)}\}_{1\le i\le M} \\ \{c_j\}_{1\le j\le N} \end{matrix} ; \{x_{s_r(i)}\}_{1\le i\le M} \right)$$
$$= \sum_{m_1,\ldots,m_M\ge 0} \left\{ \prod_{k=1}^{N} \frac{(qa_k/c_j)_{m_M}}{(qc_k/c_j)_{m_M}} \prod_{i=1}^{r-1} \frac{(b_i)_{m_i}}{(q)_{m_i}} \prod_{i=r+2}^{M} \frac{(b_i)_{m_{i-1}}}{(q)_{m_{i-1}}} \cdot \frac{(b_r)_{m_r}}{(q)_{m_r}} \frac{(c_k/(qb_r b_{r+2}\cdots b_M))_{m(r-1)}}{(c_k/(b_r\cdots b_M))_{m(r-1)}} \right.$$
$$\left. \times \prod_{i=1}^{r-1} \left(\frac{qx_i}{b_{r+1}x_{r+1}}\right)^{m_i} \prod_{i=r+2}^{M} \left(\frac{b_{r+1}x_{r+1}}{b_i x_i}\right)^{m_{i-1}} \cdot \left(\frac{b_{r+1}x_{r+1}}{b_r x_r}\right)^{m_r} \left(\frac{b_{r+1}x_{r+1}}{q}\right)^{m_M} \right\}$$
$$= \sum_{m_1,\ldots,m_{r-1},m_{r+1},\ldots,m_M\ge 0} \left\{ \prod_{k=1}^{N} \frac{(qa_k/c_j)_{m_M}}{(qc_k/c_j)_{m_M}} \prod_{i=1}^{r-1} \frac{(b_i)_{m_i}}{(q)_{m_i}} \prod_{i=r+2}^{M} \frac{(b_i)_{m_{i-1}}}{(q)_{m_{i-1}}} \cdot \frac{(c_k/(qb_r b_{r+2}\cdots b_M))_{m(r-1)'}}{(c_k/(b_r\cdots b_M))_{m(r-1)'}} \right.$$
$$\times \prod_{i=1}^{r-1} \left(\frac{qx_i}{b_{r+1}x_{r+1}}\right)^{m_i} \prod_{i=r+2}^{M} \left(\frac{b_{r+1}x_{r+1}}{b_i x_i}\right)^{m_{i-1}} \cdot \left(\frac{b_{r+1}x_{r+1}}{q}\right)^{m_M}$$
$$\left. \times {}_2\varphi_1\left(\begin{matrix} b_r, qb_r\cdots b_M/(c_k q^{m(r-1)'}) \\ q^2 b_r b_{r+2}\cdots b_M/(c_k q^{m(r-1)'}) \end{matrix} ; \frac{qx_{r+1}}{b_r x_r} \right) \right\}, \tag{4.66}$$

and by applying the formula (4.36) to ${}_2\varphi_1$, we obtain

$$\mathcal{G}_{N,M}^{M;k,r}\left(\begin{matrix} \{a_j\}_{1\le j\le N}, \{b_{s_r(i)}\}_{1\le i\le M} \\ \{c_j\}_{1\le j\le N} \end{matrix} ; \{x_{s_r(i)}\}_{1\le i\le M} \right)$$

$$
\begin{aligned}
&= \frac{(q/b_{r+1}, b_r)_\infty \theta(x_r c_k/(x_{r+1} q b_{r+1} \cdots b_M))}{(q^2 b_r b_{r+2} \cdots b_M/c_k, c_k/(q b_{r+1} \cdots b_M))_\infty \theta(x_r b_r/x_{r+1})} \mathcal{G}_{N,M}^{M;k,r}\left(\begin{matrix} \{a_j\}_{1\le j\le N}, \{b_i\}_{1\le i\le M} \\ \{c_j\}_{1\le j\le N} \end{matrix}; \{x_i\}_{1\le i\le M}\right) \\
&+ \frac{(q^2 b_{r+2} \cdots b_M/c_k, q b_r \cdots b_M/c_k)_\infty}{(q^2 b_r b_{r+2} \cdots b_M/c_k, q_{r+1} \cdots b_M/c_k)_\infty} \frac{\theta(x_r/x_{r+1})}{\theta(x_r b_r/x_{r+1})} \mathcal{G}_{N,M}^{M;j,r+1}\left(\begin{matrix} \{a_j\}_{1\le j\le N}, \{b_i\}_{1\le i\le M} \\ \{c_j\}_{1\le j\le N} \end{matrix}; \{x_i\}_{1\le i\le M}\right).
\end{aligned} \tag{4.67}
$$

Similarly, we have

$$
\begin{aligned}
&\mathcal{G}_{N,M}^{M;k,r+1}\left(\begin{matrix} \{a_j\}_{1\le j\le N}, \{b_{s_r(i)}\}_{1\le i\le M} \\ \{c_j\}_{1\le j\le N} \end{matrix}; \{x_{s_r(i)}\}_{1\le i\le M}\right) \\
&= \frac{(c_k/(b_r \cdots b_M), c_k/(q b_{r+2} \cdots b_M))_\infty}{(c_k/(b_r b_{r+2} \cdots b_M), c_k/(q b_{r+1} \cdots b_M))_\infty} \frac{\theta(x_r b_r/(x_{r+1} b_{r+1}))}{\theta(x_r b_r/x_{r+1})} \\
&\times \mathcal{G}_{N,M}^{M;k,r}\left(\begin{matrix} \{a_j\}_{1\le j\le N}, \{b_i\}_{1\le i\le M} \\ \{c_j\}_{1\le j\le N} \end{matrix}; \{x_i\}_{1\le i\le M}\right) \\
&+ \frac{(q/b_r, b_{r+1})_\infty}{(c_k/(b_r b_{r+2} \cdots b_M), q b_{r+1} \cdots b_M/c_k)_\infty} \frac{\theta(x_r q b_r b_{r+2} \cdots b_M/(x_{r+1} c_k))}{\theta(x_r b_r/x_{r+1})} \\
&\times \mathcal{G}_{N,M}^{M;k,r+1}\left(\begin{matrix} \{a_j\}_{1\le j\le N}, \{b_i\}_{1\le i\le M} \\ \{c_j\}_{1\le j\le N} \end{matrix}; \{x_i\}_{1\le i\le M}\right).
\end{aligned} \tag{4.68}
$$

Therefore, we set

$$
S_{s_r}^{M,\mathrm{id}} = S_{s_r}^{M,\mathrm{id}}\left(\begin{matrix} \{b_i\}_{1\le i\le M} \\ \{c_j\}_{1\le j\le N} \end{matrix}; \frac{x_r}{x_{r+1}}\right) = \left(\begin{array}{c:ccccc} 1 & O & \cdots & \cdots & O \\ \hdashline O & S_r^1 & O & \cdots & O \\ \vdots & O & S_r^2 & & \vdots \\ \vdots & \vdots & & \ddots & O \\ O & O & \cdots & O & S_r^N \end{array}\right). \tag{4.69}
$$

The blocks S_r^k $(1 \le k \le N)$ are given by

$$
S_r^k = \begin{pmatrix} I_{r-1} & O & O & O \\ O & S_{r,r}^k & S_{r,r+1}^k & O \\ O & S_{r+1,r}^k & S_{r+1,r+1}^k & O \\ O & O & O & I_{M-r-1} \end{pmatrix}, \tag{4.70}
$$

where $S_{r,r}^k$, $S_{r,r+1}^k$, $S_{r+1,r}^k$, $S_{r+1,r+1}^k$ are scalars given by

$$
S_{r,r}^k = \frac{(q/b_{r+1}, b_r)_\infty}{(q^2 b_r b_{r+2} \cdots b_M/c_k, c_k/(q b_{r+1} \cdots b_M))_\infty} \frac{\theta(x_r c_k/(x_{r+1} q b_{r+1} \cdots b_M))}{\theta(x_r b_r/x_{r+1})} \left(\frac{x_r}{x_{r+1}}\right)^{-1-\sum_{i=r}^M \beta_i + \gamma_k}, \tag{4.71}
$$

$$
S_{r,r+1}^k = \frac{(q^2 b_{r+2} \cdots b_M/c_k, q b_r \cdots b_M/c_k)_\infty}{(q^2 b_r b_{r+2} \cdots b_M/c_k, q b_{r+1} \cdots b_M/c_k)_\infty} \frac{\theta(x_r/x_{r+1})}{\theta(x_r b_r/x_{r+1})} \left(\frac{x_r}{x_{r+1}}\right)^{-\beta_r}, \tag{4.72}
$$

$$
S_{r+1,r}^k = \frac{(c_k/(b_r \cdots b_M), c_k/(q b_{r+2} \cdots b_M))_\infty}{(c_k/(b_r b_{r+2} \cdots b_M), c_k/(q b_{r+1} \cdots b_M))_\infty} \frac{\theta(x_r b_r/(x_{r+1} b_{r+1}))}{\theta(x_r b_r/x_{r+1})} \left(\frac{x_r}{x_{r+1}}\right)^{-\beta_{r+1}}, \tag{4.73}
$$

$$
S_{r+1,r+1}^k = \frac{(q/b_r, b_{r+1})_\infty}{(c_k/(b_r b_{r+2} \cdots b_M), q b_{r+1} \cdots b_M/c_k)_\infty} \frac{\theta(x_r q b_r b_{r+2} \cdots b_M/(x_{r+1} c_k))}{\theta(x_r b_r/x_{r+1})} \left(\frac{x_r}{x_{r+1}}\right)^{1+\sum_{i=r+2}^M \beta_i - \gamma_k} \tag{4.74}
$$

Then we have

$$
\boldsymbol{u}^{M,s_r} = S_{s_r}^{M,\mathrm{id}} \boldsymbol{u}^{M,\mathrm{id}}. \tag{4.75}
$$

In a similar reason of cases (1), (2), we have

$$\boldsymbol{u}^{M,s_r\sigma} = S_{s_r}^{M,\sigma}\boldsymbol{u}^{M,\sigma}, \tag{4.76}$$

$$S_{s_r}^{M,\sigma} = S_{s_r}^{M,\mathrm{id}}\left(\begin{array}{c}\{b_{\sigma(i)}\}_{1\le i\le M}\\ \{c_j\}_{1\le j\le N}\end{array};\frac{x_{\sigma(r)}}{x_{\sigma(r+1)}}\right) \tag{4.77}$$

for $\sigma\in\mathfrak{S}_M$. From computations of cases (1), (2), (3), we finally obtain the following theorem:

Theorem 4.9 ([45]). *For $0\le L_1,L_2\le M$ and $\sigma_1,\sigma_2\in\mathfrak{S}_M$, we have*

$$\boldsymbol{u}^{L_2,\sigma_2} = A^{L_2,\sigma_2}A^{L_2+1,\sigma_2}\cdots A^{M-1,\sigma_2}S_{s_{r_1}}^{M,s_{r_2}\cdots s_{r_I}\sigma_1}S_{s_{r_2}}^{M,s_{r_3}\cdots s_{r_I}\sigma_1}\cdots S_{s_{r_I}}^{M,\sigma_1}B^{M,\sigma_1}B^{M-1,\sigma_1}\cdots B^{L_1+1,\sigma_1}\boldsymbol{u}^{L_1,\sigma_1}, \tag{4.78}$$

if $\sigma_2 = s_{r_1}\cdots s_{r_I}\sigma_1$, where $s_r=(r,r+1)\in\mathfrak{S}_M$.

Remark 4.10. By the braid relation $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$, the matrix S satisfies the Yang-Baxter equation $S_{s_i}^{M,s_{i+1}s_i}S_{s_{i+1}}^{M,s_i}S_{s_i}^{M,\mathrm{id}} = S_{s_{i+1}}^{M,s_is_{i+1}}S_{s_i}^{M,s_{i+1}}S_{s_{i+1}}^{M,\mathrm{id}}$. For the way to get a solution for the Yang-Baxter equation by some connection problem of a q-difference system, see [6, 45].

Remark 4.11. The connection problem for the q-Appell-Lauricella system (4.10), (4.11) is solved by putting $N=1$ in Theorem 4.9. We summarize the case here. We put

$$u_0^L = \prod_{i=L+1}^{M}x_i^{-\beta_i}\cdot\varphi_{D,L}\left(\begin{array}{c}a/(b_{L+1}\cdots b_M);\{b_i\}_{1\le i\le M}\\ c/(b_{L+1}\cdots b_M)\end{array};\left\{x_1,\ldots,x_L,\frac{q}{b_{L+1}x_{L+1}},\ldots,\frac{q}{b_Mx_M}\right\}\right), \tag{4.79}$$

$$\begin{aligned}u_{l'}^L &= x_l^{1+\beta_{l+1}+\cdots+\beta_M-\gamma}\prod_{i=l+1}^{M}x_i^{-\beta_i}\\ &\times\varphi_{D,l'-1}\left(\begin{array}{c}c/(b_{l'+1}\cdots b_Mq);\{b_1,\ldots,b_{l'-1},b_{l'+1},\ldots,b_L,aq/c,b_{L+1},\ldots,b_M\}\\ c/(b_{l'}\cdots b_M)\end{array};\right.\\ &\qquad\left.\left\{\frac{qx_1}{b_{l'}x_{l'}},\ldots,\frac{qx_{l'-1}}{b_{l'}x_{l'}},\frac{b_{l'}x_{l'}}{b_{l'+1}x_{l'+1}},\ldots,\frac{b_{l'}x_{l'}}{b_Lx_L},\frac{b_{l'}x_{l'}}{q},\frac{b_{l'}x_{l'}}{b_{L+1}x_{L+1}},\ldots\frac{b_{l'}x_{l'}}{b_Mx_M}\right\}\right)\quad(1\le l'\le L),\end{aligned} \tag{4.80}$$

$$\begin{aligned}u_l^L &= x_l^{-\alpha+\beta_{l+1}+\cdots+\beta_M}\prod_{i=l+1}^{M}x_i^{-\beta_i}\\ &\times\varphi_{D,l}\left(\begin{array}{c}a/(b_{l+1}\cdots b_M);\{b_1,\ldots,b_L,aq/c,b_{L+1},\ldots,b_{l-1},b_{l+1},\ldots,b_M\}\\ aq/(b_l\cdots b_M)\end{array};\right.\\ &\qquad\left.\left\{\frac{qx_1}{b_lx_l},\ldots,\frac{qx_L}{b_lx_l},\frac{cq}{ab_lx_l},\frac{qx_{L+1}}{b_lx_l},\ldots,\frac{qx_{l-1}}{b_lx_l},\frac{b_lx_l}{b_{l+1}x_{l+1}},\ldots,\frac{b_lx_l}{b_Mx_M}\right\}\right)\quad(L+1\le l\le M),\end{aligned} \tag{4.81}$$

where $a=q^\alpha$, $b_i=q^{\beta_i}$, $c=q^\gamma$. And we put

$$\boldsymbol{u}^L = \boldsymbol{u}^L\left(\begin{array}{c}a;\{b_i\}_{1\le i\le M}\\ c\end{array};\{x_i\}_{1\le i\le M}\right) = (u_0^L,u_1^L,\ldots,u_M^L)^{\mathrm{T}}, \tag{4.82}$$

$$\boldsymbol{u}^{L,\sigma} = \boldsymbol{u}^L\left(\begin{array}{c}a;\{b_{\sigma(i)}\}_{1\le i\le M}\\ c\end{array};\{x_{\sigma(i)}\}_{1\le i\le M}\right). \tag{4.83}$$

Then $\boldsymbol{u}^{L,\sigma}$ is a fundamental solution for the q-Appell-Lauricella system in

$$D^{L,\sigma} = \left\{ |x_{\sigma(i)}| < 1 \ (1 \le i \le L), \ \left| \frac{cq}{ab_{\sigma(i)} x_{\sigma(i)}} \right| < 1 \ (L+1 \le i \le M), \right. \\ \left. \left| \frac{q x_{\sigma(i)}}{b_{\sigma(j)} x_{\sigma(j)}} \right| < 1 \ (1 \le i < j \le M) \right\}. \tag{4.84}$$

We set

$$A^L = A^L \left(\begin{matrix} a; \{b_i\}_{1\le i\le M} \\ c \end{matrix} ; x_{L+1} \right) = (A_{i,j})_{0\le i,j\le M} = \begin{pmatrix} A_{0,0} & O & A_{0,L+1} & O \\ O & I_L & O & O \\ A_{L+1,0} & O & A_{L+1,L+1} & O \\ O & O & O & I_{M-L-1} \end{pmatrix}, \tag{4.85}$$

where $A_{0,0}$, $A_{0,L+1}$, $A_{L+1,0}$, $A_{L+1,L+1}$ are scalars given by

$$A_{0,0} = \frac{(qb_{L+2}\cdots b_M/a, qb_{L+1}\cdots b_M/c)_\infty}{(qb_{L+1}\cdots b_M/a, qb_{L+2}\cdots b_M/c)_\infty} \cdot \frac{\theta(x_{L+1}a/c)}{\theta(x_{L+1}b_{L+1}a/c)} x_{L+1}^{-\beta_{L+1}}, \tag{4.86}$$

$$A_{0,L+1} = \frac{(c/a)_\infty}{(qb_{L+1}\cdots b_M/a)_\infty} \frac{(b_{L+1})_\infty}{(c/(qb_{L+2}\cdots b_M))_\infty} \frac{\theta(x_{L+1}a/(qb_{L+2}\cdots b_M))}{\theta(x_{L+1}b_{L+1}a/c)} x_{L+1}^{-1-\sum_{i=L+1}^M \beta_i+\gamma}, \tag{4.87}$$

$$A_{L+1,0} = \frac{(q/b_{L+1}, qa/c)_\infty}{(qa/(b_{L+1}\cdots b_M), qb_{L+2}\cdots b_M/c)_\infty} \frac{\theta(x_{L+1}b_{L+1}\cdots b_M c)}{\theta(x_{L+1}b_{L+1}a/c)} x_{L+1}^{-\alpha+\sum_{i=L+2}^M \beta_i}, \tag{4.88}$$

$$A_{L+1,L+1} = \frac{(c/(b_{L+1}\cdots b_M))_\infty}{(qa/(b_{L+1}\cdots b_M))_\infty} \frac{(a/(b_{L+2}\cdots b_M))_\infty}{(c/(qb_{L+2}\cdots b_M))_\infty} \frac{\theta(x_{L+1}b_{L+1}/q)}{\theta(x_{L+1}b_{L+1}a/c)} x_{L+1}^{-1-\alpha+\gamma}. \tag{4.89}$$

