Explicit description of moduli spaces of parabolic connections

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博　士　論　文

Explicit description of moduli spaces of parabolic connections
(放物接続のモジュライ空間の明示的記述)

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Chapter 1

Introduction

The Painlevé equations are second-order differential equations whose only movable singularities are poles. One of the important characteristics of the Painlevé equations is that they can be derived from the isomonodromic deformations of systems of linear differential equations. For example, the Painlevé VI equation is the isomonodromic deformation equation of a rank two linear system with four regular singularities.

Another way to obtain the Painlevé equations is by using the theory of rational surfaces. The notion of the spaces of initial conditions for the Painlevé equations was introduced by K. Okamoto [Ok1]. H. Sakai [Sa] characterized the good compactification of spaces of initial conditions as a certain projective rational surface and classified them according to some affine root systems. In his framework, the second order discrete Painlevé equations are the dynamical systems generated by the action of the translation part of the corresponding affine Weyl group on the family of rational surfaces and the Painlevé equations appear as a limit of the translation part. Saito-Takebe-Terajima [STT] also characterized the spaces of initial conditions and classified them. In their framework, the Painlevé equations arise from certain deformations of rational surfaces.

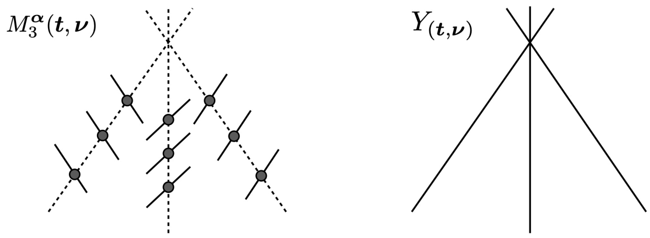
Moduli spaces of meromorphic connections connect the isomonodromic deformation and the space of initial conditions. The equations of the isomonodromic deformations can be geometrically understood as an algebraic vector field on the moduli space of meromorphic connections by Riemann-Hilbert correspondence. In particular, we can regard the moduli space of meromorphic connections as a space of initial conditions of the equation determined by the isomonodromic deformation. Giving a coordinate on the moduli space of meromorphic connections leads to giving an explicit description of the higher dimensional Painlevé equations and characterizing the space of initial conditions for them.

Moduli spaces of meromorphic connections are mainly studied in the case of rank two logarithmic connections on the projective line. The first purpose in this thesis is to give an example of the moduli space of logarithmic connections with rank ≥ 3. Specifically, we provide an explicit description of the moduli space of rank three logarithmic connections over $\mathbb{P}^1$ with three poles, considering its relation to the difference Painlevé equation. The second purpose is to give a Darboux coordinate on the moduli space of logarithmic connections over the curve with higher genus.

1.1 The moduli space of connections and difference Painlevé equations

First, we consider the higher rank case. The moduli space of parabolic logarithmic connections of rank r and degree d on the smooth irreducible projective curve C with n distinct points has dimension $2r^2(g-1)+nr(r-1)+2$. In particular, the moduli space has the even dimension. The dimension of the moduli space is two if and only if $(g,n,r)=(0,4,2),(0,3,3),(1,n,1)$. So we focus on the case $(g,n,r)=(0,3,3)$.

Rank three logarithmic connections over $\mathbb{P}^1$ with three poles do not admit nontrivial isomonodromic deformations. However it is known that discrete deformations of those connections give rise to the difference Painlevé equation associated to $A_2^{(1)*}$-surfaces. Here an $A_2^{(1)*}$-surface is a surface with a unique effective anti-canonical divisor and is obtained by blowing up $\mathbb{P}^2$ at three points on each three lines meeting in a single point, i.e. blowing up at nine points in total. So the moduli spaces of rank three logarithmic connections over $\mathbb{P}^1$ with three poles can be identified with the spaces of initial conditions of the difference Painlevé equation, i.e. $A_2^{(1)*}$-surfaces. D. Arinkin and A. Borodin [AB] proved that the moduli space of a certain type of difference connections over $\mathbb{P}^1$ for generic parameters, which is a



geometric interpretation of difference equations, is isomorphic to the surface obtained by removing the effective anti-canonical divisor from an $A_2^{(1)*}$-surface. They pointed out that the moduli space of the type of difference connections is isomorphic to the moduli space of rank three logarithmic connections over $\mathbb{P}^1$ with three poles by the Mellin transform. P. Boalch [Bo] considered the relation between $A_2^{(1)*}$-surfaces and the moduli spaces of logarithmic connections from the perspective of quiver variety and symmetry. The moduli space of rank 3 logarithmic connections on the trivial bundle over $\mathbb{P}^1$ with 3 poles is identified with the Kronheimer's E_6-type ALE space, which is obtained by blowing up $\mathbb{P}^2$ at 6 points on the smooth locus of a cuspidal cubic. Boalch explained how to obtain an $A_2^{(1)*}$-surface from the Kronheimer's E_6-type ALE space, that is, how to pratially compactify the moduli space of logarithmic connections on the trivial bundle to get the full moduli space of logarithmic connections of degree zero. On the other hand, they did not explicitly mention the correspondence between each logarithmic connection and the points on an $A_2^{(1)*}$-surface. A. Dzhamay and T. Takenawa [DT] provided a coordinate on a Zariski open subset of the moduli space of logarithmic connections by introducing rational parameters of Fuchsian systems of the spectral type $111, 111, 111$ and described the difference Painlevé equation. To obtain the whole of the moduli space of parabolic logarithmic connections, we must also consider connections on nontrivial bundles. In this thesis, we provide normal forms of $\boldsymbol{\alpha}$-stable rank three parabolic ϕ-connections over $\mathbb{P}^1$ with three poles by the apparent singularity and its dual parameter (see Section 4.5), and prove that the moduli space of $\boldsymbol{\alpha}$-stable rank three parabolic ϕ-connections over $\mathbb{P}^1$ with three poles for arbitrary local exponents is isomorphic to an $A_2^{(1)*}$-surface by using the normal forms.

Put

$$T_3 := \left\{(t_1,t_2,t_3) \in (\mathbb{P}^1)^3 \;\middle|\; t_i \neq t_j \text{ for } i \neq j\right\},$$

$$\mathcal{N}(\nu_1,\nu_2,\nu_3) := \{(\nu_{i,j}) \in \mathbb{C}^9 \mid \nu_{i,0}+\nu_{i,1}+\nu_{i,2} = \nu_i, 1 \leq i \leq 3\},$$

where $\nu_1,\nu_2,\nu_3 \in \mathbb{C}$ and $\nu_1+\nu_2+\nu_3 \in \mathbb{Z}$. Take $\boldsymbol{t} \in T_3$ and $\boldsymbol{\nu} \in \mathcal{N}(\nu_1,\nu_2,\nu_3)$. Let $M_3^{\boldsymbol{\alpha}}(\nu_1,\nu_2,\nu_3) \to T_3 \times \mathcal{N}(\nu_1,\nu_2,\nu_3)$ (resp. $\overline{M_3^{\boldsymbol{\alpha}}}(\nu_1,\nu_2,\nu_3) \to T_3 \times \mathcal{N}(\nu_1,\nu_2,\nu_3)$) be the family of moduli spaces of $\boldsymbol{\alpha}$-stable $\boldsymbol{\nu}$-parabolic connections (resp. ϕ-connections), whose fiber $M_3^{\boldsymbol{\alpha}}(\boldsymbol{t},\boldsymbol{\nu})$ (resp. $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$) at $(\boldsymbol{t},\boldsymbol{\nu}) \in T_3 \times \mathcal{N}(\nu_1,\nu_2,\nu_3)$ is the moduli space of $\boldsymbol{\alpha}$-stable $\boldsymbol{\nu}$-parabolic connections (resp. ϕ-connections) over $(\mathbb{P}^1,\boldsymbol{t})$. The existence of $M_3^{\boldsymbol{\alpha}}(\nu_1,\nu_2,\nu_3)$ is proved in [IIS1] and that of $\overline{M_3^{\boldsymbol{\alpha}}}(\nu_1,\nu_2,\nu_3)$ in Chapter 3. Let S be the family of $A_2^{(1)*}$-surfaces parametrized by $T_3 \times \mathcal{N}(0,0,2)$ defined in section 4.1.

Theorem 1.1.1. (Theorem 4.1.1) Take $\boldsymbol{\alpha} = (\alpha_{i,j})_{1\leq i,j\leq 3}$ such that $0 < \alpha_{i,j} \ll 1$ for any $1 \leq i,j \leq 3$.

(1) There exists an isomorphism $\overline{M_3^{\boldsymbol{\alpha}}}(0,0,2) \longrightarrow S$ over $T_3 \times \mathcal{N}(0,0,2)$. In particular, for each $(\boldsymbol{t},\boldsymbol{\nu}) \in T_3 \times \mathcal{N}(0,0,2)$, the fiber $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$ is isomorphic to an $A_2^{(1)*}$-surface .

(2) Let Y be the closed subscheme of $\overline{M_3^{\boldsymbol{\alpha}}}(0,0,2)$ defined by the conditions $\wedge^3\phi = 0$. Then Y is reduced and the natural morphism

$$M_3^{\boldsymbol{\alpha}}(0,0,2) \longrightarrow \overline{M_3^{\boldsymbol{\alpha}}}(0,0,2) \setminus Y, \quad (E,\nabla,l_*) \longmapsto (E,E,\mathrm{id},\nabla,l_*,l_*)$$

is an isomorphism. Moreover for each $(\boldsymbol{t},\boldsymbol{\nu}) \in T_3 \times \mathcal{N}(0,0,2)$, the fiber $Y_{(\boldsymbol{t},\boldsymbol{\nu})}$ is the anti-canonical divisor of $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$.

1.2 Moduli spaces of parabolic bundles and parabolic connections

Second, we consider the higher genus case. Let C be an irreducible smooth projective curve of genus g over the field of complex numbers $\mathbb{C}$, and let $\mathbf{t} = \{t_1,\ldots,t_n\}$ be a set of n distinct points on C. Let

$M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))$ be the moduli space of rank two $\boldsymbol{\alpha}$-stable $\boldsymbol{\nu}$-parabolic logarithmic connections over $(C, \mathbf{t})$ with fixed determinant (L, ∇_L). The moduli space of parabolic connections has the canonical symplectic structure, and providing a Darboux coordinate of such a moduli space is important from the viewpoint of the isomonodromic deformation. There are two main approaches to giving a Darboux coordinate. One is to use the apparent singularities and their dual parameters. Okamoto [Ok2] described Hamiltonian systems of the Garnier systems, which are obtained from the isomonodromic deformation of rank 2 connections on $\mathbb{P}^1$, by using the apparent singularities and their dual parameters. Iwasaki [Iw] proved that the moduli space of SL_2-connections on a Riemann surface of any genus can be locally written by the apparent singularities and their dual parameters as an analytic space and provided Hamiltonian systems of the equations obtained from the isomonodromic deformation in the case of higher genus, which is a generalization of Okamoto's result. Arinkin-Lysenko [AL], Oblezin [Ob], Inaba-Iwasaki-Saito [IIS2] and Komyo-Saito [KS] give an explicit description of the moduli space of parabolic connections on $\mathbb{P}^1$ as an algebraic variety. The other approach is to analyze the apparent singularities and underlying parabolic bundles. Loray-Saito [LS] provided an explicit description of the moduli space in the case of $g = 0$ in this way. Specifically, they proved that a Zariski-open subset of the moduli space of parabolic connections on $\mathbb{P}^1$ is isomorphic to a Zariski-open subset of the product of a projective space and the moduli space of parabolic bundles. Fassarella-Loray [FL] and Fassarella-Loray-Muniz [FLM] investigated the geometry of the moduli space in the case of $g = 1$. In this thesis, we describe the Zariski-open subset of the moduli space $M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))$ for a certain parabolic weight $\boldsymbol{\alpha}$ in the case $g \geq 2$ by using the apparent singularities and underlying parabolic bundles, which is a generalization of Loray-Saito's result.

In order to state the description of the Zariski-open subset of the moduli space precisely, we introduce some notations. Let $\boldsymbol{\nu} = (\nu_{i,j})^{i=1,\dots,n}_{j=0,1}$ be a collection of complex numbers satisfying $\sum_{i=1}^n (\nu_{i,0} + \nu_{i,1}) = -d$. Let $\boldsymbol{\alpha} = \{\alpha_{i,1}, \alpha_{i,2}\}_{1 \leq i \leq n}$ be a collection of rational numbers such that for all $i = 1, \dots, n$, $0 < \alpha_{i,1} < \alpha_{i,2} < 1$. Let (L, ∇_L) be a pair of a line bundle on C with $\deg L = d$ and a logarithmic connection ∇_L over L which has the residue data $\mathrm{res}_{t_i}(\nabla_L) = \nu_{i,0} + \nu_{i,1}$ for each i. Let $M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))$ be the moduli space of rank 2 $\boldsymbol{\alpha}$-stable $\boldsymbol{\nu}$-parabolic connections over $(C, \mathbf{t})$ whose determinant and trace connection are isomorphic to (L, ∇_L). Inaba [In] showed that $M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))$ is a smooth irreducible variety if

$$g = 1, n \geq 2 \text{ or } g \geq 2, n \geq 1. \tag{1.1}$$

By elementary transformations, we can change degree d freely. When $d = 2g - 1$, by the theory of apparent singularities [SS], we can define the rational map

$$\mathrm{App} : M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L)) \cdots \to \mathbb{P}H^0(C, L \otimes \Omega^1_C(D)).$$

The map which forgets connections induces a rational map

$$\mathrm{Bun} : M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L)) \cdots \to P^{\boldsymbol{\alpha}}(2, L).$$

Let V_0 and $M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))^0$ be the open subsets of $P^{\boldsymbol{\alpha}}(2, L)$ and $M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))$, respectively, defined in Subsection 5.1.1 . From Proposition 5.1.5, we obtain an open immersion $V_0 \hookrightarrow \mathbb{P}H^1(C, L^{-1}(-D))$. Let $\Sigma \subset \mathbb{P}H^0(C, L \otimes \Omega^1_C(D)) \times \mathbb{P}H^1(C, L^{-1}(-D))$ be the incidence variety. Then the following theorem holds.

Theorem 1.2.1. (Theorem 5.1.6 and Proposition 5.1.11) Under the condition (1.1), assume that $d = 2g - 1$, $\sum_{i=1}^n \nu_{i,0} \neq 0$ and $\sum_{i=1}^n (\alpha_{i,2} - \alpha_{i,1}) < 1$. Then the map

$$\mathrm{App} \times \mathrm{Bun}\colon M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))^0 \longrightarrow (\mathbb{P}H^0(C, L \otimes \Omega^1_C(D)) \times V_0) \setminus \Sigma$$

is an isomorphism. Hence, the rational map

$$\mathrm{App} \times \mathrm{Bun}\colon M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L)) \cdots \to |L \otimes \Omega^1_C(D)| \times P^{\boldsymbol{\alpha}}(2, L)$$

is birational. Moreover, App and Bun are Lagrangian fibrations.

From the above theorem, we wonder whether App $\times$ Bun is birational in general. So, we investigate App $\times$ Bun in the case of rank three parabolic logarithmic connections over $\mathbb{P}^1$ with three poles.

Let (E, l_*) be a parabolic bundle and ∇ be a $\boldsymbol{\nu}$-logarithmic connection over (E, l_*). All $\lambda\boldsymbol{\nu}$-logarithmic λ-connections over (E, l_*) are of the form $\lambda\nabla + \Phi$, where Φ is a parabolic Higgs field over (E, l_*). So the space of all isomorphim classes of $\lambda\boldsymbol{\nu}$-logarithmic λ-connections over (E, l_*) is $\mathbb{P}(\mathbb{C}\nabla \oplus H)$ and it can be regarded as a compactification of the space of all $\boldsymbol{\nu}$-logarithmic connections over (E, l_*). Here H is the space of all parabolic Higgs fields over (E, l_*). Let $P^w(3,-2)$ be the moduli space of rank three w-stable parabolic bundles with degree -2 over $(\mathbb{P}^1, \boldsymbol{t})$ and $\overline{M_3^w(\boldsymbol{t},\boldsymbol{\nu})^0}$ be the moduli space of $\lambda\boldsymbol{\nu}$-parabolic λ-connections over $(\mathbb{P}^1, \boldsymbol{t})$ whose underlying parabolic bundles are w-stable, that is,

$$\overline{M_3^w(\boldsymbol{t},\boldsymbol{\nu})^0} := \{(\lambda, E, \nabla, l_*) \mid (E, l_*) \in P^w(3,-2)\} / \sim .$$

Here the w-stability is a special case of the $\boldsymbol{\alpha}$-stability. Analyzing $P^w(3,-2)$, we obtain the following theorem.

Theorem 1.2.2. (Theorem 5.2.2) Assume that $2/9 < w < 1/3$. Then we have

$$\overline{M_3^w(\boldsymbol{t},\boldsymbol{\nu})^0} \cong \begin{cases} \mathbb{P}^1 \times \mathbb{P}^1 & \nu_{1,0} + \nu_{2,0} + \nu_{3,0} \neq 0 \\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) & \nu_{1,0} + \nu_{2,0} + \nu_{3,0} = 0. \end{cases}$$

Let V_0 be a Zariski open subset of $P^w(3,-2)$ defined in the Subsection 5.2.3. The following shows that App × Bun is not birational in general.

Corollary 1.2.3. (Proposition 5.2.5) Assume that $2/9 < w < 1/3$ and $\nu_{1,0} + \nu_{2,0} + \nu_{3,0} \neq 0$. Then the morphism

$$\mathrm{App} \times \mathrm{Bun} \colon \mathrm{Bun}^{-1}(V_0) \longrightarrow \mathbb{P}^1 \times V_0$$

is finite and its generic fiber consists of three points.

1.3 Outline of this paper

Chapter 2 contains a summary of parabolic bundles and parabolic connections.

In Chapter 3, we construct of the moduli space of parabolic ϕ-connections. This construction is essentially due to Inaba-Iwasaki-Saito [IIS1] and Inaba [In].

In Chapter 4, we will prove Theorem 1.1.1. First, we analyze underlying vector bundles of $\boldsymbol{\alpha}$-stable parabolic connections under the assumption of Theorem 1.1.1. Second, we define the apparent singularity of parabolic ϕ-connections. We can see that the apparent singularity of parabolic ϕ-connections with rank $\phi = 1$ is not uniquely determined. So we consider pairs of a parabolic ϕ-connection and a subbundle. Then the apparent map is defined on moduli space $\widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$ of such pairs. Third, we define a morphism $\varphi\colon \widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu}) \to \mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1})$ by using the apparent singularity and its dual parameter. Fourth, we provide a normal form of parabolic ϕ-connections. By using this form we prove the smoothness of $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$. Finally we prove Theorem 1.1.1 through φ and the normal forms. In appendix, we describe the moduli space of rank three parabolic Higgs bundles on $\mathbb{P}^1$ with three poles. We extend the Hitchin map to a map from the moduli space of $\boldsymbol{\nu}$-parabolic ϕ-Higgs bundles to a natural compactification of the Hitchin base, and we determine the singular fibers of the extended Hitchin map when $\boldsymbol{\nu} = 0$.

Chapter 5 is divided into two sections. In first section, we study the Zariski-open subset of moduli spaces of rank two parabolic connections for certain parabolic weights. Firstly, we provide the distinguished open subset V_0 of the moduli space of parabolic bundles. Secondly, we introduce the apparent map. The apparent map was defined in general genus and rank by Saito and Szabó [SS]. Thirdly, we prove the first assertion of Theorem 1.2.1. This proof is based on the proof of Theorem 4.3 in [LS]. We also give another proof that App × Bun is birational. Finally, we show that App and Bun are Lagrangian fibrations. Second section is devoted to the case of rank three parabolic logarithmic connections over $\mathbb{P}^1$ with three poles. First, we consider the moduli space of w-stable parabolic bundles. We determine the type of w-stable parabolic bundles and investigate a wall-crossing phenomenon. Second, we show Theorem 1.2.2 by writing down a $\boldsymbol{\nu}$-parabolic connection and a parabolic Higgs field. Moreover, we investigate the relation between two moduli spaces $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$ and $\overline{M_3^w(\boldsymbol{t},\boldsymbol{\nu})^0}$. Finally, we study the morphism App × Bun.

Chapter 2

General theory

2.1 Parabolic bundles

Let C be an irreducible smooth projective curve over $\mathbb{C}$ and $\boldsymbol{t} = (t_i)_{1\le i\le n}$ be n distinct points of C.

Definition 2.1.1. A quasi-parabolic bundle of rank r and degree d is a pair $(E, l_* = \{l_{i,*}\}_{1\le i\le n})$ consisting of the following data:

(1) E is a vector bundle on C of rank r and degree d and,

(2) $l_{i,*}$ is a filtration $E|_{t_i} = l_{i,0} \supsetneq \cdots \supsetneq l_{i,r-1} \supsetneq l_{i,r} = 0$

Definition 2.1.2. We say that two quasi-parabolic bundles $(E, l_*), (E, l'_*)$ are isomorphic to each other if there is an isomorphisms $\sigma\colon E \xrightarrow{\sim} E'$ such that $\sigma_{t_i}(l_{i,j}) = l'_{i,j}$ for $1 \le i \le n$ and $1 \le j \le r-1$.

Let $\boldsymbol{\alpha} = \{\alpha_{i,j}\}^{1\le i\le n}_{1\le j\le r}$ be a set of rational numbers satisfying $0 < \alpha_{i,1} < \cdots < \alpha_{i,r} < 1$ for each $i = 1,\dots,n$ and $\alpha_{i,j} \ne \alpha_{i',j'}$ for $(i,j) \ne (i',j')$.

Definition 2.1.3. A quasi-parabolic bundle (E, l_*) is said to be $\boldsymbol{\alpha}$-semistable (resp. $\boldsymbol{\alpha}$-stable) if for any nonzero subbundle $F \subsetneq E$, the inequality

$$\frac{\deg F + \sum_{i=1}^n \sum_{j=1}^r \alpha_{i,j} \dim((F|_{t_i} \cap l_{i,j-1})/(F|_{t_i} \cap l_{i,j}))}{\operatorname{rank} F} \underset{(\text{resp.}<)}{\le} \frac{\deg E + \sum_{i=1}^n \sum_{j=1}^r \alpha_{i,j}}{\operatorname{rank} E} \tag{2.1}$$

holds.

Let $P^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(r,d)$ denote the moduli space of $\boldsymbol{\alpha}$-semistable quasi-parabolic bundles over $(C,\mathbf{t})$ of rank r and degree d.

Theorem 2.1.4. (Mehta and Seshadri [Theorem 4.1 [MS]]). The moduli space $P^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(r,d)$ is an irreducible normal projective variety of dimension $r^2(g-1)+nr(r-1)/2+1$. Moreover, if (E,l_*) is $\boldsymbol{\alpha}$-stable, then $P^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(r,d)$ is smooth at the point corresponding to (E,l_*).

Let $\operatorname{Pic}^d C$ be the Picard variety of degree d, which is the set of isomorphism classes of line bundles of degree d on C. Then we can define the morphism

$$\det\colon P^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(d) \longrightarrow \operatorname{Pic}^d C,\ (E,l_*) \longmapsto \det E,$$

where $\det E = \bigwedge^r E$. For each $L \in \operatorname{Pic}^d C$, set

$$P^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(r,L) = \{(E,l_*) \in P^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(d) \mid \det E \simeq L\}.$$

2.2 Parabolic λ-connections

Put $D(\boldsymbol{t}) = t_1 + \cdots + t_n$. We take $\boldsymbol{\nu} = (\nu_{i,j})^{1\le i\le n}_{0\le j\le r-1} \in \mathbb{C}^{rn}$ and $\lambda \in \mathbb{C}$.

Definition 2.2.1. A $\boldsymbol{\nu}$-parabolic λ-connection of rank r and degree d is a collection $(E, \nabla, l_* = \{l_{i,*}\}_{1\le i\le n})$ consisting of the following data:

(1) E is a vector bundle on C of rank r and degree d,

(2) $\nabla\colon E \to E \otimes \Omega^1_C(D(\boldsymbol{t}))$ is a logarithmic λ-connection, i.e. $\nabla(fs) = s \otimes \lambda df + f\nabla(s)$ for any $f \in \mathcal{O}_C, s \in E$, and

(3) $l_{i,*}$ is a filtration $E|_{t_i} = l_{i,0} \supsetneq \cdots \supsetneq l_{i,r-1} \supsetneq l_{i,r} = 0$ satisfying $(\mathrm{res}_{t_i}(\nabla) - \nu_{i,j}\mathrm{id})(l_{i,j}) \subset l_{i,j+1}$ for $1 \le i \le n$ and $0 \le j \le r-1$.

When $\lambda = 1$, a λ-connection is a connection. When $\lambda = 0$, a λ-connection is a Higgs bundle.

Proposition 2.2.2. (Fuchs relation) Let (E, ∇, l_*) be a $\boldsymbol{\nu}$-parabolic connection of rank r and degree d. Then we have

$$\sum_{i=1}^{n}\sum_{j=0}^{r-1}\nu_{i,j} + \lambda d = 0.$$

For a integer d, we put

$$\mathcal{N}_{n,r}(d) := \left\{ (\nu_{i,j})^{1\le i\le n}_{0\le j\le r-1} \in \mathbb{C}^{rn} \;\middle|\; \sum_{i=1}^{n}\sum_{j=0}^{r-1}\nu_{i,j} + d = 0 \right\}.$$

Let us fix $\boldsymbol{\nu} = (\nu_{i,j})^{1\le i\le n}_{0\le j\le r-1} \in \mathcal{N}_{n,r}(d)$.

Definition 2.2.3. We say that two $\boldsymbol{\nu}$-parabolic λ-connections $(E, \nabla, l_*), (E, \nabla', l'_*)$ are isomorphic to each other if there is an isomorphisms $\sigma\colon E \xrightarrow{\sim} E'$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & E \otimes \Omega^1_C(D(\boldsymbol{t})) \\ \sigma\downarrow & & \downarrow\sigma\otimes\mathrm{id} \\ E' & \xrightarrow{\nabla'} & E' \otimes \Omega^1_C(D(\boldsymbol{t})) \end{array}$$

is commutative and $\sigma_{t_i}(l_{i,j}) = l'_{i,j}$ for $1 \le i \le n$ and $1 \le j \le r-1$.

Let $\boldsymbol{\alpha} = \{\alpha_{i,j}\}^{1\le i\le n}_{1\le j\le r}$ be a set of rational numbers satisfying $0 < \alpha_{i,1} < \cdots < \alpha_{i,r} < 1$ for each $i = 1, \dots, n$ and $\alpha_{i,j} \ne \alpha_{i',j'}$ for $(i,j) \ne (i',j')$.

Definition 2.2.4. A $\boldsymbol{\nu}$-parabolic λ-connection (E, ∇, l_*) is said to be $\boldsymbol{\alpha}$-stable (resp. $\boldsymbol{\alpha}$-semistable) if for any nonzero subbundle $F \subsetneq E$ satisfying $\nabla(F) \subset F \otimes \Omega^1_C(D(\boldsymbol{t}))$, the inequality

$$\frac{\deg F + \sum_{i=1}^{n}\sum_{j=1}^{r}\alpha_{i,j}\dim((F|_{t_i} \cap l_{i,j-1})/(F|_{t_i} \cap l_{i,j}))}{\operatorname{rank} F} \underset{(\text{resp. } \le)}{<} \frac{\deg E + \sum_{i=1}^{n}\sum_{j=1}^{r}\alpha_{i,j}}{\operatorname{rank} E}$$

holds.

Let $\tilde{M}_{g,n}$ be a smooth algebraic scheme which is a smooth covering of the coarse moduli space of n pointed irreducible smooth projective curves of genus g over $\mathbb{C}$ and take a universal family $(\mathcal{C}, \tilde{\boldsymbol{t}}) = (\mathcal{C}, \tilde{t}_1, \dots, \tilde{t}_n)$ over $\tilde{M}_{g,n}$.

Theorem 2.2.5. (Theorem 2.1 [In]) There exists a relative fine moduli scheme

$$M^{\boldsymbol{\alpha}}_{\mathcal{C}/\tilde{M}_{g,n}}(\tilde{\boldsymbol{t}}, r, d) \longrightarrow \tilde{M}_{g,n} \times \mathcal{N}_{n,r}(d)$$

of $\boldsymbol{\alpha}$-stable parabolic connections of rank r and degree d, which is smooth and quasi-projective. The fiber $M^{\boldsymbol{\alpha}}_{(\mathcal{C}_x, \tilde{t}_x)}(r, \boldsymbol{\nu})$ over $(x, \boldsymbol{\nu}) \in \tilde{M}_{g,n} \times \mathcal{N}_{n,r}(d)$ is the moduli space of $\boldsymbol{\alpha}$-stable $\boldsymbol{\nu}$-parabolic connections over $(\mathcal{C}_x, \tilde{t}_x)$ whose dimension is $2r^2(g-1) + nr(r-1) + 2$.

2.3 Parabolic ϕ-connections

Definition 2.3.1. For $\boldsymbol{\nu} \in \mathcal{N}_{n,r}(d)$, a $\boldsymbol{\nu}$-parabolic ϕ-connection of rank r and degree d over $(C, \boldsymbol{t})$ is a collection $(E_1, E_2, \phi, \nabla, l^{(1)}_* = \{l^{(1)}_{i,*}\}_{1\le i\le n}, l^{(2)}_* = \{l^{(2)}_{j,*}\}_{1\le j\le n})$ consisting of the following data:

(1) E_1 and E_2 are vector bundles on C of rank r and degree d,

(2) $l^{(k)}_{i,*}$ is a filtration $E_k|_{t_i} = l^{(k)}_{i,0} \supsetneq l^{(k)}_{i,1} \supsetneq \cdots \supsetneq l^{(k)}_{i,r} = 0$ for $k = 1, 2$ and $i = 1, \ldots, n$,

(3) $\phi\colon E_1 \to E_2$ is a homomorphism such that $\phi_{t_i}(l^{(1)}_{i,j}) \subset l^{(2)}_{i,j}$ for any $1 \leq i \leq n$ and $1 \leq j \leq r-1$, and

(4) $\nabla\colon E_1 \to E_2 \otimes \Omega^1_C(D(\boldsymbol{t}))$ is a logarithmic ϕ-connection, i.e. $\nabla(fs) = \phi(s) \otimes df + f\nabla(s)$ for any $f \in \mathcal{O}_C, s \in E_1$, and ∇ satisfies that $(\mathrm{res}_{t_i}\nabla - \nu_{i,j}\phi_{t_i})(l^{(1)}_{i,j}) \subset l^{(2)}_{i,j+1}$ for any $1 \leq i \leq n$ and $0 \leq j \leq r-1$.

Consider the case where $E_1 = E_2$ and $\phi = \lambda \mathrm{id}$ for $\lambda \in \mathbb{C}$. When $\lambda = 1$, the parabolic ϕ-connection is a parabolic λ-connection because $l^{(1)}_* = l^{(2)}_*$ by the condition (3). On the other hand, when $\lambda = 0$, the parabolic ϕ-connection is not a parabolic Higgs bundle in general.

Definition 2.3.2. We say that two $\boldsymbol{\nu}$-parabolic ϕ-connections $(E_1, E_2, \phi, \nabla, l^{(1)}_*, l^{(2)}_*), (E'_1, E'_2, \phi', \nabla', l'^{(1)}_*, l'^{(2)}_*)$ are isomorphic to each other if there are isomorphisms $\sigma_1\colon E_1 \xrightarrow{\sim} E'_1$ and $\sigma_2\colon E_2 \xrightarrow{\sim} E'_2$ such that the diagrams

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ {\scriptstyle\sigma_1}\downarrow & & \downarrow{\scriptstyle\sigma_2} \\ E'_1 & \xrightarrow{\phi'} & E'_2 \end{array} \qquad \begin{array}{ccc} E_1 & \xrightarrow{\nabla} & E_2 \otimes \Omega^1_C(D) \\ {\scriptstyle\sigma_1}\downarrow & & \downarrow{\scriptstyle\sigma_2\otimes\mathrm{id}} \\ E'_1 & \xrightarrow{\nabla'} & E'_2 \otimes \Omega^1_C(D) \end{array}$$

commute and $(\sigma_k)_{t_i}(l^{(k)}_{i,j}) = l'^{(k)}_{i,j}$ for $k = 1, 2$, $1 \leq i \leq n$ and $0 \leq j \leq r-1$.

Remark 2.3.3. Assume that $r = 2$. Given a parabolic ϕ-connection $(E_1, E_2, \phi, \nabla, l^{(1)}_*, l^{(2)}_*)$, we obtain a parabolic ϕ-connection in the sense of Definition 2.5 in [IIS1] by forgetting $l^{(2)}_*$. However we can not canonically obtain parabolic ϕ-connections in this paper from parabolic ϕ-connections in [IIS1]. For example, let $(E, \{l_i\}_{1\leq i\leq n})$ be a rank 2 parabolic bundle over $(C, (t_1, \ldots, t_n))$ with the determinant L and $\Phi\colon E \to E \otimes \Omega^1_C(t_1 + \cdots + t_n)$ be a parabolic Higgs bundle of rank 2. Let us fix an isomorphism $\varphi\colon \wedge^2 E \xrightarrow{\sim} L$. We put $E_1 = E_2 = E$ and $l^{(1)}_i = l_i$ for $1 \leq i \leq n$. Take a point $t_{n+1} \in C \setminus \{t_1, \ldots, t_n\}$. Let $l^{(1)}_{n+1} \subset E|_{t_{n+1}}$ be a one dimensional subspace and Ψ be the composite

$$E \xrightarrow{\Phi} E \otimes \Omega^1_C(t_1 + \cdots + t_n) \to E \otimes \Omega^1_C(t_1 + \cdots + t_n + t_{n+1}).$$

Then $(E_1, E_2, 0, \Psi, \varphi, \{l^{(i)}\}_{1\leq i\leq n+1})$ becomes a parabolic ϕ-connection in the sense of [IIS1]. However $l^{(2)}_{n+1} \subset E_2|_{t_{n+1}}$ is not uniquely determined by $(E_1, E_2, 0, \Psi, \varphi, \{l^{(i)}\}_{1\leq i\leq n+1})$.

Let γ be a positive integer. Take a set of rational numbers $\boldsymbol{\alpha} = \{\alpha^{(k)}_{i,j}\}^{k=1,2}_{1\leq i\leq n, 1\leq j\leq r}$ satisfying $0 \leq \alpha^{(k)}_{i,1} < \cdots < \alpha^{(k)}_{i,r} < 1$ for $k = 1, 2$ and $i = 1, \ldots, n$, and $\alpha^{(k)}_{i,j} \neq \alpha^{(k)}_{i',j'}$ for $(i,j) \neq (i',j')$.

Definition 2.3.4. A $\boldsymbol{\nu}$-parabolic ϕ-connection $(E_1, E_2, \phi, \nabla, l^{(1)}_*, l^{(2)}_*)$ is $\boldsymbol{\alpha}$-stable (resp. $\boldsymbol{\alpha}$-semistable) if for any subbundles $F_1 \subseteq E_1, F_2 \subseteq E_2, (F_1, F_2) \neq (0, 0)$ satisfying $\phi(F_1) \subset F_2$ and $\nabla(F_1) \subset F_2 \otimes \Omega^1_C(D(\boldsymbol{t}))$, the inequality

$$\begin{aligned} &\frac{\deg F_1 + \deg F_2(-\gamma) + \sum_{i=1}^n \sum_{j=1}^r \alpha^{(1)}_{i,j} d^{(1)}_{i,j}(F_1) + \sum_{i=1}^n \sum_{j=1}^r \alpha^{(2)}_{i,j} d^{(2)}_{i,j}(F_2)}{\operatorname{rank} F_1 + \operatorname{rank} F_2} \\ &\underset{(\text{resp. } \leq)}{<} \frac{\deg E_1 + \deg E_2(-\gamma) + \sum_{i=1}^n \sum_{j=1}^r \alpha^{(1)}_{i,j} d^{(1)}_{i,j}(E_1) + \sum_{i=1}^n \sum_{j=1}^r \alpha^{(2)}_{i,j} d^{(2)}_{i,j}(E_2)}{\operatorname{rank} E_1 + \operatorname{rank} E_2} \end{aligned}$$

holds, where $d^{(k)}_{i,j}(F) = \dim(F|_{t_i} \cap l^{(k)}_{i,j-1})/(F|_{t_i} \cap l^{(k)}_{i,j})$ for a subbundle $F \subset E_k$ and for $k = 1, 2$.