We also set

$$B^L = B^L \left(\begin{matrix} a; \{b_i\}_{1\le i\le M} \\ c \end{matrix} ; x_L \right) = (B_{i,j})_{0\le i,j\le M} = \begin{pmatrix} B_{0,0} & O & B_{0,L} & O \\ O & I_{L-1} & O & O \\ B_{L,0} & O & B_{L,L} & O \\ O & O & O & I_{M-L} \end{pmatrix}, \tag{4.90}$$

where $B_{0,0}$, $B_{0,L}$, $B_{L,0}$, $B_{L,L}$ are scalars given by

$$B_{0,0} = \frac{(a/(b_{L+1}\cdots b_M), c/(b_L\cdots b_M))_\infty}{(a/(b_L\cdots b_M), c/(b_{L+1}\cdots b_M))_\infty} \cdot \frac{\theta(x_L b_L)}{\theta(x_L)} x_L^{\beta_L}, \tag{4.91}$$

$$B_{0,L} = \frac{(c/a)_\infty}{(c/(b_{L+1}\cdots b_M))_\infty} \frac{(b_L)_\infty}{(b_L\cdots b_M/a)_\infty} \frac{\theta(x_L a/(b_{L+1}\cdots b_M))}{\theta(x_L)} x_L^{\alpha-\sum_{i=L+1}^M \beta_i}, \tag{4.92}$$

$$B_{L,0} = \frac{(q/b_L)_\infty}{(q^2 b_{L+1}\cdots b_M/c_k)_\infty} \frac{(qa/c)_\infty}{(a/(b_L\cdots b_M))_\infty} \frac{\theta(x_L q b_L \cdots b_M/c)}{\theta(x_L)} x_L^{1+\sum_{i=L}^M \beta_i-\gamma}, \tag{4.93}$$

$$B_{L,L} = \frac{(qb_{L+1}\cdots b_M/a)_\infty}{(q^2 b_{L+1}\cdots b_M/c)_\infty} \frac{(qb_L\cdots b_M/c)_\infty}{(b_L\cdots b_M/a)_\infty} \frac{\theta(x_L qa/c)}{\theta(x_L)} x_L^{1+\alpha-\gamma}. \tag{4.94}$$

Finally we set

$$S_{s_r} = S_{s_r} \left(\begin{matrix} \{b_i\}_{1\le i\le M} \\ c \end{matrix} ; \frac{x_r}{x_{r+1}} \right) = \begin{pmatrix} I_r & O & O & O \\ O & S_{r,r} & S_{r,r+1} & O \\ O & S_{r+1,r} & S_{r+1,r+1} & O \\ O & O & O & I_{M-r-1} \end{pmatrix}, \tag{4.95}$$

where $S_{r,r}$, $S_{r,r+1}$, $S_{r+1,r}$, $S_{r+1,r+1}$ are scalars given by

$$S_{r,r} = \frac{(q/b_{r+1}, b_r)_\infty}{(q^2 b_r b_{r+2} \cdots b_M/c, c/(qb_{r+1} \cdots b_M))_\infty} \frac{\theta(x_r c/(x_{r+1} q b_{r+1} \cdots b_M))}{\theta(x_r b_r/x_{r+1})} \left(\frac{x_r}{x_{r+1}}\right)^{-1-\sum_{i=r}^M \beta_i + \gamma}, \tag{4.96}$$

$$S_{r,r+1} = \frac{(q^2 b_{r+2} \cdots b_M/c, qb_r \cdots b_M/c)_\infty}{(q^2 b_r b_{r+2} \cdots b_M/c, qb_{r+1} \cdots b_M/c)_\infty} \frac{\theta(x_r/x_{r+1})}{\theta(x_r b_r/x_{r+1})} \left(\frac{x_r}{x_{r+1}}\right)^{-\beta_r}, \tag{4.97}$$

$$S_{r+1,r} = \frac{(c/(b_r \cdots b_M), c/(qb_{r+2} \cdots b_M))_\infty}{(c/(b_r b_{r+2} \cdots b_M), c/(qb_{r+1} \cdots b_M))_\infty} \frac{\theta(x_r b_r/(x_{r+1} b_{r+1}))}{\theta(x_r b_r/x_{r+1})} \left(\frac{x_r}{x_{r+1}}\right)^{-\beta_{r+1}}, \tag{4.98}$$

$$S_{r+1,r+1} = \frac{(q/b_r, b_{r+1})_\infty}{(c/(b_r b_{r+2} \cdots b_M), qb_{r+1} \cdots b_M/c)_\infty} \frac{\theta(x_r q b_r b_{r+2} \cdots b_M/(x_{r+1} c))}{\theta(x_r b_r/x_{r+1})} \left(\frac{x_r}{x_{r+1}}\right)^{1+\sum_{i=r+2}^M \beta_i - \gamma}. \tag{4.99}$$

Using the above matrices A^L, B^L, S_{s_r}, we put

$$A^{L,\sigma} = A^L \left(\begin{matrix} a; \{b_{\sigma(i)}\}_{1 \le i \le M} \\ c \end{matrix} ; x_{\sigma(L+1)} \right), \tag{4.100}$$

$$B^{L,\sigma} = B^L \left(\begin{matrix} a; \{b_{\sigma(i)}\}_{1 \le i \le M} \\ c \end{matrix} ; x_{\sigma(L)} \right), \tag{4.101}$$

$$S^{\sigma}_{s_r} = S_{s_r} \left(\begin{matrix} \{b_{\sigma(i)}\}_{1 \le i \le M} \\ c \end{matrix} ; \frac{x_{\sigma(r)}}{x_{\sigma(r+1)}} \right). \tag{4.102}$$

Then we have

$$\boldsymbol{u}^{L_2,\sigma_2} = A^{L_2,\sigma_2} A^{L_2+1,\sigma_2} \cdots A^{M-1,\sigma_2} S^{M, s_{r_2} \cdots s_{r_I} \sigma_1}_{s_{r_1}} S^{M, s_{r_3} \cdots s_{r_I} \sigma_1}_{s_{r_2}} \cdots S^{M,\sigma_1}_{s_{r_I}} B^{M,\sigma_1} B^{M-1,\sigma_1} \cdots B^{L_1+1,\sigma_1} \boldsymbol{u}^{L_1,\sigma_1}, \tag{4.103}$$

if $\sigma_2 = s_{r_1} \cdots s_{r_I} \sigma_1$ and $s_r = (r, r+1) \in \mathfrak{S}_M$.

5 A multivariable extension of the q-Riemann-Papperitz type integral

In this section, we consider the following Jackson integral:

$$\varphi_{i,j}^M = \int_{q/a_i}^{q/a_j} \prod_{k=1}^{M+3} \frac{(a_k t)_\infty}{(b_k t)_\infty} d_q t, \tag{5.1}$$

where

$$a_1 \cdots a_{M+3} = q^2 b_1 \cdots b_{M+3}. \tag{5.2}$$

This integral is a multivariable extension of the Jackson integral of Riemann-Papperitz type (2.55). First, we give a finite rank q-difference system associated with (5.1). Second, we show a transformation formula between the integral (5.1) and the Kajihara's q-hypergeometric series [29]. Using this formula, we obtain series solutions for the above system. Finally we discuss degenerations of the system and its solutions. This section is based on [48].

5.1 A system the integral satisfies

In this subsection, we derive a q-difference system that the integral $\varphi_{i,j}^M$ satisfies. As mentioned in section 2.2, the integral $\varphi_{i,j}^M$ with $a_1 = Ax$, $b_1 = Bx$, satisfies the equation

$$E_{M+2}(A, B, \{a_k\}_{2\le k\le M+3}, \{b_k\}_{2\le k\le M+3})\varphi_{i,j}^M = 0. \tag{5.3}$$

Rewriting in a_1, b_1, we have the following proposition.

Proposition 5.1. *The integral $\varphi_{i,j}^M$ satisfies the rank $M+1$ equation $\hat{E}_M \varphi_{i,j}^M = 0$. The operator $\hat{E}_M$ is given by*

$$\begin{aligned}
\hat{E}_M &= b_1^{M+2} T^{-1} \prod_{n=0}^{M} (1 - (a_1 q^n/b_1)T) \\
&\quad + \sum_{k=1}^{M+1} (-1)^k b_1^{M+2-k} [e_k(\hat{a})T^{-1} - q e_k(\hat{b})] \prod_{n=0}^{M-k} (1 - (a_1 q^n/b_1)T) \prod_{n=0}^{k-2} (1 - q^{-n}T) \\
&\quad + (-1)^{M+2} a_2 \cdots a_{M+3} T^{-1} \prod_{n=0}^{M} (1 - q^{-n}T),
\end{aligned} \tag{5.4}$$

where $T = T_{a_1}T_{b_1}$ and $\hat{a} = (a_2, \ldots, a_{M+3})$, $\hat{b} = (b_2, \ldots, b_{M+3})$.

Remark 5.2. The operator $T = T_{a_1}T_{b_1}$ preserves the condition $a_1 \cdots a_{M+3} = q^2 b_1 \cdots b_{M+3}$. Similarly, we can only consider $T_{a_k}T_{a_l}^{-1}$, $T_{b_k}T_{b_l}^{-1}$, $T_{a_k}T_{b_l}$, etc. as q-shift operators under that condition.

Proposition 5.1 means that the integral $\varphi_{i,j}^M$ satisfies a q-difference equation of rank $M+1$. The following proposition leads to a q-difference system of rank $M+1$ that the integral satisfies.

Proposition 5.3. *The integral (5.1) satisfies the following three term relations:*

$$[(a_1 - a_k q)T_{a_k}T_{a_l}^{-1} - (a_l - a_k q)T_{a_k}T_{a_1}^{-1} + (a_l - a_1)]\varphi_{i,j}^M = 0, \tag{5.5}$$

$$[(b_1 - a_1)T_{a_1}T_{a_k}^{-1} - (q^{-1}a_k - a_1)T_{a_1}T_{b_1} + (q^{-1}a_k - b_1)]\varphi_{i,j}^M = 0, \tag{5.6}$$

$$[(b_1 - q^{-1}b_l)T_{b_k}T_{b_l}^{-1} - (b_k - q^{-1}b_l)T_{b_1}T_{b_l}^{-1} + (b_k - b_1)]\varphi_{i,j}^M = 0, \tag{5.7}$$
$$[(a_1 - b_1)T_{b_k}T_{b_1}^{-1} - (b_k q - b_1)(T_{a_1}T_{b_1})^{-1} + (b_k q - a_1)]\varphi_{i,j}^M = 0, \tag{5.8}$$
$$[(b_1 - a_k)T_{a_k}T_{b_l} - (b_l - a_k)T_{a_k}T_{b_1} + (b_l - b_1)]\varphi_{i,j}^M = 0, \tag{5.9}$$
$$[(b_l - a_1)T_{a_k}^{-1}T_{b_l}^{-1} - (b_l - a_k)T_{a_1}^{-1}T_{b_l}^{-1} + (a_1 - a_k)]\varphi_{i,j}^M = 0, \tag{5.10}$$

where $1 \leq k, l \leq M+3$.

Proof. We only prove the equation (5.5). Other relations can be proved in the same way. The integrand

$$\psi^M = \prod_{i=1}^{M+3} \frac{(a_i t)_\infty}{(b_i t)_\infty}, \tag{5.11}$$

satisfies

$$[(a_1 - a_k q)T_{a_k}T_{a_l}^{-1} - (a_l - a_k q)T_{a_k}T_{a_1}^{-1} + (a_l - a_1)]\psi^M = 0. \tag{5.12}$$

This equation is derived by the relation for rational functions of t:

$$(a_1 - a_k q)\frac{1 - a_l q^{-1}t}{1 - a_k t} - (a_l - a_k q)\frac{1 - a_1 q^{-1}t}{1 - a_k t} + (a_l - a_1) = 0. \tag{5.13}$$

By integrating in t, we have the relation (5.5). □

Also the integrand ψ^M satisfies

$$T_t\psi^M = \left(\prod_{k=1}^{M+3} T_{a_k}T_{b_k}\right)\psi^M, \tag{5.14}$$

thus $\varphi_{i,j}^M$ satisfies

$$q^{-1}\varphi_{i,j}^M = \left(\prod_{k=1}^{M+3} T_{a_k}T_{b_k}\right)\varphi_{i,j}^M. \tag{5.15}$$

In conclusion, the integral $\varphi_{i,j}^M$ satisfies the following q-difference system.

Theorem 5.4. *The integral* $\varphi_{i,j}^M$ *satisfies the following system:*

$$\hat{E}_M y = 0, \tag{5.16}$$
$$[(a_1 - a_k q)T_{a_k}T_{a_l}^{-1} - (a_l - a_k q)T_{a_k}T_{a_1}^{-1} + (a_l - a_1)]y = 0, \tag{5.17}$$
$$[(b_1 - a_1)T_{a_1}T_{a_k}^{-1} - (q^{-1}a_k - a_1)T_{a_1}T_{b_1} + (q^{-1}a_k - b_1)]y = 0, \tag{5.18}$$
$$[(b_1 - q^{-1}b_l)T_{b_k}T_{b_l}^{-1} - (b_k - q^{-1}b_l)T_{b_1}T_{b_l}^{-1} + (b_k - b_1)]y = 0, \tag{5.19}$$
$$[(a_1 - b_1)T_{b_k}T_{b_1}^{-1} - (b_k q - b_1)(T_{a_1}T_{b_1})^{-1} + (b_k q - a_1)]y = 0, \tag{5.20}$$
$$[(b_1 - a_k)T_{a_k}T_{b_l} - (b_l - a_k)T_{a_k}T_{b_1} + (b_l - b_1)]y = 0, \tag{5.21}$$
$$[(b_l - a_1)T_{a_k}^{-1}T_{b_l}^{-1} - (b_l - a_k)T_{a_1}^{-1}T_{b_l}^{-1} + (a_1 - a_k)]y = 0, \tag{5.22}$$
$$\left(q\prod_{k=1}^{M+3} T_{a_k}T_{b_k} - 1\right)y = 0, \tag{5.23}$$

where $1 \leq k, l \leq M+3$. *We call this system* q-RP^M. *The rank of* q-RP^M *is* $M+1$.

Proof. We already proved that the integral $\varphi^M_{i,j}$ satisfies the system q-RP^M. We will prove that the rank of q-RP^M is $M+1$. Using the three term relations (5.17), ..., (5.22), the functions $T_{a_k}T_{a_l}^{-1}y$, $T_{b_k}T_{b_l}^{-1}y$, $T_{a_k}T_{b_l}y$ can be rewritten as linear combinations of y, $(T_{a_1}T_{b_1})^{\pm 1}$, $(T_{a_1}T_{b_1})^{\pm 2}$,... The operator $\hat{E}_M$ is an ordinary q-difference operator of rank $M+1$, so the rank of q-RP^M is $M+1$. □

Due to Theorem 2.17, the integrals $\varphi^M_{1,2},\ldots,\varphi^M_{1,M+2}$ are linearly independent over $\{C \mid T_{a_1}T_{b_1}C = C\}$. So we have the following:

Theorem 5.5. *The integrals $\varphi^M_{1,2},\ldots,\varphi^M_{1,M+2}$ are basis of the space of solutions for q-RP^M over $K = \{C \mid T_{a_k}T_{a_l}^{-1}C = T_{b_k}T_{b_l}^{-1}C = T_{a_k}T_{b_l}C = C\}$.*

Proof. Suppose

$$C_2\varphi^M_{1,2} + \cdots + C_{M+2}\varphi^M_{1,M+2} = 0, \tag{5.24}$$

where $C_i \in K$. Then every pseudo constant C_i satisfies $T_{a_1}T_{b_1}C_i = C_i$. According to Theorem 2.17, we have $C_2 = \cdots = C_{M+2} = 0$. □

Remark 5.6. By the formula (2.78) in Theorem 2.35 and the obvious relation $\varphi^M_{i,j}+\varphi^M_{j,k}+\varphi^M_{k,i} = 0$, the integral $\varphi^M_{i,j}$ can be written by a linear combination of $\varphi^M_{1,2},\ldots,\varphi^M_{1,M+2}$ explicitly. Hence, in particular, the connection problem among the integrals $\{\varphi^M_{i,j}\}$ is solved already.

5.2 Series solutions

In this section, we give multiple series solutions for the system q-RP^M. In the case of $M = 3$, the Jackson integral of Riemann-Papperitz type

$$\int_{q/a_i}^{q/a_j} \prod_{k=1}^{4} \frac{(a_k t)_\infty}{(b_k t)_\infty} d_q t, \tag{5.25}$$

can be transformed to the very-well-poised q-hypergeometric series ${}_8W_7$ by the Bailey's formula (see Lemma 2.25). We show an extension of the Bailey's formula.

Definition 5.7 (Kajihara's q-hypergeometric series [29])**.** Following Kajihara, we define the q-hypergeometric series:

$$\begin{aligned}
&W^{n,m}\left(\begin{array}{c}\{a_i\}_{1\le i\le n}\\ \{x_i\}_{1\le i\le n}\end{array}\middle|\, s; \{u_k\}_{1\le k\le m}; \{v_k\}_{1\le k\le m}; z\right)\\
&= \sum_{l\in(\mathbb{Z}_{\ge 0})^n} z^{|l|}\frac{\Delta(xq^l)}{\Delta(x)} \prod_{1\le i\le n}\frac{1-q^{|l|+l_i}sx_i/x_n}{1-sx_i/x_n}\\
&\quad \times \prod_{1\le j\le n}\left(\frac{(sx_j/x_n)_{|l|}}{((sq/a_j)x_j/x_n)_{|l|}}\prod_{1\le i\le n}\frac{(a_jx_i/x_j)_{l_i}}{(qx_i/x_j)_{l_i}}\right)\\
&\quad \times \prod_{1\le k\le m}\left(\frac{(v_k)_{|l|}}{(sq/u_k)_{|l|}}\prod_{1\le i\le n}\frac{(u_kx_i/x_n)_{l_i}}{((sq/v_k)x_i/x_n)_{l_i}}\right),
\end{aligned} \tag{5.26}$$

where $|l| = l_1 + l_2 + \cdots + l_n$, $xq^l = \{x_iq^{l_i}\}_{1\le i\le n}$ and $\Delta(x)$ is the Vandermonde determinant of x:

$$\Delta(x) = \prod_{1\le i<j\le n}(x_i - x_j). \tag{5.27}$$

The function $W^{n,m}$ is the Kajihara's very well-poised q-hypergeometric series.

Proposition 5.8 (Kajihara's transformation formula [29]). *For any $N \in \mathbb{Z}_{\geq 0}$, we have*

$$
\begin{aligned}
&W^{n,m+2}\left(\begin{matrix} \{b_i\}_{1\le i\le n} \\ \{x_i\}_{1\le i\le n} \end{matrix} \;\middle|\; a; \{c_k y_k/y_m\}_{1\le k\le m}, d, e; \{f y_m/y_k\}_{1\le k\le m}, \mu f q^N, q^{-N}; q \right) \\
&= \frac{(\mu df/a, \mu ef/a)_N}{(aq/d, aq/e)_N} \prod_{1\le k\le m} \frac{((\mu c_k f/a)y_k/y_m, f y_m/y_k)_N}{(\mu q y_k/y_m, (aq/c_k)y_m/y_k)_N} \prod_{1\le i\le n} \frac{(aqx_i/x_n, (\mu b_i f/a)x_n/x_i)_N}{((aq/b_i)x_i/x_n, (\mu f/a)x_n/x_i)_N} \\
&\quad\times W^{m,n+2}\left(\begin{matrix} \{aq/(c_k f)\}_{1\le k\le m} \\ \{y_k\}_{1\le k\le m} \end{matrix} \;\middle|\; \mu; \left\{ \frac{aq}{b_i f}\frac{x_i}{x_n} \right\}_{1\le i\le n}, \frac{aq}{df}, \frac{aq}{ef}; \left\{ \frac{\mu f}{a}\frac{x_n}{x_i} \right\}_{1\le i\le n}, \mu f q^N, q^{-N}; q \right),
\end{aligned}
\tag{5.28}
$$

where $\mu = a^{m+2}q^{m+1}/(b_1 b_2 \cdots b_n c_1 c_2 \cdots c_n d e f^{m+1})$.

For the proof, see [29].

Corollary 5.9 ([29]). *We have*

$$
\begin{aligned}
&W^{n,3}\left(\begin{matrix} \{b_i\}_{1\le i\le n} \\ \{x_i\}_{1\le i\le n} \end{matrix} \;\middle|\; a; c, d, e; f, \mu f q^N, q^{-N}; q \right) \\
&= \frac{(\mu cf/a, \mu df/a, \mu ef/a, f)_N}{(aq/c, aq/d, aq/e, q\mu)_N} \prod_{1\le i\le n} \frac{(aqx_i/x_n, (\mu b_i f/a)x_n/x_i)_N}{((aq/b_i)x_i/x_n, (\mu f/a)x_n/x_i)_N} \\
&\quad\times {}_{2n+8}W_{2n+7}\left(\mu; \left\{ \frac{aq}{b_i f}\frac{x_i}{x_n} \right\}_{1\le i\le n}, \frac{aq}{cf}, \frac{aq}{df}, \frac{aq}{ef}; \left\{ \frac{\mu f}{a}\frac{x_n}{x_i} \right\}_{1\le i\le n}, \mu f q^N, q^{-N}; q \right),
\end{aligned}
\tag{5.29}
$$

where $\mu = a^3 q^2/(b_1 b_2 \cdots b_n c d e f^2)$.

Proof. We get the desired equation by putting $m = 1$ in Kajihara's transformation formula (5.28). □

Remark 5.10. When $n = 1$, the formula (5.29) reduces a transformation formula of terminating very-well-poised q-hypergeometric series ${}_{10}W_9$ (see Gasper-Rahman [16] and references therein).

Theorem 5.11. *We have*

$$
\begin{aligned}
&W^{n,2}\left(\begin{matrix} \{b_i\}_{1\le i\le n} \\ \{x_i\}_{1\le i\le n} \end{matrix} \;\middle|\; a; f, f\mu; d, e; c/a \right) \\
&= \frac{(aq/(df), aq/(\mu df), aq/(ef), aq/(\mu ef), f, \mu f)_\infty}{(1-q)(q, c/a, aq/d, aq/e, q\mu, 1/\mu)_\infty} \prod_{1\le i\le n} \frac{(aqx_i/x_n, (aq/b_i f)x_i/x_n, (aq/\mu b_i f)x_i/x_n)_\infty}{((aq/b_i)x_i/x_n, (aq/f)x_i/x_n, (aq/\mu f)x_i/x_n)_\infty} \\
&\times \int_{1/\mu}^{1} \frac{(qt, \mu cft/a, q\mu t)_\infty}{(aqt/(df), aqt/(ef), \mu f t)_\infty} \prod_{i=1}^{n} \frac{((aq/f)x_i t/x_n)_\infty}{((aq/b_i f)x_i t/x_n)_\infty} d_q t,
\end{aligned}
\tag{5.30}
$$

where $\mu = a^3 q^2/(b_1 b_2 \cdots b_n c d e f^2)$.

Proof. The proof is based on the method similar to Bailey's one [10]. For the Bailey's idea, see Gasper-Rahman [16], section 2. The formula (5.30) is derived by considering the limit $N \to \infty$ in (5.29). First, we consider the limit

$$
\lim_{N\to\infty} W^{n,3}\left(\begin{matrix} \{b_i\}_{1\le i\le n} \\ \{x_i\}_{1\le i\le n} \end{matrix} \;\middle|\; a; cq^N, d, e; f, \mu f, q^{-N}; q \right).
\tag{5.31}
$$

We recall the definition of the function $W^{n,3}$:

$$W^{n,3}\left(\begin{array}{c}\{b_i\}_{1\le i\le n}\\ \{x_i\}_{1\le i\le n}\end{array}\middle| a; cq^N, d, e; f, \mu f, q^{-N}; q\right)$$
$$= \sum_{l\in(\mathbb{Z}_{\ge 0})^n} z^{|l|}\frac{\Delta(xq^l)}{\Delta(x)}\prod_{1\le i\le n}\frac{1-q^{|l|+l_i}ax_i/x_n}{1-ax_i/x_n}$$
$$\times\prod_{1\le j\le n}\left(\frac{(ax_j/x_n)_{|l|}}{((aq/b_j)x_j/x_n)_{|l|}}\prod_{1\le i\le n}\frac{(b_jx_i/x_j)_{l_i}}{(qx_i/x_j)_{l_i}}\right)$$
$$\times\frac{(cq^N,d,e)_{|l|}}{(aq/f,aq/(\mu f),aqq^N)_{|l|}}\prod_{1\le i\le n}\frac{(fx_i/x_n,\mu fx_i/x_n,q^{-N}x_i/x_n)_{l_i}}{((aq/d)x_i/x_n,(aq/e)x_i/x_n,(aqq^{-N}/c)x_i/x_n)_{l_i}}. \tag{5.32}$$

Since $(cq^N)_{|l|}$, $(aqq^N)_{|l|}\xrightarrow{N\to\infty}1$ and $\dfrac{(q^{-N}x_i/x_n)_{l_i}}{((aqq^{-N}/c)x_i/x_n)_{l_i}}\xrightarrow{N\to\infty}\left(\dfrac{c}{aq}\right)^{l_i}$, we have

$$\lim_{N\to\infty}W^{n,3}\left(\begin{array}{c}\{b_i\}_{1\le i\le n}\\ \{x_i\}_{1\le i\le n}\end{array}\middle| a; cq^N, d, e; f, \mu f, q^{-N}; q\right)$$
$$= \sum_{l\in(\mathbb{Z}_{\ge 0})^n}\left(\frac{c}{a}\right)^{|l|}\frac{\Delta(xq^l)}{\Delta(x)}\prod_{1\le i\le n}\frac{1-q^{|l|+l_i}ax_i/x_n}{1-ax_i/x_n}$$
$$\times\prod_{1\le j\le n}\left(\frac{(ax_j/x_n)_{|l|}}{((aq/b_j)x_j/x_n)_{|l|}}\prod_{1\le i\le n}\frac{(b_jx_i/x_j)_{l_i}}{(qx_i/x_j)_{l_i}}\right)$$
$$\times\frac{(d,e)_{|l|}}{(aq/f,aq/(\mu f))_{|l|}}\prod_{1\le i\le n}\frac{(fx_i/x_n,\mu fx_i/x_n)_{l_i}}{((aq/d)x_i/x_n,(aq/e)x_i/x_n)_{l_i}}$$
$$= W^{n,2}\left(\begin{array}{c}\{b_i\}_{1\le i\le n}\\ \{x_i\}_{1\le i\le n}\end{array}\middle| a; f, f\mu; d, e; c/a\right). \tag{5.33}$$

On the other hand, we have

$$W^{n,3}\left(\begin{array}{c}\{b_i\}_{1\le i\le n}\\ \{x_i\}_{1\le i\le n}\end{array}\middle| a; cq^N, d, e; f, \mu f, q^{-N}; q\right)$$
$$= \frac{(\mu cf/a,\mu q^{-N}df/a,\mu q^{-N}ef/a,f)_N}{(aq/(cq^N),aq/d,aq/e,q\mu q^{-N})_N}\prod_{1\le i\le n}\frac{(aqx_i/x_n,(\mu q^{-N}b_if/a)x_n/x_i)_N}{((aq/b_i)x_i/x_n,(\mu q^{-N}f/a)x_n/x_i)_N}$$
$$\times {}_{2n+8}W_{2n+7}\left(\mu q^{-N};\left\{\frac{aq}{b_if}\frac{x_i}{x_n}\right\}_{1\le i\le n},\frac{aq}{cq^Nf},\frac{aq}{df},\frac{aq}{ef};\left\{\frac{\mu q^{-N}f}{a}\frac{x_n}{x_i}\right\}_{1\le i\le n},\mu f,q^{-N};q\right). \tag{5.34}$$

Since $(A)_N\xrightarrow{N\to\infty}(A)_\infty$ and

$$\prod_{l=1}^L\frac{(A_iq^{-N})_N}{(B_iq^{-N})_N}=\prod_{l=1}^L\left(\frac{A_i}{B_i}\right)^N\frac{(q/A_i)_N}{(q/B_i)_N}\xrightarrow{N\to\infty}\prod_{l=1}^L\frac{(q/A_i)_\infty}{(q/B_i)_\infty}, \tag{5.35}$$

if $A_1\cdots A_L=B_1\cdots B_L$, we have

$$\frac{(\mu cf/a,\mu q^{-N}df/a,\mu q^{-N}ef/a,f)_N}{(aq/(cq^N),aq/d,aq/e,q\mu q^{-N})_N}\prod_{1\le i\le n}\frac{(aqx_i/x_n,(\mu q^{-N}b_if/a)x_n/x_i)_N}{((aq/b_i)x_i/x_n,(\mu q^{-N}f/a)x_n/x_i)_N}$$

$$\xrightarrow{N\to\infty} \frac{(\mu cf/a, aq/(\mu df), aq/(\mu ef), f)_\infty}{(c/a, aq/d, aq/e, 1/\mu)_\infty} \prod_{1\le i\le n} \frac{(aqx_i/x_n, (aq/\mu b_i f)x_i/x_n)_\infty}{((aq/b_i)x_i/x_n, (aq/\mu f)x_i/x_n)_\infty}. \tag{5.36}$$

We put $N = 2M-1$ and

$$C_l = \frac{1-\mu q^{-N}q^{2l}}{1-\mu q^{-N}} \frac{(\mu q^{-N}, aqq^{-N}/(cf), aq/(df), aq/(ef), \mu f, q^{-N})_l}{(q, \mu cf/a, \mu df q^{-N}/a, \mu ef q^{-N}/a, qq^{-N}/f, q\mu)_l} \prod_{i=1}^n \frac{((aq/b_i f)x_i/x_n, (\mu f q^{-N}/a)x_n/x_i)_l}{((\mu b_i f q^{-N}/a)x_n/x_i, (aq/f)x_i/x_n)_l} q^l. \tag{5.37}$$

By the definition of the very-well-poised q-hypergeometric series ${}_{2n+8}W_{2n+7}$, we have

$$\begin{aligned}
&{}_{2n+8}W_{2n+7}\left(\mu q^{-N}; \left\{\frac{aq}{b_i f}\frac{x_i}{x_n}\right\}_{1\le i\le n}, \frac{aq}{cq^N f}, \frac{aq}{df}, \frac{aq}{ef}; \left\{\frac{\mu q^{-N} f}{a}\frac{x_n}{x_i}\right\}_{1\le i\le n}, \mu f, q^{-N}; q\right) \\
&= \sum_{l=0}^N C_l = \sum_{l=0}^{M-1} C_l + \sum_{l=0}^{M-1} C_{N-l}.
\end{aligned} \tag{5.38}$$

For $0 \le l \le M-1$, we have

$$C_l \xrightarrow{M\to\infty} \frac{(aq/(df), aq/(ef), \mu f)_l}{(q, \mu cf/a, q\mu)_l} \prod_{i=1}^n \frac{((aq/b_i f)x_i/x_n)_l}{((aq/f)x_i/x_n)_l} q^l, \tag{5.39}$$

and

$$\begin{aligned}
C_{N-l} &\xrightarrow{M\to\infty} -\frac{q^k}{\mu} \frac{(aq/(df), aq/(ef), \mu f)_\infty}{(q, \mu cf/a, q\mu)_\infty} \prod_{i=1}^n \frac{((aq/b_i f)x_i/x_n)_\infty}{((aq/f)x_i/x_n)_\infty} \\
&\times \frac{(q/\mu, cf/a, q)_\infty}{(aq/(df\mu), aq/(ef\mu), f)_\infty} \prod_{i=1}^n \frac{((aq/\mu f)x_i/x_n)_\infty}{((aq/\mu b_i f)x_i/x_n)_\infty} \frac{(aq/(df\mu), aq/(ef\mu), f)_l}{(q/\mu, cf/a, q)_l} \prod_{i=1}^n \frac{((aq/\mu b_i f)x_i/x_n)_l}{((aq/\mu f)x_i/x_n)_l}.
\end{aligned} \tag{5.40}$$