Take a universal family $(\mathcal{C}, \tilde{\boldsymbol{t}}) = (\mathcal{C}, \tilde{t}_1, \ldots, \tilde{t}_n)$ over $\tilde{M}_{g,n}$ and put $D = \tilde{t}_1 + \cdots + \tilde{t}_n$. Then D is an effective Cartier divisor which is flat over $\tilde{M}_{g,n}$. For simplicity of notation, we use the same character D to denote the pull back of D by the projection $\mathcal{C} \times \mathcal{N} \to \mathcal{C}$, where $\mathcal{N} := \mathcal{N}_{n,r}(d)$. Let $\tilde{\nu}_{i,j} \subset \mathbb{C} \times \tilde{M}_{g,n} \times \mathcal{N}$ be the section defined by

$$\tilde{M}_{g,n} \times \mathcal{N} \hookrightarrow \mathbb{C} \times \tilde{M}_{g,n} \times \mathcal{N}; \quad (x, (\nu_{k,l})^{1\leq k\leq n}_{0\leq l\leq r-1}) \mapsto (\nu_{i,j}, x, (\nu_{k,l})^{1\leq k\leq n}_{0\leq l\leq r-1}).$$

Definition 2.3.5. We define the moduli functor $\overline{\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d)$ of the category of locally noetherian schemes over $\tilde{M}_{g,n} \times \mathcal{N}$ to the category of sets by

$$\overline{\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d)(S) := \{(E_1, E_2, \phi, \nabla, l^{(1)}_*, l^{(2)}_*)\}/\sim,$$

where S is a locally noetherian scheme over $\tilde{M}_{g,n} \times \mathcal{N}$ and

(1) E_1, E_2 are vector bundles on $(\mathcal{C}\times\mathcal{N})_S := (\mathcal{C}\times\mathcal{N})\times_{\tilde{M}_{g,n}\times\mathcal{N}} S$ such that for any geometric point s of S, $\mathrm{rank}\,(E_1)_s = \mathrm{rank}\,(E_2)_s = r$ and $\deg(E_1)_s = \deg(E_2)_s = d$,

(2) for each $k = 1, 2$, $E_k|_{(\tilde{t}_i)_S} = l^{(k)}_{i,0} \supsetneq \cdots \supsetneq l^{(k)}_{i,r-1} \supsetneq l^{(k)}_{i,r} = 0$ is a filtration by subbundles,

(3) $\phi\colon E_1 \to E_2$ is a homomorphism such that $\phi_{(\tilde{t}_i)_S}(l^{(1)}_{i,j}) \subset l^{(2)}_{i,j}$ for each $k = 1, 2$, $1 \le i \le n$ and $1 \le j \le r-1$,

(4) $\nabla\colon E_1 \to E_2 \otimes \Omega^1_{(\mathcal{C}\times\mathcal{N})_S/S}(D_S)$ is a relative logarithmic ϕ-connection such that $(\mathrm{res}_{(\tilde{t}_i)_S}\nabla - (\tilde{\nu}_{i,j})_S\phi_{(\tilde{t}_i)_S})(l^{(1)}_{i,j}) \subset l^{(2)}_{i,j+1}$ for each $k = 1, 2$, $1 \le i \le n$ and $0 \le j \le r-1$,

(5) for any geometric point s of S, the parabolic ϕ-connection $((E_1)_s, (E_2)_s, \phi_s, \nabla_s, (l^{(1)}_*)_s, (l^{(2)}_*)_s)$ is $\boldsymbol{\alpha}$-stable.

In Chapter 3, we prove the following theorem.

Theorem 2.3.6. (1) There exists a fine moduli scheme $\overline{M^{\boldsymbol{\alpha}}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d)$ of $\overline{\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d)$. If $\boldsymbol{\alpha}$ is generic, then $\overline{M^{\boldsymbol{\alpha}}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d)$ is projective over $\tilde{M}_{g,n}\times\mathcal{N}$.

(2) Assume that $\alpha^{(1)}_{i,j} = \alpha^{(2)}_{i,j} =: \alpha'_{i,j}$ for any $1 \le i \le n$ and $1 \le j \le r$. Then the set

$$U_{\mathrm{isom}} := \left\{(E_1, E_2, \phi, \nabla, l^{(1)}_*, l^{(2)}_*) \in \overline{M^{\boldsymbol{\alpha}}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d) \;\middle|\; \phi \text{ is an isomorphism}\right\}$$

is a Zariski open subset of $\overline{M^{\boldsymbol{\alpha}}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d)$ and it is just a moduli space of $\boldsymbol{\alpha}'$-stable parabolic connections $M^{\boldsymbol{\alpha}'}_{\mathcal{C}/\tilde{M}_{g,n}}(\tilde{\boldsymbol{t}}, r, d)$, where $\boldsymbol{\alpha}' = \{\alpha'_{i,j}\}^{1\le i\le n}_{1\le j\le r}$.

2.4 Elementary transformations of parabolic ϕ-connections

Let $(E_1, E_2, \phi, \nabla, l^{(1)}_*, l^{(2)}_*)$ be a $\boldsymbol{\nu}$-parabolic ϕ-connection of rank r and degree d over $(C, \boldsymbol{t})$. Let us fix integers $1 \le p \le n$ and $0 \le q \le r$. Put $E'_k := \ker(E_k \to E_k|_{t_p}/l^{(k)}_{p,q})$ for $k = 1, 2$. Then E'_k is a locally free sheaf of rank r and degree $d - q$, ϕ induces a homomorphism $\phi'\colon E'_1 \to E'_2$ and ∇ induces a logarithmic ϕ-connection $\nabla'\colon E'_1 \to E'_2 \otimes \Omega^1_C(D(\boldsymbol{t}))$. Put

$$l'^{(k)}_{i,j} := \begin{cases} l^{(k)}_{i,j} & i \ne p \\ (\pi^{(k)}_{p,q})|_{t_p}^{-1}(l^{(k)}_{q+j}) & i = p,\ 0 \le j \le r-q \\ \iota^{(k)}_p|_{t_p}(l^{(k)}_{p,j-r+q}/l^{(k)}_{p,q}) & i = p,\ r-q \le j \le r, \end{cases}$$

$$\nu'_{i,j} := \begin{cases} \nu_{i,j} & i \ne p \\ \nu_{i,q+j} & i = p,\ 0 \le j \le r-q-1 \\ \nu_{i,j-r+q} + 1 & i = p,\ r-q \le j \le r-1, \end{cases}$$

where

$$0 \longrightarrow E_k(-t_p) \xrightarrow{\iota^{(k)}_p} E'_k \xrightarrow{\pi^{(k)}_{p,q}} l^{(k)}_{p,q} \longrightarrow 0.$$

Then $(E'_1, E'_2, \phi', \nabla', l'^{(1)}_*, l'^{(2)}_*)$ be a $\boldsymbol{\nu}'$-parabolic ϕ-connection of rank r and degree $d-q$ over $(C, \boldsymbol{t})$. This correspondence induces a morphism

$$\mathrm{elm}_{p,q}\colon \overline{\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d) \longrightarrow \overline{\mathcal{M}^{\boldsymbol{\alpha}'}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d-q),\ (E_1, E_2, \phi, \nabla, l^{(1)}_*, l^{(2)}_*) \longmapsto (E'_1, E'_2, \phi', \nabla', l'^{(1)}_*, l'^{(2)}_*)$$

of functors. Here $\boldsymbol{\alpha}'$ is a suitable parabolic weight. Let b_p be a morphism of functors defined by tensoring with $(\mathcal{O}_C(t_p), d)$, i.e.

$$b_p\colon \overline{\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d) \longrightarrow \overline{\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d+r),\ (E_1, E_2, \phi, \nabla, l^{(1)}_*, l^{(2)}_*) \longmapsto (E_1, E_2, \phi, \nabla, l^{(1)}_*, l^{(2)}_*) \otimes (\mathcal{O}_C(t_p), d).$$

Then we can see that

$$b_p \circ \mathrm{elm}_{p,r-q} \circ \mathrm{elm}_{p,q} = \mathrm{id}, \quad \mathrm{elm}_{p,q} \circ b_p \circ \mathrm{elm}_{p,r-q} = \mathrm{id}.$$

So $\mathrm{elm}_{p,q}$ is an isomorphism. Hence we can freely change degree.

Chapter 3

Construction of the moduli space of parabolic ϕ-connections

In this chapter we construct the moduli space of parabolic ϕ-connections. The construction is based on [IIS1] and [In]. For propositions and theorems without proofs, please refer to these papers.

3.1 Parabolic Λ^1_D-triples

Let D be an effective Cartier divisor on C. We define an $\mathcal{O}_C$-bimodule structure on $\Lambda^1_D = \mathcal{O}_C \oplus (\Omega^1_C(D(\boldsymbol{t})))^\vee$ by

$$(a,v)f := (fa + \langle v, df\rangle, fv),\ f(a,v) := (fa, fv)$$

for $a, f \in \mathcal{O}_C$ and $v \in (\Omega^1_C(D))^\vee$, where $\langle\ ,\ \rangle\colon (\Omega^1_C(D))^\vee \times \Omega^1_C(D) \to \mathcal{O}_C$ is the canonical pairing. Let $\phi\colon E_1 \to E_2$ be a homomorphism of vector bundles on C and $\nabla\colon E_1 \to E_2 \otimes \Omega^1_C(D)$ be a ϕ-connection. We define $\Phi\colon \Lambda^1_D \otimes_{\mathcal{O}_X} E_1 \to E_2$ by $\Phi((a,v)\otimes s) = a\phi(s) + \langle v, \nabla s\rangle$. Then we can easily see that Φ becomes a left $\mathcal{O}_C$-homomorphism. Conversely, let $\Phi\colon \Lambda^1_D \otimes_{\mathcal{O}_X} E_1 \to E_2$ be a left $\mathcal{O}_C$-homomorphism. We define a homomorphism $\phi\colon E_1 \to E_2$ by $\phi(s) = \Phi((1,0)\otimes s)$. Let $\nabla\colon E_1 \to E_2 \otimes \Omega^1_C(D)$ be a map satisfying $\Phi((0,v)\otimes s) = \langle v, \nabla s\rangle$ for any $v \in (\Omega^1_C(D))^\vee$ and $s \in E_1$. Then ∇ is uniquely determined and ∇ becomes a ϕ-connection. The above correspondence is inverse each other.

Definition 3.1.1. A parabolic Λ^1_D-triple is a collection $(E_1, E_2, \Phi, F_*(E_1), F_*(E_2))$ consisting of the following data:

(1) E_1 and E_2 are vector bundles on C of rank r and degree d.

(2) $F_*(E_k)$ is a filtration $E_k = F_1(E_k) \supset F_2(E_k) \supset \cdots \supset F_{l_i}(E_k) \supset F_{l_i+1}(E_k) = E_k(-D)$ for $k = 1, 2$.

(3) $\Phi\colon \Lambda^1_D \otimes_{\mathcal{O}_X} E_1 \to E_2$ is a left $\mathcal{O}_C$-homomorphism.

Remark 3.1.2. A parabolic Λ^1_D-triple in [IIS1] is a collection $(E_1, E_2, \Phi, F_*(E_1))$ consisting of vector bundles E_1, E_2, a left $\mathcal{O}_C$-homomorphism $\Phi\colon \Lambda^1_D \otimes E_1 \to E_2$ and a filtration $F_*(E_1)$ of E_1. So forgetting a filtration $F_*(E_2)$ of a present parabolic Λ^1_D-triple $(E_1, E_2, \Phi, F_*(E_1), F_*(E_2))$, we obtain a parabolic Λ^1_D-triple $(E_1, E_2, \Phi, F_*(E_1))$ in their sense.

Definition 3.1.3. A parabolic Λ^1_D-triple $(E'_1, E'_2, \Phi', F_*(E'_1), F_*(E'_2))$ is said to be a parabolic Λ^1_D-subtriple of $(E_1, E_2, \Phi, F_*(E_1), F_*(E_2))$ if $E'_1 \subset E_1$, $E'_2 \subset E_2$, $\Phi' = \Phi|_{\Lambda^1_D \otimes_{\mathcal{O}_X} E'_1}$, $F_i(E'_1) \subset F_i(E_1)$ and $F_i(E'_2) \subset F_i(E_2)$.

For each $k = 1, 2$, let $\boldsymbol{\beta}^{(k)} = \{\beta^{(k)}_i\}_{1 \le i \le l_k}$ be a collection of rational numbers with $0 \le \beta^{(k)}_1 < \cdots < \beta^{(k)}_{l_k} < 1$.

Definition 3.1.4. For a parabolic Λ^1_D-triple $(E_1, E_2, \Phi, F_*(E_1), F_*(E_2))$, we put

$$\mu_{\boldsymbol{\beta}}((E_1, E_2, \Phi, F_*(E_1), F_*(E_2))) := \frac{\deg E_1(-D) + \deg E_2(-D) - \gamma \deg \mathcal{O}_X(1)\operatorname{rank} E_2}{\operatorname{rank} E_1 + \operatorname{rank} E_2} + \frac{\sum_{i=1}^{l_1} \beta^{(1)}_i \operatorname{length} F_i(E_1)/F_{i+1}(E_1) + \sum_{i=1}^{l_2} \beta^{(2)}_i \operatorname{length} F_i(E_2)/F_{i+1}(E_2)}{\operatorname{rank} E_1 + \operatorname{rank} E_2}.$$

Definition 3.1.5. A parabolic Λ^1_D-triple $(E_1, E_2, \Phi, F_*(E_1), F_*(E_2))$ is $\boldsymbol{\beta}$-stable if for any nonzero proper parabolic subtriple $(E'_1, E'_2, \Phi', F_*(E'_1), F_*(E'_2))$ of $(E_1, E_2, \Phi, F_*(E_1), F_*(E_2))$, the inequality

$$\mu_{\boldsymbol{\beta}}((E'_1, E'_2, \Phi', F_*(E'_1), F_*(E'_2))) < \mu_{\boldsymbol{\beta}}((E_1, E_2, \Phi, F_*(E_1), F_*(E_2)))$$

holds.

3.2 Properties of the moduli functor

Let S be a connected noetherian scheme and $\pi_S \colon X \to S$ be a smooth projective morphism whose geometric fibers are irreducible smooth curves of genus g. Let $D \subset X$ be a relative effective Cartier divisor for π_S.

Definition 3.2.1. We define the moduli functor $\overline{\mathcal{M}^{D,\boldsymbol{\beta}}_{X/S}}(r, d, \boldsymbol{d}_1 = \{d^{(1)}_i\}_{2\le i\le l_1}, \boldsymbol{d}_2 = \{d^{(2)}_i\}_{2\le i\le l_2})$ of the category of locally noetherian schemes over S to the category of sets by

$$\overline{\mathcal{M}^{D,\boldsymbol{\beta}}_{X/S}}(r, d, \boldsymbol{d}_1, \boldsymbol{d}_2)(T) := \{(E_1, E_2, \Phi, F_*(E_1), F_*(E_2))\}/\sim$$

where T is a locally noetherian scheme over S and

(1) E_1, E_2 are vector bundles on $X \times_S T$ such that for any geometric point s of T, $\mathrm{rank}\,(E_1)_s = \mathrm{rank}\,(E_2)_s = r$ and $\deg(E_1)_s = \deg(E_2)_s = d$,

(2) $\Phi \colon \Lambda^1_{D/S} \otimes E_1 \to E_2$ is a homomorphism of left $\mathcal{O}_{X\times_S T}$-modules,

(3) For each $k = 1, 2$, $E_k = F_1(E_k) \supset \cdots \supset F_{l_k}(E_k) \supset F_{l_k+1}(E_k) = E_k(-D_T)$ is a filtration of E_1 by coherent subsheaves such that each $E_k/F_i(E_k)$ is flat over T and for any geometric point s of T and $2 \le i \le l_k$, $\mathrm{length}\,(E_k/F_i(E_k))_s = d^{(k)}_i$,

(4) for any geometric point s of T, the parabolic $\Lambda^1_{D_s}$-triple $((E_1)_s, (E_2)_s, \Phi_s, F_*(E_1)_s, F_*(E_2)_s)$ is $\boldsymbol{\beta}$-stable.

Proposition 3.2.2. The family of geometric points of $\overline{\mathcal{M}^{D,\boldsymbol{\beta},\gamma}_{X/S}}(r, d, \boldsymbol{d}_1, \boldsymbol{d}_2)$ is bounded.

Proposition 3.2.3. Put $\beta^{(1)}_{l_1+1} = \beta^{(2)}_{l_2+1} = 1$ and $\epsilon^{(k)}_i = \beta^{(k)}_{i+1} - \beta^{(k)}_i$ for $k = 1, 2$ and $1 \le i \le l_k$. There exists an integer m_0 such that for any geometric point $(E_1, E_2, \Phi, F_*(E_1), F_*(E_2))$ of $\overline{\mathcal{M}^{D,\boldsymbol{\beta}}_{X/S}}(r, d, \boldsymbol{d}_1, \boldsymbol{d}_2)(K)$, the inequality

$$\frac{\beta^{(1)}_1 h^0(E'_1(m)) + \beta^{(2)}_1 h^0(E'_2(m-\gamma)) + \sum_{i=1}^{l_1} \epsilon^{(1)}_i h^0(F_{i+1}(E'_1)(m))) + \sum_{i=1}^{l_2} \epsilon^{(2)}_i h^0(F_{i+1}(E'_2)(m-\gamma)))}{\mathrm{rank}\, E'_1 + \mathrm{rank}\, E'_2}$$
$$< \frac{\beta^{(1)}_1 h^0(E_1(m)) + \beta^{(2)}_1 h^0(E_2(m-\gamma)) + \sum_{i=1}^{l_1} \epsilon^{(1)}_i h^0(F_{i+1}(E_1)(m))) + \sum_{i=1}^{l_2} \epsilon^{(2)}_i h^0(F_{i+1}(E_2)(m-\gamma)))}{\mathrm{rank}\, E_1 + \mathrm{rank}\, E_2}$$

holds for any proper nonzero parabolic $\Lambda^1_{D_K}$-subtriple $(E'_1, E'_2, \Phi, F_*(E'_1), F_*(E'_2))$ of $(E_1, E_2, \Phi, F_*(E_1), F_*(E_2))$ and any integer $m \ge m_0$.

Proposition 3.2.4. Let T be a noetherian scheme over S and $(E_1, E_2, \Phi, F_*(E_1), F_*(E_2))$ be a flat family of parabolic $\Lambda^1_{D_T/T}$-triples on $X \times_S T$ over T. Then there exists an open subscheme T^s of T such that

$$T^s(K) = \{s \in T(K) \mid (E_1, E_2, \Phi, F_*(E_1), F_*(E_2)) \otimes k(s) \text{ is } \boldsymbol{\beta}\text{-stable.}\}$$

for any algebraically closed field K.

3.3 Construction of the moduli spaces

We introduce a proposition and a lemma.

Proposition 3.3.1. (EGA III (7.7.8), (7.7.9) or [AK] (1.1)) Let $f \colon X \to S$ be a proper morphism of noetherian schemes, and let I and F be two coherent $\mathcal{O}_X$-modules with F flat over S. Then there exist a coherent $\mathcal{O}_S$ module $H(I, F)$ and an element $h(I, F)$ of $\mathrm{Hom}_X(I, F \otimes_S H(I, F))$ which represents the functor

$$M \longmapsto \mathrm{Hom}_X(I, F \otimes_{\mathcal{O}_S} M)$$

defined on the category of quasi-coherent $\mathcal{O}_S$-modules M, and the formation of the pair commutes with base change; in other words, the Yoneda map defined by $h(I,F)$

$$y\colon \mathrm{Hom}_T(H(I,F)_T, M) \longmapsto \mathrm{Hom}_{X_T}(I_T, F \otimes_{\mathcal{O}_S} M)$$

is an isomorphism for every S-scheme T and every quasi-coherent $\mathcal{O}_T$-module M.

Lemma 3.3.2. (Lemma 4.3 [Yo]) Let $f\colon X \to S$ be a proper morphism of noetherian schemes and let $\phi\colon I \to F$ be an $\mathcal{O}_X$-homomorphism of coherent $\mathcal{O}_S$-modules with F flat over S. Then there exists a unique closed subscheme Z of S such that for all morphism $g\colon T \to S$, $g^*(\phi) = 0$ if and only if g factors through Z.

Let $P(m) = rd_X m + d + r(1-g)$ where $d_X = \deg \mathcal{O}_{X_s}(1)$ for $s \in S$. We take an integer m_0 in Proposition 3.2.3. We may assume that for any $m \geq m_0$, $h^k(F_i(E_1)(m)) = h^k(F_j(E_2)(m-\gamma)) = 0$ for $k > 0$, $1 \leq i \leq l_1+1$, $1 \leq j \leq l_2+1$, and $F_i(E_1)(m_0), F_j(E_2)(m_0-\gamma)$ are generated by their global sections for any geometric point $(E_1, E_2, \Phi, F_*(E_1), F_*(E_2))$ of $\overline{\mathcal{M}^{D,\boldsymbol{\beta}}_{X/S}}(r,d,\boldsymbol{d}_1,\boldsymbol{d}_2)$ by Proposition 3.2.2. Put $n_1 = P(m_0)$ and $n_2 = P(m_0-\gamma)$. Let V_1, V_2 be free $\mathcal{O}_S$-modules of rank n_1, n_2, respectively. Let $Q^{(1)}$ be the Quot-scheme $\mathrm{Quot}^P_{V_1 \otimes \mathcal{O}_S(-m_0)/X/S}$ and $V_1 \otimes \mathcal{O}_{X_{Q^{(1)}}}(-m_0) \to \mathcal{E}_1$ be the universal quotient sheaf. Let $Q^{(2)} = \mathrm{Quot}^P_{V_2 \otimes \mathcal{O}_S(-m_0+\gamma)/X/S}$ and $V_2 \otimes \mathcal{O}_{X_{Q^{(2)}}}(-m_0+\gamma) \to \mathcal{E}_2$ be the universal quotient sheaf. Put $d^{(1)}_{l_1+1} = d^{(2)}_{l_2+1} = rn$. For $k = 1,2$ and $2 \leq i \leq l_k+1$, let $Q^{(k)}_i := \mathrm{Quot}^{d^{(k)}_i}_{\mathcal{E}_k/X_{Q^{(k)}}/Q^{(k)}}$ and $F_i(\mathcal{E}_k) \subset \mathcal{E}_k$ be the universal subsheaf. We define Q as the maximal closed subscheme of

$$Q^{(1)}_2 \times_{Q^{(1)}} \cdots \times_{Q^{(1)}} Q^{(1)}_{l_1+1} \times Q^{(2)}_2 \times_{Q^{(2)}} \cdots \times_{Q^{(2)}} Q^{(2)}_{l_2+1}$$

such that there exist filtrations

$$(\mathcal{E}_1)_Q \otimes \mathcal{O}_{X_Q}(-D_Q) = F_{l_1+1}(\mathcal{E}_1)_Q \subset F_{l_1}(\mathcal{E}_1)_Q \subset \cdots \subset F_2(\mathcal{E}_1)_Q \subset F_1(\mathcal{E}_1)_Q := (\mathcal{E}_1)_Q$$

and

$$(\mathcal{E}_2)_Q \otimes \mathcal{O}_{X_Q}(-D_Q) = F_{l_2+1}(\mathcal{E}_2)_Q \subset F_{l_2}(\mathcal{E}_2)_Q \subset \cdots \subset F_2(\mathcal{E}_2)_Q \subset F_1(\mathcal{E}_2)_Q := (\mathcal{E}_2)_Q.$$

By Proposition 3.3.1 there exists a coherent sheaf $\mathcal{H}$ on Q such that for any noetherian scheme T over Q and for any quasi-coherent $\mathcal{O}_T$-module $\mathcal{F}$, there exists a functorial isomorphism

$$\mathrm{Hom}_T(\mathcal{H}_T, \mathcal{F}) \cong \mathrm{Hom}_{X_T}(\Lambda^1_{D/S} \otimes_{\mathcal{O}_X} (\mathcal{E}_1)_T, (\mathcal{E}_2)_T \otimes_{\mathcal{O}_T} \mathcal{F}).$$

Let $\boldsymbol{V} = \mathrm{Spec}\,\mathrm{Sym}_{\mathcal{O}_Q}(\mathcal{H})$, where $\mathrm{Sym}_{\mathcal{O}_Q}(\mathcal{H})$ is the symmetric algebra of $\mathcal{H}$ on Q. Then the homomorphism

$$\tilde{\Phi}\colon \Lambda^1_{D/S} \otimes_{\mathcal{O}_X} (\mathcal{E}_1)_{\boldsymbol{V}} \longrightarrow (\mathcal{E}_2)_{\boldsymbol{V}}$$

corresponding to the natural homomorphism $\mathcal{H}_{\boldsymbol{V}} \to \mathcal{O}_{\boldsymbol{V}}$ is the universal homomorphism. Put

$$R^s := \left\{ s \in \boldsymbol{V} \;\middle|\; \begin{array}{l} (V_1)_s \to H^0((\mathcal{E}_1)_s(m_0)), (V_2)_s \to H^0((\mathcal{E}_2)_s(m_0-\gamma)) \text{ are isomor-} \\ \text{phisms, and } ((\mathcal{E}_1)_s, (\mathcal{E}_2)_s, \tilde{\Phi}_s, F_*(\mathcal{E}_1)_s, F_*(\mathcal{E}_2)_s) \text{ is } \boldsymbol{\beta}\text{-stable} \end{array} \right\}.$$

By Proposition 3.2.4, R^s is a open subscheme of $\boldsymbol{V}$. For $y \in R^s$ and vector subspaces $V_1' \subset V_1$ and $V_2' \subset V_2$, let $E_1'(V_1', V_2', y)$ be the image of $V_1' \otimes \mathcal{O}_X(-m_0) \to (\mathcal{E}_1)_y$ and $E_2'(V_1', V_2', y)$ be the image of $\Lambda^1_{D/S} \otimes V_1' \otimes \mathcal{O}_X(-m_0) \oplus V_2' \otimes \mathcal{O}_X(-m_0+\gamma) \to (\mathcal{E}_2)_y$. Since the family

$$\mathcal{F} = \{(E(V_1', V_2', y)_1, E(V_1', V_2', y)_2) \mid y \in R^s, V_1' \subset V_1, V_2' \subset V_2\}$$

is bounded, there exists an integer $m_1 \geq m_0$ such that for all $m \geq m_1$ and all members $(E(V_1', V_2', y)_1, E(V_1', V_2', y)_2) \in \mathcal{F}$,

$$V_1' \otimes H^0(\mathcal{O}_{X_y}(m)) \to H^0(E(V_1', V_2', y)_1(m+m_0))$$

and

$$V_1' \otimes H^0(\mathcal{O}_{X_y}(m_0+m-\gamma) \otimes \Lambda^1_{D_y} \otimes \mathcal{O}_{X_y}(-m_0)) \oplus V_2' \otimes H^0(\mathcal{O}_{X_y}(m)) \to H^0(E(V_1', V_2', y)_2(m_0+m-\gamma))$$

are surjective, $H^i(\mathcal{O}_{X_y}(m_0+m-\gamma)\otimes\Lambda^1_{D_y}\otimes\mathcal{O}_{X_y}(-m_0))=0, H^i(\mathcal{O}_{X_y}(m))=0$ for $i>0$, and the inequality

$$\begin{aligned}
&(r_1'+r_2')d_X\left\{h^0(E_1(m_0))+h^0(E_2(m_0-\gamma))-\sum_{i=1}^{l_1}\epsilon_i^{(1)}d_{i+1}^{(1)}-\sum_{j=1}^{l_2}\epsilon_j^{(2)}d_{j+1}^{(2)}\right\}\\
&\quad-2rd_X\left\{h^0(E_1'(m_0))+h^0(E_2'(m_0-\gamma))-\sum_{i=1}^{l_1}\epsilon_i^{(1)}\Big(h^0(E_1'(m_0))-h^0(F_{i+1}(E_1')(m_0))\Big)\right.\\
&\qquad\left.-\sum_{j=1}^{l_2}\epsilon_j^{(2)}\Big(h^0(E_2'(m_0-\gamma))-h^0(F_{j+1}(E_2')(m_0-\gamma))\Big)\right\}\\
&>m^{-1}\Big(\dim V_1+\dim V_2-\sum_{i=1}^{l_1}\epsilon_i^{(1)}d_{i+1}^{(1)}-\sum_{j=1}^{l_2}\epsilon_j^{(2)}d_{j+1}^{(2)}\Big)\Big(\dim V_1'+\dim V_2'-\chi(E_1'(m_0))-\chi(E_2'(m_0-\gamma))\Big)
\end{aligned} \tag{3.1}$$

holds for $(0,0)\subsetneq(V_1',V_2')\subsetneq((V_1)_y,(V_2)_y)$, where $E_k'=E(V_1',V_2',y)_k$ and $F_{i+1}(E_k')=E_k'\cap F_{i+1}(\mathcal{E}_k)_y$ for $k=1,2$ and $1\le i\le l_k$. We note that the left hand side of (3.1) is positive since m_0 is an integer in Proposition 3.2.3. The composite

$$V_1\otimes\Lambda^1_{D/S}\otimes\mathcal{O}_{X_{R^s}}(-m_0)\longrightarrow\Lambda^1_{D/S}\otimes(\mathcal{E}_1)_{R^s}\xrightarrow{\tilde{\Phi}}(\mathcal{E}_2)_{R^s}$$

induces a homomorphism

$$V_1\otimes W_1\otimes\mathcal{O}_{R^s}\longrightarrow(\pi_{R^s})_*(\mathcal{E}_2(m_0+m_1-\gamma)_{R^s}),$$

where $W_1=(\pi_S)_*(\mathcal{O}_X(m_0+m_1-\gamma)\otimes\Lambda^1_{D/S}\otimes\mathcal{O}_X(-m_0))$ and $\pi_{R^s}\colon X_{R^s}:=X\times_S R^s\to R^s$ be the projection, and the quotient $V_2\otimes\mathcal{O}_{X_{R^s}}(-m_0+\gamma)\to(\mathcal{E}_2)_{R^s}$ induces a homomorphism

$$V_2\otimes W_2\otimes\mathcal{O}_{R^s}\longrightarrow(\pi_{R^s})_*(\mathcal{E}_2(m_0+m_1-\gamma)_{R^s})$$

where $W_2=(\pi_S)_*(\mathcal{O}_X(m_1))$. These homomorphism induce a quotient bundle

$$(V_1\otimes W_1\oplus V_2\otimes W_2)\otimes\mathcal{O}_{R^s}\longrightarrow(\pi_{R^s})_*(\mathcal{E}_2(m_0+m_1-\gamma)_{R^s}). \tag{3.2}$$

Taking m_1 sufficiently large, we obtain the surjectivities of this homomorphism and the canonical homomorphism

$$V_1\otimes W_2\otimes\mathcal{O}_{R^s}\longrightarrow(\pi_{R^s})_*(\mathcal{E}_1(m_0+m_1)_{R^s}). \tag{3.3}$$

The canonical homomorphisms

$$V_1\otimes\mathcal{O}_{R^s}\longrightarrow(\pi_{R^s})_*((\mathcal{E}_1/F_i(\mathcal{E}_1))(m_0)_{R^s}), \tag{3.4}$$

$$V_2\otimes\mathcal{O}_{R^s}\longrightarrow(\pi_{R^s})_*((\mathcal{E}_2/F_i(\mathcal{E}_2))(m_0-\gamma)_{R^s}) \tag{3.5}$$

are surjective. Indeed, set

$$\mathcal{G}_1=\ker(V_1\otimes\mathcal{O}_{X_{R^s}}(-m_0)\to(\mathcal{E}_1)_{R^s}),$$
$$\mathcal{G}_i^{(1)}=\ker(V_1\otimes\mathcal{O}_{X_{R^s}}(-m_0)\to(\mathcal{E}_1/F_i(\mathcal{E}_1))_{R^s}).$$

Then we obtain a commutative diagram

$$\begin{array}{ccccc}
V_1\otimes\mathcal{O}_{R^s} & \longrightarrow & (\pi_{R^s})_*(\mathcal{E}_1(m_0))_{R^s} & \xrightarrow{\ \delta\ } & R^1\pi_{R^s*}(\mathcal{G}_1(m_0))\\
\downarrow{=} & & \downarrow & & \downarrow\\
V_1\otimes\mathcal{O}_{R^s} & \longrightarrow & (\pi_{R^s})_*(\mathcal{E}_1/F_i(\mathcal{E}_1)(m_0))_{R^s} & \longrightarrow & R^1\pi_{R^s*}(\mathcal{G}_i^{(1)}(m_0))
\end{array}.$$

Since $H^1(F_i(\mathcal{E}_1)_y(m_0))=0$ and $V_1\cong H^0((\mathcal{E}_1)_y(m_0))$ for any $y\in R^s$, the middle homomorphism is surjective and $\delta=0$. So the homomorphism $V_1\otimes\mathcal{O}_{R^s}\to(\pi_{R^s})_*(\mathcal{E}_1/F_i(\mathcal{E}_1)(m_0))_{R^s}$ is surjective. In a similar way, we obtain the surjectivity of the homomorphism $V_2 \otimes \mathcal{O}_{R^s} \to (\pi_{R^s})_*(\mathcal{E}_2/F_i(\mathcal{E}_2)(m_0-\gamma)_{R^s})$. The quotients (3.2), (3.3), (3.4) and (3.5) determine a morphism

$$\iota\colon R^s \longrightarrow \mathrm{Grass}_{r_2}(V_1\otimes W_1 \oplus V_2\otimes W_2)\times \mathrm{Grass}_{r_1}(V_1\otimes W_2)\times \prod_{i=1}^{l_1}\mathrm{Grass}_{d^{(1)}_{i+1}}(V_1)\times \prod_{i=1}^{l_2}\mathrm{Grass}_{d^{(2)}_{i+1}}(V_2),$$

where $r_1 = h^0(\mathcal{E}_1(m_0+m_1)_y), r_2 = h^0(\mathcal{E}_2(m_0+m_1-\gamma)_y)$ for any $y \in R^s$. We can see that ι is a closed immersion.

Let $G := (GL(V_1)\times_S GL(V_2))/(\boldsymbol{G}_m \times S)$. Here $\boldsymbol{G}_m\times S$ is the subgroup of $GL(V_1)\times_S GL(V_2)$ consisting of all scalar matrices. The group G acts canonically on R^s and on $\mathrm{Grass}_{r_2}(V_1\otimes W_1\oplus V_2\otimes W_2)\times\mathrm{Grass}_{r_1}(V_1\otimes W_2)\times\prod_{i=1}^{l_1}\mathrm{Grass}_{d^{(1)}_{i+1}}(V_1)\times\prod_{i=1}^{l_2}\mathrm{Grass}_{d^{(2)}_{i+1}}(V_2)$. We can see that ι is a G-equivariant immersion. Let $\mathcal{O}_{\mathrm{Grass}_{r_2}(V_1\otimes W_1\oplus V_2\otimes W_2)}(1), \mathcal{O}_{\mathrm{Grass}_{r_1}(V_1\otimes W_2)}(1), \mathcal{O}_{\mathrm{Grass}_{d^{(1)}_i}(V_1)}(1), \mathcal{O}_{\mathrm{Grass}_{d^{(2)}_i}(V_2)}(1)$ be the S-ample line bundle on $\mathrm{Grass}_{r_2}(V_1\otimes W_1\oplus V_2\otimes W_2)$, $\mathrm{Grass}_{r_1}(V_1\otimes W_2), \mathrm{Grass}_{d^{(1)}_i}(V_1), \mathrm{Grass}_{d^{(2)}_i}(V_2)$, respectively, induced by Plücker embedding. For $i = 1,\dots,l_1$ and $j = 1,\dots,l_2$, we define positive rational numbers $\xi, \xi^{(1)}_i, \xi^{(2)}_j$ by

$$\xi = P(m_0)+P(m_0-\gamma) - \sum_{i=1}^{l_1}\epsilon^{(1)}_i d^{(1)}_{i+1} - \sum_{j=1}^{l_2}\epsilon^{(2)}_j d^{(2)}_{j+1}, \quad \xi^{(1)}_i = 2rd_X m_1\epsilon^{(1)}_i, \quad \xi^{(2)}_i = 2rd_X m_1 \epsilon^{(2)}_i. \quad (3.6)$$

Put

$$L := \iota^*\Big(\mathcal{O}_{\mathrm{Grass}_{r_2}(V_1\otimes W_1\oplus V_2\otimes W_2)}(\xi)\otimes\mathcal{O}_{\mathrm{Grass}_{r_1}(V_1\otimes W_2)}(\xi)\otimes\bigotimes_{i=1}^{l_1}\mathcal{O}_{\mathrm{Grass}_{d^{(1)}_{i+1}}(V_1)}(\xi^{(1)}_i)\otimes\bigotimes_{j=1}^{l_2}\mathcal{O}_{\mathrm{Grass}_{d^{(2)}_{j+1}}(V_2)}(\xi^{(2)}_j)\Big).$$

Then L is a $\mathbb{Q}$-line bundle on R^s and for some positive integer N, $L^{\otimes N}$ becomes a G -linearized S-ample line bundle on R^s.

Proposition 3.3.3. All points of R^s are properly stable with respect to the action of G and the G-linearized S-ample line bundle $L^{\otimes N}$.

Proof. Take any geometric point x of R^s. Let y be the induced geometric point of S. We prove that x is a properly stable point of the fiber R^s_y with respect to the action of G_y and the polarization $L^{\otimes N}$. So we may assume that $S = \operatorname{Spec} K$ with K an algebraically closed field. We put

$$(E_1, E_2, \Phi, F_*(E_1), F_*(E_2)) := ((\mathcal{E}_1)_x, (\mathcal{E}_2)_x, \tilde{\Phi}_x, F_*(\mathcal{E}_1)_x, F_*(\mathcal{E}_2)_x))$$

For simplicity, we write the same character V_1, V_2, W_1, W_2 to denote $(V_1)_y, (V_2)_y, (W_1)_y, (W_2)_y$, respectively. Let

$$\pi_2\colon V_1\otimes W_1\oplus V_2\otimes W_2 \to N_2,\ \pi_1\colon V_1\otimes W_2\to N_1,\ \pi_{1,i}\colon V_1\to N^{(1)}_i,\ \pi_{2,i}\colon V_2\to N^{(2)}_i$$

be the quotients of vector spaces corresponding to $\iota(x)$. We will show that $\iota(x)$ is a properly stable point with respect to the action of G and the linearization of $L^{\otimes N}$. Consider the character

$$\chi\colon GL(V_1)\times GL(V_2)\longrightarrow \boldsymbol{G}_m;\ (g_1,g_2)\mapsto \det(g_1)\det(g_2).$$

Since the natural composite $\ker\chi\to GL(V_1)\times GL(V_2)\to G$ is an isogeny, by Theorem 2.1 [MFK] it is sufficient to show that $\mu^{L^{\otimes N}}(x,\lambda) > 0$ for any nontrivial homomorphism $\lambda\colon \boldsymbol{G}_m\to\ker\chi$,where $\mu^{L^{\otimes N}}(x,\lambda)$ is defined in Definition 2.2 [MFK]. Let $\lambda\colon\boldsymbol{G}_m\to\ker\chi$ be a nontrivial homomorphism. For a suitable basis $e^{(1)}_1,\dots,e^{(1)}_{n_1}$ (resp. $e^{(1)}_1,\dots,e^{(2)}_{n_2}$), the action of λ on V_1 (resp. V_2) is represented by

$$e^{(1)}_i\mapsto t^{u^{(1)}_i}e^{(1)}_i \text{ (resp. } e^{(2)}_i\mapsto t^{u^{(2)}_i}e^{(2)}_i) \quad (t\in\boldsymbol{G}_m),$$

where $u^{(1)}_1\le\cdots\le u^{(1)}_{n_1}$ (resp. $u^{(2)}_1\le\cdots\le u^{(2)}_{n_2}$). Then we have $\sum_{i=1}^{n_1}u^{(1)}_i+\sum_{i=1}^{n_2}u^{(2)}_i = 0$. Let $f^{(k)}_1,\dots,f^{(k)}_{b_k}$ be a basis of W_k for each $k=1,2$.

For $q = 0,1,\dots,n_1+n_2$, we define functions $a_1(q), a_2(q)$ as follows. First, we set $(a_1(q),a_2(q)) = (0,0)$ and put

$$(a_1(1),a_2(1)) = \begin{cases}(1,0) & \text{if } u^{(1)}_1\le u^{(2)}_1\\ (0,1) & \text{if } u^{(1)}_1 > u^{(2)}_1\end{cases}.$$

We inductively define

$$(a_1(q+1), a_2(q+1)) = \begin{cases} (a_1(q)+1, a_2(q)) & \text{if } u^{(1)}_{a_1(q)+1} \leq u^{(2)}_{a_2(q)+1},\ a_1(q) < n_1, \text{and } a_2(q) < n_2 \\ (a_1(q), a_2(q)+1) & \text{if } u^{(1)}_{a_1(q)+1} > u^{(2)}_{a_2(q)+1}\ a_1(q) < n_1, \text{and } a_2(q) < n_2 \\ (a_1(q)+1, a_2(q)) & \text{if } a_2(q) = n_2 \\ (a_1(q), a_2(q)+1) & \text{if } a_1(q) = n_1 \end{cases}.$$

Then $a_1(q)$ and $a_2(q)$ are integers satisfying $0 \leq a_1(q) \leq n_1$, $0 \leq a_2(q) \leq n_2$, $a_1(q) \leq a_1(q+1)$, $a_2(q) \leq a_2(q+1)$ and $a_1(q) + a_2(q) = q$. We define $v_1, \dots, v_{n_1+n_2}$ by

$$v_q = \begin{cases} u^{(1)}_{a_1(q)} & \text{if } (a_1(q), a_2(q)) = (a_1(q-1)+1, a_2(q-1)) \\ u^{(2)}_{a_2(q)} & \text{if } (a_1(q), a_2(q)) = (a_1(q-1), a_2(q-1)+1) \end{cases}.$$

For $p = 1, \dots, b_1n_1 + b_2n_2$, we can find a unique integer $q \in \{1, \dots, n_1+n_2\}$ such that

$$p = \begin{cases} (a_1(q)-1)b_1 + a_2(q)b_2 + j & \text{for some } 1 \leq j \leq b_1 \text{ if } (a_1(q), a_2(q)) = (a_1(q-1)+1, a_2(q-1)) \\ a_1(q)b_1 + (a_2(q)-1)b_2 + j & \text{for some } 1 \leq j \leq b_2 \text{ if } (a_1(q), a_2(q)) = (a_1(q-1), a_2(q-1)+1) \end{cases}.$$

For each p, we put $s^{(2)}_p := v_q$ and

$$h_p := \begin{cases} e^{(1)}_{a_1(q)} \otimes f^{(1)}_j & \text{if } (a_1(q), a_2(q)) = (a_1(q-1)+1, a_2(q-1)) \\ e^{(2)}_{a_2(q)} \otimes f^{(2)}_j & \text{if } (a_1(q), a_2(q)) = (a_1(q-1), a_2(q-1)+1) \end{cases}.$$

Put $\delta_p := (v_{q+1} - v_q)(n_1+n_2)^{-1}$. Then we have

$$v_{n_1+n_2} = \sum_{q=1}^{n_1+n_2-1} q\delta_q, \tag{3.7}$$

$$u^{(1)}_{n_2} = \sum_{\substack{1 \leq q \leq n_1+n_2-1 \\ a_1(q) < n_1}} q\delta_q + \sum_{\substack{1 \leq q \leq n_1+n_2-1 \\ a_1(q) = n_1}} (q - n_1 - n_2)\delta_q, \tag{3.8}$$

and

$$u^{(2)}_{n_2} = \sum_{\substack{1 \leq q \leq n_1+n_2-1 \\ a_2(q) < n_2}} q\delta_q + \sum_{\substack{1 \leq q \leq n_1+n_2-1 \\ a_2(q) = n_2}} (q - n_1 - n_2)\delta_q. \tag{3.9}$$

Let $U^{(2)}_p$ be the vector subspace of $V_1 \otimes W_1 \oplus V_2 \otimes W_2$ generated by $h_1, \dots, h_p$. For $i = 1, \dots, r_2$, we can find an integer $p^{(2)}_i \in \{1, \dots, b_1n_1 + b_2n_2\}$ such that $\dim \pi_2(U^{(2)}_{p^{(2)}_i}) = i$ and $\dim \pi_2(U^{(2)}_{p^{(2)}_i - 1}) = i - 1$. Then

$$\begin{aligned} \sum_{i=1}^{r_2} s^{(2)}_{p^{(2)}_i} &= \sum_{i=1}^{r_2} s^{(2)}_{p^{(2)}_i} \left(\dim \pi_2(U^{(2)}_{p^{(2)}_i}) - \dim \pi_2(U^{(2)}_{p^{(2)}_i - 1}) \right) \\ &= \sum_{p=1}^{b_1n_1+b_2n_2} s^{(2)}_p \left(\dim \pi_2(U^{(2)}_p) - \dim \pi_2(U^{(2)}_{p-1}) \right) \\ &= r_2 s^{(2)}_{b_1n_1+b_2n_2} - \sum_{p=1}^{b_1n_1+b_2n_2-1} (s^{(2)}_{p+1} - s^{(2)}_p) \dim \pi_2(U^{(2)}_p) \\ &= r_2 v_{n_1+n_2} - \sum_{q=1}^{n_1+n_2-1} (v_{q+1} - v_q) \dim \pi_2(U^{(2)}_{b_1a_1(q)+b_2a_2(q)}) \\ &\overset{(3.7)}{=} \sum_{q=1}^{n_1+n_2-1} \left(r_2 q - (n_1+n_2) \dim \pi_2(U^{(2)}_{b_1a_1(q)+b_2a_2(q))}) \right) \delta_q. \end{aligned}$$

For $p = (i-1)b_2 + j$ $(1 \leq i \leq n_1, 1 \leq j \leq b_2)$, we put $s^{(1)}_p = u^{(1)}_i$ and $h'_p = e^{(1)}_i \otimes f^{(2)}_j$. Let $U^{(1)}_p$ be the subspace of $V_1 \otimes W_2$ generated by $h'_1, \dots, h'_p$. For $i = 1, \dots, r_1$, we can find an integer $p^{(1)}_i \in \{1, \dots, b_2n_1\}$

such that $\dim \pi_1(U^{(1)}_{p_i^{(1)}}) = i$ and $\dim \pi_1(U^{(1)}_{p_i^{(1)}-1}) = i-1$. Then

$$\begin{aligned}
\sum_{i=1}^{r_1} s^{(1)}_{p_i^{(1)}} &= \sum_{i=1}^{r_1} s^{(1)}_{p_i^{(1)}} \left(\dim \pi_1(U^{(1)}_{p_i^{(1)}}) - \dim \pi_1(U^{(1)}_{p_i^{(1)}-1})\right) \\
&= \sum_{p=1}^{b_2 n_1} s^{(1)}_p \left(\dim \pi_1(U^{(1)}_p) - \dim \pi_1(U^{(1)}_{p-1})\right) \\
&= r_1 s^{(1)}_{b_2 n_1} - \sum_{p=1}^{b_2 n_1 - 1} (s^{(1)}_{p+1} - s^{(1)}_p) \dim \pi_1(U^{(1)}_p) \\
&= r_1 u^{(1)}_{n_1} - \sum_{i=1}^{n_1-1} (u^{(1)}_{i+1} - u^{(1)}_i) \dim \pi_1(U^{(1)}_{ib_2}) \\
&= r_1 u^{(1)}_{n_1} - \sum_{\substack{1 \le q \le n_1+n_2-1 \\ a_1(q) < n_1}} (v_{q+1} - v_q) \dim \pi_1(U^{(1)}_{a_1(q)b_2}) \\
&\overset{(3.8)}{=} r_1 \left(\sum_{\substack{1 \le q \le n_1+n_2-1 \\ a_1(q) < n_1}} q\delta_q + \sum_{\substack{1 \le q \le n_1+n_2-1 \\ a_1(q) = n_1}} (q - n_1 - n_2)\delta_q \right) - \sum_{\substack{1 \le q \le n_1+n_2-1 \\ a_1(q) < n_1}} (n_1+n_2)\delta_q \dim \pi_1(U^{(1)}_{a_1(q)b_2}) \\
&= \sum_{q=1}^{n_1+n_2-1} \left(r_1 q - (n_1+n_2) \dim \pi_1(U^{(1)}_{a_1(q)b_2}) \right) \delta_q.
\end{aligned}$$

Let $V^{(1)}_p$ be the subspace of V_1 generated by $e^{(1)}_1, \ldots, e^{(1)}_p$. For $i = 1, \ldots, l_1$ and for $j = 1, \ldots, d^{(1)}_i$, let $p^{(1)}_{i,j}$ be the integer such that $\dim \pi_{1,i}(V^{(1)}_{p^{(1)}_{i,j}}) = j$ and $\dim \pi_{1,i}(V^{(1)}_{p^{(1)}_{i,j}-1}) = j-1$. Then

$$\begin{aligned}
\sum_{j=1}^{d_i^{(1)}} u^{(1)}_{p^{(1)}_{i,j}} &= \sum_{j=1}^{d_i^{(1)}} u^{(1)}_{p^{(1)}_{i,j}} \left(\dim \pi_{1,i}(V^{(1)}_{p^{(1)}_{i,j}}) - \dim \pi_{1,i}(V^{(1)}_{p^{(1)}_{i,j}-1})\right) \\
&= \sum_{p=1}^{n_1} u^{(1)}_p \left(\dim \pi_{1,i}(V^{(1)}_p) - \dim \pi_{1,i}(V^{(1)}_{p-1})\right) \\
&= d^{(1)}_i u^{(1)}_{n_1} - \sum_{p=1}^{n_1-1} (u^{(1)}_{p+1} - u^{(1)}_p) \dim \pi_{1,i}(V^{(1)}_p) \\
&= d^{(1)}_i u^{(1)}_{n_1} - \sum_{a_1(q) < n_1} (v_{q+1} - v_q) \dim \pi_{1,i}(V^{(1)}_{a_1(q)}) \\
&\overset{(3.8)}{=} d^{(1)}_i \left(\sum_{\substack{1 \le q \le n_1+n_2-1 \\ a_1(q) < n_1}} q\delta_q + \sum_{\substack{1 \le q \le n_1+n_2-1 \\ a_1(q) = n_1}} (q - n_1 - n_2)\delta_q \right) - \sum_{\substack{1 \le q \le n_1+n_2-1 \\ a_1(q) < n_1}} (n_1+n_2)\delta_q \dim \pi_{1,i}(V^{(1)}_{a_1(q)}) \\
&= \sum_{q=1}^{n_1+n_2-1} \left(d^{(1)}_i q - (n_1+n_2) \dim \pi_{1,i}(V^{(1)}_{a_1(q)}) \right) \delta_q.
\end{aligned}$$

Let $V^{(2)}_p$ be the subspace of V_2 generated by $e^{(2)}_1, \ldots, e^{(2)}_p$. For $i = 1, \ldots, l_2$, and for $j = 1, \ldots, d^{(2)}_i$, let $p^{(2)}_{i,j}$ be the integer such that $\dim \pi_{2,i}(V^{(2)}_{p^{(2)}_{i,j}}) = j$ and $\dim \pi_{2,i}(V^{(2)}_{p^{(2)}_{i,j}-1}) = j-1$. Then

$$\begin{aligned}
\sum_{j=1}^{d_i^{(2)}} u^{(2)}_{p^{(2)}_{i,j}} &= \sum_{j=1}^{d_i^{(2)}} u^{(2)}_{p^{(2)}_{i,j}} \left(\dim \pi_{2,i}(V^{(2)}_{p^{(2)}_{i,j}}) - \dim \pi_{2,i}(V^{(2)}_{p^{(2)}_{i,j}-1})\right) \\
&= \sum_{p=1}^{n_2} u^{(2)}_p \left(\dim \pi_{2,i}(V^{(2)}_p) - \dim \pi_{2,i}(V^{(2)}_{p-1})\right)
\end{aligned}$$

$$= d_i^{(2)} u_{n_2}^{(2)} - \sum_{p=1}^{n_2-1} (u_{p+1}^{(2)} - u_p^{(2)}) \dim \pi_{2,i}(V_p^{(2)})$$

$$= d_i^{(2)} u_{n_2}^{(2)} - \sum_{a_2(q)<n_2} (u_{q+1}^{(2)} - u_q^{(2)}) \dim \pi_{2,i}(V_{a_2(q)}^{(2)})$$

$$\overset{(3.9)}{=} d_i^{(2)} \left(\sum_{\substack{1\le q\le n_1+n_2-1\\ a_2(q)<n_2}} q\delta_q + \sum_{\substack{1\le q\le n_1+n_2-1\\ a_2(q)=n_2}} (q-n_1-n_2)\delta_q \right) - \sum_{\substack{1\le q\le n_1+n_2-1\\ a_2(q)<n_2}} (n_1+n_2)\delta_q \dim \pi_{2,i}(V_{a_2(q)}^{(2)})$$

$$= \sum_{q=1}^{n_1+n_2-1} \left(d_i^{(2)} q - (n_1+n_2) \dim \pi_{2,i}(V_{a_2(q)}^{(2)}) \right) \delta_q.$$

So we have

$$\mu^{L^{\otimes N}}(x,\lambda) = -\left(\xi \sum_{i=1}^{r_1} s_{p_i^{(k)}}^{(k)} + \sum_{i=1}^{l_1} \xi_i^{(1)} \sum_{j=1}^{d_i^{(1)}} u_{p_{i,j}^{(1)}}^{(1)} + \sum_{i=1}^{l_2} \xi_i^{(2)} \sum_{j=1}^{d_i^{(2)}} u_{p_{i,j}^{(2)}}^{(2)} \right) N$$

$$= -\sum_{q=1}^{n_1+n_2-1} N\delta_q \Bigg\{ q \sum_{i=1}^{l_1} \xi_i^{(1)} d_i^{(1)} + q \sum_{i=1}^{l_2} \xi_i^{(2)} d_i^{(2)} - (n_1+n_2) \sum_{i=1}^{l_1} \xi_i^{(1)} \dim \pi_i^{(1)}(V_{a_1(q)}^{(1)})$$

$$- (n_1+n_2) \sum_{i=1}^{l_2} \xi_i^{(2)} \dim \pi_i^{(2)}(V_{a_2(q)}^{(2)}) + (r_1+r_2) q\xi$$

$$- (n_1+n_2)\xi \left(\dim \pi_1(U_{a_1(q)b_2}^{(1)}) + \dim \pi_2(U_{b_1a_1(q)+b_2a_2(q)}^{(2)}) \right) \Bigg\}.$$

Hence x is properly stable point if

$$- q \sum_{i=1}^{l_1} \xi_i^{(1)} d_{i+1}^{(1)} - q \sum_{i=1}^{l_2} \xi_i^{(2)} d_{i+1}^{(2)} + (n_1+n_2) \sum_{i=1}^{l_1} \xi_i^{(1)} \dim \pi_{1,i}(V_{a_1(q)}^{(1)}) + (n_1+n_2) \sum_{i=1}^{l_2} \xi_i^{(2)} \dim \pi_{2,i}(V_{a_2(q)}^{(2)})$$
$$- q\xi(r_1+r_2) + \xi(n_1+n_2) \left(\dim \pi_1(U_{a_1(q)b_2}^{(1)}) + \dim \pi_2(U_{b_1a_1(q)+b_2a_2(q)}^{(2)}) \right) > 0$$

for all $q = 1, \dots, n_1+m_2-1$.

For each $q = 1, \dots, n_1+n_2-1$, let V_k' be the vector subspace of V_k generated by $e_1^{(k)}, \dots, e_{a_k(q)}^{(k)}$ for $k = 1, 2$. We note that

$$q = \dim V_1' + \dim V_2'. \tag{3.10}$$

Then $U_{a_1(q)b_2}^{(1)} = V_1' \otimes W_2$ and $U_{b_1a_1(q)+b_2a_2(q)}^{(2)} = V_1' \otimes W_1 \oplus V_2' \otimes W_2$. Put

$$E_1' := \operatorname{Im}(V_1' \otimes \mathcal{O}_{X_y}(-m_0) \to E_1),\ E_2' := \operatorname{Im}(\Lambda_{D_y}^1 \otimes V_1' \otimes \mathcal{O}_{X_y}(-m_0) \oplus V_2' \otimes \mathcal{O}_{X_y}(-m_0+\gamma) \to E_2).$$

By the choice of m_1, we have

$$\pi_2(U_{b_1a_1(q)+b_2a_2(q)}^{(2)}) = H^0(E_2'(m_0+m_1-\gamma)), \quad \pi_1(U_{a_1(q)b_2}^{(1)}) = H^0(E_1'(m_0+m_1)). \tag{3.11}$$

Put $r_1' = \operatorname{rank} E_1', r_2' = \operatorname{rank} E_2'$. Let $\pi_{k,i}'$ be the composite $V_k' \hookrightarrow V_k \stackrel{\pi_{k,i}}{\to} N_i^{(k)}$ for $k = 1, 2$. Then we have

$$\dim V_1' \le h^0(E_1'(m_0)),\ \dim\ker \pi_{1,i} \le h^0(F_{i+1}(E_1')(m_0)),\ \dim V_2' \le h^0(E_2'(m_0)),\ \dim\ker \pi_{2,j} \le h^0(F_{j+1}(E_2')(m_0)) \tag{3.12}$$

for $1 \leq i \leq l_1$ for $1 \leq j \leq l_2$. So we obtain

$$
\begin{aligned}
& - q\xi(r_1+r_2) + \xi(n_1+n_2)\Big(\dim \pi_1(U^{(1)}_{a_1(q)b_2}) + \dim \pi_2(U^{(2)}_{b_1a_1(q)+b_2a_2(q)})\Big) \\
& - q\sum_{i=1}^{l_1}\xi^{(1)}_i d^{(1)}_{i+1} - q\sum_{j=1}^{l_2}\xi^{(2)}_j d^{(2)}_{j+1} + (n_1+n_2)\sum_{i=1}^{l_1}\xi^{(1)}_i \dim \pi_{1,i}(V^{(1)}_{a_1(q)}) + (n_1+n_2)\sum_{j=1}^{l_2}\xi^{(2)}_j \dim \pi_{2,i}(V^{(2)}_{a_2(q)}) \\
\overset{(3.10)(3.11)}{=} & \xi\Big\{ - (\dim V'_1 + \dim V'_2)(h^0(E_1(m_0+m_1)) + h^0(E_2(m_0+m_1-\gamma))) \\
& + (\dim V_1 + \dim V_2)(h^0(E'_1(m_0+m_1)) + h^0(E'_2(m_0+m_1-\gamma)))\Big\} \\
& - (\dim V'_1 + \dim V'_2)\sum_{i=1}^{l_1}\xi^{(1)}_i d^{(1)}_{i+1} + (\dim V_1+\dim V_2)\sum_{i=1}^{l_1}\xi^{(1)}_i(\dim V'_1 - \dim\ker \pi'_{1,i}) \\
& - (\dim V'_1 + \dim V'_2)\sum_{j=1}^{l_2}\xi^{(2)}_j d^{(2)}_{j+1} + (\dim V_1+\dim V_2)\sum_{j=1}^{l_2}\xi^{(2)}_j(\dim V'_2 - \dim\ker \pi'_{2,j}) \\
\overset{(3.6)}{=} & \big(\dim V_1 + \dim V_2 - \sum_{i=1}^{l_1}\epsilon^{(1)}_i d^{(1)}_{i+1} - \sum_{j=1}^{l_2}\epsilon^{(2)}_j d^{(2)}_{j+1}\big)\big\{ - (\dim V'_1+\dim V'_2)(2rd_X m_1 + \dim V_1 + \dim V_2) \\
& + (\dim V_1+\dim V_2)((r'_1+r'_2)d_X m_1 + \chi(E'_1(m_0)) + \chi(E'_2(m_0-\gamma)))\big\} \\
& - 2rd_X m_1(\dim V'_1+\dim V'_2)\sum_{i=1}^{l_1}\epsilon^{(1)}_i d^{(1)}_{i+1} + 2rd_X m_1(\dim V_1+\dim V_2)\sum_{i=1}^{l_1}\epsilon^{(1)}_i(\dim V'_1 - \dim\ker\pi'_{1,i}) \\
& - 2rd_X m_1(\dim V'_1+\dim V'_2)\sum_{j=1}^{l_2}\epsilon^{(2)}_j d^{(2)}_{j+1} + 2rd_X m_1(\dim V_1+\dim V_2)\sum_{j=1}^{l_2}\epsilon^{(2)}_j(\dim V'_2 - \dim\ker\pi'_{2,j}) \\
= & - 2rd_X m_1(\dim V_1+\dim V_2)\big\{\dim V'_1 + \dim V'_2 - \sum_{i=1}^{l_1}\epsilon^{(1)}_i(\dim V'_1 - \dim\ker\pi'_{1,i}) - \sum_{j=1}^{l_2}\epsilon^{(2)}_j(\dim V'_2 - \dim\ker\pi'_{2,j}) \\
& + (r'_1+r'_2)d_X m_1(\dim V_1+\dim V_2)\big(\dim V_1+\dim V_2 - \sum_{i=1}^{l_1}\epsilon^{(1)}_i d^{(1)}_{i+1} - \sum_{j=1}^{l_2}\epsilon^{(2)}_j d^{(2)}_{j+1}\big) \\
& + (\dim V_1+\dim V_2)\Big(\dim V_1+\dim V_2 - \sum_{i=1}^{l_1}\epsilon^{(1)}_i d^{(1)}_{i+1} - \sum_{j=1}^{l_2}\epsilon^{(2)}_j d^{(2)}_{j+1}\Big) \\
& \times \big\{ - (\dim V'_1+\dim V'_2) + \chi(E'_1(m_0)) + \chi(E'_2(m_0-\gamma))\big\} \\
\overset{(3.12)}{\geq} & (r'_1+r'_2)d_X m_1(\dim V_1+\dim V_2)\left\{h^0(E_1(m_0)) + h^0(E_2(m_0-\gamma)) - \sum_{i=1}^{l_1}\epsilon^{(1)}_i d^{(1)}_{i+1} - \sum_{j=1}^{l_2}\epsilon^{(2)}_j d^{(2)}_{j+1}\right\} \\
& - 2rd_X m_1(\dim V_1+\dim V_2)\Big\{h^0(E'_1(m_0)) + h^0(E'_2(m_0-\gamma)) \\
& - \sum_{i=1}^{l_1}\epsilon^{(1)}_i\big(h^0(E'_1(m_0)) - h^0(F_{i+1}(E'_1)(m_0))\big) - \sum_{j=1}^{l_2}\epsilon^{(2)}_j\big(h^0(E'_2(m_0-\gamma)) - h^0(F_{j+1}(E'_2)(m_0-\gamma))\big)\Big\} \\
& - (\dim V_1+\dim V_2)\left(\dim V_1+\dim V_2 - \sum_{i=1}^{l_1}\epsilon^{(1)}_i d^{(1)}_{i+1} - \sum_{j=1}^{l_2}\epsilon^{(2)}_j d^{(2)}_{j+1}\right) \\
& \times(\dim V'_1+\dim V'_2 - \chi(E'_1(m_0)) - \chi(E'_2(m_0-\gamma))) \\
\overset{(3.1)}{>} & 0.
\end{aligned}
$$

Hence x is a properly stable point. □

By Proposition 3.3.3, there exists a geometric quotient R^s/G.

Theorem 3.3.4. $\overline{M^{D,\boldsymbol{\beta}}_{X/S}}(r,d,\boldsymbol{d}_1,\boldsymbol{d}_2) := R^s/G$ is a coarse moduli scheme of $\overline{\mathcal{M}^{D,\boldsymbol{\beta}}_{X/S}}(r,d,\boldsymbol{d}_1,\boldsymbol{d}_2)$.

Lemma 3.3.5. Take any geometric point $(E_1, E_2, \Phi, F_*(E_1), F_*(E_2)) \in \overline{\mathcal{M}^{D,\boldsymbol{\beta}}_{X/S}}(r, d, \boldsymbol{d}_1, \boldsymbol{d}_2)(K)$. Then for any endomorphisms $f_1 \colon E_1 \to E_1, f_2 \colon E_2 \to E_2$ satisfying $\Phi \circ (1 \otimes f_1) = f_2 \circ \Phi$, $f_1(F_{j+1}(E_1)) \subset F_{j+1}(E_1)$ $(1 \leq j \leq l_1)$ and $f_2(F_{j+1}(E_2)) \subset F_{j+1}(E_2)$ $(1 \leq j \leq l_2)$, there exists $c \in K$ such that $(f_1, f_2) = (c \cdot \mathrm{id}_{E_1}, c \cdot \mathrm{id}_{E_2})$.

Proposition 3.3.6. Let R be a discrete valuation ring over S with the residue field $k = R/\mathfrak{m}$ and the quotient field K. Let $(E_1, E_2, \Phi, F_*(E_1), F_*(E_2))$ be a semistable parabolic $\Lambda^1_{D_K}$-triple on X_K. Then there exists a flat family $(\tilde{E}_1, \tilde{E}_2, \tilde{\Phi}, F_*(\tilde{E}_1), F_*(\tilde{E}_2))$ of parabolic $\Lambda^1_{D_R}$-triples on X_R over R such that $(E_1, E_2, \Phi, F_*(E_1), F_*(E_2)) \cong (\tilde{E}_1, \tilde{E}_2, \tilde{\Phi}, F_*(\tilde{E}_1), F_*(\tilde{E}_2)) \otimes_R K$ and $(\tilde{E}_1, \tilde{E}_2, \tilde{\Phi}, F_*(\tilde{E}_1), F_*(\tilde{E}_2)) \otimes_R k$ is semistable.

Proof of Theorem 2.3.6. Put $l_1 = l_2 = rn$ and $d^{(1)}_i = d^{(2)}_i = i - 1$ for $2 \leq i \leq rn + 1$. Put $\{\beta^{(k)}_i\}_{1 \leq i \leq rn} = \{\alpha^{(k)}_{i,j}\}^{1 \leq i \leq n}_{1 \leq j \leq r}$ for each $k = 1, 2$. For a parabolic ϕ-connection $(E_1, E_2, \phi, \nabla, l^{(1)}_*, l^{(2)}_*)$ over $(C, \boldsymbol{t})$, we define a parabolic Λ^1_D-triple $(E_1, E_2, \Phi, F_*(E_1), F_*(E_2))$ as follows: Let $\Phi \colon \Lambda^1_D \otimes E_1 \to E_2$ be a left $\mathcal{O}_C$-homomorphism induced by ϕ and ∇. For each $1 \leq p \leq rn$, there exists a unique pair of integers (i, j) such that $1 \leq i \leq n$, $1 \leq j \leq r$ and $\beta^{(1)}_p = \alpha^{(1)}_{i,j}$. Then we put $F_1(E_1) := E_1$ and $F_{p+1}(E_1) := \ker(F_p(E_1) \to E_1|_{t_i}/l^{(1)}_{i,j})$. In a similar way we define $F_p(E_2)$ for $1 \leq p \leq rn + 1$. By the definition of the stability we can see that $(E_1, E_2, \phi, \nabla, l^{(1)}_*, l^{(2)}_*)$ is $\boldsymbol{\alpha}$-stable if and only if $(E_1, E_2, \Phi, F_*(E_1), F_*(E_2))$ is $\boldsymbol{\beta}$-stable. The above correspondence determines a morphism of functors

$$\iota \colon \overline{\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d) \longrightarrow \overline{\mathcal{M}^{D,\boldsymbol{\beta}}_{\mathcal{C}\times\mathcal{N}/\tilde{M}_{g,n}\times\mathcal{N}}}(r, d, \boldsymbol{d}_1, \boldsymbol{d}_2).$$

We can see that ι is a closed immersion by Lemma 3.3.2. So there exists a closed subscheme $Z \subset R^s$ such that

$$h_Z = h_{R^s} \times_{\overline{\mathcal{M}^{D,\boldsymbol{\beta}}_{\mathcal{C}\times\mathcal{N}/\tilde{M}_{g,n}\times\mathcal{N}}}(r,d,\boldsymbol{d}_1,\boldsymbol{d}_2)} \overline{\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d),$$

where $h_Z = \mathrm{Hom}_{\tilde{M}_{g,n}\times\mathcal{N}}(-, Z)$. Z is invariant by the action of G. By Lemma 3.3.5, the quotient $R^s \to \overline{M^{D,\boldsymbol{\beta}}_{\mathcal{C}\times\mathcal{N}/\tilde{M}_{g,n}\times\mathcal{N}}}(r, d, \boldsymbol{d}_1, \boldsymbol{d}_2)$ is a principal G-bundle. So Z/G is a closed subscheme of $\overline{M^{D,\boldsymbol{\beta}}_{\mathcal{C}\times\mathcal{N}/\tilde{M}_{g,n}\times\mathcal{N}}}(r, d, \boldsymbol{d}_1, \boldsymbol{d}_2)$ which is just the coarse moduli scheme of $\overline{M^{\boldsymbol{\alpha}}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d)$.

When r and d are coprime, we can see that $\overline{M^{\boldsymbol{\alpha}}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d)$ is fine by Lemma 3.3.5 and the standard argument. For general d, there is an isomorphism $\sigma \colon \overline{M^{\boldsymbol{\alpha}}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d) \to \overline{M^{\boldsymbol{\alpha}'}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d')$ induced an elementary transformation, where r and d' are coporime. Then we obtain a universal family over $\overline{M^{\boldsymbol{\alpha}}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d) \times_{\tilde{M}_{g,n}\times\mathcal{N}} (\mathcal{C}\times\mathcal{N})$ by pulling back a universal family over $\overline{M^{\boldsymbol{\alpha}'}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d') \times_{\tilde{M}_{g,n}\times\mathcal{N}} (\mathcal{C}\times\mathcal{N})$ through σ. So $\overline{M^{\boldsymbol{\alpha}}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d)$ is fine for arbitrary d.

It follows from Proposition 3.3.6 that $\overline{M^{\boldsymbol{\alpha}}_{\mathcal{C}/\tilde{M}_{g,n}}}(\tilde{\boldsymbol{t}}, r, d) \to \tilde{M}_{g,n} \times \mathcal{N}$ is projective for generic $\boldsymbol{\alpha}$. □

Chapter 4

Moduli space of rank three logarithmic connections on the projective line with three poles

In this chapter, we describe the moduli space of rank 3 parabolic logarithmic connections on $\mathbb{P}^1$ with 3 poles. Through this chapter, we may assume that $\boldsymbol{\alpha} = (\alpha_{i,j})_{1 \le i,j \le 3}$ and γ satisfy $0 < \alpha_{i,j} \ll 1$ for any $1 \le i, j \le 3$ and $\gamma \gg 0$. We put

$$T_3 := \left\{ (t_1, t_2, t_3) \in (\mathbb{P}^1)^3 \;\middle|\; t_i \neq t_j \text{ for } i \neq j \right\},$$

$$\mathcal{N}(\nu_1, \nu_2, \nu_3) := \{(\nu_{i,j}) \in \mathbb{C}^9 \mid \nu_{i,0} + \nu_{i,1} + \nu_{i,2} = \nu_i, 1 \le i \le 3\},$$

where $\nu_1, \nu_2, \nu_3 \in \mathbb{C}$ and $\nu_1 + \nu_2 + \nu_3 \in \mathbb{Z}$.

Let $M_3^{\boldsymbol{\alpha}}(\nu_1, \nu_2, \nu_3) \to T_3 \times \mathcal{N}(\nu_1, \nu_2, \nu_3)$ (resp. $\overline{M_3^{\boldsymbol{\alpha}}}(\nu_1, \nu_2, \nu_3) \to T_3 \times \mathcal{N}(\nu_1, \nu_2, \nu_3)$) be the family of moduli spaces of $\boldsymbol{\alpha}$-stable $\boldsymbol{\nu}$-parabolic connections (resp. ϕ-connections), whose fiber $M_3^{\boldsymbol{\alpha}}(\boldsymbol{t}, \boldsymbol{\nu})$ (resp. $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$) at $(\boldsymbol{t}, \boldsymbol{\nu}) \in T_3 \times \mathcal{N}(\nu_1, \nu_2, \nu_3)$ is the moduli space of $\boldsymbol{\alpha}$-stable $\boldsymbol{\nu}$-parabolic connections (resp. ϕ-connections) over $(\mathbb{P}^1, \boldsymbol{t})$. Here a parabolic ϕ-connection is said to be $\boldsymbol{\alpha}$-stable if a parabolic ϕ-connection is $\{\boldsymbol{\alpha}, \boldsymbol{\alpha}\}$-stable.