The limit of C_{N-l} can be calculated by using formulas $(a)_{N-l} = (a)_N/(aq^{N-l})_l$, $(aq^{-l})_l = (-a)^l q^{-l(l-1)/2}(q/a)_l$ and the condition $\mu = a^3q^2/(b_1\cdots b_n cdef^2)$. So we have

$$\begin{aligned}
\lim_{M\to\infty} \sum_{l=0}^{M-1} C_l &= \sum_{l=0}^\infty \frac{(aq/(df), aq/(ef), \mu f)_l}{(q, \mu cf/a, q\mu)_l} \prod_{i=1}^n \frac{((aq/b_i f)x_i/x_n)_l}{((aq/f)x_i/x_n)_l} q^l \\
&= \frac{(aq/(df), aq/(ef), \mu f)_\infty}{(q, \mu cf/a, q\mu)_\infty} \prod_{i=1}^n \frac{((aq/b_i f)x_i/x_n)_\infty}{((aq/f)x_i/x_n)_\infty} \\
&\times \frac{1}{1-q} \int_0^1 \frac{(qt, \mu cft/a, q\mu t)_\infty}{(aqt/(df), aqt/(ef), \mu ft)_\infty} \prod_{i=1}^n \frac{((aq/f)x_i t/x_n)_\infty}{((aq/b_i f)x_i t/x_n)_\infty}, d_q t
\end{aligned} \tag{5.41}$$

and

$$\begin{aligned}
\lim_{M\to\infty} \sum_{l=0}^{M-1} C_{N-l} &= -\frac{(aq/(df), aq/(ef), \mu f, cf/a, q/\mu)_\infty}{(aq/(df\mu), aq/(ef\mu), f, \mu cf/a, q\mu)_\infty} \prod_{i=1}^n \frac{((aq/b_i f)x_i/x_n, (aq/\mu f)x_i/x_n)_\infty}{((aq/\mu b_i f)x_i/x_n, (aq/f)x_i/x_n)_\infty} \\
&\times \left(\frac{1}{\mu}\right) \sum_{l=0}^\infty \frac{(aq/(df\mu), aq/(ef\mu), f)_l}{(q/\mu, cf/a, q)_l} \prod_{i=1}^n \frac{((aq/\mu b_i f)x_i/x_n)_l}{((aq/\mu f)x_i/x_n)_l} \\
&= -\frac{(aq/(df), aq/(ef), \mu f)_\infty}{(q, \mu cf/a, q\mu)_\infty} \prod_{i=1}^n \frac{((aq/b_i f)x_i/x_n)_\infty}{((aq/f)x_i/x_n)_\infty}
\end{aligned}$$

$$\times \frac{1}{1-q}\int_0^{1/\mu} \frac{(qt,\mu cft/a,q\mu t)_\infty}{(aqt/(df),aqt/(ef),\mu ft)_\infty}\prod_{i=1}^{n}\frac{((aq/f)x_it/x_n)_\infty}{((aq/b_if)x_it/x_n)_\infty}d_qt. \tag{5.42}$$

This completes the proof. □

Using the formula (5.30), we give series solutions for the system q-RP^M. We put

$$\begin{aligned}
&W\left(\begin{array}{c}\{a_i\}_{1\le i\le M+3}\\ \{b_i\}_{1\le i\le M+3}\end{array}\right)\\
&=\frac{(a_{M+1}/b_{M+3},b_{M+1}b_{M+3}q^2/(a_{M+2}a_{M+3}),b_{M+2}b_{M+3}q^2/(a_{M+2}a_{M+3}),a_{M+2}q/a_{M+3},a_{M+3}/a_{M+2})_\infty}{a_{M+3}(b_{M+1}q/a_{M+2},b_{M+1}q/a_{M+3},b_{M+2}q/a_{M+2},b_{M+2}q/a_{M+3},b_{M+3}q/a_{M+2},b_{M+3}q/a_{M+3})_\infty}\\
&\times\prod_{i=1}^{M}\frac{(b_ib_{M+3}q^2/(a_{M+2}a_{M+3}),a_iq/a_{M+2},a_iq/a_{M+3})_\infty}{(a_ib_{M+3}q^2/(a_{M+2}a_{M+3}),b_iq/a_{M+2},b_iq/a_{M+3})_\infty}\\
&\times W^{M,2}\left(\begin{array}{c}\{a_i/b_i\}_{1\le i\le M}\\ \{a_i\}_{1\le i\le M}\end{array}\middle|\,\frac{a_Mb_{M+3}q}{a_{M+2}a_{M+3}};\frac{b_{M+3}q}{a_{M+2}},\frac{b_{M+3}q}{a_{M+3}};\frac{a_M}{b_{M+1}},\frac{a_M}{b_{M+2}};\frac{a_{M+1}}{b_{M+3}}\right).
\end{aligned} \tag{5.43}$$

Theorem 5.12. *The function*

$$W\left(\begin{array}{c}\{a_{\sigma_1(i)}\}_{1\le i\le M+3}\\ \{b_{\sigma_2(i)}\}_{1\le i\le M+3}\end{array}\right), \tag{5.44}$$

is a solution of the system q-RP^M. Here, $\sigma_1,\sigma_2\in\mathfrak{S}_{M+3}$.

Proof. The integral $\varphi_{M+2,M+3}$ is rewritten as

$$\begin{aligned}
&\int_{q/a_{M+2}}^{q/a_{M+3}}\prod_{i=1}^{M+3}\frac{(a_it)_\infty}{(b_it)_\infty}d_qt\\
&=\frac{q}{a_{M+3}}\int_{a_{M+3}/a_{M+2}}^{1}\frac{(qt,a_{M+1}qt/a_{M+3},a_{M+2}qt/a_{M+3})_\infty}{(b_{M+1}qt/a_{M+3},b_{M+2}qt/a_{M+3},b_{M+3}qt/a_{M+3})_\infty}\prod_{i=1}^{M}\frac{(a_iqt/a_{M+3})_\infty}{(b_iqt/a_{M+3})_\infty}d_qt.
\end{aligned} \tag{5.45}$$

Applying the transformation formula (5.30), we have

$$\begin{aligned}
&\frac{1}{a_{M+3}(1-q)(q)_\infty}\int_{a_{M+3}/a_{M+2}}^{1}\frac{(qt,a_{M+1}qt/a_{M+3},a_{M+2}qt/a_{M+3})_\infty}{(b_{M+1}qt/a_{M+3},b_{M+2}qt/a_{M+3},b_{M+3}qt/a_{M+3})_\infty}\prod_{i=1}^{M}\frac{(a_iqt/a_{M+3})_\infty}{(b_iqt/a_{M+3})_\infty}d_qt\\
&=\frac{(a_{M+1}/b_{M+3},b_{M+1}b_{M+3}q^2/(a_{M+2}a_{M+3}),b_{M+2}b_{M+3}q^2/(a_{M+2}a_{M+3}),a_{M+2}q/a_{M+3},a_{M+3}/a_{M+2})_\infty}{a_{M+3}(b_{M+1}q/a_{M+2},b_{M+1}q/a_{M+3},b_{M+2}q/a_{M+2},b_{M+2}q/a_{M+3},b_{M+3}q/a_{M+2},b_{M+3}q/a_{M+3})_\infty}\\
&\times\prod_{i=1}^{M}\frac{(b_ib_{M+3}q^2/(a_{M+2}a_{M+3}),a_iq/a_{M+2},a_iq/a_{M+3})_\infty}{(a_ib_{M+3}q^2/(a_{M+2}a_{M+3}),b_iq/a_{M+2},b_iq/a_{M+3})_\infty}\\
&\times W^{M,2}\left(\begin{array}{c}\{a_i/b_i\}_{1\le i\le M}\\ \{a_i\}_{1\le i\le M}\end{array}\middle|\,\frac{a_Mb_{M+3}q}{a_{M+2}a_{M+3}};\frac{b_{M+3}q}{a_{M+2}},\frac{b_{M+3}q}{a_{M+3}};\frac{a_M}{b_{M+1}},\frac{a_M}{b_{M+2}};\frac{a_{M+1}}{b_{M+3}}\right)\\
&=W\left(\begin{array}{c}\{a_i\}_{1\le i\le M+3}\\ \{b_i\}_{1\le i\le M+3}\end{array}\right).
\end{aligned} \tag{5.46}$$

□

Remark 5.13. In the general theory of q-difference systems in several variables, local solutions should be characterized by the asymptotic behavior near the singularity, i.e. origin or infinity, in some prescribed sector. For the general theory, see [4, 5]. A characterization of solutions for the q-Appell-Lauricella system around origin or infinity was obtained in [45]. See also section 4, Proposition 4.6.

However, origin and infinity are essentially non-singular for the equation $\hat{E}_M y = 0$. So it is very difficult to characterize series solutions by the asymptotic behavior, though the characterization is a very important problem. Note that even characterizations for the series solution (2.61) of the one variable equation $\mathcal{H}_3 y = 0$ is an open problem.

Using the integral representation for the Kajihara's q-hypergeometric series $W^{M,2}$, we have linear relations for $W^{M,2}$.

Theorem 5.14. *Suppose $a_1 \cdots a_{M+3} = q^2 b_1 \cdots b_{M+3}$. We have*

$$W\left(\begin{array}{c}\{a_{\sigma(i)}\}_{1\le i\le M+1}, a_{M+2}, a_{M+3}\\ \{b_i\}_{1\le i\le M+3}\end{array}\right) = W\left(\begin{array}{c}\{a_i\}_{1\le i\le M+3}\\ \{b_{\sigma'(i)}\}_{1\le i\le M+3}\end{array}\right) = W\left(\begin{array}{c}\{a_i\}_{1\le i\le M+3}\\ \{b_i\}_{1\le i\le M+3}\end{array}\right), \tag{5.47}$$

$$\begin{aligned} &W\left(\begin{array}{c}\{a_i\}_{1\le i\le M}, a_{M+1}, a_{M+2}, a_{M+3}\\ \{b_i\}_{1\le i\le M+3}\end{array}\right) + W\left(\begin{array}{c}\{a_i\}_{1\le i\le M}, a_{M+2}, a_{M+3}, a_{M+1}\\ \{b_i\}_{1\le i\le M+3}\end{array}\right)\\ &\qquad + W\left(\begin{array}{c}\{a_i\}_{1\le i\le M}, a_{M+3}, a_{M+1}, a_{M+2}\\ \{b_i\}_{1\le i\le M+3}\end{array}\right) = 0, \end{aligned} \tag{5.48}$$

$$\sum_{i=1}^{M+2} C_k W\left(\begin{array}{c}a_1, \ldots, a_{k-1}, a_{M+2}, a_{k+1}, \ldots, a_{M+1}, a_k, a_{M+3}\\ \{b_i\}_{1\le i\le M+3}\end{array}\right) = 0, \tag{5.49}$$

where $\sigma \in \mathfrak{S}_{M+1}$, $\sigma' \in \mathfrak{S}_{M+3}$ and

$$C_k = \left(\frac{a_k}{a_{M+3}}\right)^2 \prod_{i=1}^{M+3} \frac{\theta(a_k/b_i)}{\theta(a_{M+3}/b_i)} \prod_{\substack{1\le i\le M+3\\ i\ne M+3}} \theta(a_{M+3}/a_i) \prod_{\substack{1\le i\le M+3\\ i\ne k}} \theta(a_k/a_i)^{-1}. \tag{5.50}$$

Proof. We already know by equations (5.45), (5.46) that

$$W\left(\begin{array}{c}\{a_i\}_{1\le i\le M+3}\\ \{b_i\}_{1\le i\le M+3}\end{array}\right) = \frac{1}{q(1-q)(q)_\infty} \int_{q/a_{M+2}}^{q/a_{M+3}} \prod_{i=1}^{M+3} \frac{(a_i t)_\infty}{(b_i t)_\infty} d_q t. \tag{5.51}$$

The right-hand-side is symmetric for $a_1, \ldots, a_{M+1}$ and $b_1, \ldots, b_{M+3}$. So we get (5.47). Due to the trivial relation $\displaystyle\int_{q/a_{M+2}}^{q/a_{M+3}} + \int_{q/a_{M+3}}^{q/a_{M+1}} + \int_{q/a_{M+1}}^{q/a_{M+2}} = 0$, we obtain (5.48). Applying the connection formula (2.78), we have (5.49). □

5.3 Degeneration to the q-Appell-Lauricella system

In this subsection, we consider a degeneration for the system q-RP^M. We show that the degeneration of q-RP^M includes the q-Appell-Lauricella system. First, we put $b_{M+3} = q^\lambda a_{M+3} = q^{-n}$ ($n \in \mathbb{Z}$), and consider the limit $a_{M+3} \to \infty$, i.e. $n \to +\infty$. In the following, we suppose $|q^\lambda| < 1$. We define a pseudo constant

$$C_0(t) = t^\lambda \frac{\theta(b_{M+3} t)}{\theta(a_{M+3} t)}. \tag{5.52}$$

For $1 \le j \le M+2$, we have

$$\lim_{a_{M+3}\to\infty} C_0\left(\frac{q}{a_j}\right)\int_0^{q/a_j}\prod_{i=1}^{M+3}\frac{(a_it)_\infty}{(b_it)_\infty}d_qt = \lim_{a_{M+3}\to\infty}\int_0^{q/a_j} C_0(t)\prod_{i=1}^{M+3}\frac{(a_it)_\infty}{(b_it)_\infty}d_qt$$
$$= \int_0^{q/a_j}\prod_{i=1}^{M+2} t^\lambda\frac{(a_it)_\infty}{(b_it)_\infty}d_qt. \tag{5.53}$$

On the other hand, we have

$$\int_0^{q/a_{M+3}}\prod_{i=1}^{M+3}\frac{(a_it)_\infty}{(b_it)_\infty}d_qt = \frac{q}{a_{M+3}}\int_0^1\prod_{i=1}^{M+3}\frac{(a_iqt/a_{M+3})_\infty}{(b_iqt/a_{M+3})_\infty}d_qt$$
$$= \frac{q}{a_{M+3}}\left(\int_0^1\frac{(qt)_\infty}{(q^{\lambda+1}t)_\infty}d_qt + O\left(\frac{1}{a_{M+3}}\right)\right). \tag{5.54}$$

According to the condition $|q^\lambda| < 1$, we have

$$\frac{1}{a_{M+3}}C_0\left(\frac{q}{a_j}\right) = \frac{1}{q^{-n}q^{-\lambda}}\left(\frac{q}{a_j}\right)^\lambda\frac{\theta(q^{-n}q/a_j)}{\theta(q^{-n}q^{-\lambda}q/a_j)} = (q^{1+\lambda})^n\left(\frac{q^2}{a_j}\frac{\theta(q/a_j)}{\theta(q^{-\lambda}q/a_j)}\right)\xrightarrow{n\to+\infty} 0. \tag{5.55}$$

Here we used $\theta(xq^{-1}) = -xq^{-1}\theta(x)$. So we have

$$\lim_{n\to+\infty} C_0\left(\frac{q}{a_j}\right)\int_{q/a_{M+3}}^{q/a_j}\prod_{i=1}^{M+3}\frac{(a_it)_\infty}{(b_it)_\infty}d_qt = \int_0^{q/a_j}\prod_{i=1}^{M+2} t^\lambda\frac{(a_it)_\infty}{(b_it)_\infty}d_qt. \tag{5.56}$$

Next we consider the limit $a_{M+2} \to 0$ with $b_{M+2} = q^\beta a_{M+2}$ ($a_1\cdots a_{M+1} = q^2q^\lambda q^\beta b_1\cdots b_{M+1}$). For $1 \le j \le M+1$, we easily find

$$\lim_{a_{M+2}\to 0}\int_0^{q/a_j}\prod_{i=1}^{M+2} t^\lambda\frac{(a_it)_\infty}{(b_it)_\infty}d_qt = \int_0^{q/a_j}\prod_{i=1}^{M+1} t^\lambda\frac{(a_it)_\infty}{(b_it)_\infty}d_qt. \tag{5.57}$$

We consider the same degenerations for the system q-RP^M. First we put $b_{M+3} = q^\lambda a_{M+3}$. We put $z = C_0(\tau)y$ ($\tau = q/a_1, \ldots, q/a_{M+2}$), and we should consider the degeneration $a_{M+3} \to \infty$ for the system associated with z. The function $C_0(\tau)$, given by (5.52), is a pseudo constant for a_i, b_i ($1 \le i \le M+2$), and

$$C_0(\tau)T_{a_{M+3}}T_{b_{M+3}}y = q^\lambda T_{a_{M+3}}T_{b_{M+3}}C_0(\tau)y. \tag{5.58}$$

So the function z satisfies

$$\hat{E}_Mz = 0, \tag{5.59}$$
$$[(a_1 - a_kq)T_{a_k}T_{a_l}^{-1} - (a_l - a_kq)T_{a_k}T_{a_1}^{-1} + (a_l - a_1)]z = 0, \tag{5.60}$$
$$[(b_1 - a_1)T_{a_1}T_{a_k}^{-1} - (q^{-1}a_k - a_1)T_{a_1}T_{b_1} + (q^{-1}a_k - b_1)] = 0, \tag{5.61}$$
$$[(b_1 - q^{-1}b_l)T_{b_k}T_{b_l}^{-1} - (b_k - q^{-1}b_l)T_{b_1}T_{b_l}^{-1} + (b_k - b_1)]z = 0, \tag{5.62}$$
$$[(a_1 - b_1)T_{b_k}T_{b_1}^{-1} - (b_kq - b_1)(T_{a_1}T_{b_1})^{-1} + (b_kq - a_1)]z = 0, \tag{5.63}$$
$$[(b_1 - a_k)T_{a_k}T_{b_l} - (b_l - a_k)T_{a_k}T_{b_1} + (b_l - b_1)]z = 0, \tag{5.64}$$
$$[(b_l - a_1)T_{a_k}^{-1}T_{b_l}^{-1} - (b_l - a_k)T_{a_1}^{-1}T_{b_l}^{-1} + (a_1 - a_k)]z = 0, \tag{5.65}$$

$$\left(q^{\lambda+1}\prod_{k=1}^{M+3} T_{a_k}T_{b_k}-1\right)z=0, \tag{5.66}$$

for $1\leq k,l\leq M+2$. By the definition of $\hat{E}_M$, we have

$$\lim_{a_{M+3}\to\infty}\frac{1}{a_{M+3}}\hat{E}_M z=\hat{E}'_M z, \tag{5.67}$$

where $\hat{a}'=(a_2,\dots,a_{M+2})$ $\hat{b}'=(b_2,\dots,b_{M+2})$, $T=T_{a_1}T_{b_1}$ and

$$\begin{aligned}\hat{E}'_M=&\sum_{k=1}^{M+1}(-1)^k b_1^{M+2-k}[e_{k-1}(\hat{a}')T^{-1}-qq^\lambda e_{k-1}(\hat{b}')]\prod_{n=0}^{M-k}(1-(a_1q^n/b_1)T)\prod_{n=0}^{k-2}(1-q^{-n}T)\\&+(-1)^{M+2}a_2\cdots a_{M+2}T^{-1}\prod_{n=0}^{M}(1-q^{-n}T).\end{aligned} \tag{5.68}$$