4.1 The family of $A_2^{(1)*}$-surfaces and the main theorem

In this section, we construct a family of $A_2^{(1)*}$-surfaces parameterized by $T_3 \times \mathcal{N}(0, 0, 2)$ and state the main theorem. We put $\mathcal{N} := \mathcal{N}(0, 0, 2)$.

Let $\tilde{t}_i \subset \mathbb{P}^1 \times T_3 \times \mathcal{N}$ be the section defined by

$$T_3 \times \mathcal{N} \hookrightarrow \mathbb{P}^1 \times T_3 \times \mathcal{N}; \quad ((t_j)_{1 \le j \le 3}, (\nu_{m,n})_{0 \le n \le 2}^{1 \le m \le 3}) \mapsto (t_i, (t_j)_{1 \le j \le 3}, (\nu_{m,n})_{0 \le n \le 2}^{1 \le n \le 3})$$

for $i = 1, 2, 3$ and $D(\tilde{\boldsymbol{t}}) = \tilde{t}_1 + \tilde{t}_2 + \tilde{t}_3$ be a relative effective Cartier divisor for the projection $\mathbb{P}^1 \times T_3 \times \mathcal{N} \to T_3 \times \mathcal{N}$. Put

$$\mathcal{E} := \Omega^1_{\mathbb{P}^1 \times T_3 \times \mathcal{N} / T_3 \times \mathcal{N}}(D(\tilde{\boldsymbol{t}})) \oplus \mathcal{O}_{\mathbb{P}^1 \times T_3 \times \mathcal{N}}.$$

Let

$$\pi \colon \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}^1 \times T_3 \times \mathcal{N}$$

be the projection, where $\mathbb{P}(\mathcal{E}) := \operatorname{Proj} \operatorname{Sym}(\mathcal{E}^\vee)$. We note that for each $x \in T_3 \times \mathcal{N}$, there is an isomorphism $(\Omega^1_{\mathbb{P}^1 \times T_3 \times \mathcal{N} / T_3 \times \mathcal{N}}(D(\tilde{\boldsymbol{t}})))_x \cong \Omega^1_{\mathbb{P}^1}(D(\tilde{\boldsymbol{t}})_x) \cong \mathcal{O}_{\mathbb{P}^1}(1)$ and so $\mathbb{P}(\mathcal{E}_x)$ is a Hirzebruch surface of degree 1. Let $\tilde{D}_0 \subset \mathbb{P}(\mathcal{E})$ be the section over $\mathbb{P}^1 \times T_3 \times \mathcal{N}$ defined by the injection $\Omega^1_{\mathbb{P}^1 \times T_3 \times \mathcal{N} / T_3 \times \mathcal{N}}(D(\tilde{\boldsymbol{t}})) \hookrightarrow \mathcal{E}$ and $\tilde{D}_i \subset \mathbb{P}(\mathcal{E})$ be the inverse image of $\tilde{t}_i$. Put $\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\tilde{D}_0 + \tilde{D}_1)$. Let

$$\varpi \colon \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}^1 \times T_3 \times \mathcal{N} \longrightarrow T_3 \times \mathcal{N}$$

be the projection and take a closed point $x \in T_3 \times \mathcal{N}$. Since $\tilde{D}_0$ and $\tilde{D}_1$ are flat over $T_3 \times \mathcal{N}$, $(\tilde{D}_0)_x$ and $(\tilde{D}_1)_x$ are effective Cartier divisors on $\mathbb{P}(\mathcal{E}_x)$, and so $\mathcal{L}_x \cong \mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}((\tilde{D}_0)_x + (\tilde{D}_1)_x)$. The section

$(\tilde{D}_0)_x \subset \mathbb{P}(\mathcal{E}_x)$ is a (-1)-curve by definition, so we get a morphism $f\colon \mathbb{P}(\mathcal{E}_x) \to \mathbb{P}^2$ by contracting $(\tilde{D}_0)_x$. By the projection formula $R^i f_* \mathcal{L}_x \cong \mathcal{O}_{\mathbb{P}^2}(1) \otimes R^i f_* \mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}$, we have $H^i(\mathbb{P}(\mathcal{E}_x), \mathcal{L}_x) \cong H^i(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) = 0$ for any $i > 0$, which leads to $\dim H^0(\mathbb{P}(\mathcal{E}_x), \mathcal{L}_x) = 3$ by Riemann-Roch theorem. Hence $\varpi_* \mathcal{L}$ is a rank 3 locally free sheaf on $T_3 \times \mathcal{N}$. Since $\mathcal{L}_x$ is generated by global section, the canonical homomorphism $\varpi^* \varpi_* \mathcal{L} \to \mathcal{L}$ is surjective, so we obtain a morphism $\rho\colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\varpi_* \mathcal{L})$ over $T_3 \times \mathcal{N}$. Let W be the scheme theoretic image of $\rho\colon \tilde{D}_0 \to \mathbb{P}(\varpi_* \mathcal{L})$. Since $\tilde{D}_0$ is proper over $T_3 \times \mathcal{N}$, W is a closed subvariety of $\mathbb{P}(\varpi_* \mathcal{L})$. W_x consists of one point because $\deg_{(\tilde{D}_0)_x} \mathcal{L}|_{(\tilde{D}_0)_x} = (\tilde{D}_0)_x.((\tilde{D}_0)_x + (\tilde{D}_1)_x) = 0$. We can see that $\mathbb{P}(\mathcal{E}) \setminus (\tilde{D}_0) \to \mathbb{P}(\varpi_* \mathcal{L}) \setminus W$ is an isomorphism by the proof of Theorem V.2.17. in [Ha], and $\mathbb{P}(\mathcal{E})$ is isomorphic to the blow-up of $\mathbb{P}(\varpi_* \mathcal{L})$ along W. By the residue map

$$\mathrm{res}_{\tilde{t}_i} : \Omega^1_{\mathbb{P}^1 \times T_3 \times \mathcal{N}/T_3 \times \mathcal{N}}(D(\tilde{\boldsymbol{t}}))|_{\tilde{t}_i} \longrightarrow \mathcal{O}_{\tilde{t}_i},$$

we obtain an isomorphism $\tilde{D}_i \xrightarrow{\sim} \mathbb{P}^1 \times T_3 \times \mathcal{N}$. For each $i = 1, 2, 3$ and $j = 0, 1, 2$, let $\tilde{b}_{i,j}$ be the section of $\tilde{D}_i$ over $T_3 \times \mathcal{N}$ defined by

$$\{((\nu_{i,j} + \mathrm{res}_{t_i}(\tfrac{dz}{z - t_3}) : 1), (t_k)_k, (\nu_{m,n})_{m,n})\} \subset \mathbb{P}^1 \times T_3 \times \mathcal{N}.$$

Let $\tilde{\mathcal{B}}_j$ denote the reduced induced structure on $\tilde{b}_{1,j} \cup \tilde{b}_{2,j} \cup \tilde{b}_{3,j}$ for $j = 0, 1, 2$. Then we can naturally regard $\rho(\tilde{\mathcal{B}}_i)$ as a closed subvariety of $\mathbb{P}(\varpi_* \mathcal{L})$, and it is isomorphic to $\tilde{\mathcal{B}}_i$. So we use the same character $\tilde{\mathcal{B}}_i$ to denote $\rho(\tilde{\mathcal{B}}_i)$ for simplicity of notation. Let $g_2\colon S_2 \to \mathbb{P}(\varpi_* \mathcal{L})$ be the blow-up along $\tilde{\mathcal{B}}_2$, $g_1\colon S_1 \to S_2$ be the blow-up along the strict transform of $\tilde{\mathcal{B}}_1$ and $g\colon S \to S_1$ be the blow-up along the strict transform of $\tilde{\mathcal{B}}_0$. Then for each closed point $(\boldsymbol{t}, \boldsymbol{\nu}) \in T_3 \times \mathcal{N}$, the fiber $S_{(\boldsymbol{t},\boldsymbol{\nu})}$ is a surface obtained by blowing up three points on each of three lines meeting at a single point on $\mathbb{P}((\varpi_* \mathcal{L})_{(\boldsymbol{t},\boldsymbol{\nu})}) \cong \mathbb{P}^2$. Let $Bl_W\colon Z \to S$ be the blow-up along W. Z is also obtained by repeating the blow-up of $\mathbb{P}(\mathcal{E})$.

Let $\widehat{M_3^{\boldsymbol{\alpha}}}(0,0,2)$ be the moduli space of pairs of an $\boldsymbol{\alpha}$-stable parabolic ϕ-connection and a certain subbundle (see Section 4.3), and $\mathrm{PC}\colon \widehat{M_3^{\boldsymbol{\alpha}}}(0,0,2) \to \overline{M_3^{\boldsymbol{\alpha}}}(0,0,2)$ be the morphism defined by forgetting subbundles. Our aim is to prove the following theorem.

Theorem 4.1.1. Take $\boldsymbol{\alpha} = (\alpha_{i,j})_{1 \le i,j \le 3}$ and γ such that $0 < \alpha_{i,j} \ll 1$ for any $1 \le i, j \le 3$ and $\gamma \gg 0$.

(1) The closed subscheme $Y_{\le 1}$ defined by $\operatorname{rank} \phi \le 1$ is reduced. The forgetful map $\mathrm{PC}\colon \widehat{M_3^{\boldsymbol{\alpha}}}(0,0,2) \to \overline{M_3^{\boldsymbol{\alpha}}}(0,0,2)$ is the blow-up along $Y_{\le 1}$.

(2) There exists an isomorphism $\widehat{M_3^{\boldsymbol{\alpha}}}(0,0,2) \xrightarrow{\sim} Z$ and $\overline{M_3^{\boldsymbol{\alpha}}}(0,0,2) \xrightarrow{\sim} S$ over $T_3 \times \mathcal{N}$ such that the diagram

$$\begin{array}{ccc} \widehat{M_3^{\boldsymbol{\alpha}}}(0,0,2) & \xrightarrow{\sim} & Z \\ {\scriptstyle \mathrm{PC}}\downarrow & & \downarrow{\scriptstyle Bl_W} \\ \overline{M_3^{\boldsymbol{\alpha}}}(0,0,2) & \xrightarrow{\sim} & S \end{array}$$

commutes. In particular, $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$ is isomorphic to an $A_2^{(1)*}$-surface for each $(\boldsymbol{t}, \boldsymbol{\nu}) \in T_3 \times \mathcal{N}$.

(3) Let Y be the closed subscheme of $\overline{M_3^{\boldsymbol{\alpha}}}(0,0,2)$ defined by the conditions $\wedge^3 \phi = 0$. Then Y is reduced and $M_3^{\boldsymbol{\alpha}}(0,0,2) \cong \overline{M_3^{\boldsymbol{\alpha}}}(0,0,2) \setminus Y$. Moreover, for each $(\boldsymbol{t}, \boldsymbol{\nu}) \in T_3 \times \mathcal{N}$, the fiber $Y_{(\boldsymbol{t},\boldsymbol{\nu})}$ is the anti-canonical divisor of $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$.

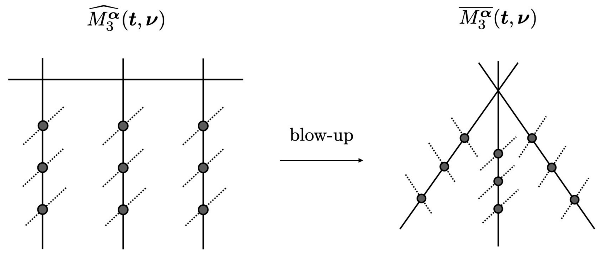
Remark 4.1.2. Theorem 4.1.1 implies a description for all $\boldsymbol{\nu}$. Take $\nu_1, \nu_2, \nu_3 \in \mathbb{C}$ satisfying $\nu_1 + \nu_2 + \nu_3 = 2$. Put $L := \mathcal{O}_{\mathbb{P}^1}$ and

$$\nabla_L := d + \frac{1}{3}\left(\frac{\nu_1}{z - t_1} + \frac{\nu_2}{z - t_2} + \frac{\nu_3 - 2}{z - t_3}\right) dz.$$

Then the morphism defined by

$$\overline{M_3^{\boldsymbol{\alpha}}}(0,0,2) \longrightarrow \overline{M_3^{\boldsymbol{\alpha}}}(\nu_1, \nu_2, \nu_3),\ (E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)}) \longmapsto (E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)}) \otimes (L, \nabla_L)$$

is an isomorphism. When $\deg E_1 = \deg E_2 \neq -2$, elementary transformations give isomorphisms of moduli spaces (see section 2.4).



4.2 Types of underlying vector bundles

In this section, we investigate types of underlying vector bundles. Take $\boldsymbol{t} = (t_i)_{1\le i\le 3} \in T_3, \boldsymbol{\nu} \in \mathcal{N}$ and put $D(\boldsymbol{t}) = t_1 + t_2 + t_3$. Let $(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)})$ be a $\boldsymbol{\nu}$-parabolic ϕ-connection. We assume that $0 < \alpha_{i,j} \ll 1$ for any $1 \le i, j \le 3$ and $\gamma \gg 0$. Let $(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)})$ be a $\boldsymbol{\nu}$-parabolic ϕ-connection, and $F_1 \subset E_1$ and $F_2 \subset E_2$ be subbundles such that $(F_1, F_2) \neq (0, 0)$. We put

$$\mu_{\boldsymbol{\alpha}}(F_1, F_2) := \frac{\deg F_1(-D(\boldsymbol{t})) + \deg F_2(-D(\boldsymbol{t})) - \gamma \operatorname{rank} F_2 + \sum_{i=1}^3 \sum_{j=1}^3 \alpha_{i,j}(d_{i,j}^{(1)}(F_1) + d_{i,j}^{(2)}(F_2))}{\operatorname{rank} F_1 + \operatorname{rank} F_2},$$

where $d_{i,j}^{(k)}(F) = \dim(F|_{t_i} \cap l_{i,j-1}^{(k)})/(F|_{t_i} \cap l_{i,j}^{(k)})$.

Lemma 4.2.1. Let $(F_1, F_2) \subset (E_1, E_2)$ be a pair of subbundles with non-negative degree. If (F_1, F_2) satisfies $\phi(F_1) \subset F_2$, $\nabla(F_1) \subset F_2 \otimes \Omega_{\mathbb{P}^1}^1(D(\boldsymbol{t}))$ and $\operatorname{rank} F_1 > \operatorname{rank} F_2$, then (F_1, F_2) is an $\boldsymbol{\alpha}$-destabilizing pair of $(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)})$.

Proof. We have

$$\begin{aligned}\mu_{\boldsymbol{\alpha}}(F_1, F_2) - \mu_{\boldsymbol{\alpha}}(E_1, E_2) &= \frac{\operatorname{rank} F_1 - \operatorname{rank} F_2}{2(\operatorname{rank} E_1 + \operatorname{rank} E_2)}\gamma + \frac{\deg F_1 + \deg F_2}{\operatorname{rank} F_1 + \operatorname{rank} F_2} - \frac{\deg E_1}{\operatorname{rank} E_1} \\ &\quad + \frac{\sum_{k=1}^2 \sum_{i=1}^3 \sum_{j=1}^3 \alpha_{i,j} d_{i,j}^{(k)}(F_k)}{\operatorname{rank} F_1 + \operatorname{rank} F_2} - \frac{\sum_{k=1}^2 \sum_{i=1}^3 \sum_{j=1}^3 \alpha_{i,j}}{\operatorname{rank} E_1 + \operatorname{rank} E_2}.\end{aligned}$$

Now $\gamma \gg 0$, so under the assumption, we obtain $\mu_{\boldsymbol{\alpha}}(F_1, F_2) - \mu_{\boldsymbol{\alpha}}(E_1, E_2) > 0$. □

Lemma 4.2.2. Let $(F_1, F_2) \subset (E_1, E_2)$ be a pair of non-zero subbundles of rank $r' < r$. If (F_1, F_2) satisfy $\phi(F_1) \subset F_2$, $\nabla(F_1) \subset F_2 \otimes \Omega_{\mathbb{P}^1}^1(D(\boldsymbol{t}))$ and $\mu(F_1) + \mu(F_2) \ge -1$, then (F_1, F_2) is an $\boldsymbol{\alpha}$-destabilizing pair of $(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)})$. Here for nonzero vector bundle F, $\mu(F) = \deg F/\operatorname{rank} F$.

Proof. We have

$$\mu_{\boldsymbol{\alpha}}(F_1, F_2) - \mu_{\boldsymbol{\alpha}}(E_1, E_2) = \frac{1}{2}\left\{\mu(F_1) + \mu(F_2) + \frac{4}{3} + \frac{\sum_{k=1}^2 \sum_{i=1}^3 \sum_{j=1}^3 \alpha_{i,j}(3d_{i,j}^{(k)}(F_k) - r')}{3r'}\right\}.$$

If $\mu(F_1) + \mu(F_2) \ge -1$, we obtain $\mu_{\boldsymbol{\alpha}}(F_1, F_2) - \mu_{\boldsymbol{\alpha}}(E_1, E_2) > 0$. □

Proposition 4.2.3. For any $\boldsymbol{\alpha}$-stable $\boldsymbol{\nu}$-parabolic ϕ-connection $(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)})$ of rank 3 and degree -2, we have

$$E_1 \cong E_2 \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Proof. Take decompositions

$$\begin{aligned} E_1 &= \mathcal{O}_{\mathbb{P}^1}(l_1) \oplus \mathcal{O}_{\mathbb{P}^1}(l_2) \oplus \mathcal{O}_{\mathbb{P}^1}(l_3) \qquad (l_1 + l_2 + l_3 = -2,\ l_1 \ge l_2 \ge l_3) \\ E_2 &= \mathcal{O}_{\mathbb{P}^1}(m_1) \oplus \mathcal{O}_{\mathbb{P}^1}(m_2) \oplus \mathcal{O}_{\mathbb{P}^1}(m_3) \qquad (m_1 + m_2 + m_3 = -2,\ m_1 \ge m_2 \ge m_3). \end{aligned}$$

If a triple of integers (n_1, n_2, n_3) satisfies $n_1 + n_2 + n_3 = -2$ and $n_1 \ge n_2 \ge n_3$, then (n_1, n_2, n_3) satisfies one of the following conditions:

(i) $n_1 \ge n_2 \ge 0 > n_3$,

(ii) $n_1 \ge 1,\ 0 > n_2 \ge n_3$,

(iii) $n_1=0,\ n_2=n_3=-1$.

If (l_1,l_2,l_3) and (m_1,m_2,m_3) satisfy the condition (i), then we have $\phi(\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus\mathcal{O}_{\mathbb{P}^1}(l_2))\subset\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2)$. The composite

$$\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus\mathcal{O}_{\mathbb{P}^1}(l_2)\to E_1\xrightarrow{\nabla}E_2\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))\to\mathcal{O}_{\mathbb{P}^1}(m_3)\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))\cong\mathcal{O}_{\mathbb{P}^1}(m_3+1)$$

becomes a homomorphism and must be zero since $m_3+1=-1-m_1-m_2\le -1$. So we have $\nabla(\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus\mathcal{O}_{\mathbb{P}^1}(l_2))\subset(\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$. Since $\mu(\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus\mathcal{O}_{\mathbb{P}^1}(l_2))+\mu(\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))\ge 0$, the pair $(\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus\mathcal{O}_{\mathbb{P}^1}(l_2),\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))$ breaks the stability of $(E_1,E_2,\phi,\nabla,l_*^{(1)},l_*^{(2)})$.

Suppose that (l_1,l_2,l_3) satisfies (i) and (m_1,m_2,m_3) satisfies (ii). Since $m_3\le -2$, we have $\phi(\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus\mathcal{O}_{\mathbb{P}^1}(l_2))\subset\mathcal{O}_{\mathbb{P}^1}(m_1)$ and $\nabla(\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus\mathcal{O}_{\mathbb{P}^1}(l_2))\subset(\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))\otimes\Omega^1_C(D(\boldsymbol{t}))$. Since $m_1+m_2=-2-m_3\ge 0$, the pair $(\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus\mathcal{O}_{\mathbb{P}^1}(l_2),\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))$ breaks the stability of $(E_1,E_2,\phi,\nabla,l_*^{(1)},l_*^{(2)})$.

Suppose that (l_1,l_2,l_3) satisfies (i) and (m_1,m_2,m_3) satisfies (iii). Then we have $\phi(\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus\mathcal{O}_{\mathbb{P}^1}(l_2))\subset\mathcal{O}_{\mathbb{P}^1}(m_1)$. If $l_1\ge 1$, then $\nabla(\mathcal{O}_{\mathbb{P}^1}(l_1))\subset\mathcal{O}_{\mathbb{P}^1}(m_1)\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$. The pair $(\mathcal{O}_{\mathbb{P}^1}(l_1),\mathcal{O}_{\mathbb{P}^1}(m_1))$ breaks the stability. If $l_1=0$, then we have $l_2=0$. Put $F_1=\operatorname{Ker}\phi|_{\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus\mathcal{O}_{\mathbb{P}^1}(l_2)}$. Then the composite

$$f\colon F_1\longrightarrow E_1\xrightarrow{\nabla}E_2\otimes\Omega^1_C(D(\boldsymbol{t}))$$

becomes a homomorphism. Put $F_2=(\operatorname{Im}f)\otimes\Omega^1_C(D(\boldsymbol{t}))^\vee$. The pair (F_1,F_2) breaks the stability.

Suppose that (l_1,l_2,l_3) satisfies (ii) and (m_1,m_2,m_3) satisfies (i). If $l_1>m_1$, then the composite $\mathcal{O}_{\mathbb{P}^1}(l_1)\to E_1\xrightarrow{\nabla}E_2\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$ becomes a homomorphism. Put $F_2=(\operatorname{Im}\nabla|_{\mathcal{O}_{\mathbb{P}^1}(l_1)})\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))^\vee$, then $(\mathcal{O}_{\mathbb{P}^1}(l_1),F_2)$ breaks the stability of $(E_1,E_2,\phi,\nabla,l_*^{(1)},l_*^{(2)})$. If $l_1\le m_1$, then we have $l_2-2\ge l_2+l_3=m_1-l_1+m_2+m_3\ge m_3$ since $l_3\le -2$. So we have $\phi(\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus\mathcal{O}_{\mathbb{P}^1}(l_2))\subset\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2)$ and $\nabla(\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus\mathcal{O}_{\mathbb{P}^1}(l_2))\subset(\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$. The pair $(\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus\mathcal{O}_{\mathbb{P}^1}(l_2),\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))$ breaks the stability of $(E_1,E_2,\phi,\nabla,l_*^{(1)},l_*^{(2)})$ because

$$\mu(\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus\mathcal{O}_{\mathbb{P}^1}(l_2))+\mu(\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))=\frac{l_1+l_2-2-m_3}{2}\ge\frac{1}{2}.$$

If (l_1,l_2,l_3) satisfies (ii) and (m_1,m_2,m_3) satisfies (ii) or (iii), then $\phi(\mathcal{O}_{\mathbb{P}^1}(l_1))\subset\mathcal{O}_{\mathbb{P}^1}(m_1)$ and $\nabla(\mathcal{O}_{\mathbb{P}^1}(l_1))\subset\mathcal{O}_{\mathbb{P}^1}(m_1)\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$. $(\mathcal{O}_{\mathbb{P}^1}(l_1),\mathcal{O}_{\mathbb{P}^1}(m_1))$ breaks the stability of $(E_1,E_2,\phi,\nabla,l_*^{(1)},l_*^{(2)})$.

Suppose that (l_1,l_2,l_3) satisfies (iii) and (m_1,m_2,m_3) satisfies (i), then $m_3=-2-m_1-m_2\le -2$. If $m_3<-2$, then $\phi(E_1)\subset\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2)$ and $\nabla(E_1)\subset(\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$. The pair $(E_1,\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))$ breaks the stability of $(E_1,E_2,\phi,\nabla,l_*^{(1)},l_*^{(2)})$. If $m_3=-2$, then $m_1=m_2=0$ and $\phi(\mathcal{O}_{\mathbb{P}^1}(l_2)\oplus\mathcal{O}_{\mathbb{P}^1}(l_3))\subset\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2)$. Moreover the composite

$$f:\mathcal{O}_{\mathbb{P}^1}(l_2)\oplus\mathcal{O}_{\mathbb{P}^1}(l_3)\to E_1\xrightarrow{\nabla}E_2\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))\to\mathcal{O}_{\mathbb{P}^1}(m_3)\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$$

becomes a homomorphism. Let $F_1=\operatorname{Ker}f$. If $F_1=\mathcal{O}_{\mathbb{P}^1}(l_2)\oplus\mathcal{O}_{\mathbb{P}^1}(l_3)$, then $\phi(E_1)\subset\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2)$ and $\nabla(E_1)\subset(\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$. The pair $(E_1,\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))$ breaks the stability of $(E_1,E_2,\phi,\nabla,l_*^{(1)},l_*^{(2)})$. If $F_1\ne\mathcal{O}_{\mathbb{P}^1}(l_2)\oplus\mathcal{O}_{\mathbb{P}^1}(l_3)$, then we have $F_1\cong\mathcal{O}_{\mathbb{P}^1}(-1)$ since $\mathcal{O}_{\mathbb{P}^1}(l_2)\cong\mathcal{O}_{\mathbb{P}^1}(l_3)\cong\mathcal{O}_{\mathbb{P}^1}(m_3)\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))\cong\mathcal{O}_{\mathbb{P}^1}(-1)$. So we obtain $\phi(\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus F_1)\subset\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2)$ and $\nabla(\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus F_1)\subset(\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$. The pair $(\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus F_1,\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))$ breaks the stability of $(E_1,E_2,\phi,\nabla,l_*^{(1)},l_*^{(2)})$.

Suppose that (l_1,l_2,l_3) satisfies (iii) and (m_1,m_2,m_3) satisfies (ii). If $m_2<-1$, then $\phi(\mathcal{O}_{\mathbb{P}^1}(l_1))\subset\mathcal{O}_{\mathbb{P}^1}(m_1)$ and $\nabla(\mathcal{O}_{\mathbb{P}^1}(l_1))\subset\mathcal{O}_{\mathbb{P}^1}(m_1)\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$. The pair $(\mathcal{O}_{\mathbb{P}^1}(l_1),\mathcal{O}_{\mathbb{P}^1}(m_1))$ breaks the stability of $(E_1,E_2,\phi,\nabla,l_*^{(1)},l_*^{(2)})$. If $m_2=-1$ and $m_3<-2$, then $\phi(E_1)\subset\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2)$ and $\nabla(E_1)\subset(\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$. The pair $(E_1,\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))$ breaks the stability of $(E_1,E_2,\phi,\nabla,l_*^{(1)},l_*^{(2)})$. If $m_2=-1$ and $m_3=-2$, then we have $\phi(\mathcal{O}_{\mathbb{P}^1}(l_2)\oplus\mathcal{O}_{\mathbb{P}^1}(l_3))\subset\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2)$ and so the composite

$$f:\mathcal{O}_{\mathbb{P}^1}(l_2)\oplus\mathcal{O}_{\mathbb{P}^1}(l_3)\to E_1\xrightarrow{\nabla}E_2\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))\to\mathcal{O}_{\mathbb{P}^1}(m_3)\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$$

becomes a homomorphism. Let $F_1=\operatorname{Ker}f$. If $F_1=\mathcal{O}_{\mathbb{P}^1}(l_2)\oplus\mathcal{O}_{\mathbb{P}^1}(l_3)$, then $\phi(E_1)\subset\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2)$ and $\nabla(E_1)\subset(\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$. The pair $(E_1,\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))$ breaks the stability of $(E_1,E_2,\phi,\nabla,l_*^{(1)},l_*^{(2)})$. If $F_1\ne\mathcal{O}_{\mathbb{P}^1}(l_2)\oplus\mathcal{O}_{\mathbb{P}^1}(l_3)$, then we have $F_1\cong\mathcal{O}_{\mathbb{P}^1}(-1)$ since $\mathcal{O}_{\mathbb{P}^1}(l_2)\cong\mathcal{O}_{\mathbb{P}^1}(l_3)\cong\mathcal{O}_{\mathbb{P}^1}(m_3)\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))\cong\mathcal{O}_{\mathbb{P}^1}(-1)$. So we obtain $\phi(\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus F_1)\subset\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2)$ and

$\nabla(\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus F_1) \subset (\mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))\otimes\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$. The pair $(\mathcal{O}_{\mathbb{P}^1}(l_1)\oplus F_1, \mathcal{O}_{\mathbb{P}^1}(m_1)\oplus\mathcal{O}_{\mathbb{P}^1}(m_2))$ breaks the stability of $(E_1, E_2, \phi, \nabla, l^{(1)}_*, l^{(2)}_*)$.

Hence we have $E_1 \cong E_2 \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. □

Lemma 4.2.4. Let F be a subbundle of E_1 which is isomorphic to the trivial bundle. If $\phi|_F = 0$, then $(E_1, E_2, \phi, \nabla, l^{(1)}_*, l^{(2)}_*)$ is $\boldsymbol{\alpha}$-unstable. In particular, if $\phi = 0$, then $(E_1, E_2, \phi, \nabla, l^{(1)}_*, l^{(2)}_*)$ is $\boldsymbol{\alpha}$-unstable.

Proof. If $\phi|_F = 0$, then the composite

$$f\colon F \longrightarrow E_1 \xrightarrow{\nabla} E_2 \otimes \Omega^1_C(D(\boldsymbol{t}))$$

becomes a homomorphism. If $f = 0$, then $(F, 0)$ breaks the stability. If $f \neq 0$, then $(F, (\operatorname{Im} f) \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))^\vee)$ breaks the stability. □

4.3 The apparent map

Proposition 4.3.1. Take $(E_1, E_2, \phi, \nabla, l^{(1)}_*, l^{(2)}_*) \in \overline{M^{\boldsymbol{\alpha}}_3}(\boldsymbol{t}, \boldsymbol{\nu})$. Then there exists a filtration $E_k = F^{(k)}_0 \supsetneq F^{(k)}_1 \supsetneq F^{(k)}_2 \supsetneq F^{(k)}_3 = 0$ by subbundles for $k = 1, 2$ such that

$$F^{(1)}_1 \cong F^{(2)}_1 \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1),\ F^{(1)}_2 \cong F^{(2)}_2 \cong \mathcal{O}_{\mathbb{P}^1}, \tag{4.1}$$

and

$$\phi(F^{(1)}_i) \subset F^{(2)}_i,\ \nabla(F^{(1)}_{i+1}) \subset F^{(2)}_i \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \tag{4.2}$$

for any $0 \leq i \leq 2$. Subbundles $F^{(1)}_2, F^{(2)}_1, F^{(2)}_2$ satisfying the above conditions are uniquely determined. If $\operatorname{rank}\phi = 2$ and 3, then $F^{(1)}_1$ is also unique. If $\operatorname{rank}\phi = 1$, then there is a one-to-one correspondence between the set of all such $F^{(1)}_1$ and $\mathbb{P}^1$.

Proof. By Proposition 4.2.3, E_1 and E_2 have a unique line subbundle which is isomorphic to the trivial line bundle. Let $F^{(k)}_2$ be the such line subbundle of E_k for $k = 1, 2$. Then we have $\phi(F^{(1)}_2) \subset F^{(2)}_2$ by Proposition 4.2.3, and so the composite

$$f_2\colon \mathcal{O}_{\mathbb{P}^1} \cong F^{(1)}_2 \hookrightarrow E_1 \xrightarrow{\nabla} E_2 \otimes \Omega^1_{\mathbb{P}^1}(D) \to E_2/F^{(2)}_2 \otimes \Omega^1_{\mathbb{P}^1}(D) \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$$

becomes a homomorphism. If $f_2 = 0$, then $(F^{(1)}_2, F^{(2)}_2)$ breaks the stability of $(E_1, E_2, \phi, \nabla, l^{(1)}_*, l^{(2)}_*)$. So f_2 is not zero. Let

$$F^{(2)}_1 = \ker(E_2 \otimes \Omega^1_{\mathbb{P}^1}(D) \to (E_2/F^{(2)}_2 \otimes \Omega^1_{\mathbb{P}^1}(D))/\operatorname{Im} f_2) \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))^\vee.$$

Then we have $F^{(2)}_1 \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and $\nabla(F^{(1)}_2) \subset F^{(2)}_1 \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$. Let $K := \ker(\phi\colon E_1 \to E_2/F^{(2)}_1)$. If $\operatorname{rank}\phi = 2, 3$, then we have $K \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Put $F^{(1)}_1 = K$. We then have desire filtrations. The uniqueness of a filtration satisfying the above condition is clear. If $\operatorname{rank}\phi = 1$, then $K = E_1$ by Lemma 4.2.4. Take a subbundle $F^{(1)}_1 \subset E_1$ which is isomorphic to $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Then we have $\phi(F^{(1)}_1) \subset F^{(2)}_1$. We can see that there is a one-to-one correspondence between the set of such subbundles $F^{(1)}_1$ and $\mathbb{P}\operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}(-1), E_1/F^{(1)}_2) \cong \mathbb{P}^1$. □

Let $E_k = F^{(k)}_0 \supsetneq F^{(k)}_1 \supsetneq F^{(k)}_2 \supsetneq F^{(k)}_3 = 0$ be a filtration in Proposition 4.3.1. We define f_1 by

$$f_1\colon F^{(1)}_1 \hookrightarrow E_1 \xrightarrow{\nabla} E_2 \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \to E_2/F^{(2)}_1 \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})).$$

Then f_1 becomes a homomorphism. If $f_1 = 0$, then $(F^{(1)}_1, F^{(2)}_1)$ breaks the stability. So f_1 is not zero, and it implies that the induced homomorphism

$$u\colon \mathcal{O}_{\mathbb{P}^1}(-1) \cong F^{(1)}_1/F^{(1)}_2 \to E_1 \xrightarrow{\nabla} E_2 \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \to E_2/F^{(2)}_1 \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \cong \mathcal{O}_{\mathbb{P}^1}$$

is also not zero because $\nabla(F^{(1)}_2) \subset F^{(2)}_1 \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$. Since $u \in \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, there exists a unique point $q \in \mathbb{P}^1$ such that $u_q = 0$.

Definition 4.3.2. We call the zero q of u the apparent singularity of $(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)}, F_1^{(1)})$, and let q denote $\mathrm{App}(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)}, F_1^{(1)})$.

Let $\widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$ be the moduli space of pairs of a parabolic ϕ-connections and a subbundle $F_1^{(1)}$, i.e.

$$\widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu}) := \{(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)}, F_1^{(1)})\}/\sim.$$

We can construct $\widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$ as follows. Let $(\tilde{E}_1, \tilde{E}_2, \tilde{\phi}, \tilde{\nabla}, \tilde{l}_*^{(1)}, \tilde{l}_*^{(2)})$ be a universal family over $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu}) \times \mathbb{P}^1$ and $\tilde{F}_2^{(k)} \subset \tilde{E}_k$ be a unique subbundle such that $(\tilde{F}_2^{(k)})_x \cong \mathcal{O}_{\mathbb{P}^1}$ for each $x \in \overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$. Put

$$\tilde{f}_2 \colon \tilde{F}_2^{(1)} \hookrightarrow \tilde{E}_1 \xrightarrow{\tilde{\nabla}} \tilde{E}_2 \otimes \Omega^1_{\mathbb{P}^1}(D) \to \tilde{E}_2/\tilde{F}_2^{(2)} \otimes \Omega^1_{\mathbb{P}^1}(D)$$

and

$$\tilde{F}_1^{(2)} = \ker(\tilde{E}_2 \otimes \Omega^1_{\mathbb{P}^1}(D) \to (\tilde{E}_2/\tilde{F}_2^{(2)} \otimes \Omega^1_{\mathbb{P}^1}(D))/\mathrm{Im}\, \tilde{f}_2) \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))^{\vee}.$$

Let $p_1 \colon \overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu}) \times \mathbb{P}^1 \to \overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$ and $p_2 \colon \overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu}) \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection and put $\mathcal{G} := (p_1)_* \mathcal{H}om(p_2^* \mathcal{O}_{\mathbb{P}^1}(-1), \tilde{E}_1/\tilde{F}_2^{(1)})$. Then we have the natural isomorphism

$$\mathcal{G}|_x \cong \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-1), (\tilde{E}_1/\tilde{F}_2^{(1)})_x) \cong \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}).$$

Let $\varpi \colon \mathbb{P}(\mathcal{G}) = \mathrm{Proj}\,\mathrm{Sym}\,(\mathcal{G}^{\vee}) \to \overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$ be the projection and $[\sigma]$ be the homothety class of a nonzero element $\sigma \in \mathcal{G}|_x$. Put

$$\widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu}) := \left\{ [\sigma] \in \mathbb{P}(\mathcal{G}) \;\middle|\; \begin{array}{l} \text{the composite } \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\sigma} (\tilde{E}_1/\tilde{F}_2^{(1)})_x \xrightarrow{\phi} (\tilde{E}_2/\tilde{F}_1^{(2)})_x \\ \text{is zero, where } x = \varpi([\sigma]) \end{array} \right\}.$$

Then $\widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$ is a closed subscheme of $\mathbb{P}(\mathcal{G})$ and desired one.

4.4 Construction of the morphism $\varphi \colon \widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu}) \to \mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1})$

Take $(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)}, F_1^{(1)}) \in \widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$ and put $q := \mathrm{App}(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)}, F_1^{(1)})$. Let $p_2 \colon E_2 \to E_2/F_1^{(2)}$ be the quotient and let us fix an isomorphism $E_2/F_1^{(2)} \cong \mathcal{O}_{\mathbb{P}^1}(-t_3)$. We define a homomorphism $B \colon E_1 \to E_2/F_1^{(2)} \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$ by $B(a) := (p_2 \otimes \mathrm{id})\nabla(a) - d(p_2\phi(a))$ for $a \in E_1$, where d is the canonical connection on $\mathcal{O}_{\mathbb{P}^1}(-t_3)$. Since $\nabla(F_2^{(1)}) \subset F_1^{(2)} \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$ and $u_q = 0$, B_q induces a homomorphism $h_1 \colon (E_1/F_1^{(1)})|_q \to (E_2/F_1^{(2)} \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})))|_q$ which makes the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1^{(1)}|_q & \longrightarrow & E_1|_q & \longrightarrow & (E_1/F_1^{(1)})|_q & \longrightarrow & 0 \\ & & & \searrow^{0} & \downarrow{\scriptstyle B_q} & \swarrow_{h_1} & & & \\ & & & & (E_2/F_1^{(2)} \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})))|_q & & & & \end{array} \tag{4.3}$$

commute. Let $h_2 \colon (E_1/F_1^{(1)})|_q \to (E_2/F_1^{(2)})|_q$ be the homomorphism induced by ϕ. Then h_1, h_2 determine a homomorphism

$$\iota \colon (E_1/F_1^{(1)})|_q \longrightarrow ((E_2/F_1^{(2)} \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))) \oplus E_2/F_1^{(2)})|_q, \quad a \mapsto (h_1(a), h_2(a)).$$

Lemma 4.4.1. ι is injective.

Proof. If $\mathrm{rank}\,\phi = 3$, then h_2 is not zero. In fact, if $h_2 = 0$, then $\phi(E_1) \subset F_1^{(2)}$ since $\phi \colon \mathcal{O}_{\mathbb{P}^1}(-1) \cong E_1/F_1^{(1)} \to E_2/F_1^{(2)} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ is zero. It is a contradiction. So ι is injective.

Consider the case $\mathrm{rank}\,\phi = 2$. Assume that $h_2 = 0$. We take a local basis $e_0^{(1)}, e_1^{(1)}, e_2^{(1)}$ (resp. $e_0^{(2)}, e_1^{(2)}, e_2^{(2)}$) of E_1 (resp. E_2) such that $e_2^{(1)}$ generates $F_2^{(1)}$ and $e_1^{(1)}, e_2^{(1)}$ generate $F_1^{(1)}$ (resp. $e_2^{(2)}$ generates $F_2^{(2)}$ and $e_1^{(2)}, e_2^{(2)}$ generate $F_1^{(2)}$). By taking bases well, ϕ and ∇ are represented by matrices

$$\phi(e_2^{(1)}, e_1^{(1)}, e_0^{(1)}) = (e_2^{(2)}, e_1^{(2)}, e_0^{(2)}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \phi_{22} & \phi_{23} \\ 0 & 0 & 0 \end{pmatrix},$$

$$\nabla(e_2^{(1)}, e_1^{(1)}, e_0^{(1)}) = (e_2^{(2)}, e_1^{(2)}, e_0^{(2)}) \begin{pmatrix} 0 & a_{12}(z) & a_{13}(z) \\ 1 & a_{22}(z) & a_{23}(z) \\ 0 & a_{32}(z) & a_{33}(z) \end{pmatrix} \frac{dz}{h(z)},$$

where z is an inhomogeneous coordinate on $\mathbb{P}^1 = \operatorname{Spec} \mathbb{C}[z] \cup \{\infty\}$ and $h(z) = (z-t_1)(z-t_2)(z-t_3)$ and $\phi_{22}, \phi_{23} \in \mathbb{C}$. Suppose that $\phi_{22} = 0$. Then we may assume that $\phi_{23} = 1$. For each $i = 1, 2, 3$, $a_{32}(t_i)$ must be zero because the polynomial

$$|\operatorname{res}_{t_i} \nabla - \lambda \phi| = \frac{1}{h'(t_i)} \begin{vmatrix} -h'(t_i)\lambda & a_{12}(t_i) & a_{13}(t_i) \\ 1 & a_{22}(t_i) & a_{23}(t_i) - h'(t_i)\lambda \\ 0 & a_{32}(t_i) & a_{33}(t_i) \end{vmatrix}$$

in λ is identically zero by Lemma 4.4.2 and $h'(t_i)a_{32}(t_i)$ is the second order coefficient of $|\operatorname{res}_{t_i} \nabla - \lambda\phi|$. Here $' = d/dz$. Since $a_{32}(z) \in H^0(\mathcal{O}_{\mathbb{P}^1}(1))$, we obtain $a_{32}(z) = 0$. Then $(F_1^{(1)}, F_1^{(2)})$ breaks the stability of $(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)})$. Suppose that $\phi_{22} \neq 0$. Then we may assume that $\phi_{23} = 0$. In the same way as the above, we can see that $a_{33}(z) = 0$. So $(F_2^{(1)} \oplus E_1/F_1^{(1)}, F_1^{(2)})$ breaks the stability of $(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)})$. Hence $h_2 \neq 0$ and so ι is injective.

Finally, we consider the case $\operatorname{rank} \phi = 1$. Let $f \colon E_1/F_2^{(1)} \to E_2/F_1^{(2)} \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$ be the homomorphism induced by ∇. Since $\phi(E_1) \subset F_2^{(2)} \subset F_1^{(2)}$, the map f becomes a homomorphism. If $h_1 = 0$, then we have $f|_q = 0$ by the diagram (4.3). If $f = 0$, then $(E_1, F_1^{(2)})$ breaks the stability, so $f \neq 0$. Since $E_1/F_2^{(1)} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, $E_2/F_1^{(2)} \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \cong \mathcal{O}_{\mathbb{P}^1}$ and $f|_q = 0$, we have $\ker f \cong \mathcal{O}_{\mathbb{P}^1}(-1)$. Put $G := \ker(E_1 \to (E_1/F_2^{(1)})/\ker f)$. Then $G \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and so $(G, F_1^{(2)})$ breaks the stability. Hence $h_1 \neq 0$ and so ι is injective. □

Lemma 4.4.2. For each i, the polynomial $|\operatorname{res}_{t_i} \nabla - \lambda \phi_{t_i}|$ in λ has the form

$$(\wedge^3 \phi_{t_i})(\nu_{i,0} - \lambda)(\nu_{i,1} - \lambda)(\nu_{i,2} - \lambda).$$

Proof. We take a basis $v_0^{(1)}, v_1^{(1)}, v_2^{(1)}$ (resp. $v_0^{(2)}, v_1^{(2)}, v_2^{(2)}$) of $E_1|_{t_i}$ (resp. $E_2|_{t_i}$) such that $v_2^{(1)}$ generates $l_2^{(1)}$ and $v_1^{(1)}, v_2^{(1)}$ generate $l_1^{(1)}$ (resp. $v_2^{(2)}$ generates $l_2^{(2)}$ and $v_1^{(2)}, v_2^{(2)}$ generate $l_1^{(2)}$). Then ϕ_{t_i} and $\operatorname{res}_{t_i}\nabla$ are represented by matrices

$$\phi_{t_i}(v_2^{(1)}, v_1^{(1)}, v_0^{(1)}) = (v_2^{(2)}, v_1^{(2)}, v_0^{(2)}) \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ 0 & \phi_{22} & \phi_{23} \\ 0 & 0 & \phi_{33} \end{pmatrix},$$

$$\operatorname{res}_{t_i}\nabla(v_2^{(1)}, v_1^{(1)}, v_0^{(1)}) = (v_2^{(2)}, v_1^{(2)}, v_0^{(2)}) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

because ϕ_{t_i} and $\operatorname{res}_{t_i}\nabla$ are parabolic. Since $(\operatorname{res}_{t_i}\nabla - \nu_{i,j}\phi_{t_i})(l_{i,j}^{(1)}) \subset l_{i,j+1}^{(2)}$ for $j = 0, 1, 2$, we have $a_{11} = \nu_{i,0}\phi_{11}$, $a_{22} = \nu_{i,1}\phi_{22}$ and $a_{33} = \nu_{i,2}\phi_{33}$. So we have

$$|\operatorname{res}_{t_i}\nabla - \lambda\phi_{t_i}| = \phi_{11}\phi_{22}\phi_{33}(\nu_{i,0} - \lambda)(\nu_{i,1} - \lambda)(\nu_{i,2} - \lambda).$$

□

By Lemma 4.4.1, the map ι determines a point $\varphi(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)}, F_1^{(1)})$ of $\mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1})$. We can see that the map

$$\varphi \colon \widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu}) \longrightarrow \mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1}) \tag{4.4}$$

is a morphism.

4.5 Normal forms of $\boldsymbol{\alpha}$-stable parabolic ϕ-connections

Take $(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)}, F_1^{(1)}) \in \widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$. For $k = 1, 2$, let $E_k \supsetneq F_1^{(k)} \supsetneq F_2^{(k)} \supsetneq 0$ be a filtration in Proposition 4.3.1. We take a local basis $e_0^{(1)}, e_1^{(1)}, e_2^{(1)}$ (resp. $e_0^{(2)}, e_1^{(2)}, e_2^{(2)}$) of E_1 (resp. E_2) such that $e_2^{(1)}$

generates $F_2^{(1)}$ and $e_1^{(1)}, e_2^{(1)}$ generate $F_1^{(1)}$ (resp. $e_2^{(2)}$ generates $F_2^{(2)}$ and $e_1^{(2)}, e_2^{(2)}$ generate $F_1^{(2)}$). Let z be a fixed inhomogeneous coordinate on $\mathbb{P}^1 = \operatorname{Spec}\mathbb{C}[z] \cup \{\infty\}$. Then ϕ and ∇ are represented by matrices

$$\phi(e_2^{(1)}, e_1^{(1)}, e_0^{(1)}) = (e_2^{(2)}, e_1^{(2)}, e_0^{(2)}) \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ 0 & \phi_{22} & \phi_{23} \\ 0 & 0 & \phi_{33} \end{pmatrix},$$

$$\nabla(e_2^{(1)}, e_1^{(1)}, e_0^{(1)}) = (e_2^{(2)}, e_1^{(2)}, e_0^{(2)}) \begin{pmatrix} a_{11}(z) & a_{12}(z) & a_{13}(z) \\ a_{21} & \phi_{22}(z-t_1)(z-t_2) + a_{22}(z) & \phi_{23}(z-t_1)(z-t_2) + a_{23}(z) \\ 0 & a_{32}(z) & \phi_{33}(z-t_1)(z-t_2) + a_{33}(z) \end{pmatrix} \frac{dz}{h(z)},$$

where $\phi_{11}, \phi_{22}, \phi_{23}, \phi_{33} \in H^0(\mathcal{O}_{\mathbb{P}^1}), \phi_{12}, \phi_{13} \in H^0(\mathcal{O}_{\mathbb{P}^1}(1))$, $a_{11}, a_{22}, a_{23}, a_{32}, a_{33} \in H^0(\mathcal{O}_{\mathbb{P}^1}(1))$,$a_{21} \in H^0(\mathcal{O}_{\mathbb{P}^1})$, and $h(z) = (z-t_1)(z-t_2)(z-t_3)$. By taking $e_0^{(1)}, e_1^{(1)}, e_0^{(2)}, e_1^{(2)}$ well, we may assume that $\phi_{12} = \phi_{13} = 0, a_{11}(z) = 0$ and $a_{21} = 1$. Then we have $a_{12}, a_{13} \in H^0(\mathcal{O}_{\mathbb{P}^1}(2))$. Let q be the apparent singular point of $(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)}, F_1^{(1)})$.

Lemma 4.5.1. Assume that $\wedge^3\phi \neq 0$. Then ϕ and ∇ have the forms

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \nabla = d + \begin{pmatrix} 0 & a_{12}(z) & a_{13}(z) \\ 1 & (z-t_1)(z-t_2) - p & 0 \\ 0 & z-q & (z-t_1)(z-t_2)+p \end{pmatrix} \frac{dz}{h(z)}, \tag{4.5}$$

respectively, where $p \in \mathbb{C}$ and $a_{12}(z), a_{13}(z)$ are quadratic polynomials in z satisfying

$$a_{12}(t_i) = -h'(t_i)^2(\nu_{i,0}\nu_{i,1} + \nu_{i,1}\nu_{i,2} + \nu_{i,2}\nu_{i,0} - (\operatorname{res}_{t_i}(\tfrac{dz}{z-t_3}))^2) - p^2, \tag{4.6}$$

$$(t_i - q)a_{13}(t_i) = \prod_{j=0}^{2}(h'(t_i)(\nu_{i,j} - \operatorname{res}_{t_i}(\tfrac{dz}{z-t_3})) - p) \tag{4.7}$$

for any $i = 1, 2, 3$. Here $' = d/dz$.

Proof. Applying ϕ^{-1} to E_2, we may assume that $\phi = \mathrm{id}$. Put

$$C = \begin{pmatrix} 1 & 0 & c_{13}(z) \\ 0 & 1 & c_{23} \\ 0 & 0 & 1 \end{pmatrix},$$

where $c_{13}(z) \in H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ and $c_{23} \in H^0(\mathcal{O}_{\mathbb{P}^1})$. Then we have

$$\begin{aligned} & C \circ \nabla \circ C^{-1} \\ =& d + \begin{pmatrix} 0 & a_{12}(z) + c_{13}(z-q) & a_{13}(z) - c_{23}a_{12}(z) + c_{13}(z)a_{33}(z) - c_{13}(z)c_{23}(z-q) - h(z)c'_{13}(z) \\ 1 & a_{22}(z) + c_{23}(z-q) & a_{23}(z) - c_{23}a_{22}(z) - c_{13}(z) + c_{23}a_{33}(z) - c_{23}^2(z-q) \\ 0 & z-q & a_{33}(z) - c_{23}(z-q) \end{pmatrix} \frac{dz}{h(z)}. \end{aligned}$$

So we may assume that $a_{23}(z) = 0$ and $a_{33}(z)$ changes into the form $(z-t_1)(z-t_2)+p$. Since $\operatorname{res}_{t_i}\operatorname{tr}\nabla = 2\operatorname{res}_{t_i}(\frac{dz}{z-t_3})$, we have $a_{22}(z) = (z-t_1)(z-t_2) - p$. So we obtain the desire form

$$\nabla = d + \begin{pmatrix} 0 & a_{12}(z) & a_{13}(z) \\ 1 & (z-t_1)(z-t_2) - p & 0 \\ 0 & z-q & (z-t_1)(z-t_2)+p \end{pmatrix} \frac{dz}{h(z)}.$$

By Lemma 4.4.2, we can see that $a_{12}(z)$ and $a_{13}(z)$ satisfy the conditions (4.6) and (4.7) for each $i = 1, 2, 3$. □

Remark 4.5.2. The polynomial $a_{12}(z)$ is uniquely determined by p. When $q \neq t_1, t_2, t_3$, $a_{13}(z)$ is also uniquely determined by q and p. When $q = t_i$, p is equal to one of $h'(t_i)(\nu_{i,0} - \operatorname{res}_{t_i}(\frac{dz}{z-t_3})), h'(t_i)(\nu_{i,1} - \operatorname{res}_{t_i}(\frac{dz}{z-t_3})), h'(t_i)(\nu_{i,2} - \operatorname{res}_{t_i}(\frac{dz}{z-t_3}))$ and $a_{13}(t_i)$ takes any complex number. When $p = h'(t_i)(\nu_{i,j} - \operatorname{res}_{t_i}(\frac{dz}{z-t_3}))$, we have $(\operatorname{res}_{t_i} \oplus \mathrm{id})(\varphi(E, \nabla, l_*)) = (\nu_{i,j} - \operatorname{res}_{t_i}(\frac{dz}{z-t_3}) : 1)$, where $\operatorname{res}_{t_i} \oplus \mathrm{id} \colon \mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1})|_{t_i} \to \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}|_{t_i} \oplus \mathcal{O}_{\mathbb{P}^1}|_{t_i})$ is a natural isomorphism. The choice of $a_{13}(t_i)$ gives an exceptional curves of the first kind on the moduli space of parabolic connections (see Proposition 4.7.2, 4.7.3, and 4.7.4).

Lemma 4.5.3. Assume that $\operatorname{rank}\phi = 2$. Then ϕ and ∇ have the forms

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \nabla = \phi \otimes d + \begin{pmatrix} 0 & 0 & \prod_{j\neq i}(z-t_j) \\ 1 & 0 & 0 \\ 0 & z-t_i & (z-t_1)(z-t_2)+p \end{pmatrix} \frac{dz}{h(z)}, \tag{4.8}$$

respectively.

Proof. By the proof of Lemma 4.4.1, we have $\phi_{33} \neq 0$. So we may assume that

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Applying an automorphism of E_1, E_2 given by the form

$$\begin{pmatrix} 1 & 0 & -a_{23}(z) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

∇ changes into the form

$$\begin{pmatrix} 0 & a_{12}(z)+a_{23}(z)a_{32}(z) & a_{13}(z)+a_{23}(z)a_{33}(z)-h(z)a'_{23}(z) \\ 1 & a_{22}(z) & 0 \\ 0 & a_{32}(z) & a_{33}(z) \end{pmatrix} \frac{dz}{h(z)}.$$

So we may assume without loss of generality that $a_{23}(z) = 0$. Using an argument of the proof of Lemma 4.4.1, we obtain $a_{12}(z) = a_{22}(z) = 0$ and $a_{32}(t_i)a_{13}(t_i) = 0$ for $i = 1,2,3$. If $a_{32}(z)$ is identically zero, then $(F_1^{(1)}, F_1^{(2)})$ breaks the stability. If $a_{13}(z)$ is identically zero, then $(E_1/F_2^{(1)}, E_2/F_1^{(2)})$ breaks the stability. So there exists unique $i \in \{1,2,3\}$ such that $a_{32}(t_i) = 0$, which implies $a_{13}(t_j) = 0$ for $j \neq i$. Applying suitable automorphisms, we obtain the desire form

$$\nabla = \phi \otimes d + \begin{pmatrix} 0 & 0 & \prod_{j\neq i}(z-t_j) \\ 1 & 0 & 0 \\ 0 & z-t_i & (z-t_1)(z-t_2)+p \end{pmatrix} \frac{dz}{h(z)}.$$

□

Lemma 4.5.4. Assume that $\operatorname{rank}\phi = 1$. Then ϕ and ∇ have the forms

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \nabla = \phi \otimes d + \begin{pmatrix} 0 & \prod_{j\neq i}(z-t_j) & 0 \\ 1 & 0 & 0 \\ 0 & z-q & z-t_i \end{pmatrix} \frac{dz}{h(z)}, \tag{4.9}$$

respectively, where $t_i \neq q$.

Proof. By Lemma 4.2.4 and the assumption, ϕ and ∇ have the forms

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \nabla = \phi \otimes d + \begin{pmatrix} 0 & a_{12}(z) & a_{13}(z) \\ 1 & a_{22}(z) & a_{23}(z) \\ 0 & z-q & a_{33}(z) \end{pmatrix} \frac{dz}{h(z)},$$

where $a_{12}, a_{13} \in H^0(\mathcal{O}_{\mathbb{P}^1}(2))$ and $a_{22}, a_{23}, a_{33} \in H^0(\mathcal{O}_{\mathbb{P}^1}(1))$. If $a_{33}(q) = 0$, then we may assume that $a_{33}(z) = 0$ by applying an automorphism of E_1, which implies that $(F_2^{(1)} \oplus E_1/F_1^{(1)}, F_1^{(2)})$ breaks the stability of $(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)})$. Hence we have $a_{33}(q) \neq 0$. Let us fix $i \in \{1,2,3\}$ satisfying $t_i \neq q$. Applying an automorphism of E_1 given by the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1-a_{33}(q)^{-1}a'_{33}(q)(q-t_i) \\ 0 & 0 & a_{33}(q)^{-1}(q-t_i) \end{pmatrix},$$

the ϕ-connection ∇ changes into the form

$$\phi \otimes d + \begin{pmatrix} 0 & a_{12}(z) & a_{13}(z) \\ 1 & a_{22}(z) & a_{23}(z) \\ 0 & z-q & z-t_i \end{pmatrix}.$$

We consider the polynomial

$$\left|\mathrm{res}_{t_j}\nabla - \lambda\phi_{t_j}\right| = \frac{1}{h'(t_j)^3}\begin{vmatrix} -h'(t_j)\lambda & a_{12}(t_j) & a_{13}(t_j) \\ 1 & a_{22}(t_j) & a_{23}(t_j) \\ 0 & t_j - q & t_j - t_i \end{vmatrix} \tag{4.10}$$

in λ. By Lemma 4.4.2, the polynomial (4.10) is identically zero, that is, we have

$$(t_j - t_i)a_{22}(t_j) - (t_j - q)a_{23}(t_j) = 0, \tag{4.11}$$

$$(t_j - t_i)a_{12}(t_j) - (t_j - q)a_{13}(t_j) = 0 \tag{4.12}$$

for any j. By (4.11) and (4.12), we have $a_{13}(t_i) = a_{23}(t_i) = 0$. Applying a suitable automorphism of E_2, we may assume without loss of generality that $a_{13}(z) = a_{23}(z) = 0$. Then we have $a_{22}(t_j) = 0$ for $j \neq i$ by (4.11), and it implies that $a_{22}(z) = 0$. By (4.12), we have $a_{12}(t_i) = 0$ for $j \neq i$. If $a_{12}(z)$ is identically zero, then $(E_1/F_2^{(1)}, E_2/F_1^{(2)})$ breaks the stability. So ϕ and ∇ have the forms

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \nabla = \phi \otimes d + \begin{pmatrix} 0 & \prod_{j\neq i}(z - t_j) & 0 \\ 1 & 0 & 0 \\ 0 & z - q & z - t_i \end{pmatrix} \frac{dz}{h(z)}.$$

□

Remark 4.5.5. Let $(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)}, F_1^{(1)})$, $(E_1', E_2', \phi', \nabla', l_*'^{(1)}, l_*'^{(2)}, F_1'^{(1)})$ be $\boldsymbol{\nu}$-parabolic ϕ-connections such that $\operatorname{rank}\phi = \operatorname{rank}\phi' = 1$. Then $(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)})$ and $(E_1', E_2', \phi', \nabla', l_*'^{(1)}, l_*'^{(2)})$ are isomorphic to each other. In other words, the locus on $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$ defined by $\operatorname{rank}\phi = 1$ consists of one point. In fact, applying automorphisms of E_1, E_2, ϕ and ∇ change into the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \phi \otimes d + \begin{pmatrix} 0 & (z - t_2)(z - t_3) & 0 \\ 1 & 0 & 0 \\ 0 & z - t_2 & z - t_1 \end{pmatrix} \frac{dz}{h(z)}.$$

By the proof of Proposition 4.5.6, it follows that parabolic structures $l_{i,*}^{(1)}$ and $l_{i,*}^{(2)}$ satisfying the conditions $\phi_{t_i}(l_{i,j}^{(1)}) \subset l_{i,j}^{(2)}$ and $(\mathrm{res}_{t_i}\nabla - \nu_{i,j}\phi_{t_i})(l_{i,j}^{(1)}) \subset l_{i,j+1}^{(2)}$ are uniquely determined.

Proposition 4.5.6. Let $Y_{(\boldsymbol{t},\boldsymbol{\nu})}$ be the closed subscheme of $\widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$ defined by the condition $\wedge^3\phi = 0$. Then the restriction morphism $\varphi\colon Y_{(\boldsymbol{t},\boldsymbol{\nu})} \longrightarrow \mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1})$ is injective.

Proof. Take a point $x = (E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)}, F_1^{(1)}) \in Y_{(\boldsymbol{t},\boldsymbol{\nu})}$. Then $\operatorname{rank}\phi$ must be one or two by Lemma 4.2.4. Let D_0 be the section of $\mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1})$ over $\mathbb{P}^1$ defined by the injection $\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \hookrightarrow \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1}$, that is, D_0 is the section defined by $h_2 = 0$, where h_2 is defined in section 4.4. Let $D_i \subset \mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1})$ be the fiber over $t_i \in \mathbb{P}^1$. By the proof of Lemma 4.5.3 and Lemma 4.5.4, $\varphi(x) \in \cup_{i=1}^3 D_i \setminus D_0$ if and only if $\operatorname{rank}\phi = 2$, and $\varphi(x) \in D_0$ if and only if $\operatorname{rank}\phi = 1$.

First, we consider the case of $\operatorname{rank}\phi = 2$. By Lemma 4.5.3, a pair (ϕ, ∇) is uniquely determined up to isomorphism by $\varphi(x)$. By Proposition 4.3.1, $F_1^{(1)}$ is also uniquely determined by (E_1, E_2, ϕ, ∇). Moreover, we can check that parabolic structures $l_*^{(1)}$ and $l_*^{(2)}$ are uniquely determined by (E_1, E_2, ϕ, ∇). For example, when $\varphi(x) \in D_1$, $l_*^{(1)}$ and $l_*^{(2)}$ are given by the following;

$$l_{1,2}^{(1)} = \mathbb{C}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad l_{1,1}^{(1)} = \mathbb{C}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mathbb{C}\begin{pmatrix} h'(t_1) \\ 0 \\ p - h'(t_1)\nu_{1,0} \end{pmatrix},$$

$$l_{1,2}^{(2)} = \mathbb{C}\begin{pmatrix} h'(t_1)p - h'(t_1)^2\nu_{1,0} - h'(t_1)\nu_{1,1} \\ h'(t_1) \\ (p - h'(t_1)\nu_{1,0})(p - h'(t_1)\nu_{1,1}) \end{pmatrix}, \quad l_{1,1}^{(2)} = \mathbb{C}\begin{pmatrix} -h'(t_1)\nu_{1,0} \\ 1 \\ 0 \end{pmatrix} + \mathbb{C}\begin{pmatrix} h'(t_1) \\ 0 \\ p - h'(t_1)\nu_{1,0} \end{pmatrix},$$

$$l_{2,2}^{(1)} = \mathbb{C}\begin{pmatrix} 0 \\ p - h'(t_2)\nu_{2,2} \\ -(t_2 - t_1) \end{pmatrix}, \ l_{2,1}^{(1)} = \mathbb{C}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mathbb{C}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ l_{2,2}^{(2)} = \mathbb{C}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ l_{2,1}^{(2)} = \mathbb{C}\begin{pmatrix} -h'(t_2)\nu_{2,0} \\ 1 \\ 0 \end{pmatrix} + \mathbb{C}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$l_{3,2}^{(1)} = \mathbb{C}\begin{pmatrix} 0 \\ p + h'(t_3) - h'(t_3)\nu_{3,2} \\ -(t_3 - t_1) \end{pmatrix}, \ l_{3,1}^{(1)} = \mathbb{C}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mathbb{C}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ l_{3,2}^{(2)} = \mathbb{C}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ l_{3,1}^{(2)} = \mathbb{C}\begin{pmatrix} -h'(t_3)\nu_{3,0} \\ 1 \\ 0 \end{pmatrix} + \mathbb{C}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Next we consider the case of rank $\phi = 1$. By Proposition 4.3.1 and Lemma 4.5.4, a triple $(\phi, \nabla, F_1^{(1)})$ is uniquely determined up to isomorphism by the apparent singularity q. We can see that parabolic structures $l_*^{(1)}$ and $l_*^{(2)}$ are determined by ϕ and ∇. In fact, we have

$$l_{i,2}^{(1)} = \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix},\ l_{i,1}^{(1)} = \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix},\ l_{i,2}^{(2)} = \mathbb{C}\begin{pmatrix}h'(t_i)\\0\\t_i - q\end{pmatrix},\ l_{i,1}^{(2)} = \mathbb{C}\begin{pmatrix}-h'(t_1)\nu_{1,0}\\1\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}h'(t_i)\\0\\t_i - q\end{pmatrix},$$

and

$$l_{j,2}^{(1)} = \mathbb{C}\begin{pmatrix}0\\t_j - t_i\\-(t_j - q)\end{pmatrix},\ l_{j,1}^{(1)} = \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix},\ l_{j,2}^{(2)} = \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix},\ l_{j,1}^{(2)} = \mathbb{C}\begin{pmatrix}-h'(t_j)\nu_{j,0}\\1\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix},$$

for $j \neq i$. So $\varphi|_{Y_{(\boldsymbol{t},\boldsymbol{\nu})}}$ is injective. □

4.6 Smoothness of moduli space of parabolic ϕ-connections

Let $\tilde{t}_i \subset \mathbb{P}^1 \times T_3 \times \mathcal{N}$ be the section defined by

$$T_3 \times \mathcal{N} \hookrightarrow \mathbb{P}^1 \times T_3 \times \mathcal{N}; \quad ((t_j)_{1\le j\le 3}, (\nu_{m,n})_{0\le n\le 2}^{1\le m\le 3}) \mapsto (t_i, (t_j)_{1\le j\le 3}, (\nu_{m,n})_{0\le n\le 2}^{1\le m\le 3})$$

for $i = 1, 2, 3$ and $D(\tilde{\boldsymbol{t}}) = \tilde{t}_1 + \tilde{t}_2 + \tilde{t}_3$ be a relative effective Cartier divisor for the projection $\mathbb{P}^1 \times T_3 \times \mathcal{N} \to T_3 \times \mathcal{N}$. For each $1 \le i \le 3$ and $0 \le j \le 2$, let

$$\tilde{\nu}_{i,j} := \{(\nu_{i,j}, (t_k)_k, (\nu_{m,n})_{m,n})\} \subset \mathbb{C} \times T_3 \times \mathcal{N}.$$

Proposition 4.6.1. $\overline{M_3^{\boldsymbol{\alpha}}}(0,0,2)$ is smooth over $T_3 \times \mathcal{N}$.

Proof. Let A be an artinian local ring with the residue field $A/\mathfrak{m} = k$ and I be an ideal of A such that $\mathfrak{m}I = 0$. Let $\operatorname{Spec} A \to T_3 \times \mathcal{N}$ be a morphism and $t_i \in \mathbb{P}^1_A, \nu_{i,j} \in A$ be the elements obtained by the pull back of the sections $\tilde{t}_i, \tilde{\nu}_{i,j}$, respectively. By the definition of $\mathcal{N}$, we have

$$\nu_{i,0} + \nu_{i,1} + \nu_{i,2} = 2\operatorname{res}_{t_i}(\tfrac{dz}{z - t_3}). \tag{4.13}$$

We take an open subset $U \subset \mathbb{P}^1_A$ such that $U \cong \operatorname{Spec} A[z]$ and $t_1, t_2, t_3 \in U$. We show that

$$\overline{M_3^{\boldsymbol{\alpha}}}(0,0,2)(A) \longrightarrow \overline{M_3^{\boldsymbol{\alpha}}}(0,0,2)(A/I) \tag{4.14}$$

is surjective. Put $K := \Omega_{\mathbb{P}^1_{A/I}/(A/I)}(D(\tilde{\boldsymbol{t}})_{A/I})$ and take $(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)}) \in \overline{M_3^{\boldsymbol{\alpha}}}(0,0,2)(A/I)$. Then $E_1 \cong E_2 \cong \mathcal{O}_{\mathbb{P}^1_{A/I}} \oplus \mathcal{O}_{\mathbb{P}^1_{A/I}}(-1) \oplus \mathcal{O}_{\mathbb{P}^1_{A/I}}(-1)$. The homomorphism ϕ can be written by the form

$$\phi = \begin{pmatrix}\phi_{11} & \phi_{12} & \phi_{13}\\ 0 & \phi_{22} & \phi_{23}\\ 0 & \phi_{32} & \phi_{33}\end{pmatrix},$$

where $\phi_{11}, \phi_{22}, \phi_{23}, \phi_{32}, \phi_{33} \in H^0(\mathcal{O}_{\mathbb{P}^1_{A/I}}) \cong A/I$ and $\phi_{12}, \phi_{13} \in H^0(\mathcal{O}_{\mathbb{P}^1_{A/I}}(1))$. By Lemma 4.2.4, ϕ_{11} is a unit, so we may assume that $\phi_{12} = \phi_{13} = 0$. Then ∇ can be written by

$$\nabla = \phi \otimes d + \begin{pmatrix}0 & 0 & 0\\ 0 & \phi_{22} & \phi_{23}\\ 0 & \phi_{32} & \phi_{33}\end{pmatrix}\frac{dz}{z - t_3} + \begin{pmatrix}\omega_{11} & \omega_{12} & \omega_{13}\\ \omega_{21} & \omega_{22} & \omega_{23}\\ \omega_{31} & \omega_{32} & \omega_{33}\end{pmatrix},$$

where $\omega_{21}, \omega_{31} \in H^0(K(-1)) \cong A/I$, $\omega_{11}, \omega_{22}, \omega_{23}, \omega_{32}, \omega_{33} \in H^0(K)$, and $\omega_{12}, \omega_{13} \in H^0(K(1))$. Taking decompositions $E_1 \cong E_2 \cong \mathcal{O}_{\mathbb{P}^1_{A/I}} \oplus \mathcal{O}_{\mathbb{P}^1_{A/I}}(-1) \oplus \mathcal{O}_{\mathbb{P}^1_{A/I}}(-1)$ well, we may assume that $\omega_{11} = \omega_{31} = 0$ and $\operatorname{res}_{t_i}\omega_{21} \in (A/I)^\times$ for any $i = 1, 2, 3$. The smoothness of the map $M_3^{\boldsymbol{\alpha}}(0,0,2) \to T_3 \times \mathcal{N}$ is proved in [In], which means the map (4.14) is surjective when $\wedge^3\phi \notin \mathfrak{m}/I$. So we consider the case $\wedge^3\phi \in \mathfrak{m}/I$.