Note that

$$e_k(\hat{a})=e_k(\hat{a}')+a_{M+3}e_{k-1}(\hat{a}'). \tag{5.69}$$

Next we put $b_{M+2}=q^\beta a_{M+2}$, then we easily find

$$\lim_{a_{M+2}\to 0}\hat{E}'_M z=b_1\hat{E}''_M z, \tag{5.70}$$

where $\hat{a}''=(a_2,\dots,a_{M+1})$, $\hat{b}''=(b_2,\dots,b_{M+1})$,

$$\hat{E}''_M=\sum_{k=1}^{M+1}(-1)^k b_1^{M+1-k}[e_{k-1}(\hat{a}'')T^{-1}-qq^\lambda e_{k-1}(\hat{b}'')]\prod_{n=0}^{M-k}(1-(a_1q^n/b_1)T)\prod_{n=0}^{k-2}(1-q^{-n}T).$$

In addition, three term relations (5.17), ..., (5.22) are not changed for $1\leq k,l\leq M+1$. Moreover, the q-shift operators $T_{a_{M+3}}$, $T_{b_{M+3}}$, $T_{a_{M+2}}$, $T_{b_{M+2}}$ become identities by taking the limits $a_{M+3}\to\infty$, $a_{M+2}\to 0$ because

$$\lim_{a_{M+3}\to\infty}f(qa_{M+3})=\lim_{a_{M+3}\to\infty}f(a_{M+3}),\quad \lim_{a_{M+2}\to 0}f(qa_{M+2})=\lim_{a_{M+2}\to 0}f(a_{M+2}), \tag{5.71}$$

if these limits converge. Therefore the system q-RP^M degenerates to the following system q-RP^M_{degene}:

$$\begin{aligned}&\hat{E}''_M z=0, &(5.72)\\&[(a_1-a_k)T_{a_l}^{-1}-(a_l-a_k)T_{a_1}^{-1}+(a_l-a_1)T_{a_k}^{-1}]z=0, &(5.73)\\&[(b_1-b_l)T_{b_k}-(b_k-b_l)T_{b_1}+(b_k-b_1)T_{b_l}]z=0, &(5.74)\\&[(b_1-q^{-1}a_k)T_{b_l}-(b_l-q^{-1}a_k)T_{b_1}+(b_l-b_1)T_{a_k}^{-1}]z=0, &(5.75)\\&[(qb_l-a_1)T_{a_k}^{-1}-(qb_l-a_k)T_{a_1}^{-1}+(a_1-a_k)T_{b_l}]z=0, &(5.76)\\&\left(q^{\lambda+1}\prod_{k=1}^{M+1}T_{a_k}T_{b_k}-1\right)z=0, &(5.77)\end{aligned}$$

where $1\leq k,l\leq M+1$. As a corollary, the integral

$$\int_0^{q/a_j}t^\lambda\prod_{i=1}^{M+1}\frac{(a_it)_\infty}{(b_it)_\infty}d_qt, \tag{5.78}$$

satisfies the above system.

Remark 5.15. The rank of q-RP^M_{degene} is $M+1$. This can be proved in the same way as the proof of Theorem 5.4. In [35, 38], the Pfaff system related to the Jackson integral (5.78) was studied.

Remark 5.16. The relation (5.2) reduces $a_1 \cdots a_{M+1} = q^2 q^\lambda q^\beta b_1 \cdots b_{M+1}$ by the degenerations. Since the parameter β is independent from the system q-RP^M_{degene}, there is no condition among $a_1, \dots, a_{M+1}, b_1, \dots, b_{M+1}, \lambda$.

We will check that the system q-RP^M_{degene} is an extension of the q-Appell-Lauricella system.

Proposition 5.17. *We put $a_{M+1} = q$, $b_{M+1} = C/A$, $q^\lambda = A/q$, $a_i = B_i x_i$, $b_i = x_i$ $(1 \le i \le M)$. Then the system q-RP^M_{degene} includes the q-Appell-Lauricella system*

$$[x_i(1 - T_{x_j})(1 - B_i T_{x_i}) - x_j(1 - T_{x_i})(1 - B_j T_{x_j})]z = 0, \tag{5.79}$$

$$[(1 - T_{x_i})(1 - Cq^{-1} T_{x_1} \cdots T_{x_M}) - x_i(1 - B_i T_{x_i})(1 - A T_{x_1} \cdots T_{x_M})]z = 0, \tag{5.80}$$

i.e. if the function z satisfies q-RP^M_{degene}, then z is a solution for the q-Appell-Lauricella system.

Proof. For $1 \le i \le M$, the q-shift operator T_{x_i} is rewritten as $T_{x_i} = T_{a_i} T_{b_i}$. We have

$$\begin{aligned}
&[(1 - T_{x_i})(1 - Cq^{-1} T_{x_1} \cdots T_{x_M}) - x_i(1 - B_i T_{x_i})(1 - A T_{x_1} \cdots T_{x_M})]z \\
&= [(1 - T_{a_i} T_{b_i})(1 - a_{M+1}^{-1} b_{M+1} q^{\lambda+1} T_{a_1} T_{b_1} \cdots T_{a_M} T_{b_M}) \\
&\quad - b_i(1 - a_i b_i^{-1} T_{a_i} T_{b_i})(1 - q^{\lambda+1} T_{a_1} T_{b_1} \cdots T_{a_M} T_{b_M})]z.
\end{aligned} \tag{5.81}$$

Using (5.77), we have

$$\begin{aligned}
&(1 - a_{M+1}^{-1} b_{M+1} q^{\lambda+1} T_{a_1} T_{b_1} \cdots T_{a_M} T_{b_M})z = (1 - a_{M+1}^{-1} b_{M+1} T_{a_{M+1}}^{-1} T_{b_{M+1}}^{-1})z \\
&= -T_{a_{M+1}}^{-1} T_{b_{M+1}}^{-1} (a_{M+1}^{-1}(b_{M+1} - a_{M+1} T_{a_{M+1}} T_{b_{M+1}}))z,
\end{aligned} \tag{5.82}$$

$$(1 - q^{\lambda+1} T_{a_1} T_{b_1} \cdots T_{a_M} T_{b_M})z = -T_{a_{M+1}}^{-1} T_{b_{M+1}}^{-1} (b_{M+1} - a_{M+1} T_{a_{M+1}} T_{b_{M+1}}). \tag{5.83}$$

By the relations (5.74) and (5.76), we have

$$\begin{aligned}
&(qb_1 - a_1)(1 - T_{a_i} T_{b_i})z = T_{a_i}(qb_1 - a_1)(T_{a_i}^{-1} - T_{b_i})z \\
&= T_{a_i}[(qb_1 - a_i)T_{a_1}^{-1} - (a_1 - a_i)T_{b_1} - (qb_i - a_1)T_{b_1} - q(b_i - b_1)T_{a_1}^{-1}]z \\
&= T_{a_i}(qb_i - a_i)(T_{a_1}^{-1} - T_{b_1})z.
\end{aligned} \tag{5.84}$$

Similarly we find

$$(qb_1 - a_1)(b_i - a_i T_{a_i} T_{b_i})z = T_{a_i}(qb_i - a_i)(b_1 T_{a_1}^{-1} - a_1 q^{-1} T_{b_1})z. \tag{5.85}$$

Therefore we have

$$\begin{aligned}
&(qb_1 - a_1)^2[(1 - T_{a_i} T_{b_i}) a_{M+1}^{-1} (b_{M+1} - a_{M+1} T_{a_{M+1}} T_{b_{M+1}}) - (b_i - a_i T_{a_i} T_{b_i})(1 - T_{a_{M+1}} T_{b_{M+1}})] \\
&= T_{a_i} T_{a_{M+1}} (qb_i - a_i)(qb_{M+1} - a_{M+1}) \\
&\quad \times [q a_{M+1}^{-1} (T_{a_1}^{-1} - T_{b_1})(b_1 T_{a_1}^{-1} - a_1 q^{-1} T_{b_1}) - (b_1 T_{a_1}^{-1} - a_1 q^{-1} T_{b_1})(T_{a_1}^{-1} - T_{b_1})] \\
&= T_{a_i} T_{a_{M+1}} (qb_i - a_i)(qb_{M+1} - a_{M+1})(q^2 a_{M+1}^{-1} - 1)(b_1 T_{a_1}^{-1} - a_1 q^{-1} T_{b_1})(T_{a_1}^{-1} - T_{b_1}) \\
&= (qb_i - qa_i)(qb_{M+1} - qa_{M+1})(q a_{M+1}^{-1} - 1) T_{a_i} T_{a_{M+1}} (b_1 T_{a_1}^{-1} - a_1 q^{-1} T_{b_1})(T_{a_1}^{-1} - T_{b_1}).
\end{aligned} \tag{5.86}$$

Since $a_{M+1} = q$ and $qb_1 - a_1 \neq 0$, the function z satisfies

$$[(1 - T_{x_i})(1 - Cq^{-1} T_{x_1} \cdots T_{x_M}) - x_i(1 - B_i T_{x_i})(1 - A T_{x_1} \cdots T_{x_M})]z = 0. \tag{5.87}$$

Also we have

$$\begin{aligned}&[x_i(1-T_{x_j})(1-B_iT_{x_i})-x_j(1-T_{x_i})(1-B_jT_{x_j})]z\\&=[(1-T_{a_j}T_{b_j})(b_i-a_iT_{a_i}T_{b_i})-(1-T_{a_i}T_{b_i})(b_j-a_jT_{a_j}T_{b_j})]z.\end{aligned} \tag{5.88}$$

Similar to the above calculation, we have

$$[x_i(1-T_{x_j})(1-B_iT_{x_i})-x_j(1-T_{x_i})(1-B_jT_{x_j})]z=0. \tag{5.89}$$

□

Remark 5.18. In the above proof, we did not use the equation (5.72). In general, the equation (5.72) may be derived by (5.73),...,(5.77).

Corollary 5.19. *Suppose $\tau\in\{1,q/(B_1x_1),\dots,q/(B_Mx_M),d\infty\}$, where $d\in\mathbb{C}$ is an arbitrary point. Then the integral*

$$\int_0^\tau t^{\alpha-1}\frac{(qt)_\infty}{(Ct/A)_\infty}\prod_{i=1}^M\frac{(B_ix_it)_\infty}{(x_it)_\infty}d_qt, \tag{5.90}$$

is a solution of the q-Appell-Lauricella system.

Finally we consider the degenerations for the series solution $W\left(\begin{array}{c}\{a_i\}_{1\le i\le M+3}\\\{b_i\}_{1\le i\le M+3}\end{array}\right)$ of the system q-RP^M. We recall the definition of W (5.46):

$$\begin{aligned}&W\left(\begin{array}{c}\{a_i\}_{1\le i\le M+3}\\\{b_i\}_{1\le i\le M+3}\end{array}\right)\\&=\frac{(a_{M+1}/b_{M+3},b_{M+1}b_{M+3}q^2/(a_{M+2}a_{M+3}),b_{M+2}b_{M+3}q^2/(a_{M+2}a_{M+3}),a_{M+2}q/a_{M+3},a_{M+3}/a_{M+2})_\infty}{a_{M+3}(b_{M+1}q/a_{M+2},b_{M+1}q/a_{M+3},b_{M+2}q/a_{M+2},b_{M+2}q/a_{M+3},b_{M+3}q/a_{M+2},b_{M+3}q/a_{M+3})_\infty}\\&\times\prod_{i=1}^M\frac{(b_ib_{M+3}q^2/(a_{M+2}a_{M+3}),a_iq/a_{M+2},a_iq/a_{M+3})_\infty}{(a_ib_{M+3}q^2/(a_{M+2}a_{M+3}),b_iq/a_{M+2},b_iq/a_{M+3})_\infty}\\&\times W^{M,2}\left(\begin{array}{c}\{a_i/b_i\}_{1\le i\le M}\\\{a_i\}_{1\le i\le M}\end{array}\middle|\,\frac{a_Mb_{M+3}q}{a_{M+2}a_{M+3}};\frac{b_{M+3}q}{a_{M+2}},\frac{b_{M+3}q}{a_{M+3}};\frac{a_M}{b_{M+1}},\frac{a_M}{b_{M+2}};\frac{a_{M+1}}{b_{M+3}}\right),\end{aligned} \tag{5.91}$$

$$\begin{aligned}&W^{M,2}\left(\begin{array}{c}\{a_i/b_i\}_{1\le i\le M}\\\{a_i\}_{1\le i\le M}\end{array}\middle|\,\frac{a_Mb_{M+3}q}{a_{M+2}a_{M+3}};\frac{b_{M+3}q}{a_{M+2}},\frac{b_{M+3}q}{a_{M+3}};\frac{a_M}{b_{M+1}},\frac{a_M}{b_{M+2}};\frac{a_{M+1}}{b_{M+3}}\right)\\&=\sum_{l\in(\mathbb{Z}_{\ge0})^M}\left(\frac{a_{M+1}}{b_{M+3}}\right)^{|l|}\frac{\Delta(\{a_iq^{l_i}\}_{1\le i\le M})}{\Delta(\{a_i\}_{1\le i\le M})}\prod_{1\le i\le M}\frac{1-q^{|l|+l_i}a_ib_{M+3}/(a_{M+2}a_{M+3})}{1-a_ib_{M+3}/(a_{M+2}a_{M+3})}\\&\times\prod_{1\le j\le M}\left(\frac{(qa_jb_{M+3}/(a_{M+2}a_{M+3}))_{|l|}}{(q^2b_jb_{M+3}/(a_{M+2}a_{M+3}))_{|l|}}\prod_{1\le i\le M}\frac{(a_i/b_j)_{l_i}}{(qa_i/a_j)_{l_i}}\right)\\&\times\frac{(a_M/b_{M+1},a_M/b_{M+2})_{|l|}}{(a_Mq/a_{M+2},a_Mq/a_{M+3})_{|l|}}\prod_{1\le i\le M}\frac{(a_ib_{M+3}q/(a_Ma_{M+2}),a_ib_{M+3}q/(a_Ma_{M+3}))_{l_i}}{(q^2a_ib_{M+1}b_{M+3}/(a_Ma_{M+2}a_{M+3}),q^2a_ib_{M+2}b_{M+3}/(a_Ma_{M+2}a_{M+3}))_{l_i}}.\end{aligned} \tag{5.92}$$

Due to the integral representation of W (5.51):

$$W\left(\begin{array}{c}\{a_i\}_{1\le i\le M+3}\\\{b_i\}_{1\le i\le M+3}\end{array}\right)=\frac{1}{q(1-q)(q)_\infty}\int_{q/a_{M+2}}^{q/a_{M+3}}\prod_{k=1}^{M+3}\frac{(a_kt)_\infty}{(b_kt)_\infty}d_qt, \tag{5.93}$$

and the equation (5.56), we will consider the similar degeneration of $C_0\left(\dfrac{q}{a_{M+2}}\right) W\left(\begin{array}{c}\{a_i\}_{1\le i\le M+3}\\ \{b_i\}_{1\le i\le M+3}\end{array}\right)$. By the definition of the q-shifted factorial, we have

$$\lim_{x\to\infty}\frac{(ax)_m}{(bx)_m}=\left(\frac{a}{b}\right)^m,\ \lim_{x\to\infty}(a/x)_m=1,\ \lim_{x\to\infty}\frac{(ax)_m}{x^m}=(-a)^m q^{\binom{m}{2}}. \tag{5.94}$$

Using these formulas, we have

$$\begin{aligned}
&\lim_{a_{M+3}\to\infty} W^{M,2}\left(\begin{array}{c}\{a_i/b_i\}_{1\le i\le M}\\ \{a_i\}_{1\le i\le M}\end{array}\middle|\ \frac{a_M b_{M+3} q}{a_{M+2}a_{M+3}};\frac{b_{M+3}q}{a_{M+2}},\frac{b_{M+3}q}{a_{M+3}};\frac{a_M}{b_{M+1}},\frac{a_M}{b_{M+2}};\frac{a_{M+1}}{b_{M+3}}\right)\Bigg|_{b_{M+3}=q^\lambda a_{M+3}}\\
&=\sum_{l\in(\mathbb{Z}_{\ge0})^M}\left(-\frac{a_{M+1}q}{a_M a_{M+2}}\right)^{|l|}\frac{\Delta(\{a_iq^{l_i}\}_{1\le i\le M})}{\Delta(\{a_i\}_{1\le i\le M})}\prod_{1\le i\le M}\frac{1-q^{|l|+l_i}a_iq^\lambda/a_{M+2}}{1-a_iq^\lambda/a_{M+2}}\\
&\times\prod_{1\le j\le M}\left(\frac{(qa_jq^\lambda/a_{M+2})_{|l|}}{(q^2b_jq^\lambda/a_{M+2})_{|l|}}\prod_{1\le i\le M}\frac{(a_i/b_j)_{l_i}}{(qa_i/a_j)_{l_i}}\right)\\
&\times\frac{(a_M/b_{M+1},a_M/b_{M+2})_{|l|}}{(a_Mq/a_{M+2})_{|l|}}\prod_{1\le i\le M}a_i^{l_i}q^{\binom{l_i}{2}}\frac{(a_iq^\lambda q/a_M)_{l_i}}{(q^2a_ib_{M+1}q^\lambda/(a_Ma_{M+2}),q^2a_ib_{M+2}q^\lambda/(a_Ma_{M+2}))_{l_i}}.
\end{aligned}\tag{5.95}$$

Next we consider the limit $a_{M+2}\to 0$ with $b_{M+2}=q^\beta a_{M+2}$. Due to the condition $a_1\cdots a_{M+3}=q^2b_1\cdots b_{M+3}$, β satisfies $a_1\cdots a_{M+1}=q^2q^\lambda q^\beta b_1\cdots b_{M+1}$. Using this relation partly, we have

$$\begin{aligned}
&\lim_{a_{M+2}\to0}(\text{right-hand-side of (5.95)})|_{b_{M+2}=q^\beta a_{M+2}}\\
&=\sum_{l\in(\mathbb{Z}_{\ge0})^M}q^{|l|}\frac{\Delta(\{a_iq^{l_i}\}_{1\le i\le M})}{\Delta(\{a_i\}_{1\le i\le M})}\prod_{1\le j\le M}\left(\prod_{1\le i\le M}\frac{(a_i/b_j)_{l_i}}{(qa_i/a_j)_{l_i}}\right)\\
&\times(a_M/b_{M+1})_{|l|}\prod_{1\le i\le M}\frac{(a_iq^\lambda q/a_M)_{l_i}}{(q^2a_iq^\beta q^\lambda/a_M)_{l_i}},
\end{aligned}\tag{5.96}$$

$$\begin{aligned}
&\lim_{a_{M+2}\to0}(s(a_{M+1},a_{M+2}).\ (\text{right-hand-side of (5.95)}))|_{b_{M+2}=q^\beta a_{M+2}}\\
&=\sum_{l\in(\mathbb{Z}_{\ge0})^M}\left(\frac{q}{q^\beta a_{M+1}}\right)^{|l|}q^{\binom{|l|}{2}}\frac{\Delta(\{a_iq^{l_i}\}_{1\le i\le M})}{\Delta(\{a_i\}_{1\le i\le M})}\prod_{1\le i\le M}\frac{1-q^{|l|+l_i}a_iq^\lambda/a_{M+1}}{1-a_iq^\lambda/a_{M+1}}\\
&\times\prod_{1\le j\le M}\left(\frac{(qa_jq^\lambda/a_{M+1})_{|l|}}{(q^2b_jq^\lambda/a_{M+1})_{|l|}}\prod_{1\le i\le M}\frac{(a_i/b_j)_{l_i}}{(qa_i/a_j)_{l_i}}\right)\\
&\times\frac{(a_M/b_{M+1})_{|l|}}{(a_Mq/a_{M+1})_{|l|}}\prod_{1\le i\le M}a_i^{l_i}q^{\binom{l_i}{2}}\frac{(a_iq^\lambda q/a_M)_{l_i}}{(q^2a_ib_{M+1}q^\lambda/(a_Ma_{M+1}))_{l_i}}.
\end{aligned}\tag{5.97}$$

Here $s(x,y): x\leftrightarrow y$. We put

$$\begin{aligned}
g=&\frac{(a_{M+1}/b_{M+3},b_{M+1}b_{M+3}q^2/(a_{M+2}a_{M+3}),b_{M+2}b_{M+3}q^2/(a_{M+2}a_{M+3}),a_{M+2}q/a_{M+3},a_{M+3}/a_{M+2})_\infty}{a_{M+3}(b_{M+1}q/a_{M+2},b_{M+1}q/a_{M+3},b_{M+2}q/a_{M+2},b_{M+2}q/a_{M+3},b_{M+3}q/a_{M+2},b_{M+3}q/a_{M+3})_\infty}\\
&\times\prod_{i=1}^M\frac{(b_ib_{M+3}q^2/(a_{M+2}a_{M+3}),a_iq/a_{M+2},a_iq/a_{M+3})_\infty}{(a_ib_{M+3}q^2/(a_{M+2}a_{M+3}),b_iq/a_{M+2},b_iq/a_{M+3})_\infty}.
\end{aligned}\tag{5.98}$$

By taking the same degenerations for the corresponding gauge factors $C_1 g$, $C_2 s(a_{M+1}, a_{M+2}).\, g$ with suitable pseudo constants C_i $(i = 1, 2)$, we get series solutions for the system $q\text{-}RP^M_{\text{degene}}$. We only show the calculation for $C_1 g$. Another limit can be calculated similarly. The factor g is rewritten as

$$
\begin{aligned}
C_0\left(\frac{q}{a_{M+2}}\right) g =& \frac{(b_{M+1}b_{M+3}q^2/(a_{M+2}a_{M+3}), b_{M+2}b_{M+3}q^2/(a_{M+2}a_{M+3}))_\infty}{(b_{M+1}q/a_{M+2}, b_{M+2}q/a_{M+2}, b_{M+3}q/a_{M+3})_\infty}\left(\frac{q}{a_{M+2}}\right)^\lambda \\
&\times \prod_{i=1}^{M} \frac{(b_i b_{M+3}q^2/(a_{M+2}a_{M+3}), a_i q/a_{M+2})_\infty}{(a_i b_{M+3}q^2/(a_{M+2}a_{M+3}), b_i q/a_{M+2})_\infty} \\
&\times \frac{(a_{M+1}/b_{M+3}, a_{M+2}q/a_{M+3})_\infty}{(b_{M+1}q/a_{M+3}, b_{M+2}q/a_{M+3})_\infty} \prod_{i=1}^{M} \frac{(a_i q/a_{M+3})_\infty}{(b_i q/a_{M+3})_\infty} \cdot \frac{(a_{M+2}/b_{M+3})_\infty}{(a_{M+2}/a_{M+3})_\infty} \\
&\times \frac{(a_{M+3}/a_{M+2})_\infty}{a_{M+3}(a_{M+3}q/a_{M+2})_\infty}. \qquad (5.99)
\end{aligned}
$$

By the definition of the q-shifted factorial, we have

$$
\lim_{x\to\infty}(1/x)_\infty = 1, \ \ \frac{(x)_\infty}{(qx)_\infty} = 1 - x. \qquad (5.100)
$$

Thus we have

$$
\begin{aligned}
&\lim_{a_{M+3}\to\infty}\left(C_0\left(\frac{q}{a_{M+2}}\right) g\right)\bigg|_{b_{M+3}=q^\lambda a_{M+3}} \\
&= -\frac{1}{a_{M+2}} \frac{(b_{M+1}q^\lambda q^2/a_{M+2}, b_{M+2}q^\lambda q^2/a_{M+2})_\infty}{(b_{M+1}q/a_{M+2}, b_{M+2}q/a_{M+2}, q^\lambda q)_\infty}\left(\frac{q}{a_{M+2}}\right)^\lambda \prod_{i=1}^{M} \frac{(b_i q^\lambda q^2/a_{M+2}, a_i q/a_{M+2})_\infty}{(a_i q^\lambda q^2/a_{M+2}, b_i q/a_{M+2})_\infty}.
\end{aligned} \qquad (5.101)
$$

Also we have

$$
\begin{aligned}
&\lim_{a_{M+2}\to 0}\Bigg(-q\left(\frac{b_{M+1}q}{a_{M+2}}\right)^{-\lambda-1} \frac{\theta(b_{M+1}q/a_{M+2})}{\theta(b_{M+1}qq^{\lambda+1}/a_{M+2})} \prod_{i=1}^{M} \frac{\theta(b_i q^\lambda q^2/a_{M+2})\theta(a_i q/a_{M+2})}{\theta(a_i q^\lambda q^2/a_{M+2})\theta(b_i q/a_{M+2})} \\
&\qquad \times (\text{right-hand-side of (5.101)})\Bigg) \\
&= b_{M+1}^{-\lambda-1} \frac{(q^\lambda q^\beta q^2)_\infty}{(q^\lambda q, q^\beta q)_\infty}. \qquad (5.102)
\end{aligned}
$$

Therefore the function

$$
\begin{aligned}
&b_{M+1}^{-\lambda-1} \sum_{l\in(\mathbb{Z}_{\geq 0})^M} q^{|l|} \frac{\Delta(\{a_i q^{l_i}\}_{1\leq i\leq M})}{\Delta(\{a_i\}_{1\leq i\leq M})} \prod_{1\leq j\leq M}\left(\prod_{1\leq i\leq M} \frac{(a_i/b_j)_{l_i}}{(qa_i/a_j)_{l_i}}\right) \\
&\quad \times (a_M/b_{M+1})_{|l|} \prod_{1\leq i\leq M} \frac{(a_i q^\lambda q/a_M)_{l_i}}{(q^2 a_i q^\beta q^\lambda/a_M)_{l_i}}, \qquad (5.103)
\end{aligned}
$$

is a solution of $q\text{-}RP^M_{\text{degene}}$. By the similar method, we get solutions for $q\text{-}RP^M_{\text{degene}}$.

Proposition 5.20. *The following functions satisfy the system $q\text{-}RP^M_{\text{degene}}$:*

$$
b_{M+1}^{-\lambda-1} \sum_{l\in(\mathbb{Z}_{\geq 0})^M} q^{|l|} \frac{\Delta(\{a_i q^{l_i}\}_{1\leq i\leq M})}{\Delta(\{a_i\}_{1\leq i\leq M})} \prod_{1\leq j\leq M}\left(\prod_{1\leq i\leq M} \frac{(a_i/b_j)_{l_i}}{(qa_i/a_j)_{l_i}}\right)
$$

$$\times (a_M/b_{M+1})_{|l|} \prod_{1\le i\le M} \frac{(a_i q^\lambda q/a_M)_{l_i}}{(q^2 a_i q^\beta q^\lambda/a_M)_{l_i}}, \tag{5.104}$$

$$a_{M+1}^{-\lambda-1} \frac{(b_{M+1}q^{\lambda+2}/a_{M+1})_\infty}{(b_{M+1}q/a_{M+1})_\infty} \prod_{i=1}^{M} \frac{(b_i q^\lambda q^2/a_{M+1}, a_i q/a_{M+1})_\infty}{(a_i q^\lambda q^2/a_{M+1}, b_i q/a_{M+1})_\infty}$$

$$\times \sum_{l\in(\mathbb{Z}_{\ge0})^M} \left(\frac{q}{q^\beta a_{M+1}}\right)^{|l|} q^{\binom{|l|}{2}} \frac{\Delta(\{a_i q^{l_i}\}_{1\le i\le M})}{\Delta(\{a_i\}_{1\le i\le M})} \prod_{1\le i\le M} \frac{1-q^{|l|+l_i} a_i q^\lambda/a_{M+1}}{1-a_i q^\lambda/a_{M+1}}$$

$$\times \prod_{1\le j\le M} \left(\frac{(q a_j q^\lambda/a_{M+1})_{|l|}}{(q^2 b_j q^\lambda/a_{M+1})_{|l|}} \prod_{1\le i\le M} \frac{(a_i/b_j)_{l_i}}{(q a_i/a_j)_{l_i}} \right)$$

$$\times \frac{(a_M/b_{M+1})_{|l|}}{(a_M q/a_{M+1})_{|l|}} \prod_{1\le i\le M} a_i^{l_i} q^{\binom{l_i}{2}} \frac{(a_i q^\lambda q/a_M)_{l_i}}{(q^2 a_i b_{M+1} q^\lambda/(a_M a_{M+1}))_{l_i}}. \tag{5.105}$$

Here, $q^\beta = a_1 \cdots a_{M+1}/(q^2 q^\lambda b_1 \cdots b_{M+1})$.

Corollary 5.21. *The functions*

$$\sum_{l\in(\mathbb{Z}_{\ge0})^M} q^{|l|} \frac{\Delta(\{B_i x_i q^{l_i}\}_{1\le i\le M})}{\Delta(\{B_i x_i\}_{1\le i\le M})} \prod_{1\le j\le M} \left(\prod_{1\le i\le M} \frac{(B_i x_i/x_j)_{l_i}}{(q B_i x_i/(B_j x_j))_{l_i}} \right)$$

$$\times (B_M x_M A/C)_{|l|} \prod_{1\le i\le M} \frac{(A B_i x_i/(B_M x_M))_{l_i}}{(A B_i x_i B_1 \cdots B_M q/(C B_M x_M))_{l_i}}, \tag{5.106}$$

$$\prod_{i=1}^{M} \frac{(A x_i, B_i x_i)_\infty}{(A B_i x_i, x_i)_\infty}$$

$$\times \sum_{l\in(\mathbb{Z}_{\ge0})^M} \left(\frac{C}{B_1\cdots B_M}\right)^{|l|} q^{\binom{|l|}{2}} \frac{\Delta(\{B_i x_i q^{l_i}\}_{1\le i\le M})}{\Delta(\{B_i x_i\}_{1\le i\le M})} \prod_{1\le i\le M} \frac{1-q^{|l|+l_i} A B_i x_i}{1-A B_i x_i}$$

$$\times \prod_{1\le j\le M} \left(\frac{(A B_j x_j/q)_{|l|}}{(A x_j)_{|l|}} \prod_{1\le i\le M} \frac{(B_i x_i/x_j)_{l_i}}{(q B_i x_i/(B_j x_j))_{l_i}} \right)$$

$$\times \frac{(A B_M x_M/C)_{|l|}}{(B_M x_M)_{|l|}} \prod_{1\le i\le M} (B_i x_i)^{l_i} q^{\binom{l_i}{2}} \frac{(A B_i x_i/(B_M x_M))_{l_i}}{(C B_i x_i/(B_M x_M))_{l_i}}. \tag{5.107}$$

are solutions of the q-Appell-Lauricella system (5.79), (5.80).

Proof. We get desired results by putting $a_i = B_i x_i$, $b_i = x_i$ $(1 \le i \le M)$, $a_{M+1} = q$, $b_{M+1} = C/A$, $q^\lambda = A/q$. Note that $q^\beta = q^{-2} q^{-\lambda} a_1 \cdots a_{M+1}/(b_1 \cdots b_{M+1}) = B_1 \cdots B_M/C$ □

Remark 5.22. In section 2.5, we get series solutions for the Heine's equation, which are q-analogs of the Kummer's 24 solutions for the Gauss hypergeometric equation. However it seems to be difficult to obtain similar results in multivariable cases.

In section 4, a characterization of solutions for the q-Appell-Lauricella system was obtained. Using this characterization, we find a relation between the above solution (5.107) and the q-Appell-Lauricella series φ_D. We remark that a similar characterization of the solution (5.106) is not obtained.

Proposition 5.23. *We have*

$$\prod_{i=1}^{M}\frac{(Ax_i,B_ix_i)_\infty}{(AB_ix_i,x_i)_\infty}$$
$$\times\sum_{l\in(\mathbb{Z}_{\geq 0})^M}\left(\frac{C}{B_1\cdots B_M}\right)^{|l|}q^{\binom{|l|}{2}}\frac{\Delta(\{B_ix_iq^{l_i}\}_{1\leq i\leq M})}{\Delta(\{B_ix_i\}_{1\leq i\leq M})}\prod_{1\leq i\leq M}\frac{1-q^{|l|+l_i}AB_ix_i}{1-AB_ix_i}$$
$$\times\prod_{1\leq j\leq M}\left(\frac{(AB_jx_j/q)_{|l|}}{(Ax_j)_{|l|}}\prod_{1\leq i\leq M}\frac{(B_ix_i/x_j)_{l_i}}{(qB_ix_i/(B_jx_j))_{l_i}}\right)$$
$$\times\frac{(AB_Mx_M/C)_{|l|}}{(B_Mx_M)_{|l|}}\prod_{1\leq i\leq M}(B_ix_i)^{l_i}q^{\binom{l_i}{2}}\frac{(AB_ix_i/(B_Mx_M))_{l_i}}{(CB_ix_i/(B_Mx_M))_{l_i}}$$
$$=\varphi_D\left(\begin{matrix}A;B_1,\dots,B_M\\ C\end{matrix};x_1,\dots,x_M\right). \tag{5.108}$$

Proof. The q-Appell-Lauricella series

$$\varphi_D\left(\begin{matrix}A;B_1,\dots,B_M\\ C\end{matrix};x_1,\dots,x_M\right), \tag{5.109}$$

is the unique solution for the q-Appell-Lauricella system, which has the asymptotic behavior

$$1+O(|x_1/x_2|+|x_2/x_3|+\cdots+|x_{M-1}/x_M|+|x_M|). \tag{5.110}$$

This characterization is obtained in Proposition 4.6. We put $X_1=x_1/x_2$, $X_2=x_2/x_3$, $\dots$, $X_{M-1}=x_{M-1}/x_M$, $X_M=x_M$, then we have

$$\Delta(\{B_ix_i\}_{1\leq i\leq M})=\prod_{1\leq i<j\leq M}(B_iX_i\cdots X_M-B_jX_j\cdots X_M)=\prod_{1\leq i<j\leq M}X_j\cdots X_M(B_iX_i\cdots X_{j-1}-B_j). \tag{5.111}$$

Thus we have

$$\frac{\Delta(\{B_ix_iq^{l_i}\}_{1\leq i\leq M})}{\Delta(\{B_ix_i\}_{1\leq i\leq M})}=\prod_{1\leq i<j\leq M}\frac{q^{l_j}B_j-q^{l_i}B_iX_i\cdots X_{j-1}}{B_j-B_iX_i\cdots X_{j-1}}=\prod_{1\leq i<j\leq M}q^{l_j}+O(|X_1|+\cdot+|X_M|). \tag{5.112}$$