Assume that $\operatorname{rank}\phi \otimes \mathrm{id}_k = 2$. Then applying certain automorphisms of E_1 and E_2, we may assume that $\phi \otimes \mathrm{id}_k$ and $\nabla \otimes \mathrm{id}_k$ have the form (4.8). Then we may also assume that $\phi_{11} = \phi_{33} = 1$ and $\phi_{23} = \phi_{32} = 0$ and $\omega_{23} = 0$. We note that $\phi_{22} \in \mathfrak{m}/I$. In the same way of the proof Lemma 4.4.2, we obtain $|\mathrm{res}_{t_i}\nabla - \lambda\phi_{t_i}| = (\wedge^3\phi_{t_i})(\nu_{i,0}-\lambda)(\nu_{i,1}-\lambda)(\nu_{i,2}-\lambda)$. By comparing the coefficients on both sides and using (4.13), we have

$$\omega_{22}(t_i) + \phi_{22}\omega_{33}(t_i) = 0, \tag{4.15}$$

$$\omega_{22}(t_i)\omega_{33}(t_i) - \omega_{21}(t_i)\omega_{12}(t_i) = \phi_{22}(\nu_{i,0}\nu_{i,1} + \nu_{i,0}\nu_{i,2} + \nu_{i,1}\nu_{i,2} - (\mathrm{res}_{t_i}(\tfrac{dz}{z-t_3}))^2), \tag{4.16}$$

$$-\omega_{21}(t_i)(\omega_{12}(t_i)(\omega_{33}(t_i) + \mathrm{res}_{t_i}(\tfrac{dz}{z-t_3})) - \omega_{13}(t_i)\omega_{32}(t_i)) = \phi_{22}\nu_{i,0}\nu_{i,1}\nu_{i,2}, \tag{4.17}$$

for each $i = 1,2,3$, where $\omega_{ij}(t_m) := \mathrm{res}_{t_m}\omega_{ij}$. From the form (4.8), we have $\omega_{13}(t_i) \in (A/I)^\times$ and $\omega_{32}(t_j) \in (A/I)^\times$ for $j \neq i$. Put

$$v_{i,2}^{(1)} = \begin{pmatrix} \phi_{22}\omega_{13}(t_i)(\omega_{33}(t_i) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - (\nu_{i,0}+\nu_{i,1})) \\ \omega_{13}(t_i)\omega_{21}(t_i) \\ \phi_{22}(\omega_{33}(t_i) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{i,0})(\omega_{33}(t_i) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{i,1}) \end{pmatrix}, \; v_{i,1}^{(1)} = \begin{pmatrix} \omega_{13}(t_i) \\ 0 \\ \omega_{33}(t_i) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{i,0} \end{pmatrix},$$

$$v_{i,2}^{(2)} = \begin{pmatrix} \omega_{13}(t_i)(\omega_{33}(t_i) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - (\nu_{i,0}+\nu_{i,1})) \\ \omega_{13}(t_i)\omega_{21}(t_i) \\ (\omega_{33}(t_i) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{i,0})(\omega_{33}(t_i) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{i,1}) \end{pmatrix}, \; v_{i,1}^{(2)} = \begin{pmatrix} \omega_{13}(t_i) \\ 0 \\ \omega_{33}(t_i) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{i,0} \end{pmatrix}$$

and

$$v_{j,2}^{(1)} = \begin{pmatrix} (\omega_{22}(t_j) + \phi_{22}(\mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{j,2}))(\omega_{33}(t_j) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{j,2}) \\ -\omega_{21}(t_j)(\omega_{33}(t_j) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{i,2}) \\ \omega_{21}(t_j)\omega_{32}(t_j) \end{pmatrix}, \; v_{j,1}^{(1)} = \begin{pmatrix} -\phi_{22}\nu_{j,0} \\ \omega_{21}(t_j) \\ 0 \end{pmatrix},$$

$$v_{j,2}^{(2)} = \begin{pmatrix} (\omega_{22}(t_j) + \phi_{22}(\mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{j,2}))(\omega_{33}(t_j) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{j,2}) \\ -\phi_{22}\omega_{21}(t_j)(\omega_{33}(t_j) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{j,2}) \\ \omega_{21}(t_j)\omega_{32}(t_j) \end{pmatrix}, \; v_{j,1}^{(2)} = \begin{pmatrix} -\nu_{j,0} \\ \omega_{21}(t_j) \\ 0 \end{pmatrix}$$

for $j \neq i$. Then we can see that

$$l_{j,2}^{(1)} = (A/I)v_{j,2}^{(1)}, \quad l_{j,1}^{(1)} = (A/I)v_{j,1}^{(1)} + (A/I)v_{j,2}^{(1)}, \quad l_{j,2}^{(2)} = (A/I)v_{j,2}^{(2)}, \quad l_{j,1}^{(2)} = (A/I)v_{j,1}^{(2)} + (A/I)v_{j,2}^{(2)}$$

for any $j = 1,2,3$ by the conditions $\phi_{t_i}(l_{i,j}^{(1)}) \subset l_{i,j}^{(2)}$, $(\mathrm{res}_{t_i}\nabla - \nu_{i,j}\phi_{t_i})(l_{i,j}^{(1)}) \subset l_{i,j+1}^{(2)}$ and the relations (4.15), (4.16), (4.17). We take lifts $\tilde{\phi}_{22} \in A, \tilde{\omega}_{21} \in H^0(\Omega^1_{\mathbb{P}^1_A/A}(D(\boldsymbol{t})_A)(-1)), \tilde{\omega}_{33} \in H^0(\Omega^1_{\mathbb{P}^1_A/A}(D(\boldsymbol{t})_A))$ and $\tilde{\omega}_{13}^{(i)} \in A^\times$ of $\phi_{22}, \omega_{21}, \omega_{33}$ and $\omega_{13}(t_i)$, respectively. Put $\tilde{\omega}_{22} := -\tilde{\phi}_{22}\tilde{\omega}_{33}$ and let $\tilde{\omega}_{12} \in H^0(\Omega^1_{\mathbb{P}^1_A/A}(D(\boldsymbol{t})_A)(1))$ be a lift of ω_{12} satisfying

$$\tilde{\omega}_{21}(t_i)\tilde{\omega}_{12}(t_i) = \tilde{\omega}_{22}\tilde{\omega}_{33} - \tilde{\phi}_{22}(\nu_{i,0}\nu_{i,1} + \nu_{i,0}\nu_{i,2} + \nu_{i,1}\nu_{i,2} - (\mathrm{res}_{t_i}(\tfrac{dz}{z-t_3}))^2).$$

Then we can find a lift $\tilde{\omega}_{32} \in H^0(\Omega^1_{\mathbb{P}^1_A/A}(D(\boldsymbol{t})_A))$ of ω_{32} satisfying

$$\tilde{\omega}_{21}(t_i)(\tilde{\omega}_{12}(t_i)(\tilde{\omega}_{33}(t_i) + \mathrm{res}_{t_i}(\tfrac{dz}{z-t_3})) - \tilde{\omega}_{13}^{(i)}\tilde{\omega}_{32}(t_i)) = \tilde{\phi}_{22}\nu_{i,0}\nu_{i,1}\nu_{i,2}.$$

Let $\tilde{\omega}_{13}$ be the element of $H^0(\Omega^1_{\mathbb{P}^1_A/A}(D(\boldsymbol{t})_A)(1))$ satisfying

$$-\tilde{\omega}_{21}(t_j)(\tilde{\omega}_{12}(t_j)(\tilde{\omega}_{33}(t_i) + \mathrm{res}_{t_i}(\tfrac{dz}{z-t_3})) - \tilde{\omega}_{13}(t_j)\tilde{\omega}_{32}(t_j)) = \tilde{\phi}_{22}\nu_{j,0}\nu_{j,1}\nu_{j,2}.$$

for $j \neq i$ and $\tilde{\omega}_{13}(t_i) = \tilde{\omega}_{13}^{(i)}$. Put

$$\tilde{\phi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tilde{\phi}_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{\nabla} = \tilde{\phi} \otimes d + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{\phi}_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{dz}{z-t_3} + \begin{pmatrix} 0 & \tilde{\omega}_{12} & \tilde{\omega}_{13} \\ \tilde{\omega}_{21} & \tilde{\omega}_{22} & 0 \\ 0 & \tilde{\omega}_{32} & \tilde{\omega}_{33} \end{pmatrix},$$

$$\tilde{v}_{i,2}^{(1)} = \begin{pmatrix} \tilde{\phi}_{22}\tilde{\omega}_{13}(t_i)(\tilde{\omega}_{33}(t_i) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - (\nu_{i,0}+\nu_{i,1})) \\ \tilde{\omega}_{13}(t_i)\tilde{\omega}_{21}(t_i) \\ \tilde{\phi}_{22}(\tilde{\omega}_{33}(t_i) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{i,0})(\tilde{\omega}_{33}(t_i) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{i,1}) \end{pmatrix}, \; \tilde{v}_{i,1}^{(1)} = \begin{pmatrix} \tilde{\omega}_{13}(t_i) \\ 0 \\ \tilde{\omega}_{33}(t_i) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{i,0} \end{pmatrix},$$

$$\tilde{v}_{i,2}^{(2)} = \begin{pmatrix} \tilde{\omega}_{13}(t_i)(\tilde{\omega}_{33}(t_i) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - (\nu_{i,0}+\nu_{i,1})) \\ \tilde{\omega}_{13}(t_i)\tilde{\omega}_{21}(t_i) \\ (\tilde{\omega}_{33}(t_i) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{i,0})(\tilde{\omega}_{33}(t_i) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{i,1}) \end{pmatrix}, \; \tilde{v}_{i,1}^{(2)} = \begin{pmatrix} \tilde{\omega}_{13}(t_i) \\ 0 \\ \tilde{\omega}_{33}(t_i) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{i,0} \end{pmatrix}$$

and

$$\tilde{v}_{j,2}^{(1)} = \begin{pmatrix} (\tilde{\omega}_{22}(t_j) + \tilde{\phi}_{22}(\mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{j,2}))(\tilde{\omega}_{33}(t_j) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{j,2}) \\ -\tilde{\omega}_{21}(t_j)(\tilde{\omega}_{33}(t_j) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{i,2}) \\ \tilde{\omega}_{21}(t_j)\tilde{\omega}_{32}(t_j) \end{pmatrix},\ \tilde{v}_{j,1}^{(1)} = \begin{pmatrix} -\tilde{\phi}_{22}\nu_{j,0} \\ \tilde{\omega}_{21}(t_j) \\ 0 \end{pmatrix},$$

$$\tilde{v}_{j,2}^{(2)} = \begin{pmatrix} (\tilde{\omega}_{22}(t_j) + \tilde{\phi}_{22}(\mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{j,2}))(\tilde{\omega}_{33}(t_j) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{j,2}) \\ -\tilde{\phi}_{22}\tilde{\omega}_{21}(t_j)(\tilde{\omega}_{33}(t_j) + \mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{j,2}) \\ \tilde{\omega}_{21}(t_j)\tilde{\omega}_{32}(t_j) \end{pmatrix},\ \tilde{v}_{j,1}^{(2)} = \begin{pmatrix} -\nu_{j,0} \\ \tilde{\omega}_{21}(t_j) \\ 0 \end{pmatrix}$$

for $j \neq i$. Let $\tilde{l}_{j,2}^{(m)} = A\tilde{v}_{j,2}^{(m)} \subset A^{\oplus 3}$ and $\tilde{l}_{j,1}^{(m)} = A\tilde{v}_{j,1}^{(m)} + A\tilde{v}_{j,2}^{(m)} \subset A^{\oplus 3}$ for $m = 1, 2$ and $j = 1, 2, 3$. Then we can see that $A^{\oplus 3}/l_{j,n}^{(m)}$ is flat over A and $(\mathrm{res}_{t_j}\tilde{\nabla} - \nu_{j,n}\tilde{\phi}_{t_j})(l_{j,n}^{(1)}) \subset l_{j,n+1}^{(2)}$ for any $j = 1, 2, 3$ and $n = 0, 1, 2$ by the way of taking lifts $\tilde{\omega}_{12}, \tilde{\omega}_{13}, \tilde{\omega}_{22}, \tilde{\omega}_{32}$. So $\tilde{\phi}, \tilde{\nabla}, \tilde{l}_{i,j}^{(1)}$ and $\tilde{l}_{i,j}^{(2)}$ are desire lifts.

Next we consider the case $\mathrm{rank}\,\phi \otimes \mathrm{id}_k = 1$. Then applying certain automorphisms of E_1 and E_2, we may assume that $\phi \otimes \mathrm{id}_k$ and $\nabla \otimes \mathrm{id}_k$ have the form (4.9). In particular, we may assume that $\omega_{32}(t_i) \in (A/I)^{\times}$. In the same way of the proof Lemma 4.4.2, we also obtain $|\mathrm{res}_{t_i}\nabla - \lambda\phi_{t_i}| = (\wedge^3\phi)(\nu_{i,0} - \lambda)(\nu_{i,1} - \lambda)(\nu_{i,2} - \lambda)$, and by comparing the coefficients on both sides and using (4.13), we have

$$\phi_{22}\omega_{33}(t_i) + \phi_{33}\omega_{22}(t_i) - \phi_{23}\omega_{32}(t_i) - \phi_{32}\omega_{23}(t_i) = 0, \tag{4.18}$$

$$\begin{aligned} &(\omega_{22}(t_i)\omega_{33}(t_i) - \omega_{23}(t_i)\omega_{32}(t_i)) - \omega_{21}(t_i)(\omega_{12}(t_i)\phi_{33} - \omega_{13}(t_i)\phi_{32}) \\ &\quad = (\phi_{22}\phi_{33} - \phi_{23}\phi_{32})(\nu_{i,0}\nu_{i,1} + \nu_{i,0}\nu_{i,2} + \nu_{i,1}\nu_{i,2} - (\mathrm{res}_{t_i}(\tfrac{dz}{z-t_3}))^2), \end{aligned} \tag{4.19}$$

$$\begin{aligned} &-\omega_{21}(t_i)(\omega_{12}(t_i)(\omega_{33}(t_i) + \phi_{33}\mathrm{res}_{t_i}(\tfrac{dz}{z-t_3})) - \omega_{13}(t_i)(\omega_{32}(t_i) + \phi_{32}\mathrm{res}_{t_i}(\tfrac{dz}{z-t_3}))) \\ &\quad = (\phi_{22}\phi_{33} - \phi_{23}\phi_{32})\nu_{i,0}\nu_{i,1}\nu_{i,2}. \end{aligned} \tag{4.20}$$

Put

$$v_{j,2}^{(1)} := \begin{pmatrix} \omega_{22}(t_j)\omega_{33}(t_j) - \omega_{32}(t_j)\omega_{23}(t_j) + (\phi_{22}\phi_{33} - \phi_{23}\phi_{32})(\mathrm{res}_{t_j}(\frac{dz}{z-t_3}) - \nu_{j,2})^2 \\ -\omega_{21}(t_j)(\omega_{33}(t_j) + \phi_{33}(\mathrm{res}_{t_j}(\frac{dz}{z-t_3}) - \nu_{j,2})) \\ \omega_{21}(t_j)(\omega_{32}(t_j) + \phi_{32}(\mathrm{res}_{t_j}(\frac{dz}{z-t_3}) - \nu_{j,2})) \end{pmatrix},$$

$$v_{j,1}^{(1)} := \begin{pmatrix} -\nu_{j,0}(\phi_{22}\omega_{32}(t_j) - \phi_{32}\omega_{22}(t_j)) + \omega_{21}(t_j)\omega_{12}(t_j)\phi_{32} \\ (\omega_{32}(t_j) + \phi_{32}(\mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{j,0}))\omega_{21}(t_j) \\ 0 \end{pmatrix},$$

$$v_{j,2}^{(2)} := (\mathrm{res}_{t_i}\nabla - \nu_{j,1}\phi_{t_j})(v_{j,1}^{(1)}),\ v_{j,1}^{(2)} := \begin{pmatrix} -\nu_{i,0} \\ \omega_{21}(t_j) \\ 0 \end{pmatrix}.$$

Then we can see that $l_{j,2}^{(1)} = (A/I)v_{j,2}^{(1)}, l_{j,1}^{(1)} = (A/I)v_{j,1}^{(1)} + (A/I)v_{j,2}^{(1)}, l_{j,2}^{(2)} = (A/I)v_{j,2}^{(2)}$ and $l_{j,1}^{(2)} = (A/I)v_{j,1}^{(2)} + (A/I)v_{j,2}^{(2)}$ for any $j = 1, 2, 3$ by the conditions $\phi_{t_j}(l_{j,m}^{(1)}) \subset l_{j,m}^{(2)}$ and $(\mathrm{res}_{t_j}\nabla - \nu_{j,m}\phi_{t_j})(l_{j,m}^{(1)}) \subset l_{j,m+1}^{(2)}$, and the relations (4.18), (4.19), (4.20). We take lifts $\psi_{22}, \psi_{23}, \tilde{\phi}_{32}, \tilde{\phi}_{33} \in A, \tilde{\omega}_{21} \in H^0(\Omega^1_{\mathbb{P}^1_A/A}(D(\boldsymbol{t})_A)(-1)), \tilde{\omega}_{32}, \tilde{\omega}_{33} \in H^0(\Omega^1_{\mathbb{P}^1_A/A}(D(\boldsymbol{t})_A))$ and $\tilde{\omega}_{12} \in H^0(\Omega^1_{\mathbb{P}^1_A/A}(D(\boldsymbol{t})_A)(1))$ of $\phi_{22}, \phi_{23}, \phi_{32}, \phi_{33}, \omega_{21}, \omega_{32}, \omega_{33}, \omega_{12}$, respectively. We take lifts $\tilde{\omega}_{13} \in H^0(\Omega^1_{\mathbb{P}^1_A/A}(D(\boldsymbol{t})_A)(1))$, $\tilde{\omega}_{22}, \tilde{\omega}_{23} \in H^0(\Omega^1_{\mathbb{P}^1_A/A}(D(\boldsymbol{t})_A))$ of $\omega_{13}, \omega_{22}, \omega_{23}$, respectively, satisfying

$$\begin{aligned} &-\tilde{\omega}_{21}(t_j)(\tilde{\omega}_{12}(t_j)(\tilde{\omega}_{33}(t_j) + \tilde{\phi}_{33}\mathrm{res}_{t_j}(\tfrac{dz}{z-t_3})) - \tilde{\omega}_{13}(t_j)(\tilde{\omega}_{32}(t_j) + \tilde{\phi}_{32}\mathrm{res}_{t_j}(\tfrac{dz}{z-t_3}))) \\ &\quad = (\psi_{22}\tilde{\phi}_{33} - \psi_{23}\tilde{\phi}_{32})\nu_{j,0}\nu_{j,1}\nu_{j,2}, \end{aligned}$$

$$\begin{aligned} &-\tilde{\omega}_{23}(t_i)\tilde{\omega}_{32}(t_i) - \tilde{\omega}_{21}(t_i)(\tilde{\omega}_{12}(t_i)\tilde{\phi}_{33} - \tilde{\omega}_{13}(t_i)\tilde{\phi}_{32}) \\ &\quad = (\psi_{22}\tilde{\phi}_{33} - \psi_{23}\tilde{\phi}_{32})(\nu_{i,0}\nu_{i,1} + \nu_{i,0}\nu_{i,2} + \nu_{i,1}\nu_{i,2} - (\mathrm{res}_{t_i}(\tfrac{dz}{z-t_3}))^2), \end{aligned}$$

$$\begin{aligned} &(\tilde{\omega}_{22}(t_j)\tilde{\omega}_{33}(t_j) - \tilde{\omega}_{23}(t_j)\tilde{\omega}_{32}(t_j)) - \tilde{\omega}_{21}(t_j)(\tilde{\omega}_{12}(t_j)\tilde{\phi}_{33} - \tilde{\omega}_{13}(t_j)\tilde{\phi}_{32}) \\ &\quad = (\psi_{22}\tilde{\phi}_{33} - \psi_{23}\tilde{\phi}_{32})(\nu_{i,0}\nu_{i,1} + \nu_{i,0}\nu_{i,2} + \nu_{i,1}\nu_{i,2} - (\mathrm{res}_{t_j}(\tfrac{dz}{z-t_3}))^2) \end{aligned}$$

for any $j = 1, 2, 3$. Put

$$\eta := \psi_{22}\tilde{\omega}_{33} + \tilde{\phi}_{33}\tilde{\omega}_{22} - \psi_{23}\tilde{\omega}_{32} - \tilde{\phi}_{32}\tilde{\omega}_{23}.$$

Since $\tilde{\omega}_{32}(t_i) \neq 0$ and $\tilde{\omega}_{33}(t_i) = 0$, $\tilde{\omega}_{32}$ and $\tilde{\omega}_{33}$ generate $H^0(\Omega^1_{\mathbb{P}^1_A/A}(D(\boldsymbol{t})_A)) \cong A^{\oplus 2}$ as A-module. In particular, η can be written by the form $b_1\tilde{\omega}_{32} + b_2\tilde{\omega}_{33}$, where $b_1, b_2 \in A$. Since $\eta \mod I$ is zero by (4.18), we have $b_1, b_2 \in I$. Put $\tilde{\phi}_{22} = \psi_{22} - b_2, \tilde{\phi}_{23} = \psi_{23} + b_1$. Then we have

$$\tilde{\phi}_{22}\tilde{\omega}_{33} + \tilde{\phi}_{33}\tilde{\omega}_{22} - \tilde{\phi}_{23}\tilde{\omega}_{32} - \tilde{\phi}_{32}\tilde{\omega}_{23} = 0, \tag{4.21}$$

$$\begin{aligned}(\tilde{\omega}_{22}(t_j)\tilde{\omega}_{33}(t_j) - \tilde{\omega}_{23}(t_i)\tilde{\omega}_{32}(t_i)) - \tilde{\omega}_{21}(t_i)(\tilde{\omega}_{12}(t_i)\tilde{\phi}_{33} - \tilde{\omega}_{13}(t_i)\tilde{\phi}_{32})\\ = (\tilde{\phi}_{22}\tilde{\phi}_{33} - \tilde{\phi}_{23}\tilde{\phi}_{32})(\nu_{i,0}\nu_{i,1} + \nu_{i,0}\nu_{i,2} + \nu_{i,1}\nu_{i,2} - (\mathrm{res}_{t_i}(\tfrac{dz}{z-t_3}))^2),\end{aligned} \tag{4.22}$$

$$\begin{aligned}&- \tilde{\omega}_{21}(t_j)(\tilde{\omega}_{12}(t_j)(\tilde{\omega}_{33}(t_j) + \tilde{\phi}_{33}\mathrm{res}_{t_j}(\tfrac{dz}{z-t_3})) - \tilde{\omega}_{13}(t_j)(\tilde{\omega}_{32}(t_j) + \tilde{\phi}_{32}\mathrm{res}_{t_j}(\tfrac{dz}{z-t_3})))\\ &= (\tilde{\phi}_{22}\tilde{\phi}_{33} - \tilde{\phi}_{23}\tilde{\phi}_{32})\nu_{j,0}\nu_{j,1}\nu_{j,2}\end{aligned} \tag{4.23}$$

for any $j = 1, 2, 3$ because $\mathfrak{m}I = 0$. Put

$$\tilde{\phi} = \begin{pmatrix}1 & 0 & 0\\ 0 & \tilde{\phi}_{22} & \tilde{\phi}_{23}\\ 0 & \tilde{\phi}_{32} & \tilde{\phi}_{33}\end{pmatrix}, \quad \tilde{\nabla} = \tilde{\phi} \otimes d + \begin{pmatrix}0 & 0 & 0\\ 0 & \tilde{\phi}_{22} & \tilde{\phi}_{23}\\ 0 & \tilde{\phi}_{32} & \tilde{\phi}_{33}\end{pmatrix}\frac{dz}{z-t_3} + \begin{pmatrix}0 & \tilde{\omega}_{12} & \tilde{\omega}_{13}\\ \tilde{\omega}_{21} & \tilde{\omega}_{22} & \tilde{\omega}_{23}\\ 0 & \tilde{\omega}_{32} & \tilde{\omega}_{33}\end{pmatrix},$$

$$\tilde{v}^{(1)}_{j,2} := \begin{pmatrix}\tilde{\omega}_{22}(t_j)\tilde{\omega}_{33}(t_j) - \tilde{\omega}_{32}(t_j)\tilde{\omega}_{23}(t_j) + (\tilde{\phi}_{22}\tilde{\phi}_{33} - \tilde{\phi}_{23}\tilde{\phi}_{32})(\mathrm{res}_{t_j}(\frac{dz}{z-t_3}) - \nu_{j,2})^2\\ -\tilde{\omega}_{21}(t_j)(\tilde{\omega}_{33}(t_j) + \tilde{\phi}_{33}(\mathrm{res}_{t_j}(\frac{dz}{z-t_3}) - \nu_{j,2}))\\ \tilde{\omega}_{21}(t_j)(\tilde{\omega}_{32}(t_j) + \tilde{\phi}_{32}(\mathrm{res}_{t_j}(\frac{dz}{z-t_3}) - \nu_{j,2}))\end{pmatrix},$$

$$\tilde{v}^{(1)}_{j,1} := \begin{pmatrix}-\nu_{j,0}(\tilde{\phi}_{22}\tilde{\omega}_{32}(t_j) - \tilde{\phi}_{32}\tilde{\omega}_{22}(t_j)) + \tilde{\omega}_{21}(t_j)\tilde{\omega}_{12}(t_j)\tilde{\phi}_{32}\\ (\tilde{\omega}_{32}(t_j) + \tilde{\phi}_{32}(\mathrm{res}_{t_i}(\frac{dz}{z-t_3}) - \nu_{j,0}))\tilde{\omega}_{21}(t_j)\\ 0\end{pmatrix},$$

$$\tilde{v}^{(2)}_{j,2} := (\mathrm{res}_{t_i}\tilde{\nabla} - \nu_{j,1}\tilde{\phi}_{t_j})(\tilde{v}^{(1)}_{j,1}),\ \tilde{v}^{(2)}_{j,1} := \begin{pmatrix}-\nu_{i,0}\\ \tilde{\omega}_{21}(t_j)\\ 0\end{pmatrix}.$$

Let $\tilde{l}^{(m)}_{j,2} := A\tilde{v}^{(m)}_{j,2} \subset A^{\oplus 3}$ and $\tilde{l}^{(m)}_{j,1} = A\tilde{v}^{(m)}_{j,1} + A\tilde{v}^{(m)}_{j,2} \subset A^{\oplus 3}$ for $m = 1, 2$ and $j = 1, 2, 3$. Then we can see that $A^{\oplus 3}/l^{(m)}_{j,n}$ is flat over A and $(\mathrm{res}_{t_j}\tilde{\nabla} - \nu_{j,n}\tilde{\phi}_{t_j})(l^{(1)}_{j,n}) \subset l^{(2)}_{j,n+1}$ for any $j = 1, 2, 3$ and $n = 0, 1, 2$ by the way of taking lifts $\tilde{\omega}_{12}, \tilde{\omega}_{13}, \tilde{\omega}_{22}, \tilde{\omega}_{32}$. So $\tilde{\phi}, \tilde{\nabla}, \tilde{l}^{(1)}_{i,j}$ and $\tilde{l}^{(2)}_{i,j}$ are desire lifts. □

4.7 Proof of Theorem 4.1.1

To prove Theorem 4.1.1, we consider $\widehat{M^{\boldsymbol{\alpha}}_3}(\boldsymbol{t}, \boldsymbol{\nu})$ and $\overline{M^{\boldsymbol{\alpha}}_3}(\boldsymbol{t}, \boldsymbol{\nu})$ for $(\boldsymbol{t}, \boldsymbol{\nu}) \in T_3 \times \mathcal{N}$. Let D_0 be the section of $\mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1})$ over $\mathbb{P}^1$ defined by the injection $\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \hookrightarrow \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1}$, and D_i be the fiber of $\mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1})$ over $t_i \in \mathbb{P}^1$. Let $b_{i,j}$ be the point of D_i corresponding to $\nu_{i,j}$. We put $B := \{b_{i,j} \mid 1 \leq i \leq 3, 0 \leq j \leq 2\}$.

Proposition 4.7.1. The restriction morphism

$$\varphi\colon \widehat{M^{\boldsymbol{\alpha}}_3}(\boldsymbol{t}, \boldsymbol{\nu}) \setminus \varphi^{-1}(B) \longrightarrow \mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1}) \setminus B \tag{4.24}$$

is an isomorphism.

Proof. Let z be a fixed inhomogeneous coordinate on $\mathbb{P}^1 = \mathrm{Spec}\,\mathbb{C}[z] \cup \{\infty\}$. Let D_∞ be the fiber of $\mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1})$ over $\infty \in \mathbb{P}^1$. Put $Y = \bigcup_{i=0}^3 D_i \cup D_\infty$. Then the morphism

$$(\mathbb{P}^1 \setminus \{t_1, t_2, t_3, \infty\}) \times \mathbb{C} \longrightarrow \mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1}) \setminus Y; \quad (q, p) \longmapsto \mathbb{C}(p\tfrac{dz}{h(z)}, 1) \subset \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))|_q \oplus \mathcal{O}_{\mathbb{P}^1}|_q$$

becomes an isomorphism. By this isomorphism, we regard (q, p) as a coordinate on $\mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1}) \setminus Y$. We define a family of $\boldsymbol{\nu}$-parabolic connections (E, ∇, l_*) on $\mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1}) \setminus Y \times \mathbb{P}^1$ as follows. Let $E = p_2^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$, where $p_2\colon \mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1}) \setminus Y \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection. We define a relative logarithmic connection $\nabla\colon E \to E \otimes p_2^*\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$ by

$$\nabla := d + \begin{pmatrix}0 & a_{12}(p; z) & a_{13}(q, p; z)\\ 1 & (z-t_1)(z-t_2) - p & 0\\ 0 & z - q & (z-t_1)(z-t_2) + p\end{pmatrix}\frac{dz}{h(z)},$$

where $a_{12}(p;z), a_{13}(q,p;z)$ are the quadratic polynomials in z satisfying

$$a_{12}(p;t_i) = (t_i - t_1)^2(t_i - t_2)^2 - p^2 - h'(t_i)^2(\nu_{i,0}\nu_{i,1} + \nu_{i,1}\nu_{i,2} + \nu_{i,2}\nu_{i,0})$$

$$(t_i - q)a_{13}(q,p;t_i) = \prod_{j=0}^{2}(h'(t_i)(\nu_{i,j} - (\mathrm{res}_{t_i}(\tfrac{dz}{z-t_3}))) - p)$$

for any $i = 1,2,3$. Let $E|_{t_i} \supsetneq l_{i,1} \supsetneq l_{i,2} \supsetneq 0$ be a filtration by subbundles such that $(\mathrm{res}_{t_i}\nabla - \nu_{i,j}\mathrm{id})(l_{i,j}) \subset l_{i,j+1}$ for any $j = 0,1,2$. Then we have

$$l_{i,2} = \mathbb{C}\begin{pmatrix} (p + h'(t_i)(\nu_{i,2} - \mathrm{res}_{t_i}(\frac{dz}{z-t_3})))(h'(t_i)(\nu_{i,2} - \mathrm{res}_{t_i}(\frac{dz}{z-t_3})) - p) \\ (h'(t_i)(\nu_{i,2} - \mathrm{res}_{t_i}(\frac{dz}{z-t_3})) - p) \\ t_i - q \end{pmatrix}, \tag{4.25}$$

$$l_{i,1} = \mathbb{C}\begin{pmatrix} (p + h'(t_i)(\nu_{i,2} - \mathrm{res}_{t_i}(\frac{dz}{z-t_3})))(h'(t_i)(\nu_{i,2} - \mathrm{res}_{t_i}(\frac{dz}{z-t_3})) - p) \\ (h'(t_i)(\nu_{i,2} - \mathrm{res}_{t_i}(\frac{dz}{z-t_3})) - p) \\ t_i - q \end{pmatrix} + \mathbb{C}\begin{pmatrix} -h'(t_i)\nu_{i,0} \\ 1 \\ 0 \end{pmatrix} \tag{4.26}$$

For any $(q,p) \in \mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1}) \setminus Y$, the corresponding $\boldsymbol{\nu}$-parabolic connection $(E_{(q,p)}, \nabla_{(q,p)}, (l_*)_{(q,p)})$ is $\boldsymbol{\alpha}$-stable. So we obtain a morphism

$$\mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1}) \setminus Y \longrightarrow \widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu}) \setminus \varphi^{-1}(Y),$$

which is just the inverse of the morphism

$$\varphi\colon \widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu}) \setminus \varphi^{-1}(Y) \longrightarrow \mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1}) \setminus Y.$$

Hence the morphism (4.24) is a birational morphism. By Proposition 4.5.6 and Zariski's main theorem, the morphism (4.24) is an isomorphism. □

Proposition 4.7.2. $\widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$ is a smooth variety.

Proof. By Remark 4.5.5, the locus on $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$ defined by $\mathrm{rank}\,\phi = 1$ consists of one point p_0. Let $\mathrm{PC}\colon \widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu}) \to \overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$ be the forgetful map. Then, by Proposition 4.3.1, the restriction map

$$\mathrm{PC}\colon \widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu}) \setminus \mathrm{PC}^{-1}(p_0) \longrightarrow \overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu}) \setminus \{p_0\}$$

becomes an isomorphism. So it sufficient to proof that $\widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$ is smooth at any point in $\mathrm{PC}^{-1}(p_0)$, and it follows from Proposition 4.7.1. □

Proposition 4.7.3. If $\nu_{i,0} \neq \nu_{i,1} \neq \nu_{i,2} \neq \nu_{i,0}$, then $\varphi^{-1}(b_{i,j}) \cong \mathbb{P}^1$ for any $j = 0,1,2$ and these are (-1)-curves.

Proof. Let $E_1 = E_2 = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $p = h'(t_i)(\nu_{i,j} - \mathrm{res}_{t_i}(\frac{dz}{z-t_3}))$ and $h(z) = (z-t_1)(z-t_2)(z-t_3)$. Let $a(z)$ be the quadratic polynomial satisfying

$$a(t_m) = (t_m - t_1)^2(t_m - t_2)^2 - p^2 - h'(t_m)^2(\nu_{m,0}\nu_{m,1} + \nu_{m,1}\nu_{m,2} + \nu_{m,2}\nu_{m,0})$$

for $m = 1,2,3$. Let $b(z)$ be the quadratic polynomial satisfying $b(t_i) = 0$ and

$$(t_m - t_i)b(t_m) = (h'(t_m)(\nu_{m,0} - \mathrm{res}_{t_m}(\tfrac{dz}{z-t_3})) - p)(h'(t_m)(\nu_{m,1} - \mathrm{res}_{t_m}(\tfrac{dz}{z-t_3})) - p)(h'(t_m)(\nu_{m,2} - \mathrm{res}_{t_m}(\tfrac{dz}{z-t_3})) - p)$$

for $m \neq i$. Put

$$\phi_\mu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \nabla_{(\mu,\eta)} = \phi_\mu \otimes d + \begin{pmatrix} 0 & \mu a(z) & \mu b(z) + \eta\prod_{m\neq i}(z - t_m) \\ 1 & \mu(z-t_1)(z-t_2) - \mu p & 0 \\ 0 & z - t_i & (z-t_1)(z-t_2) + p \end{pmatrix}\frac{dz}{h(z)}, \tag{4.27}$$

where $\mu, \eta \in \mathbb{C}$. When $\mu = \eta = 0$, the ϕ_μ-connection $(E_1, E_2, \phi_\mu, \nabla_{(\mu,\eta)})$ becomes $\boldsymbol{\alpha}$-unstable for any parabolic structures. Assume that $(\mu,\eta) \neq (0,0)$. Then parabolic structures $l^{(1)}_{i,*}$ and $l^{(2)}_{i,*}$ of E_1 and E_2, respectively, satisfying the conditions $(\phi_\mu)_{t_i}(l^{(1)}_{i,j}) \subset l^{(2)}_{i,j}$ and $(\mathrm{res}_{t_i}(\nabla_{(\mu,\eta)}) - \nu_{i,j}(\phi_\mu)_{t_i})(l^{(1)}_{i,j}) \subset l^{(2)}_{i,j+1}$ are uniquely determined. In fact, when $\mu = 0$, it is proved in the proof of Proposition 4.5.6. When $\mu \neq 0$, we may assume that $\mu = 1$. For $m \neq i$, parabolic structures $l^{(1)}_{m,*}$ and $l^{(2)}_{m,*}$ are given by (4.25) and (4.26). $l^{(1)}_{i,*}$ and $l^{(2)}_{i,*}$ are of the following form. When $p = h'(t_i)(\nu_{i,2} - \mathrm{res}_{t_i}(\frac{dz}{z-t_3}))$,

$$l^{(1)}_{i,2} = \mathbb{C}\begin{pmatrix} 0 \\ \eta \\ h'(t_i)(\nu_{i,0}-\nu_{i,2})(\nu_{i,1}-\nu_{i,2}) \end{pmatrix},\ l^{(1)}_{i,1} = \mathbb{C}\begin{pmatrix} 0 \\ \eta \\ h'(t_i)(\nu_{i,0}-\nu_{i,2})(\nu_{i,1}-\nu_{i,2}) \end{pmatrix} + \mathbb{C}\begin{pmatrix} -h'(t_i)\nu_{i,0} \\ 1 \\ 0 \end{pmatrix},$$

$l^{(2)}_{i,2} = \phi_{t_i}(l^{(1)}_{i,2})$ and $l^{(2)}_{i,1} = \phi_{t_i}(l^{(1)}_{i,1})$. When $p = h'(t_i)(\nu_{i,1} - \mathrm{res}_{t_i}(\frac{dz}{z-t_3}))$,

$$l^{(1)}_{i,2} = \mathbb{C}\begin{pmatrix} -h'(t_i)\nu_{i,0} \\ 1 \\ 0 \end{pmatrix},\ l^{(1)}_{i,1} = \mathbb{C}\begin{pmatrix} -h'(t_i)\nu_{i,0} \\ 1 \\ 0 \end{pmatrix} + \mathbb{C}\begin{pmatrix} 0 \\ \eta \\ h'(t_i)(\nu_{i,0}-\nu_{i,2})(\nu_{i,1}-\nu_{i,2}) \end{pmatrix},$$

$l^{(2)}_{i,2} = \phi_{t_i}(l^{(1)}_{i,2})$ and $l^{(2)}_{i,1} = \phi_{t_i}(l^{(1)}_{i,1})$. When $p = h'(t_i)(\nu_{i,0} - \mathrm{res}_{t_i}(\frac{dz}{z-t_3}))$,

$$l^{(1)}_{i,2} = \mathbb{C}\begin{pmatrix} -h'(t_i)\nu_{i,1} \\ 1 \\ 0 \end{pmatrix},\ l^{(1)}_{i,1} = \mathbb{C}\begin{pmatrix} -h'(t_i)\nu_{i,1} \\ 1 \\ 0 \end{pmatrix} + \mathbb{C}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$l^{(2)}_{i,2} = \phi_{t_i}(l^{(1)}_{i,2})$ and $l^{(2)}_{i,1} = \phi_{t_i}(l^{(1)}_{i,1})$. We can see that $(E_1, E_2, \phi_\mu, \nabla_{(\mu,\eta)}, l^{(1)}_*, l^{(2)}_*)$ is $\boldsymbol{\alpha}$-stable if and only if $(\mu, \eta) \neq (0,0)$. We can also see that $(E_1, E_2, \phi_{\mu_1}, \nabla_{(\mu_1,\eta_1)})$ and $(E_1, E_2, \phi_{\mu_2}, \nabla_{(\mu_2,\eta_2)})$ are isomorphic to each other if and only if there exists $c \in \mathbb{C}^\times$ such that $(\mu_1, \eta_1) = c(\mu_2, \eta_2)$. So we obtain the morphism

$$\mathbb{P}^1 \longrightarrow \varphi^{-1}(b_{i,j});\ (\mu : \eta) \longmapsto (E_1, E_2, \phi_{\mu_1}, \nabla_{(\mu_1,\eta_1)}, l^{(1)}_*, l^{(2)}_*),$$

which is an isomorphism by Lemma 4.5.1 and Lemma 4.5.3. Since $\widehat{M^{\boldsymbol{\alpha}}_3}(\boldsymbol{t}, \boldsymbol{\nu})$ and $\mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1})$ are smooth, $\varphi^{-1}(b_{i,j})$ is a (-1)-curve. □

Let $N_3(\boldsymbol{t}, \boldsymbol{\nu})$ be the moduli space of rank 3 stable $\boldsymbol{\nu}$-logarithmic connections over $(\mathbb{P}, \boldsymbol{t})$. A connection (E, ∇) is said to be stable if for any nonzero subbundle $F \subsetneq E$ preserved by ∇, the inequality

$$\frac{\deg F}{\operatorname{rank} F} < \frac{\deg E}{\operatorname{rank} E}$$

holds. Under the assumption in this section, a $\boldsymbol{\nu}$-parabolic connection (E, ∇, l_*) is $\boldsymbol{\alpha}$-stable if and only if (E, ∇) is stable. So we have the surjective morphism $M^{\boldsymbol{\alpha}}_3(\boldsymbol{t}, \boldsymbol{\nu}) \to N_3(\boldsymbol{t}, \boldsymbol{\nu})$ by forgetting parabolic structures.