For $i<j$, we have

$$\frac{(B_ix_i/x_j)_{l_i}}{(qB_ix_i/(B_jx_j))_{l_i}}=\frac{(B_iX_i\cdots X_{j-1})_{l_i}}{(qB_iX_i\cdots X_{j-1}/B_j)_{l_i}}=1+O(|X_1|+\cdots+|X_M|). \tag{5.113}$$

For $i>j$, we have

$$\frac{(B_ix_i/x_j)_{l_i}}{(qB_ix_i/(B_jx_j))_{l_i}}=\left(\frac{B_j}{q}\right)^{l_i}\frac{(qx_j/(B_ix_i))_{l_i}}{(qB_jx_j/(B_ix_i))_{l_i}}=\left(\frac{B_j}{q}\right)^{l_i}+O(|X_1|+\cdots+|X_M|). \tag{5.114}$$

Therefore we obtain

$$(\text{series }(5.107))=1+O(|x_1/x_2|+|x_2/x_3|+\cdots+|x_{M-1}/x_M|+|x_M|). \tag{5.115}$$

This completes the proof. □

Remark 5.24. As a similar but different formula to (5.108), the following transformation formula was known in [29]:

$$\frac{(u)_\infty}{(a_1\cdots a_M u)_\infty}\prod_{1\le i\le M}\frac{(cx_i/x_M)_\infty}{((c/a_i)x_i/x_M)_\infty}\sum_{l\in(\mathbb{Z}_{\ge 0})^M}u^{|l|}\frac{\Delta(xq^l)}{\Delta(x)}\prod_{1\le i,j\le M}\frac{(a_jx_i/x_j)_{l_i}}{(qx_i/x_j)_{l_i}}\prod_{1\le i\le M}\frac{(bx_i/x_M)_{l_i}}{(cx_i/x_M)_{l_i}}$$
$$=\varphi_D\left(\begin{matrix}a_1\cdots a_M bu/c; a_1,\ldots,a_M\\ a_1\cdots a_M u\end{matrix};\frac{c}{a_1}\frac{x_1}{x_M},\ldots,\frac{c}{a_{M-1}}\frac{x_{M-1}}{x_M},\frac{c}{a_M}\right). \tag{5.116}$$

This formula was obtained by using some transformation formula for multiple q-hypergeometric series and the Andrews' formula [3]. Putting $a_i=B_i$, $u=C/(B_1\cdots B_M)$, $c=x_M$, $b=Ax_M/C$, and $x_i\to B_ix_i$, the right-hand-side becomes

$$\varphi_D\left(\begin{matrix}A;B_1,\ldots,B_M\\ C\end{matrix};x_1,\ldots,x_M\right). \tag{5.117}$$

Therefore we find that the series

$$\prod_{1\le i\le M}\frac{(B_ix_i)_\infty}{(x_i)_\infty}\sum_{l\in(\mathbb{Z}_{\ge 0})^M}u^{|l|}\frac{\Delta(B_ix_iq^{l_i})}{\Delta(B_ix_i)}\prod_{1\le i,j\le M}\frac{(B_ix_i/x_j)_{l_i}}{(qB_ix_i/(B_jx_j))_{l_i}}\prod_{1\le i\le M}\frac{(AB_ix_i/C)_{l_i}}{(B_ix_i)_{l_i}}, \tag{5.118}$$

satisfies the q-Appell-Lauricella system. If we directly prove that the series (5.118) satisfies the q-Appell-Lauricella system, we get another proof of the transformation formula (5.116) by the characterization, like the proof of Proposition 5.23.

6 Summary and discussions

In this paper, we considered several extensions for the Heine's q-hypergeometric equation (1.5). We summarize the contents by each section.

In section 2, we discussed the variant of q-hypergeometric equation of degree three $\mathcal{H}_3$. Main results for the q-Riemann-Papperitz system (2.1) are Theorem 2.20 and Theorem 2.27, which give integral solutions and series solutions for the equation $\mathcal{H}_3$. The key point is to regard the equation as a q-Riemann-Papperitz system. In the theory of q-difference equations, we can not apply Möbius transformations, however we can consider the q-Riemann-Papperitz system and its solutions. More precisely, the series solution (2.61) is a q-analog of ${}_2F_1\left(\dfrac{x-t_1}{x-t_3}\dfrac{t_2-t_3}{t_2-t_1}\right)$. To get those results, we focused on the following type integral:

$$\int_C \frac{(a_1t, a_2t, a_3t, a_4t)_\infty}{(b_1t, b_2t, b_3t, b_4t)_\infty} d_qt, \tag{6.1}$$

where $a_1a_2a_3a_4 = q^2b_1b_2b_3b_4$. This is a Riemann-Papperitz type extension of Heine's integral:

$$\int_C t^{\alpha-1} \frac{(qt)_\infty}{(ct/a)_\infty} \frac{(bxt)_\infty}{(xt)_\infty} d_qt. \tag{6.2}$$

A connection formula for the solutions of the q-Riemann-Papperitz system (2.1) is given in section 2.4. By certain degenerations, we obtained several solutions for the variant of q-hypergeometric equation of degree two $\mathcal{H}_2$ and the Heine's q-hypergeometric equation in section 2.5.

In section 3, we studied the properties of the Euler-type integral transformation

$$E^\lambda(f)(x;\sigma) = x^{\lambda-1} \int_0^{\sigma\infty} f(t) \frac{(qt/x)_\infty}{(q^\lambda t/x)_\infty} d_qt. \tag{6.3}$$

Applying this transformation to solutions for q-difference equations, we can regard E^λ as a transformation among q-difference equations. We constructed an integral transformation for the q-Heun equation (3.11), defined by Hahn [19] and also studied by Takemura [61, 62]. Using this transformation and a suitable gauge transformation, we gave a representation of affine Weyl group of type $D_5^{(1)}$. We also constructed representations of affine Weyl groups of types $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$ by integral transformations for suitable q-difference equations. Main result in section 3 is Theorem 3.16, which gives a representation for the case of $E_8^{(1)}$. The equations for the cases of $E_6^{(1)}$, $E_7^{(1)}$ are equivalent to variants of the q-Heun equation [62]. The q-difference operator for the case of $E_8^{(1)}$ is studied by Moriyama-Yamada [41] in order to give quantum representations for affine Weyl groups. Note that their representation is constructed in a formal algebraic sense, and our representation is given analytically.

In section 4, we gave the fundamental solution $\boldsymbol{u}^{L,\sigma}$, for the q-difference system $E_{N,M}$ defined by Park [51, 52]. This solution converges in $\{0 \ll |x_{\sigma(1)}| \ll \cdots \ll |x_{\sigma(L)}| \ll 1 \ll |x_{\sigma(L+1)}| \ll \cdots \ll |x_{\sigma(M)}|\}$. Every component of our solution $\boldsymbol{u}^{L,\sigma}$ is characterized by an asymptotic behavior of the form

$$x^\delta(1+O(||z||)), \quad z = \left(\frac{x_{\sigma(1)}}{x_{\sigma(2)}}, \ldots, \frac{x_{\sigma(L-1)}}{x_{\sigma(L)}}, x_{\sigma(L)}, \frac{1}{x_{\sigma(L+1)}}, \frac{x_{\sigma(L+1)}}{x_{\sigma(L+2)}}, \ldots, \frac{x_{\sigma(M-1)}}{x_{\sigma(M)}}\right). \tag{6.4}$$

Main result in section 4 is Theorem 4.9, which gives a connection matrix between $\boldsymbol{u}^{L_1,\sigma_1}$ and $\boldsymbol{u}^{L_2,\sigma_2}$. We obtained this matrix by using the connection formula [64, 67] for the generalized q-hypergeometric series ${}_{N+1}\varphi_N$ inductively.

In section 5, we considered the system that the following integral satisfies:

$$\int_C \prod_{k=1}^{M+3} \frac{(a_k t)_\infty}{(b_k t)_\infty} d_q t, \tag{6.5}$$

where $a_1 \cdots a_{M+3} = q^2 b_1 \cdots b_{M+3}$. This is a multivariable extension of the Riemann-Papperitz type integral (6.1). Main results for this integral are Theorem 5.4 and Theorem 5.12, which gives the q-difference system q-RP^M associated with the integral and its solutions in terms of the Kajihara's q-hypergeometric series [29]. Theorem 5.11 gives a transformation formula between the above integral and the Kajihara's q-hypergeometric series $W^{M,2}$. Due to this formula, we obtained linear relations among $W^{M,2}$, which are written in Theorem 5.14. These relations are connection formulas for solutions of q-RP^M. We also showed that the system q-RP^M is an extension of the q-Appell-Lauricella system. As an application we gave a transformation formula, which seems to be new, between the q-Appell-Lauricella series φ_D and a degeneration of $W^{M,2}$ by using the characterization of solutions for the q-Appell-Lauricella system.

There are many related open problems.

(1) For the Appell-Lauricella differential system, hypergeometric series solutions are known, besides near origin and infinity. For example, the series

$$F_D\left(\begin{matrix} \alpha; \{\beta_i\}_{1\le i\le M} \\ \alpha+\beta_1+\cdots+\beta_M-\gamma+1 \end{matrix} ; 1-x_1, \ldots, 1-x_M \right), \tag{6.6}$$

satisfies the Appell-Lauricella system. For the case of $M = 2$, see [8, 14, 66], and see also recent reference [40]. It is expected that the q-Appell-Lauricella system has also a solution in terms of a q-analog of the above series F_D. To get such a solution is interesting. We discussed the case of $M = 1$ in section 2.5.

(2) In sections 2.3 and 5.2, we showed that q-hypergeometric series ${}_8W_7$ and $W^{M,2}$ satisfy the corresponding q-difference system. The way to find these series solutions is due to a transformation formula of the Jackson integral:

$$\int_C \prod_{k=1}^{M+3} \frac{(a_i t)_\infty}{(b_i t)_\infty} d_q t. \tag{6.7}$$

It is important to find series solutions directly. If we can derive the system directly from the definition of ${}_8W_7$, $W^{M,2}$, it would be extended more general q-/elliptic hypergeometric series. Note that an elliptic analog of Kajihara's q-hypergeometric series and its transformation formula are studied in [30].

(3) The Heine's q-hypergeometric series

$${}_2\varphi_1\left(\begin{matrix} a, b \\ c \end{matrix} ; x \right), \tag{6.8}$$

is characterized as the solution of the form $1 + O(x)$ for Heine's equation. Also the q-Appell-Lauricella series

$$\varphi_D\left(\begin{matrix} a; \{b_i\}_{1\le i\le M} \\ c \end{matrix} ; \{x_i\}_{1\le i\le M} \right), \tag{6.9}$$

is characterized as the solution of the form $1 + O(|x_1| + \cdots + |x_M|)$ for the q-Appell-Lauricella system. A characterization of solutions for the q-Appell-Lauricella system was be characterized by some conditions. It is an important problem and would be related to the above problem (2). In the last of section 5.3, we get a relation among the q-Appell-Lauricella series φ_D and a degeneration of $W^{M,2}$ by using a characterization of solutions for the q-Appell-Lauricella system. So a characterization for solutions ${}_8W_7$ and $W^{M,2}$ may be also useful to get some transformation formulas.

(4) In this paper, we studied the Jackson integral (6.7). There are many interesting hypergeometric type Jackson integral. For example, a q-Selberg type integral

$$\int_C \prod_{i=1}^{n}\left(t_i^{\alpha}\prod_{r=1}^{m}\frac{(qa_r^{-1}t_i)_\infty}{(b_rt_i)_\infty}\right)\prod_{1\le j<k\le n}t_j^{2\tau-1}\frac{(q^{1-\tau}t_k/t_j)_\infty}{(q^{\tau}t_k/t_j)_\infty}\frac{d_qt_1}{t_1}\wedge\cdots\wedge\frac{d_qt_n}{t_n}, \tag{6.10}$$

is an active area of research for special functions, combinatorics, mathematical physics and orthogonal polynomials. For details and histories, see [24, 25]. To consider a "Riemann-Papperitz type extension of the q-Selberg integral" is interesting and important. In addition, there are many important integrals for the differential case. For example, the Aomoto-Gelfand hypergeometric integrals and the GKZ hypergeometric integrals are so (see [7, 18, 34, 69]). To study q-analogs of these integrals is also interesting and important.

(5) In [42, 43], the q-Garnier system, which is a multivariable extension of the q-Painlevé VI equation, is considered by means of the Padé interpolation of some product ψ of the q-shifted factorial. For the Padé method to Painlevé equations, see also a recent book [44]. In [57], the Padé interpolation of some product of Jacobi's theta function is expressed explicitly in terms of the very-well-poised elliptic hypergeometric series ${}_{12}E_{11}$. This function ${}_{12}E_{11}$ has a transformation formula of the form:

$${}_{12}E_{11} = (\text{product of theta functions}) \times {}_{12}E_{11}. \tag{6.11}$$

That is the reason why the Padé problem is solved by using ${}_{12}E_{11}$. In this paper we got some transformation formulas for the Kajihara's q-hypergeometric function $W^{M,2}$. The Padé problem or its certain generalization associated with ψ may be solved by using $W^{M,2}$, similar to Spiridonov-Zhedanov's method [57], in particular, we will get a special solution for the q-Garnier system in terms of $W^{M,2}$.

(6) In section 2.3, we found that the very-well-poised q-hypergeometric function ${}_8W_7$ satisfies the q-Riemann-Papperitz system (2.1). As mentioned in Remark 2.31, the function ${}_8W_7$ is also known as the Askey-Wilson function, related to orthogonal polynomials. The Koornwinder polynomial is a multivariable extension of the Askey-Wilson polynomial, and it is an eigenfunction of Koornwinder's q-difference operator. To give a solution for the bispectral problem of Koornwinder operator explicitly is important for mathematical physics (see [36, 37, 58], and [50] for an explicit form of Macdonald's case). In this paper we discussed a certain extension $W^{M,2}$ of the Askey-Wilson function from the viewpoint of the integral representation of it. We hope that the series $W^{M,2}$ is an eigenfunction of Koornwinder operator of special eigenvalue. Note that an integral solution for the bispectral problem of the Macdonald's operator is given by a multiple Jackson integral in [50]. For general eigenvalue, a multiple extension of (5.1) would be needed.

(7) The variants of q-hypergeometric equation [21] are introduced as special cases of degenerations for the eigenvalue problem of Ruijsenaars-van Diejen operator of one variable case. In this paper, we introduced a multivariable extension of this variant from the viewpoint of the integral. Its relation to the Ruijsenaars-van Diejen operator [13, 53] is an interesting problem.

A Kummer's 24 solutions for the Gauss hypergeometric equation

In this appendix, we give a list of Kummer's 24 solutions for the Gauss hypergeometric equation

$$\left[x(1-x)\frac{d^2}{dx^2} + (\gamma - (\alpha+\beta+1)x)\frac{d}{dx} - \alpha\beta\right] y = 0. \tag{A.1}$$

As mentioned in section 1, this equation is characterized by the Riemann scheme

$$\left\{ \begin{array}{ccc} x=0 & x=1 & x=\infty \\ 0 & 0 & \alpha \\ 1-\gamma & \gamma-\alpha-\beta & \beta \end{array} \right\}. \tag{A.2}$$

By the gauge transformation $y_1 = x^{\gamma-1}y$, the Riemann scheme (A.2) is transformed to

$$\left\{ \begin{array}{ccc} x=0 & x=1 & x=\infty \\ \gamma-1 & 0 & \alpha-\gamma+1 \\ 0 & \gamma-\alpha-\beta & \beta-\gamma+1 \end{array} \right\}. \tag{A.3}$$

So the function $y_1 = {}_2F_1\left(\begin{array}{c} \alpha-\gamma+1, \beta-\gamma+1 \\ 2-\gamma \end{array} ; x \right)$ satisfies the equation corresponding to this Riemann scheme. Therefore $y = x^{1-\gamma}{}_2F_1\left(\begin{array}{c} \alpha-\gamma+1, \beta-\gamma+1 \\ 2-\gamma \end{array} ; x \right)$ is a solution of the Gauss hypergeometric equation. Also, by the gauge transformation $y_2 = (1-x)^{\alpha+\beta-\gamma}y$, the Riemann scheme (A.2) is transformed to

$$\left\{ \begin{array}{ccc} x=0 & x=1 & x=\infty \\ 0 & \alpha+\beta-\gamma & \gamma-\beta \\ 1-\gamma & 0 & \gamma-\alpha \end{array} \right\}. \tag{A.4}$$

So the function $y_2 = {}_2F_1\left(\begin{array}{c} \gamma-\alpha, \gamma-\beta \\ \gamma \end{array} ; x \right)$ satisfies the equation corresponding to this Riemann scheme. Therefore $y = (1-x)^{\gamma-\alpha-\beta}{}_2F_1\left(\begin{array}{c} \gamma-\alpha, \gamma-\beta \\ \gamma \end{array} ; x \right)$ is a solution of the Gauss hypergeometric equation. In a similar way, we find that the function $x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}{}_2F_1\left(\begin{array}{c} 1-\alpha, 1-\beta \\ 2-\gamma \end{array} ; x \right)$ is a solution of the Gauss' equation. Therefore we get 4 solutions by the gauge transformations.

Next we put $z = 1-x$, then the Riemann scheme (A.2) is transformed to

$$\left\{ \begin{array}{ccc} z=1 & z=0 & z=\infty \\ 0 & 0 & \alpha \\ 1-\gamma & \gamma-\alpha-\beta & \beta \end{array} \right\}. \tag{A.5}$$

So the function ${}_2F_1\left(\begin{array}{c} \alpha, \beta \\ \alpha+\beta-\gamma+1 \end{array} ; 1-x \right)$ satisfies the Gauss hypergeometric equation. There are 6 Möbius transformations $x \mapsto x, 1-x, \frac{1}{x}, \frac{1}{1-x}, \frac{x}{x-1}, \frac{x-1}{x}$ that the points $x = 0, 1, \infty$ are transformed to each other.

In conclusion, we get 24 solutions for the Gauss hypergeometric equation, which are called Kummer's 24 solutions. We list them as follows:

$${}_2F_1\left(\begin{array}{c} \alpha, \beta \\ \gamma \end{array} ; x \right), \quad x^{1-\gamma}{}_2F_1\left(\begin{array}{c} \alpha-\gamma+1, \beta-\gamma+1 \\ 2-\gamma \end{array} ; x \right),$$

$$(1-x)^{\gamma-\alpha-\beta}{}_2F_1\left(\begin{matrix}\gamma-\alpha,\gamma-\beta\\ \gamma\end{matrix};x\right),\ x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}{}_2F_1\left(\begin{matrix}1-\alpha,1-\beta\\ 2-\gamma\end{matrix};x\right),$$

$${}_2F_1\left(\begin{matrix}\alpha,\beta\\ \alpha+\beta-\gamma+1\end{matrix};1-x\right),\ x^{1-\gamma}{}_2F_1\left(\begin{matrix}\alpha-\gamma+1,\beta-\gamma+1\\ \alpha+\beta-\gamma+1\end{matrix};1-x\right),$$
$$(1-x)^{\gamma-\alpha-\beta}{}_2F_1\left(\begin{matrix}\gamma-\alpha,\gamma-\beta\\ \gamma-\alpha-\beta+1\end{matrix};1-x\right),\ x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}{}_2F_1\left(\begin{matrix}1-\alpha,1-\beta\\ \gamma-\alpha-\beta+1\end{matrix};1-x\right),$$

$$x^{-\alpha}{}_2F_1\left(\begin{matrix}\alpha,\alpha-\gamma+1\\ \alpha-\beta+1\end{matrix}\frac{1}{x}\right),\ x^{-\beta}{}_2F_1\left(\begin{matrix}\beta,\beta-\gamma+1\\ \beta-\alpha+1\end{matrix}\frac{1}{x}\right),$$
$$x^{-\alpha}\left(1-\frac{1}{x}\right)^{\gamma-\alpha-\beta}{}_2F_1\left(\begin{matrix}1-\beta,\gamma-\beta\\ \alpha-\beta+1\end{matrix}\frac{1}{x}\right),\ x^{-\beta}\left(1-\frac{1}{x}\right)^{\gamma-\alpha-\beta}{}_2F_1\left(\begin{matrix}1-\alpha,\gamma-\alpha\\ \beta-\alpha+1\end{matrix}\frac{1}{x}\right),$$

$$x^{-\alpha}\left(1-\frac{1}{x}\right)^{-\alpha}{}_2F_1\left(\begin{matrix}\alpha,\gamma-\alpha\\ \alpha-\beta+1\end{matrix};\frac{1}{1-x}\right),\ x^{-\beta}\left(1-\frac{1}{x}\right)^{-\beta}{}_2F_1\left(\begin{matrix}\beta,\gamma-\beta\\ \beta-\alpha+1\end{matrix};\frac{1}{1-x}\right),$$
$$x^{-\alpha}\left(1-\frac{1}{x}\right)^{\gamma-\alpha-1}{}_2F_1\left(\begin{matrix}1-\beta,\alpha-\gamma+1\\ \alpha-\beta+1\end{matrix};\frac{1}{1-x}\right),\ x^{-\beta}\left(1-\frac{1}{x}\right)^{\gamma-\beta-1}{}_2F_1\left(\begin{matrix}1-\alpha,\beta-\gamma+1\\ \beta-\alpha+1\end{matrix};\frac{1}{1-x}\right),$$

$$(1-x)^{-\alpha}{}_2F_1\left(\begin{matrix}\alpha,\gamma-\beta\\ \gamma\end{matrix};\frac{x}{x-1}\right),\ (1-x)^{-\beta}{}_2F_1\left(\begin{matrix}\beta,\gamma-\alpha\\ \gamma\end{matrix};\frac{x}{x-1}\right),$$
$$x^{1-\gamma}(1-x)^{\gamma-\alpha-1}{}_2F_1\left(\begin{matrix}\alpha-\gamma+1,1-\beta\\ 2-\gamma\end{matrix};\frac{x}{x-1}\right),\ x^{1-\gamma}(1-x)^{\gamma-\beta-1}{}_2F_1\left(\begin{matrix}\beta-\gamma+1,1-\alpha\\ 2-\gamma\end{matrix};\frac{x}{x-1}\right),$$

$$x^{-\alpha}{}_2F_1\left(\begin{matrix}\alpha,\alpha-\gamma+1\\ \alpha+\beta-\gamma+1\end{matrix};\frac{x-1}{x}\right),\ x^{-\beta}{}_2F_1\left(\begin{matrix}\beta,\beta-\gamma+1\\ \alpha+\beta-\gamma+1\end{matrix};\frac{x-1}{x}\right),$$
$$x^{\alpha-\gamma}(1-x)^{\gamma-\alpha-\beta}{}_2F_1\left(\begin{matrix}1-\alpha,\gamma-\alpha\\ \gamma-\alpha-\beta+1\end{matrix};\frac{x-1}{x}\right),\ x^{\beta-\gamma}(1-x)^{\gamma-\alpha-\beta}{}_2F_1\left(\begin{matrix}1-\beta,\gamma-\beta\\ \gamma-\alpha-\beta+1\end{matrix};\frac{x-1}{x}\right).$$

The dimension of the space of holomorphic solutions at $x=0$ is 1. So we have

$$\begin{aligned}{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};x\right)&=(1-x)^{\gamma-\alpha-\beta}{}_2F_1\left(\begin{matrix}\gamma-\alpha,\gamma-\beta\\ \gamma\end{matrix};x\right)\\ &=(1-x)^{-\alpha}{}_2F_1\left(\begin{matrix}\alpha,\gamma-\beta\\ \gamma\end{matrix};\frac{x}{x-1}\right)=(1-x)^{-\beta}{}_2F_1\left(\begin{matrix}\beta,\gamma-\alpha\\ \gamma\end{matrix};\frac{x}{x-1}\right).\end{aligned}\tag{A.6}$$

Similarly we have

$$\begin{aligned}&x^{1-\gamma}{}_2F_1\left(\begin{matrix}\alpha-\gamma+1,\beta-\gamma+1\\ 2-\gamma\end{matrix};x\right)=x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}{}_2F_1\left(\begin{matrix}1-\alpha,1-\beta\\ 2-\gamma\end{matrix};x\right)\\ &=x^{1-\gamma}(1-x)^{\gamma-\alpha-1}{}_2F_1\left(\begin{matrix}\alpha-\gamma+1,1-\beta\\ 2-\gamma\end{matrix};\frac{x}{x-1}\right)\\ &=x^{1-\gamma}(1-x)^{\gamma-\beta-1}{}_2F_1\left(\begin{matrix}\beta-\gamma+1,1-\alpha\\ 2-\gamma\end{matrix};\frac{x}{x-1}\right),\end{aligned}\tag{A.7}$$

$${}_2F_1\left(\begin{matrix}\alpha,\beta\\ \alpha+\beta-\gamma+1\end{matrix};1-x\right)=x^{1-\gamma}{}_2F_1\left(\begin{matrix}\alpha-\gamma+1,\beta-\gamma+1\\ \alpha+\beta-\gamma+1\end{matrix};1-x\right)$$

$$
= x^{-\alpha}{}_2F_1\left(\begin{matrix}\alpha,\alpha-\gamma+1\\ \alpha+\beta-\gamma+1\end{matrix};\frac{x-1}{x}\right) = x^{-\beta}{}_2F_1\left(\begin{matrix}\beta,\beta-\gamma+1\\ \alpha+\beta-\gamma+1\end{matrix};\frac{x-1}{x}\right), \tag{A.8}
$$

$$
\begin{aligned}
&(1-x)^{\gamma-\alpha-\beta}{}_2F_1\left(\begin{matrix}\gamma-\alpha,\gamma-\beta\\ \gamma-\alpha-\beta+1\end{matrix};1-x\right) = x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}{}_2F_1\left(\begin{matrix}1-\alpha,1-\beta\\ \gamma-\alpha-\beta+1\end{matrix};1-x\right)\\
&= x^{\alpha-\gamma}(1-x)^{\gamma-\alpha-\beta}{}_2F_1\left(\begin{matrix}1-\alpha,\gamma-\alpha\\ \gamma-\alpha-\beta+1\end{matrix};\frac{x-1}{x}\right)\\
&= x^{\beta-\gamma}(1-x)^{\gamma-\alpha-\beta}{}_2F_1\left(\begin{matrix}1-\beta,\gamma-\beta\\ \gamma-\alpha-\beta+1\end{matrix};\frac{x-1}{x}\right),
\end{aligned} \tag{A.9}
$$

$$
\begin{aligned}
&x^{-\alpha}{}_2F_1\left(\begin{matrix}\alpha,\alpha-\gamma+1\\ \alpha-\beta+1\end{matrix}\ \frac{1}{x}\right) = x^{-\alpha}\left(1-\frac{1}{x}\right)^{\gamma-\alpha-\beta}{}_2F_1\left(\begin{matrix}1-\beta,\gamma-\beta\\ \alpha-\beta+1\end{matrix}\ \frac{1}{x}\right)\\
&= x^{-\alpha}\left(1-\frac{1}{x}\right)^{-\alpha}{}_2F_1\left(\begin{matrix}\alpha,\gamma-\alpha\\ \alpha-\beta+1\end{matrix};\frac{1}{1-x}\right)\\
&= x^{-\alpha}\left(1-\frac{1}{x}\right)^{\gamma-\alpha-1}{}_2F_1\left(\begin{matrix}1-\beta,\alpha-\gamma+1\\ \alpha-\beta+1\end{matrix};\frac{1}{1-x}\right),
\end{aligned} \tag{A.10}
$$

$$
\begin{aligned}
&x^{-\beta}{}_2F_1\left(\begin{matrix}\beta,\beta-\gamma+1\\ \beta-\alpha+1\end{matrix}\ \frac{1}{x}\right) = x^{-\beta}\left(1-\frac{1}{x}\right)^{\gamma-\alpha-\beta}{}_2F_1\left(\begin{matrix}1-\alpha,\gamma-\alpha\\ \beta-\alpha+1\end{matrix}\ \frac{1}{x}\right)\\
&= x^{-\beta}\left(1-\frac{1}{x}\right)^{-\beta}{}_2F_1\left(\begin{matrix}\beta,\gamma-\beta\\ \beta-\alpha+1\end{matrix};\frac{1}{1-x}\right) = x^{-\beta}\left(1-\frac{1}{x}\right)^{\gamma-\beta-1}{}_2F_1\left(\begin{matrix}1-\alpha,\beta-\gamma+1\\ \beta-\alpha+1\end{matrix};\frac{1}{1-x}\right).
\end{aligned} \tag{A.11}
$$

Remark A.1. In [68], 24 solutions are obtained from the point of the Riemann-Papperitz equation.

In section 2.5, we gave Kummer type solutions for the Heine's equation

$$
[(1-T_x)(1-cq^{-1}T_x) - x(1-aT_x)(1-bT_x)]y = 0, \tag{A.12}
$$

by the degeneration of the q-Riemann-Papperitz system (2.1). The list of them is in Theorem 2.49. A part of this list can be also obtained in a similar method for the Gauss' equation. We apply the gauge transformation $y_1 = x^{\gamma-1}y$ where $q^\gamma = c$, then we have

$$
[(1-c^{-1}qT_x)(1-T_x) - x(1-ac^{-1}qT_x)(1-bc^{-1}T_x)]y_1 = 0. \tag{A.13}
$$

Therefore the function $x^{1-\gamma}{}_2\varphi_1\left(\begin{matrix}aq/c,bq/c\\ q^2/c\end{matrix};x\right)$ satisfies the Heine's q-hypergeometric equation. The Heine's equation is rewritten as

$$
[(x-q)T_x^{-1} + (abx-c)T_x - ((a+b)x-(c+q))]y = 0. \tag{A.14}
$$

The function $\dfrac{(Ax)_\infty}{(Bx)_\infty}$ satisfies

$$
T_x\frac{(Ax)_\infty}{(Bx)_\infty} = \frac{1-Bx}{1-Ax}\frac{(Ax)_\infty}{(Bx)_\infty},\ T_x^{-1}\frac{(Ax)_\infty}{(Bx)_\infty} = \frac{1-Aq^{-1}x}{1-Bq^{-1}x}\frac{(Ax)_\infty}{(Bx)_\infty}. \tag{A.15}
$$

We put $y_2 = \dfrac{(Bx)_\infty}{(Ax)_\infty} y$, then we have

$$\left[(x-q)\frac{1-Aq^{-1}x}{1-Bq^{-1}x} T_x^{-1} + (abx-c)\frac{1-Bx}{1-Ax} T_x - ((a+b)x-(c+q)) \right] y_2 = 0. \tag{A.16}$$

We put $A = ab/c$, $B = 1$ and $z = abx/c$, then we have

$$[(z-q)T_z^{-1} + ((c/a)(c/b)z - c)T_z - ((c/a+c/b)z-(c+q))]y_2 = 0. \tag{A.17}$$

Therefore the function $\dfrac{(abx/c)_\infty}{(x)_\infty} {}_2\varphi_1 \left(\begin{matrix} c/a, c/b \\ c \end{matrix} \; \dfrac{abx}{c} \right)$ is a solution of the Heine's equation.

We can not apply Möbius transformations for q-difference equations except the scale transformation and inversion. So we can not get q-analogs of Kummer's 24 solutions by the Möbius transformations. The inversion $x \mapsto x^{-1}$ changes the points $x = 0, \infty$ to each other, so we can apply it to the Heine's equation. We put $z = dx^{-1}$ where d is a constant, then we have $T_x = T_z^{-1}$. Thus we have

$$\begin{aligned} &[(1-T_x)(1-cq^{-1}T_x) - x(1-aT_x)(1-bT_x)]y \\ &= -\frac{ab}{dz} T_z^{-2} [(1-a^{-1}T_z)(1-b^{-1}T_z) - \frac{cqd}{ab} z(1-T_z)(1-c^{-1}qT_z)]y. \end{aligned} \tag{A.18}$$

We put $d = ab/(cq)$, $\tilde{y} = z^{-\alpha} y$ where $a = q^\alpha$, then $\tilde{y}$ satisfies

$$[(1-T_z)(1-ab^{-1}T_z) - z(1-aT_z)(1-ac^{-1}qT_z)]\tilde{y} = 0. \tag{A.19}$$

We know a solution $\tilde{y} = {}_2\varphi_1 \left(\begin{matrix} a, aq/c \\ aq/b \end{matrix} ; z \right)$ for this equation, so the function $x^{-\alpha} {}_2\varphi_1 \left(\begin{matrix} a, aq/c \\ aq/b \end{matrix} ; \dfrac{cq}{abx} \right)$ is a solution of Heine's equation.

In this way, we see that the following functions satisfy the Heine's equation:

$$ {}_2\varphi_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; x \right), \tag{A.20}$$

$$ \frac{(abx/c)_\infty}{(x)_\infty} {}_2\varphi_1 \left(\begin{matrix} c/a, c/b \\ c \end{matrix} ; \frac{abx}{c} \right), \tag{A.21}$$

$$ x^{1-\gamma} {}_2\varphi_1 \left(\begin{matrix} aq/c, bq/c \\ q^2/c \end{matrix} ; x \right), \tag{A.22}$$

$$ x^{1-\gamma} \frac{(abx/c)_\infty}{(x)_\infty} {}_2\varphi_1 \left(\begin{matrix} q/a, q/b \\ q^2/c \end{matrix} ; \frac{abx}{c} \right), \tag{A.23}$$

$$ x^{-\alpha} {}_2\varphi_1 \left(\begin{matrix} a, aq/c \\ aq/b \end{matrix} ; \frac{cq}{abx} \right), \tag{A.24}$$

$$ x^{-\alpha} \frac{(q/x)_\infty}{(cq/(abx))_\infty} {}_2\varphi_1 \left(\begin{matrix} q/b, c/b \\ aq/b \end{matrix} ; \frac{q}{x} \right), \tag{A.25}$$

$$ x^{-\beta} {}_2\varphi_1 \left(\begin{matrix} b, bq/c \\ bq/a \end{matrix} ; \frac{cq}{abx} \right), \tag{A.26}$$

$$ x^{-\beta} \frac{(q/x)_\infty}{(cq/(abx))_\infty} {}_2\varphi_1 \left(\begin{matrix} q/a, c/a \\ bq/a \end{matrix} ; \frac{q}{x} \right). \tag{A.27}$$

Each solution is well known though they are scattered in literature. In the main text, we gave more solutions in Theorem 2.49.

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