Proposition 4.7.4. Let j_0, j_1 and j_2 be distinct elements of $\{0,1,2\}$. Assume that $\nu_{i,j_0} = \nu_{i,j_1} \neq \nu_{i,j_2}$. Then $\varphi^{-1}(b_{i,j_0})$ is the union of two projective lines C_1, C_2 such that $Y_{(\boldsymbol{t},\boldsymbol{\nu})} \cap C_1$ and $C_1 \cap C_2$ consist of one point, respectively, and $Y_{(\boldsymbol{t},\boldsymbol{\nu})} \cap C_2 = \emptyset$. Moreover, self-intersection numbers of C_1 and C_2 are -1 and -2, respectively.

Proof. Assume that $j_0 = 0, j_1 = 1, j_2 = 2$. Put $\nu_i := \nu_{i,0} = \nu_{i,1}, \nu'_i := \nu_{i,2}$ and $p := h'(t_i)(\nu_i - \mathrm{res}_{t_i}(\frac{dz}{z-t_3}))$. Let $a(z), b(z), h(z)$ be the polynomials defined in the proof of Proposition 4.7.3. Then we can see that any element $(E_1, E_2, \phi, \nabla, l^{(1)}_*, l^{(2)}_*) \in \varphi^{-1}(b_{i,0})$ have the forms

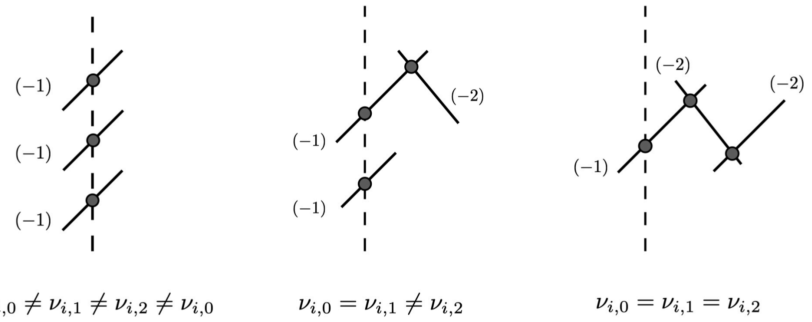
$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \nabla = \phi \otimes d + \begin{pmatrix} 0 & \mu a(z) & \mu b(z) + \eta \prod_{m\neq i}(z-t_m) \\ 1 & \mu(z-t_1)(z-t_2) - \mu p & 0 \\ 0 & z-t_i & (z-t_1)(z-t_2)+p \end{pmatrix} \frac{dz}{h(z)},$$

where $(\mu : \eta) \in \mathbb{P}^1$. So we have

$$\mathrm{res}_{t_i}\nabla - \nu_i \phi_{t_i} = \frac{1}{h'(t_i)} \begin{pmatrix} -h'(t_i)\nu_i & \mu a(t_i) & \eta \prod_{m\neq i}(t_i - t_m) \\ 1 & -\mu h'(t_i)\nu'_i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathrm{res}_{t_i}\nabla - \nu'_i \phi_{t_i} = \frac{1}{h'(t_i)} \begin{pmatrix} -h'(t_i)\nu'_i & \mu a(t_i) & \eta \prod_{m\neq i}(t_i - t_m) \\ 1 & -\mu h'(t_i)\nu_i & 0 \\ 0 & 0 & h'(t_i)(\nu_i - \nu'_i) \end{pmatrix}.$$



By definition, we have $a(t_i) = -h'(t_i)^2\nu_i\nu_i'$. If $\eta = 0$, then $l_{i,*}^{(1)}$ and $l_{i,*}^{(2)}$ are of the form

$$l_{i,2}^{(1)} = \begin{pmatrix} -h'(t_i)\nu_i\mu \\ 1 \\ 0 \end{pmatrix},\ l_{i,1}^{(1)} = \mathbb{C}\begin{pmatrix} -h'(t_i)\nu_i\mu \\ 1 \\ 0 \end{pmatrix} + \mathbb{C}\begin{pmatrix} s \\ 0 \\ t \end{pmatrix},\ l_{i,2}^{(2)} = \mathbb{C}\begin{pmatrix} -h'(t_i)\nu_i \\ 1 \\ 0 \end{pmatrix},\ l_{i,1}^{(2)} = \mathbb{C}\begin{pmatrix} -h'(t_i)\nu_i \\ 1 \\ 0 \end{pmatrix} + \mathbb{C}\begin{pmatrix} s \\ 0 \\ t \end{pmatrix},$$

where $(s:t) \in \mathbb{P}^1$. If $\eta \neq 0$, then

$$l_{i,2}^{(1)} = \mathbb{C}\begin{pmatrix} -h'(t_i)\nu_i\mu \\ 1 \\ 0 \end{pmatrix}, l_{i,1}^{(1)} = \mathbb{C}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mathbb{C}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ l_{i,2}^{(2)} = \mathbb{C}\begin{pmatrix} -h'(t_i)\nu_i \\ 1 \\ 0 \end{pmatrix}, l_{i,1}^{(2)} = \mathbb{C}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mathbb{C}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

By the above argument, we have

$$C_1 := \overline{\{\eta \neq 0\} \cap \varphi^{-1}(b_{i,j_0})} \cong \mathbb{P}^1, C_2 := \{\eta = 0\} \cong \mathbb{P}^1, \varphi^{-1}(b_{i,j_0}) = C_1 \cup C_2$$

and we find that $C_1 \cap Y_{(\boldsymbol{t},\boldsymbol{\nu})}$ and $C_1 \cap C_2$ consist of one point, respectively.

Next we consider self-intersection numbers. Let $a_{12}(p;z)$ be the quadratic polynomial satisfying

$$a_{12}(p;t_m) = (t_m - t_1)^2(t_m - t_2)^2 - p^2 - h'(t_m)^2(\nu_{m,0}\nu_{m,1} + \nu_{m,1}\nu_{m,2} + \nu_{m,2}\nu_{m,0})$$

for $m = 1,2,3$. Let $a_{13}(q,p,\eta;z)$ be the quadratic polynomial satisfying $a_{13}(q,p,\eta;t_i) = \eta$ and

$$(t_m-q)a_{13}(q,p,\eta;t_m) = (h'(t_m)(\nu_{m,0}-\mathrm{res}_{t_m}(\tfrac{dz}{z-t_3}))-p)(h'(t_m)(\nu_{m,1}-\mathrm{res}_{t_m}(\tfrac{dz}{z-t_3}))-p)(h'(t_m)(\nu_{m,2}-\mathrm{res}_{t_m}(\tfrac{dz}{z-t_3}))-p)$$

for $m \neq i$. Put $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$,

$$\nabla_{(q,p,\eta)} = d + \begin{pmatrix} 0 & a_{12}(p;z) & a_{13}(q,p,\eta;z) \\ 1 & (z-t_1)(z-t_2) - p & 0 \\ 0 & z-q & (z-t_1)(z-t_2)+p \end{pmatrix} \frac{dz}{h(z)},$$

$$f(q,p,\eta) = (t_i-q)\eta-(h'(t_i)(\nu_{i,0}-\mathrm{res}_{t_i}(\tfrac{dz}{z-t_3}))-p)(h'(t_i)(\nu_{i,1}-\mathrm{res}_{t_i}(\tfrac{dz}{z-t_3}))-p)(h'(t_i)(\nu_{i,2}-\mathrm{res}_{t_i}(\tfrac{dz}{z-t_3}))-p),$$

and

$$X = \{f(q,p,\eta) = 0\} \subset (\mathbb{C} \setminus \{t_m\}_{m\neq i}) \times \mathbb{C} \times \mathbb{C}.$$

Then $(E, \nabla_{(q,p,\eta)})$ is a stable $\boldsymbol{\nu}$-connection, which induces the morphism $X \to N_3(\boldsymbol{t},\boldsymbol{\nu})$. We can see that this morphism is an open immersion, which implies that the point in $N_3(\boldsymbol{t},\boldsymbol{\nu})$ corresponding to $(q,p,\eta) = (t_i, h'(t_i)(\nu_i - \mathrm{res}_{t_i}(\frac{dz}{z-t_3})), 0)$ is an A_1-singularity. Since C_2 is the fiber of the map $M_3^{\boldsymbol{\alpha}}(\boldsymbol{t},\boldsymbol{\nu}) \to N_3(\boldsymbol{t},\boldsymbol{\nu})$ over $(t_i, h'(t_i)(\nu_i - \mathrm{res}_{t_i}(\frac{dz}{z-t_3})), 0)$, we have $C_2^2 = -2$. The morphism φ can be factored into a composition of blow-ups, so C_1 must be a (-1)-curve.

We can also prove the case of $j_2 = 0, 1$ in the same manner. □

Proposition 4.7.5. Assume that $\nu_{i,0} = \nu_{i,1} = \nu_{i,2}$. Then $\varphi^{-1}(b_{i,j})$ is the union of three projective lines C_1, C_2, C_3 such that $C_1 \cap Y_{(\boldsymbol{t},\boldsymbol{\nu})}$, $C_1 \cap C_2$, and $C_2 \cap C_3$ consist of one point, $C_1 \cap C_3 = \emptyset$, and self-intersection numbers of C_1, C_2 and C_3 are -1, -2, and -2, respectively.

Proof. Put $\nu_i := \nu_{i,0} = \nu_{i,1} = \nu_{i,2}$ and $p := h'(t_i)(\nu_i - \mathrm{res}_{t_i}(\frac{dz}{z-t_3}))$. Let $a(z), b(z), h(z)$ be the polynomials defined in the proof of Proposition 4.7.3. Then we can see that any element $(E_1, E_2, \phi, \nabla, l_*^{(1)}, l_*^{(2)}) \in \varphi^{-1}(b_{i,j})$ have the forms

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \nabla = \phi \otimes d + \begin{pmatrix} 0 & \mu a(z) & \mu b(z) + \eta \prod_{m \neq i}(z - t_m) \\ 1 & \mu(z-t_1)(z-t_2) - \mu p & 0 \\ 0 & z - t_i & (z-t_1)(z-t_2) + p \end{pmatrix} \frac{dz}{h(z)},$$

where $(\mu : \eta) \in \mathbb{P}^1$. So we have

$$\mathrm{res}_{t_i} \nabla - \nu_i \phi_{t_i} = \frac{1}{h'(t_i)} \begin{pmatrix} -h'(t_i)\nu_i & \mu a(t_i) & \eta \prod_{m \neq i}(t_i - t_m) \\ 1 & \mu h'(t_i)\nu_i & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Assume that $\eta = 0$. Then $l_{i,2}^{(1)}$ has the following form

$$l_{i,2}^{(1)} = \mathbb{C} \begin{pmatrix} -\mu h'(t_i)\nu_i s \\ s \\ t \end{pmatrix}, \quad l_{i,2}^{(2)} = \mathbb{C} \begin{pmatrix} -\mu h'(t_i)\nu_i s \\ \mu s \\ t \end{pmatrix},$$

where $(s : t) \in \mathbb{P}^1$. If $t \neq 0$, then $l_{i,1}^{(1)}$ and $l_{i,1}^{(2)}$ are of the form

$$l_{i,1}^{(1)} = \mathbb{C} \begin{pmatrix} -\mu h'(t_i)\nu_i \\ 1 \\ 0 \end{pmatrix} + \mathbb{C} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad l_{i,1}^{(2)} = \mathbb{C} \begin{pmatrix} -h'(t_i)\nu_i \\ 1 \\ 0 \end{pmatrix} + \mathbb{C} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

If $t = 0$, then $l_{i,1}^{(1)}$ and $l_{i,1}^{(2)}$ are of the form

$$l_{i,1}^{(1)} = \mathbb{C} \begin{pmatrix} -\mu h'(t_i)\nu_i \\ 1 \\ 0 \end{pmatrix} + \mathbb{C} \begin{pmatrix} u \\ 0 \\ v \end{pmatrix}, \quad l_{i,1}^{(2)} = \mathbb{C} \begin{pmatrix} -h'(t_i)\nu_i \\ 1 \\ 0 \end{pmatrix} + \mathbb{C} \begin{pmatrix} u \\ 0 \\ v \end{pmatrix},$$

where $(u : v) \in \mathbb{P}^1$. If $\eta \neq 0$, then $l_{i,*}^{(1)}$ and $l_{i,*}^{(2)}$ are given by the following;

$$l_{i,2}^{(1)} = \mathbb{C} \begin{pmatrix} -\mu h'(t_i)\nu_i \\ 1 \\ 0 \end{pmatrix}, \ l_{i,1}^{(1)} = \mathbb{C} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mathbb{C} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ l_{i,2}^{(2)} = \mathbb{C} \begin{pmatrix} -h'(t_i)\nu_i \\ 1 \\ 0 \end{pmatrix}, \ l_{i,1}^{(1)} = \mathbb{C} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mathbb{C} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

By the above argument, we have

$$C_1 := \overline{\{\eta \neq 0\} \cap \varphi^{-1}(b_{i,j})} \cong \mathbb{P}^1, C_2 := \{t = 0\} \cong \mathbb{P}^1, C_3 := \overline{\{t \neq 0\}} \cong \mathbb{P}^1, \varphi^{-1}(b_{i,j}) = C_1 \cup C_2 \cup C_3,$$

and we find that $C_1 \cap Y_{(\boldsymbol{t},\boldsymbol{\nu})}$, $C_1 \cap C_2$, and $C_2 \cap C_3$ consist of one point, respectively, and $C_1 \cap C_3 = \emptyset$.

Next we consider self-intersection numbers. Let $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and $\nabla_{(q,p,\eta)}$ be the logarithmic connection defined in the proof of Proposition 4.7.4. Put

$$X = \{(q, p, \eta) \in (\mathbb{C} \setminus \{t_m\}_{m \neq i}) \times \mathbb{C} \times \mathbb{C} \mid (t_i - q)\eta - (h'(t_i)(\nu_i - \mathrm{res}_{t_i}(\tfrac{dz}{z-t_3})) - p)^3 = 0\}.$$

Then we can construct an open immersion $X \hookrightarrow N_3(\boldsymbol{t}, \boldsymbol{\nu})$ as in the proof of Proposition 4.7.4. Since X has an A_2-singularity at $(q, p, \eta) = (t_i, h'(t_i)(\nu_i - \mathrm{res}_{t_i}(\frac{dz}{z-t_3})), 0)$, we have $C_2^2 = C_3^2 = -2$, and so $C_1^2 = -1$. □

Proof of Theorem 2.1. We prove (2) first. The morphism (4.4) extends to the morphism

$$\varphi \colon \widehat{M_3^{\boldsymbol{\alpha}}}(0, 0, 2) \longrightarrow \mathbb{P}(\mathcal{E}).$$

Let $\tilde{\mathcal{B}}$ be the reduced induced structure on $\tilde{\mathcal{B}}_0 \cup \tilde{\mathcal{B}}_1 \cup \tilde{\mathcal{B}}_2$. Then we can see that the restriction morphism

$$\varphi \colon \widehat{M_3^{\boldsymbol{\alpha}}}(0, 0, 2) \setminus \varphi^{-1}(\tilde{\mathcal{B}}) \longrightarrow \mathbb{P}(\mathcal{E}) \setminus \tilde{\mathcal{B}}$$

is an isomorphism by Proposition 4.7.1. Any irreducible component of the inverse image $\varphi^{-1}(\tilde{\mathcal{B}})$ has codimension one by Zariski's main theorem. In particular, the inverse image $\varphi^{-1}(\tilde{\mathcal{B}}_2)$ is a Cartier divisor on $\widehat{M_3^{\boldsymbol{\alpha}}}(0,0,2)$, so φ induces the morphism

$$f_2\colon \widehat{M_3^{\boldsymbol{\alpha}}}(0,0,2) \longrightarrow Z_2,$$

where Z_2 is the blow-up of $\mathbb{P}(\mathcal{E})$ along $\tilde{\mathcal{B}}_2$. Let Z_1 is the blow-up of Z_2 along the strict transform of $\tilde{\mathcal{B}}_1$. In the same way, we obtain the morphisms $f_1\colon \widehat{M_3^{\boldsymbol{\alpha}}}(0,0,2) \to Z_1$ and $f\colon \widehat{M_3^{\boldsymbol{\alpha}}}(0,0,2) \to Z$. By Proposition 4.7.1, 4.7.3, 4.7.4, and 4.7.5, the morphism $f_{(\boldsymbol{t},\boldsymbol{\nu})}\colon \widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu}) \to Z_{(\boldsymbol{t},\boldsymbol{\nu})}$ is an isomorphism for any $(\boldsymbol{t},\boldsymbol{\nu}) \in T_3 \times \mathcal{N}$. So f is an isomorphism. Let $(Y_{\leq 1})_{\mathrm{red}}$ be the reduction of $Y_{\leq 1}$. Then the composite

$$Bl_W \circ f \circ \mathrm{PC}^{-1}\colon \overline{M_3^{\boldsymbol{\alpha}}}(0,0,2) \setminus (Y_{\leq 1})_{\mathrm{red}} \longrightarrow S \setminus W$$

is an isomorphism, where $Bl_W\colon Z \to S$ is the blow-up along W. By Hartogs' theorem, the above morphism extends to the morphism $f'\colon \overline{M_3^{\boldsymbol{\alpha}}}(0,0,2) \to S$ and it becomes an isomorphism by Zariski's main theorem. By the construction of f', the diagram

$$\begin{array}{ccc} \widehat{M_3^{\boldsymbol{\alpha}}}(0,0,2) & \xrightarrow{f} & Z \\ {\scriptstyle \mathrm{PC}}\big\downarrow & & \big\downarrow{\scriptstyle Bl_W} \\ \overline{M_3^{\boldsymbol{\alpha}}}(0,0,2) & \xrightarrow{f'} & S \end{array}$$

becomes commutative.

To prove (1), it is sufficient to show that $Y_{\leq 1}$ is reduced. Let us fix $\boldsymbol{t} = (t_i)_{1\leq i\leq 3} \in T_3$. Take a Zariski open subset $U \subset \mathbb{P}^1$ such that $U \cong \operatorname{Spec}\mathbb{C}[z]$ and $t_1, t_2, t_3 \in U \setminus \{0\} \cong \operatorname{Spec}\mathbb{C}[z, \frac{1}{z}]$. Let $a_{12}(u;z)$ and $a_{13}(u,v;z)$ be the quadratic polynomials in z satisfying

$$a_{12}(u;t_i) = u^2(t_i-t_1)^2(t_i-t_2)^2 - 1 - u^2h'(t_i)^2(\nu_{i,0}\nu_{i,1} + \nu_{i,1}\nu_{i,2} + \nu_{i,2}\nu_{i,0})$$

$$a_{13}(u,v;t_i) = \prod_{j=0}^{2}((\nu_{i,j} - \operatorname{res}_{t_i}(\tfrac{dz}{z-t_3}))h'(t_i)u - 1)\prod_{m\neq i}(t_m v - u)$$

for $i = 1,2,3$. Put $E_1 = E_2 = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\mu(u,v) = (t_1v-u)(t_2v-u)(t_3v-u)$

$$\phi_{(u,v)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u^2\mu(u,v) & 0 \\ 0 & 0 & u \end{pmatrix}, \quad \nabla_{(u,v)} = \begin{pmatrix} 0 & \mu(u,v)a_{12}(u;z) & a_{13}(u,v;z) \\ 1 & u^2\mu(u,v)(z-t_1)(z-t_2) - u\mu(u,v) & 0 \\ 0 & vz-u & u(z-t_1)(z-t_2)+1 \end{pmatrix}$$

and

$$X = \left\{ (u,v,\boldsymbol{t},\boldsymbol{\nu}) \in \mathbb{C}^2 \times T_3 \times \mathcal{N} \;\middle|\; \begin{array}{l} (\nu_{i,j} - \operatorname{res}_{t_i}(\frac{dz}{z-t_3}))h'(t_i)u - 1 \neq 0 \text{ for any } 1 \leq i \leq 3 \\ \text{and } 0 \leq j \leq 2 \text{ and } \boldsymbol{t} \in (U\setminus\{0\})^3 \end{array} \right\}.$$

Then we can see that parabolic structures of $(l_*^{(1)})_{(u,v)}$ and $(l_*^{(2)})_{(u,v)}$ of E_1 and E_2, respectively, satisfying $\phi_{(u,v)}((l_{i,j}^{(1)})_{(u,v)}) \subset (l_{i,j}^{(2)})_{(u,v)}$ and $(\operatorname{res}_{t_i}\nabla_{(u,v)} - \nu_{i,j}\phi_{(u,v)})((l_{i,j}^{(1)})_{(u,v)}) \subset (l_{i,j+1}^{(2)})_{(u,v)}$ are unique. So we obtain an open immersion $X \hookrightarrow \overline{M_3^{\boldsymbol{\alpha}}}(0,0,2)$. Since $Y_{\leq 1}$ is defined by $u=0$, $Y_{\leq 1}$ is reduced.

Finally, we prove (3). Let $\rho\colon \mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1}) \to \mathbb{P}^2$ be the blow-down of D_0 and $H_i = \rho(D_i)$. Then there is a morphism $\varphi'\colon \overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu}) \to \mathbb{P}^2$ such that the diagram

$$\begin{array}{ccc} \widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu}) & \xrightarrow{\varphi} & \mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1}) \\ {\scriptstyle \mathrm{PC}}\big\downarrow & & \big\downarrow{\scriptstyle \rho} \\ \overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu}) & \xrightarrow{\varphi'} & \mathbb{P}^2 \end{array} \tag{4.28}$$

commutes. The morphism φ' can be factored into a composition of blow-ups at a point. Let $\hat{H}_i$ be the strict transform of H_i under φ', respectively. Then we have $-K_{\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})} = \hat{H}_1 + \hat{H}_2 + \hat{H}_3$. So it is sufficient to show that $Y_{(\boldsymbol{t},\boldsymbol{\nu})}$ on $\overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$ has multiplicity one along $\hat{H}_i$ for each $i = 1,2,3$, which is equivalent to that the strict transform $\hat{Y}_{(\boldsymbol{t},\boldsymbol{\nu})}$ of $Y_{(\boldsymbol{t},\boldsymbol{\nu})}$ under PC on $\widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$ has multiplicity one along

$\hat{D}_i$ for $i = 1, 2, 3$, where $\hat{D}_i$ is that the strict transform of D_i under φ. Let $b_{12}(p; z)$ be the quadratic polynomial in z satisfying

$$b_{12}(p; t_m) = (t_m - t_1)^2(t_m - t_2)^2 - p^2 - h'(t_m)^2(\nu_{i,0}\nu_{i,1} + \nu_{i,1}\nu_{i,2} + \nu_{i,2}\nu_{i,0})$$

for $m = 1, 2, 3$. Let $b_{13}(q, p; z)$ be the quadratic polynomial in z satisfying $b_{13}(q, p; t_i) = 0$ and

$$(t_m - q)b_{13}(q, p; t_m) = (h'(t_m)(\nu_{m,0} - \mathrm{res}_{t_m}(\tfrac{dz}{z-t_3})) - p)(h'(t_m)(\nu_{m,1} - \mathrm{res}_{t_m}(\tfrac{dz}{z-t_3})) - p)(h'(t_m)(\nu_{m,2} - \mathrm{res}_{t_m}(\tfrac{dz}{z-t_3})) - p)$$

for $m \neq i$. Put

$$f(q, p, \mu) = h'(t_i)(t_i - q) - \mu(h'(t_i)(\nu_{i,0} - \mathrm{res}_{t_i}(\tfrac{dz}{z-t_3})) - p)(h'(t_i)(\nu_{i,1} - \mathrm{res}_{t_i}(\tfrac{dz}{z-t_3})) - p)(h'(t_i)(\nu_{i,2} - \mathrm{res}_{t_i}(\tfrac{dz}{z-t_3})) - p)$$

and

$$X = \{f(q, p, \mu) = 0\} \subset (\mathbb{C} \setminus \{t_m\}_{m\neq i}) \times (\mathbb{C} \setminus \{h'(t_i)(\nu_{i,j} - \mathrm{res}_{t_i}(\tfrac{dz}{z-t_3}))\}_{0\leq j\leq 2}) \times \mathbb{C}.$$

Then the family of parabolic ϕ-connections defined by

$$\phi_\mu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \nabla_{(q,p,\mu)} = \phi_\mu \otimes d + \begin{pmatrix} 0 & \mu b_{12}(p; z) & \mu b_{13}(q, p; z) + \prod_{m\neq i}(z - t_m) \\ 1 & \mu(z - t_1)(z - t_2) - \mu p & 0 \\ 0 & z - q & (z - t_1)(z - t_2) + p \end{pmatrix} \frac{dz}{h(z)} \tag{4.29}$$

gives an open immersion $\iota\colon X \hookrightarrow \widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$. In particular, $\iota^*\hat{Y}_{(\boldsymbol{t},\boldsymbol{\nu})}$ is defined by $\mu = 0$. So $\hat{Y}_{(\boldsymbol{t},\boldsymbol{\nu})}$ on $\widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$ has multiplicity one along D_i. □

4.8 The moduli space of parabolic Higgs bundles and Hitchin fibration

Take $\boldsymbol{t} \in T_3$, $\lambda \in \mathbb{C}$ and $\boldsymbol{\nu} \in \mathcal{N}(0, 0, 2\lambda)$.

Definition 4.8.1. A $\boldsymbol{\nu}$-parabolic ϕ-λ-connection of rank 3 and degree d over $(\mathbb{P}^1, \boldsymbol{t})$ is a collection $(E_1, E_2, \phi, \nabla, l_*^{(1)} = \{l_{i,*}^{(1)}\}_{i=1}^3, l_*^{(2)} = \{l_{i,*}^{(2)}\}_{i=1}^3)$ consisting of the following data:

(1) E_1 and E_2 are vector bundles on $\mathbb{P}^1$ of rank 3 and degree d,

(2) $\phi\colon E_1 \to E_2$ is a homomorphism and $\nabla\colon E_1 \to E_2 \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$ is a λ-twisted logarithmic ϕ-connection, i.e. $\phi(fa) = f\phi(a)$ and $\nabla(fa) = \phi(a) \otimes \lambda df + f\nabla(a)$ for any $f \in \mathcal{O}_{\mathbb{P}^1}, a \in E_1$, and

(3) For each $k = 1, 2$, $l_{i,*}^{(k)}$ is a filtration $E_k|_{t_i} = l_{i,0}^{(k)} \supsetneq l_{i,1}^{(k)} \supsetneq l_{i,2}^{(k)} \supsetneq l_{i,3}^{(k)} = 0$ satisfying $\phi_{t_i}(l_{i,j}^{(1)}) \subset l_{i,j}^{(2)}$ and $(\mathrm{res}_{t_i}(\nabla) - \nu_{i,j}\phi_{t_i})(l_{i,j}^{(1)}) \subset l_{i,j+1}^{(2)}$ for $1 \leq i \leq 3$ and $0 \leq j \leq 2$.

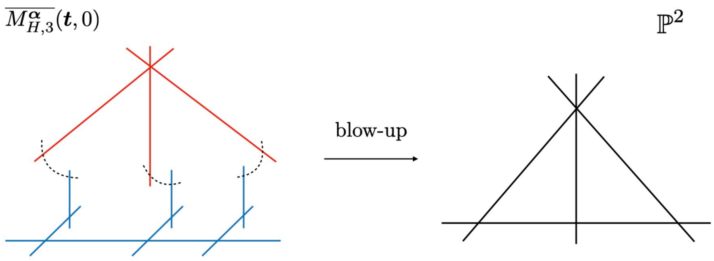
Remark 4.8.2. When $E_1 = E_2$ and $\phi = \mathrm{id}$, a $\boldsymbol{\nu}$-parabolic ϕ-λ-connection is a $\boldsymbol{\nu}$-parabolic λ-connection. When $\lambda = 0$, we call ϕ-λ-connections ϕ-Higgs bundles. If $\phi = \mathrm{id}$, then a $\boldsymbol{\nu}$-parabolic ϕ-Higgs bundle is a $\boldsymbol{\nu}$-parabolic Higgs bundle.

We define the $\boldsymbol{\alpha}$-stability for $\boldsymbol{\nu}$-parabolic ϕ-λ-connections by the same condition of Definition ??. Let $M_3^{\boldsymbol{\alpha}}(\lambda, \boldsymbol{t}, \boldsymbol{\nu})$ and $\overline{M_3^{\boldsymbol{\alpha}}}(\lambda, \boldsymbol{t}, \boldsymbol{\nu})$ be the moduli space of rank 3 $\boldsymbol{\nu}$-parabolic λ-connections with 3 poles and $\boldsymbol{\nu}$-parabolic ϕ-λ-connections, respectively. If $\lambda \neq 0$, then we have $M_3^{\boldsymbol{\alpha}}(\lambda, \boldsymbol{t}, \lambda\boldsymbol{\nu}) \cong M_3^{\boldsymbol{\alpha}}(1, \boldsymbol{t}, \boldsymbol{\nu}) = M_3^{\boldsymbol{\alpha}}(\boldsymbol{t}, \boldsymbol{\nu})$ and $\overline{M_3^{\boldsymbol{\alpha}}}(\lambda, \boldsymbol{t}, \lambda\boldsymbol{\nu}) \cong \overline{M_3^{\boldsymbol{\alpha}}}(1, \boldsymbol{t}, \boldsymbol{\nu}) = \overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$ for any $\boldsymbol{t} \in T_3$ and $\boldsymbol{\nu} \in \mathcal{N}_3(0, 0, 2)$. Put

$$M_{H,3}^{\boldsymbol{\alpha}}(\boldsymbol{t}, \boldsymbol{\nu}) := M_3^{\boldsymbol{\alpha}}(0, \boldsymbol{t}, \boldsymbol{\nu}), \quad \overline{M_{H,3}^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu}) := \overline{M_3^{\boldsymbol{\alpha}}}(0, \boldsymbol{t}, \boldsymbol{\nu})$$

for $\boldsymbol{t} \in T_3$ and $\boldsymbol{\nu} \in \mathcal{N}_3(0, 0, 0)$. In the same way of the case of connections, we can also provide an explicit description of $M_{H,3}^{\boldsymbol{\alpha}}(\boldsymbol{t}, \boldsymbol{\nu})$ and $\overline{M_{H,3}^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$. Specifically, $\overline{M_{H,3}^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$ is obtained by blowing up $\mathbb{P}^2$ at 9 points including infinitely near points such that a cubic curve passing through those 9 points is not unique, which means that the complete linear system of an anti-canonical divisor has dimension one. $M_{H,3}^{\boldsymbol{\alpha}}(\boldsymbol{t}, \boldsymbol{\nu})$ is obtained by removing an anti-canonical divisor of $\overline{M_{H,3}^{\boldsymbol{\alpha}}}(\boldsymbol{t}, \boldsymbol{\nu})$. In the same manner as Lemma 4.5.1, Lemma 4.5.3, and Lemma 4.5.4, we have a normal form of $\boldsymbol{\alpha}$-stable $\boldsymbol{\nu}$-parabolic ϕ-Higgs bundles.

Lemma 4.8.3. Take $\boldsymbol{\alpha} = (\alpha_{i,j})_{1\leq i,j\leq 3}$ and γ such that $|\alpha_{i,j}| \ll 1$ for any $1 \leq i, j \leq 3$ and $\gamma \gg 0$. Let $(E_1, E_2, \phi, \Phi, l_*^{(1)}, l_*^{(2)})$ be a $\boldsymbol{\nu}$-parabolic ϕ-Higgs bundle.



(1) Assume that $\wedge^3\phi \neq 0$. Then ϕ and Φ have the forms

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \Phi = \begin{pmatrix} 0 & a_{12}(z) & a_{13}(z) \\ 1 & -p & 0 \\ 0 & z-q & p \end{pmatrix} \frac{dz}{h(z)}, \tag{4.30}$$

respectively, where $q, p \in \mathbb{C}$ and $a_{12}(z), a_{13}(z)$ are the quadratic polynomial in z satisfying

$$a_{12}(t_i) = -h'(t_i)^2(\nu_{i,0}\nu_{i,1} + \nu_{i,1}\nu_{i,2} + \nu_{i,2}\nu_{i,0}) - p^2, \tag{4.31}$$

$$(t_i - q)a_{13}(t_i) = \prod_{j=0}^{2}(h'(t_i)\nu_{i,j} - p) \tag{4.32}$$

for any $i = 1, 2, 3$.

(2) Assume that $\operatorname{rank}\phi = 2$. Then ϕ and Φ have the forms

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \Phi = \begin{pmatrix} 0 & 0 & \prod_{j\neq i}(z-t_j) \\ 1 & 0 & 0 \\ 0 & z-t_i & p \end{pmatrix} \frac{dz}{h(z)}, \tag{4.33}$$

respectively.

(3) Assume that $\operatorname{rank}\phi = 1$. Then ϕ and Φ have the forms

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \Phi = \begin{pmatrix} 0 & \prod_{j\neq i}(z-t_j) & 0 \\ 1 & 0 & 0 \\ 0 & z-q & z-t_i \end{pmatrix} \frac{dz}{h(z)}, \tag{4.34}$$

respectively.

(4) Assume that $\phi = 0$. Then $(E_1, E_2, \phi, \Phi, l_*^{(1)}, l_*^{(2)})$ is $\boldsymbol{\alpha}$-unstable.

Take a $\boldsymbol{\nu}$-parabolic ϕ-Higgs bundle $\boldsymbol{E} = (E_1, E_2, \phi, \Phi, l_*^{(1)}, l_*^{(2)})$. For each $0 \leq i \leq 3$, let $c_i(\boldsymbol{E}) \in H^0(\mathbb{P}^1, \mathcal{H}om(\wedge^3 E_1, \wedge^3 E_2) \otimes (\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})))^{\otimes i}) \cong H^0(\mathbb{P}^1, (\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})))^{\otimes i})$ be the coefficient of the polynomial $\wedge^3(t\phi - \Phi)$ in t, that is,

$$\wedge^3(t\phi - \Phi) = c_0(\boldsymbol{E})t^3 + c_1(\boldsymbol{E})t^2 + c_2(\boldsymbol{E})t + c_3(\boldsymbol{E}).$$

In other words, $c_i(\boldsymbol{E})$ is the homomorphism defined by

$$c_0(\boldsymbol{E})(v_1 \wedge v_2 \wedge v_3) = \phi(v_1) \wedge \phi(v_2) \wedge \phi(v_3),$$

$$c_1(\boldsymbol{E})(v_1 \wedge v_2 \wedge v_3) = -(\Phi(v_1) \wedge \phi(v_2) \wedge \phi(v_3) + \phi(v_1) \wedge \Phi(v_2) \wedge \phi(v_3) + \phi(v_1) \wedge \phi(v_2) \wedge \Phi(v_3)),$$

$$c_2(\boldsymbol{E})(v_1 \wedge v_2 \wedge v_3) = \phi(v_1) \wedge \Phi(v_2) \wedge \Phi(v_3) + \Phi(v_1) \wedge \phi(v_2) \wedge \Phi(v_3) + \Phi(v_1) \wedge \Phi(v_2) \wedge \phi(v_3),$$

$$c_3(\boldsymbol{E})(v_1 \wedge v_2 \wedge v_3) = -\Phi(v_1) \wedge \Phi(v_2) \wedge \Phi(v_3),$$

where $v_1, v_2, v_3 \in E_1$. Put $\mathcal{H} = \oplus_{k=0}^{3} H^0((\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})))^{\otimes k})$. Let us define the morphism $\overline{\mathrm{Hit}}$ by

$$\overline{\mathrm{Hit}} \colon \overline{M^{\boldsymbol{\alpha}}_{H,3}}(\boldsymbol{t}, \boldsymbol{\nu}) \longrightarrow \mathbb{P}\mathcal{H}, \quad x \longmapsto [(c_0(x), c_1(x), c_2(x), c_3(x))],$$

which is well-defined by Lemma 4.8.3. Here for a nonzero element $\sigma \in \mathcal{H}$, $[\sigma]$ is the homothety class of σ. The restriction $\overline{\mathrm{Hit}}$ on $M^{\boldsymbol{\alpha}}_{H,3}(\boldsymbol{t}, \boldsymbol{\nu})$ is just the parabolic Hitchin map. We can see that for any

$x \in \overline{M^{\boldsymbol{\alpha}}_{H,3}}(\boldsymbol{t},\boldsymbol{\nu})$, $c_1(x)=0, c_2(x)=(\wedge^3\phi)f(\boldsymbol{\nu};z)$, and $c_3(x)$ has the form $bh(z)+(\wedge^3\phi)g(\boldsymbol{\nu};z)$ by Lemma 4.4.2, where $b \in \mathbb{C}$, and $f(\boldsymbol{\nu};z)$ and $g(\boldsymbol{\nu};z)$ are the quadratic polynomials satisfying the condition

$$f(\boldsymbol{\nu};t_i)=\nu_{i,0}\nu_{i,1}+\nu_{i,1}\nu_{i,2}+\nu_{i,2}\nu_{i,0}, \quad g(\boldsymbol{\nu};t_i)=-\nu_{i,0}\nu_{i,1}\nu_{i,2}$$

for $i=1,2,3$. So the image $\overline{\mathrm{Hit}}(\overline{M^{\boldsymbol{\alpha}}_{H,3}}(\boldsymbol{t},\boldsymbol{\nu}))$ is the locus defined by

$$\left\{\left[\left(a,0,af(\boldsymbol{\nu};z)(\tfrac{dz}{h(z)})^{\otimes 2},(bh(z)+ag(\boldsymbol{\nu};z))(\tfrac{dz}{h(z)})^{\otimes 3}\right)\right] \;\middle|\; (a:b)\in\mathbb{P}^1\right\} \subset \mathbb{P}\mathcal{H}.$$

Let us consider the fiber $\overline{\mathrm{Hit}}^{-1}(a:b)$. When $a=0$, $\overline{\mathrm{Hit}}^{-1}(a:b)$ is the boundary of $M^{\boldsymbol{\alpha}}_{H,3}(\boldsymbol{t},\boldsymbol{\nu})$. Assume that $a=1$. The form (4.30) provides an open immersion $\mathbb{P}^1\setminus\{t_1,t_2,t_3,\infty\}\times\mathbb{C}\hookrightarrow M^{\boldsymbol{\alpha}}_{H,3}(\boldsymbol{t},\boldsymbol{\nu})$. Since $\det\Phi=((z-q)a_{13}(z)-pa_{12}(z))(\frac{dz}{h(z)})^{\otimes 3}$, the fiber $\overline{\mathrm{Hit}}^{-1}(1:b)$ is the locus defined by the equation

$$pa_{12}(q)=bh(q)+g(\boldsymbol{\nu};q) \tag{4.35}$$

on $\mathbb{P}^1\setminus\{t_1,t_2,t_3,\infty\}\times\mathbb{C}$. Consider the case $\boldsymbol{\nu}=0$. Since $f(0;z)=g(0;z)=0$, we can replace $\mathbb{P}\mathcal{H}$ with $\mathbb{P}(H^0(\mathcal{O}^1_{\mathbb{P}})\oplus H^0((\Omega^1_{\mathbb{P}^1})^3(2D(\boldsymbol{t}))))\cong\mathbb{P}^1$, and the equation (4.35) becomes

$$p^3+b(q-t_1)(q-t_2)(q-t_3)=0.$$

So we obtain the following proposition.

Proposition 4.8.4. The morphism $\overline{\mathrm{Hit}}\colon \overline{M^{\boldsymbol{\alpha}}_{H,3}}(\boldsymbol{t},0)\longrightarrow\mathbb{P}(H^0(\mathcal{O}^1_{\mathbb{P}})\oplus H^0((\Omega^1_{\mathbb{P}^1})^3(2D(\boldsymbol{t}))))$ is an elliptic fibration and has singular fibers of type IV^* and IV over $(1:0)$ and $(0:1)$, respectively.

Chapter 5

Moduli space of parabolic bundles and parabolic connections

5.1 Rank 2 case

In this section, we describe the birational structure of moduli spaces of rank 2 parabolic connections. Let C be an irreducible smooth projective curve over $\mathbb{C}$ of genus $g \geq 1$ and $\boldsymbol{t} = (t_i)_{1\leq i\leq n}$ be n distinct points of C. Let us fix a line bundle L with degree $d := 2g-1$. Then we have $H^1(C, L) = \{0\}$ and by Riemann-Roch theorem, $\dim H^0(C, L) = d + 1 - g = g$. Let us fix a weight $\boldsymbol{\alpha} = \{\alpha_{i,1}, \alpha_{i,2}\}_{1\leq i\leq n}$ and set $w_i = \alpha_{i,2} - \alpha_{i,1}$.

5.1.1 The distinguished open subset of the moduli space of parabolic bundles

Lemma 5.1.1. Assume that $\sum_{i=1}^{n} w_i < 1$. For a quasi-parabolic bundle (E, l_*) of rank 2 and odd degree, the following conditions are equivalent:

(i) (E, l_*) is $\boldsymbol{\alpha}$-semistable.

(ii) (E, l_*) is $\boldsymbol{\alpha}$-stable.

(iii) E is stable.

Proof. If (E, l_*) is $\boldsymbol{\alpha}$-semistable but not $\boldsymbol{\alpha}$-stable, then there is a sub line bundle $F \subset E$ such that

$$\deg E - 2\deg F = \sum_{F|_{t_i}=l_1^{(i)}} w_i - \sum_{F|_{t_i}\neq l_1^{(i)}} w_i.$$

The left hand side is odd, but

$$\left| \sum_{F|_{t_i}=l_1^{(i)}} w_i - \sum_{F|_{t_i}\neq l_1^{(i)}} w_i \right| \leq \sum_{i=1}^{n} w_i < 1. \tag{5.1}$$

It is a contradiction. So conditions (i) and (ii) are equivalent.

If (E, l_*) is $\boldsymbol{\alpha}$-stable, then for all sub line bundle $F \subset E$, the inequality

$$2\deg F < \deg E + \sum_{F|_{t_i}\neq l_1^{(i)}} w_i - \sum_{F|_{t_i}=l_1^{(i)}} w_i \tag{5.2}$$

holds. From (5.1), it follows that

$$\deg E - 1 < \deg E + \sum_{F|_{t_i}\neq l_1^{(i)}} w_i - \sum_{F|_{t_i}=l_1^{(i)}} w_i < \deg E + 1,$$

and so we have $2\deg F \leq \deg E$ by (5.2). Since $\deg E$ is odd, we obtain

$$2\deg F \leq \deg E - 1 < \deg E.$$

Hence, E is stable. Conversely, if E is stable, then we can prove that (E, l_*) is $\boldsymbol{\alpha}$-stable by the above argument. □

Lemma 5.1.2. Suppose that a vector bundle E on C satisfies the following conditions:

(i) E is an extension of L by $\mathcal{O}_C$, that is, E fits into an exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow L \longrightarrow 0.$$

(ii) $\dim H^0(C, E) = 1$.

Then E is stable.

Proof. If E is not stable, then there exists a sub line bundle $F \subset E$ such that $\deg F \geq g$. Since $\dim H^0(C, F) - \dim H^1(C, F) = \deg F + 1 - g \geq 1$, we have $\dim H^0(C, F) \geq 1$, hence we have an inclusion $\mathcal{O}_C \hookrightarrow F$. By assumption (ii), we have a unique inclusion $\mathcal{O}_C \subset F \subset E$, and this inclusion induces the injection $F/\mathcal{O}_C \hookrightarrow E/\mathcal{O}_C \simeq L$. Since L is torsion free, one concludes that $F/\mathcal{O}_C = 0$, that is, $F \simeq \mathcal{O}_C$. This contradicts the fact that $\deg F \geq g \geq 1$. □

Proposition 5.1.3. For an element $b \in H^1(C, L^{-1})$, let

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E_b \longrightarrow L \longrightarrow 0 \tag{5.3}$$

be the exact sequence obtained by the extension of L by $\mathcal{O}_C$ with the extension class b. Then $\dim H^0(C, E_b) = 1$ if and only if the natural cup-product map

$$\langle\ , b\rangle \colon H^0(C, L) \longrightarrow H^1(C, \mathcal{O}_C)$$

is an isomorphism. Moreover, $\dim H^0(C, E_b) = 1$ for a generic element $b \in H^1(C, L^{-1})$.

Proof. Since $H^1(C, L) = \{0\}$, from the exact sequence (5.3), we obtain the following exact sequence

$$0 \longrightarrow H^0(C, \mathcal{O}_C) \longrightarrow H^0(C, E_b) \longrightarrow H^0(C, L) \stackrel{\langle\ ,b\rangle}{\longrightarrow} H^1(C, \mathcal{O}_C) \longrightarrow H^1(C, E_b) \longrightarrow 0$$

Here we note that by definition of the extension with b the connecting homomorphism $\delta \colon H^0(C, L) \to H^1(C, \mathcal{O}_C)$ is given by $\langle\ , b\rangle$. Since $\dim H^0(C, E_b) = \dim H^1(C, E_b) + \deg E_b + 2(1-g) = \dim H^1(C, E_b) + 1$, the first assertion follows from the above exact sequence.

We show the second assertion. We set

$$Z := \{(s, b) \in H^0(C, L) \times H^1(C, L^{-1}) \mid \langle s, b\rangle = 0\}.$$

Since $\deg L \otimes \Omega^1_C = 4g - 3 \geq 2g - 1$, we have $H^1(C, L \otimes \Omega^1_C) = \{0\}$ and

$$\dim H^1(C, L^{-1}) = \dim H^0(C, L \otimes \Omega^1_C)^* = \deg L \otimes \Omega^1_C + 1 - g = 3g - 2.$$

Hence, it is sufficient to show that $\dim Z = 3g - 2$. In fact, if $\dim Z = 3g - 2$, then for generic $b \in H^1(C, L^{-1})$, we have $\dim q^{-1}(b) = 0$ and it means $q^{-1}(b) = \{(0, b)\}$. Here $q \colon Z \to H^1(C, L^{-1})$ is the projection.

Let $p \colon Z \to H^0(C, L)$ be the projection. We show that for any $s \in H^0(C, L) \setminus \{0\}$, $\dim p^{-1}(s) = 2g - 2$.

A section $\sigma \in H^0(C, \Omega^1_C)$ induces the diagram

$$\begin{array}{ccc} H^0(C, L) \times H^1(C, L^{-1}) & \xrightarrow{\langle\ ,\ \rangle} & H^1(C, \mathcal{O}_C) \\ \downarrow{\scriptstyle \otimes\sigma\times\mathrm{id}} & & \downarrow{\scriptstyle \otimes\sigma} \\ H^0(C, L \otimes \Omega^1_C) \times H^1(C, L^{-1}) & \xrightarrow{\langle\ ,\ \rangle'} & H^1(C, \Omega^1_C) \end{array}$$

where the above and below map are natural cup-products and the left and right map are natural maps induced by σ. Note that $\langle\ ,\ \rangle'$ is nondegenerate. Set $s \in H^0(C, L) \setminus \{0\}$. For $b \in H^1(C, L^{-1})$, $\langle s, b\rangle = 0$ if and only if for all $\sigma \in H^0(C, \Omega^1_C)$, $\langle s \otimes \sigma, b\rangle' = \langle s, b\rangle \otimes \sigma = 0$. Since the set

$$\{s \otimes \sigma \mid \sigma \in H^0(C, \Omega^1_C)\} \simeq H^0(C, \Omega^1_C)$$

is a g dimensional subspace of $H^0(C, L \otimes \Omega^1_C)$ and by the nondegeneracy of $\langle\ ,\ \rangle'$, the set

$$\{b \in H^1(C, L^{-1}) \mid \langle s, b\rangle = 0\}$$

defines a $2g - 2$ dimensional subspace of $H^1(C, L^{-1})$. We therefore obtain $\dim p^{-1}(s) = 2g - 2$. So we conclude $\dim Z = 3g - 2$. □

Proposition 5.1.4. Let $\sum_{i=1}^{n} w_i < 1$. Let $V_0 \subset P^{\boldsymbol{\alpha}}(L) = P^{\boldsymbol{\alpha}}_{(C,\boldsymbol{t})}(L)$ be the subset which consists of all elements $(E, l_*) \in P^{\boldsymbol{\alpha}}(L)$ satisfying following conditions:

(i) E is an extension of L by $\mathcal{O}_C$.

(ii) $\dim H^0(C, E) = 1$.

(iii) For any i, $\mathcal{O}_C|_{t_i} \neq l_{i,1}$. Here $\mathcal{O}_C|_{t_i}$ is identified with the image by an injection $\mathcal{O}_C|_{t_i} \hookrightarrow E|_{t_i}$.

Then V_0 is a nonempty Zariski open subset of $P^{\boldsymbol{\alpha}}(L)$.

Proof. Let E be a vector bundle on C satisfying conditions (i) and (ii). Then we have $\det E \simeq L$ from (i) and E is stable by Lemma 5.1.2. Let M_L denote the moduli space of rank 2 stable vector bundles on C with the determinant L.

First, we show that the subset of M_L consisting of vector bundles satisfying (i) and (ii) is open. Since rank and degree are coprime, M_L has the universal family $\mathcal{E}$. Set

$$V = \{x \in M_L \mid \dim H^0(C, \mathcal{E}|_{C\times x}) = 1\},$$

then V is an open subset of M_L by the upper semicontinuity of dimensions. Let $q\colon C \times V \to V$ be the natural projection. By Corollary 12.9 in [Ha], $q_*\mathcal{E}$ is an invertible sheaf on V and for any $x \in V$, $(q_*\mathcal{E})|_x$ is naturally isomorphic to $H^0(C, \mathcal{E}|_{C\times x})$. Hence $q^*q_*\mathcal{E}$ is an invertible sheaf on $C \times V$ and a natural homomorphism $\iota\colon q^*q_*\mathcal{E} \to \mathcal{E}$ is injective. By definition, for any $x \in V$, we have $(q^*q_*\mathcal{E})|_{C\times x} \simeq H^0(C, \mathcal{E}|_{C\times x}) \otimes_{\mathbb{C}} \mathcal{O}_{C\times x} \simeq \mathcal{O}_C$ and $\iota|_{c\times x}\colon \mathcal{O}_C \simeq (q^*q_*\mathcal{E})|_{C\times x} \to \mathcal{E}|_{C\times x}$ is not zero. Set

$$Y = \{(c, x) \in C \times V \mid \iota|_{(c,x)}\colon \mathcal{O}_C|_c \simeq (q^*q_*\mathcal{E})|_{(c\times x)} \to \mathcal{E}|_{(c,x)} \text{ is zero.}\}$$

and $V' = V \setminus q(Y)$, then Y is a closed subset of $C \times V$ and V' is an open subset of V. If $x \in V'$, then we obtain $\mathcal{E}|_{C\times x}/\mathcal{O}_C \simeq L$, that is, $\mathcal{E}|_{C\times x}$ is an extension of L by $\mathcal{O}_C$. Therefore, V' consists of all isomorphism classes of vector bundles satisfying the conditions (i) and (ii), and V' is an open subset of M_L. Moreover, V' is not empty by Proposition 5.1.3.

Second, we prove that V_0 is open. By Lemma 5.1.1, we obtain

$$P^{\boldsymbol{\alpha}}(L) \simeq \mathbb{P}(\mathcal{E}|_{t_1\times M_L}) \times_{M_L} \mathbb{P}(\mathcal{E}|_{t_2\times M_L}) \times_{M_L} \cdots \times_{M_L} \mathbb{P}(\mathcal{E}|_{t_n\times M_L}).$$

For each t_i, by projectivization of $\iota|_{t_i\times V'}\colon (q^*q_*\mathcal{E})|_{t_i\times V'} \to \mathcal{E}|_{t_i\times V'}$, we obtain a morphism $\hat{l}_{i,1}\colon V' \to \mathbb{P}(\mathcal{E}|_{t_i\times V'})$ such that for all $x \in V'$, $\hat{l}_{i,1}(x)$ is the point associated with the image by the immersion $\mathcal{O}_C \hookrightarrow \mathcal{E}|_{C\times x}$ at t_i. Let $\varpi\colon P^{\boldsymbol{\alpha}}(L) \to M_L$ be the natural forgetful map and $p_i\colon P^{\boldsymbol{\alpha}}(L) \to \mathbb{P}(\mathcal{E}|_{t_i\times M_L})$ be the natural projection. Set

$$V_0 = \varpi^{-1}(V') \setminus \bigcup_{i=1}^{n} p_i^{-1}(\hat{l}_{i,1}(V')).$$

Then V_0 is an open subset of $P^{\boldsymbol{\alpha}}(L)$ and V_0 is the set of all isomorphism classes of parabolic bundles satisfying (i), (ii), and (iii). □

We introduce another expression of V_0. For $b \in H^1(C, L^{-1})$, let

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E_b \longrightarrow L \longrightarrow 0$$

be the exact sequence associated with b. We set

$$U := \{b \in H^1(C, L^{-1}) \mid \dim H^0(C, E_b) = 1\}$$

and then U is an open subset and $0 \notin U$ by Proposition 5.1.3.

The natural homomorphism $\psi\colon H^1(C, L^{-1}(-D)) \to H^1(C, L^{-1})$ induces the morphism

$$\tilde{\psi}\colon \mathbb{P}H^1(C, L^{-1}(-D)) \setminus \mathbb{P}\mathrm{Ker}\,\psi \longrightarrow \mathbb{P}H^1(C, L^{-1}).$$

Let $\tilde{U} \subset \mathbb{P}H^1(C, L^{-1})$ be the open subset associated with U and $\tilde{V} = \tilde{\psi}^{-1}(\tilde{U})$.

Suppose that $(E, l_*) \in P^{\boldsymbol{\alpha}}(L)$ satisfies conditions (i), (ii), and (iii) of Proposition 5.1.4. Let $b \in H^1(C, L^{-1})$ be the element associated with an exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow L \longrightarrow 0$$

and $[b]$ be the point in $\mathbb{P}H^1(C, L^{-1})$ associated with the subspace generated by b. By assumption, we have $b \in U$. Let $\{U_i\}_i$ be an open covering of C and $(c_{ij})_{i,j}, c_{ij} = c_i/c_j$ be transition functions of L over $\{U_i\}_i$. Let e_1^i be the restriction of a global section $\mathcal{O}_C \hookrightarrow E$ on U_i and e_2^i be a local section of E on U_i whose image by the natural map $E \to E|_{t_i}$ generates $l_{k,1}$ at each $t_k \in U_i$. For generators e_1^i and e_2^i, transition matrices $M_{i,j}$ is denoted by

$$M_{ij} = \begin{pmatrix} 1 & b'_{ij} \\ 0 & c_{ij} \end{pmatrix}$$

where $b' = (b'_{ij}c_j)_{i,j} \in H^1(C, L^{-1}(-D))$. Then we have $\tilde{\psi}([b']) = [b]$, and so $[b'] \in \tilde{V}$. By using the above argument, we can correspond $[b'] \in \tilde{V}$ to an isomorphism class of a parabolic bundle satisfying all conditions of Proposition 5.1.4. Thus we conclude $V_0 \simeq \tilde{V}$.

Putting together the above argument, we get the following proposition.

Proposition 5.1.5. Suppose that $\sum_{i=1}^n w_i < 1$. Let $V_0 \subset P^{\boldsymbol{\alpha}}(L)$ be the subset defined in Proposition 5.1.4. Then there is an open immersion $V_0 \hookrightarrow \mathbb{P}H^1(C, L^{-1}(-D))$.

5.1.2 The apparent map

Let us fix $\boldsymbol{\nu} = (\nu_{i,j})_{j=0,1}^{i=1,\dots,n} \in \mathcal{N}_{n,2}(d)$ and a $\mathrm{tr}(\boldsymbol{\nu})$-parabolic connection ∇_L over L. Let V_0 be the open subset of $P^{\boldsymbol{\alpha}}(L)$ defined in Proposition 5.1.4. We set

$$\begin{aligned} M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L)) &:= M^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(2, \boldsymbol{\nu}, (L, \nabla_L)), \\ M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))^0 &:= \{(E, \nabla, l_*) \in M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L)) \mid (E, l_*) \in V_0\}. \end{aligned}$$

For each $(E, \nabla, l_*) \in M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))^0$, E has the unique sub line bundle which is isomorphic to the trivial line bundle. We define the section $\varphi_\nabla \in H^0(C, L \otimes \Omega_C^1(D))$ by the composite

$$\mathcal{O}_C \hookrightarrow E \xrightarrow{\nabla} E \otimes \Omega_C^1(D) \to E/\mathcal{O}_C \otimes \Omega_C^1(D) \simeq L \otimes \Omega_C^1(D).$$

Suppose that $\varphi_\nabla = 0$, i.e. $\nabla(\mathcal{O}_C) \subset \mathcal{O}_C \otimes \Omega_C^1(D)$. Then we obtain $\sum_{i=1}^n \nu_{i,0} = 0$ by Fuchs relation because $\mathcal{O}_C|_{t_i} \cap l_{i,1} = \{0\}$ for any i. So if $\sum_{i=1}^n \nu_{i,0} \neq 0$, then $\varphi_\nabla \neq 0$ and we therefore define the morphism

$$\begin{aligned} \mathrm{App}\colon M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))^0 &\longrightarrow \mathbb{P}H^0(C, L \otimes \Omega_C^1(D)) \simeq |L \otimes \Omega_C^1(D)|. \\ (E, \nabla, l_*) &\longmapsto [\varphi_\nabla] \end{aligned}$$

Here $[\varphi_\nabla]$ is the point in $\mathbb{P}H^0(C, L \otimes \Omega_C^1(D))$ associated with the subspace of $H^0(C, L \otimes \Omega_C^1(D))$ generated by φ_∇. We can extend this map to the rational map

$$\mathrm{App}\colon M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L)) \cdots \to |L \otimes \Omega_C^1(D)|.$$

5.1.3 Parabolic bundles and the apparent singularities

Let

$$\mathrm{Bun}\colon M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))^0 \longrightarrow V_0$$

be the forgetful map which sends (E, ∇, l_*) to (E, l_*). We can extend this map to the rational map

$$\mathrm{Bun}\colon M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L)) \cdots \to P^{\boldsymbol{\alpha}}(L).$$

Let

$$\langle\ ,\ \rangle\colon H^0(C, L \otimes \Omega_C^1(D)) \times H^1(C, L^{-1}(-D)) \longrightarrow H^1(C, \Omega_C^1)$$

be the natural cup-product. This cup-product is nondegenerate.

Theorem 5.1.6. Assume that $\sum_{i=1}^n \nu_{i,0} \neq 0$ and $\sum_{i=1}^n w_i < 1$. Let us define the subvariety $\Sigma \subset \mathbb{P}H^0(C, L \otimes \Omega_C^1(D)) \times \mathbb{P}H^1(C, L^{-1}(-D))$ by

$$\Sigma = \{([s], [b]) \mid \langle s, b \rangle = 0\}.$$

Then the map

$$\mathrm{App} \times \mathrm{Bun}\colon M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))^0 \longrightarrow (\mathbb{P}H^0(C, L \otimes \Omega_C^1(D)) \times V_0) \setminus \Sigma$$

is an isomorphism. Therefore, the rational map

$$\mathrm{App} \times \mathrm{Bun}\colon M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L)) \ \cdots \to |L \otimes \Omega_C^1(D)| \times P^{\boldsymbol{\alpha}}(L)$$

is birational. In particular, $M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))$ is a rational variety.

Before showing this theorem, we prove the following lemma.

Lemma 5.1.7. Let $(E, l_*) \in V_0$ and $b \in H^1(C, L^{-1})$ be an element associated with an extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow L \longrightarrow 0.$$

Then the natural cup-product map

$$\langle\ , b\rangle' \colon H^0(C, \Omega^1_C) \longrightarrow H^1(C, L^{-1} \otimes \Omega^1_C)$$

is an isomorphism. In particular, for an element $b' \in H^1(C, L^{-1}(-D))$ associated with (E, l_*), the composite of the natural cup-product map and the natural homomorphism

$$H^0(C, \Omega^1_C) \xrightarrow{\langle\ , b'\rangle''} H^1(C, L^{-1}(-D) \otimes \Omega^1_C) \longrightarrow H^1(C, L^{-1} \otimes \Omega^1_C)$$

is also an isomorphism.

Proof. By Serre duality, we have $H^0(C, \Omega^1_C) \simeq H^1(C, \mathcal{O}_C)^*$ and $H^1(C, L^{-1} \otimes \Omega^1_C) \simeq H^0(C, L)^*$. So it suffices to prove that the natural cup-product map

$$\langle\ , b\rangle''' \colon H^0(C, L) \longrightarrow H^1(C, \mathcal{O}_C)$$

is an isomorphism, and it is nothing but the first assertion of Proposition 5.1.3.

The second assertion follows from the following diagram.

$$\begin{array}{ccc}
H^0(C, \Omega^1_C) \times H^1(C, L^{-1}(-D)) & \xrightarrow{\langle\ ,\ \rangle''} & H^1(C, L^{-1}(-D) \otimes \Omega^1_C) \\
\downarrow & & \downarrow \\
H^0(C, \Omega^1_C) \times H^1(C, L^{-1}) & \xrightarrow{\langle\ ,\ \rangle'} & H^1(C, L^{-1} \otimes \Omega^1_C)
\end{array}$$

□

Proof. (Proof of Theorem 5.1.6)

Firstly, we show that for any $\gamma \in H^0(C, L \otimes \Omega^1_C(D))$ and $b \in H^1(C, L^{-1}(-D))$ such that the quasi-parabolic bundle (E, l_*) associated with b is in V_0, there exist a unique complex number λ and a unique $\lambda\boldsymbol{\nu}$-parabolic λ-connection (E, ∇, l_*) such that $\mathrm{tr}\nabla = \lambda\nabla_L$ and $\varphi_\nabla = \gamma$.

Let $\{U_i\}_i$ be an open covering of C and $(c_{ij})_{i,j}, c_{ij} = c_i/c_j$ be transition functions of L over $\{U_i\}_i$. Let e^i_1 be the restriction of a global section $\mathcal{O}_C \hookrightarrow E$ on U_i and e^i_2 be a local section of E on U_i whose image $\bar{e}^i_2$ by the natural map $E \to E|_{t_i}$ generates $l_{k,1}$ at each $t_k \in U_i$. For local generators e^i_1 and e^i_2, we can denote transition matrices of E by

$$M_{ij} = \begin{pmatrix} 1 & b_{ij} \\ 0 & c_{ij} \end{pmatrix},$$

where $b = (b_{ij}c_j)_{i,j} \in H^1(C, L^{-1}(-D))$ is the cocycle corresponding to an extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow L \longrightarrow 0.$$

A logarithmic λ-connection ∇ is given in U_i by $\lambda d + A_i$

$$A_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \in M_2(\Omega^1_C(D)(U_i))$$

with the compatibility condition

$$\lambda dM_{ij} + A_i M_{ij} = M_{ij} A_j$$

on each intersection $U_i \cap U_j$. By using elements of matrices, this condition is written by

$$\begin{cases} \dfrac{\gamma_i}{c_i} - \dfrac{\gamma_j}{c_j} = 0 \\ \alpha_i - \alpha_j = b_{ij}\gamma_j \\ \delta_i - \delta_j = -b_{ij}\gamma_j - \lambda \dfrac{dc_{ij}}{c_{ij}} \\ c_i\beta_i - c_j\beta_j = -(\lambda c_j db_{ij} + (b_{ij}c_j)(\alpha_i - \delta_j)). \end{cases} \tag{5.4}$$

If (E, ∇, l_*) is a $\lambda\boldsymbol{\nu}$-parabolic λ-connection, then for each point t_i, ∇ satisfies the residual condition

$$\mathrm{res}_{t_k}(A_i) = \begin{pmatrix} \lambda\nu_{k,0} & 0 \\ * & \lambda\nu_{k,1} \end{pmatrix} \tag{5.5}$$

at each $t_k \in U_i$ because $\vec{e}_2^{\,i}$ generates $l_{k,1}$. ∇_L is denoted in U_i by $d + \omega_i$ with the compatibility condition

$$dc_{ij} + c_{ij}\omega_i = c_{ij}\omega_j \tag{5.6}$$

on each $U_i \cap U_j$. If $\mathrm{tr}\nabla = \lambda\nabla_L$, then the equation

$$\alpha_i + \delta_i = \lambda\omega_i \tag{5.7}$$

holds. When ∇ is denoted in U_i by $\lambda d + A_i$, we have $\varphi_\nabla = (\gamma_i/c_i)_i \in H^0(C, L \otimes \Omega^1_C(D))$. So if $\varphi_\nabla = \gamma$, then we have

$$(\gamma_i/c_i)_i = \gamma. \tag{5.8}$$

We show that there exist $\lambda \in \mathbb{C}$ and $\alpha_i, \beta_i, \gamma_i, \delta_i \in \Omega^1_C(D)(U_i)$ satisfying the conditions (5.4), (5.5), (5.7) and (5.8) uniquely.

Step 1: we find γ_i. From (5.8), we have to set $\gamma_i = c_i\gamma$.

Step 2: we find α_i. Fix a section $\alpha_i^0 \in \Omega^1_C(D)(U_i)$ which has the residue data $\mathrm{res}_{t_k}(\alpha_i^0) = \nu_{k,0}$ at each $t_k \in U_i$. The cocycle $(\alpha_i^0 - \alpha_j^0)_{i,j}$ defines an element of $H^1(C, \Omega^1_C)$. If $(\alpha_i^0 - \alpha_j^0)_{i,j}$ is zero in $H^1(C, \Omega^1_C)$, then there exist sections $\tilde{\alpha}_i \in \Omega^1_C(U_i)$ on each i such that $\alpha_i^0 - \alpha_j^0 = \tilde{\alpha}_i - \tilde{\alpha}_j$ for any i, j. $(\alpha_i^0 - \tilde{\alpha}_i)_i$ defines a global logarithmic 1-form whose sum of residues $\sum_{i=1}^n \nu_{i,0}$ is not zero. This contradicts the residue theorem. Therefore, the cocycle $(\alpha_i^0 - \alpha_j^0)_{i,j}$ is a generator of $H^1(C, \Omega^1_C)$ and there is a unique complex number λ such that $\lambda(\alpha_i^0 - \alpha_j^0)_{i,j} = (b_{ij}\gamma_j)_{i,j}$. Let $\tilde{\alpha}_i \in \Omega^1_C(U_i)$ be a section such that

$$\tilde{\alpha}_i - \tilde{\alpha}_j = b_{ij}\gamma_j - \lambda(\alpha_i^0 - \alpha_j^0)$$

for any i, j. Set $\alpha_i = \lambda\alpha_i^0 + \tilde{\alpha}_i$, then $(\alpha_i)_i$ is a solution of the second equation of (5.4) and has the residue data $\mathrm{res}_{t_k}(\alpha_i) = \lambda\nu_{k,0}$. Note that $(\alpha_i)_i$ is still not uniquely determined. Actually, the difference of two solutions of the second equation of (5.4) having the same residue data defines a global 1-form and now $\dim H^0(C, \Omega^1_C) \geq g \geq 1$.

Step 3: we find δ_i. From (5.7), we have to set $\delta_i = \lambda\omega_i - \alpha_i$. It is clear that $(\delta_i)_i$ is a solution of the third equation of (5.4) and has the residue data $\mathrm{res}_{t_k}(\delta_i) = \lambda\nu_{k,1}$. δ_i is uniquely determined by α_i.

Step 4: we find β_i and show that α_i is uniquely determined. From the cocycle condition of $(b_{ij}c_j)_{i,j}$ and the first, second, and third equations of (5.4), we obtain

$$\begin{aligned}
&(\lambda c_j db_{ij} + (b_{ij}c_j)(\alpha_i - \delta_j)) + (\lambda c_k db_{jk} + (b_{jk}c_k)(\alpha_j - \delta_k)) \\
=& -\lambda b_{ij}c_k dc_{jk} + \lambda c_k db_{ik} + (b_{ik}c_k - b_{jk}c_k)\alpha_i - b_{ij}c_j\delta_j + (b_{jk}c_k)(\alpha_j - \delta_k) \\
=& -\lambda b_{ij}c_k dc_{jk} + \lambda c_k db_{ik} + b_{ik}c_k(\alpha_i - \delta_k) - b_{jk}c_k(\alpha_i - \alpha_j) - b_{ij}c_j(\delta_j - \delta_k) \\
=& \lambda c_k db_{ik} + b_{ik}c_k(\alpha_i - \delta_k).
\end{aligned}$$

So $(-(\lambda c_j db_{ij} + (b_{ij}c_j)(\alpha_i - \delta_j)))_{i,j}$ defines a cocycle of $H^1(C, L^{-1} \otimes \Omega^1_C)$. Note that a solution of the fourth equation of (5.4) exists if and only if $(-(\lambda c_j db_{ij} + (b_{ij}c_j)(\alpha_i - \delta_j)))_{i,j}$ is trivial. We denote the image of b by the natural homomorphism $H^1(C, L^{-1}(-D)) \to H^1(C, L^{-1})$ by the same character b. Since the linear map $\langle\ , b\rangle'' \colon H^0(C, \Omega^1_C) \to H^1(C, L^{-1} \otimes \Omega^1_C)$ is an isomorphism by Lemma 5.1.7, there exists a unique global 1-form $\zeta = (\zeta_i/c_i)_i \in H^0(C, \Omega^1_C)$ such that

$$(2b_{ij}\zeta_j)_{i,j} = \langle 2\zeta, b\rangle'' = -(\lambda c_j db_{ij} + (b_{ij}c_j)(\alpha_i - \delta_j))_{i,j},$$

that is,

$$-(\lambda c_j db_{ij} + (b_{ij}c_j)((\alpha_i + \zeta_i/c_i) - (\delta_j - \zeta_j/c_j)))_{i,j} = 0$$

in $H^1(C, L^{-1} \otimes \Omega^1_C)$. So there exist unique $(\alpha_i)_i$ and $(\delta_i)_i$ satisfying the condition (5.7) and

$$-(\lambda c_j db_{ij} + (b_{ij}c_j)(\alpha_i - \delta_j))_{i,j} = 0,$$

and there exists a solution of the fourth equation $(\beta_i)_i$ of (5.4) such that $\mathrm{res}_{t_k}(\beta_i) = 0$ for any i and $t_k \in U_i$. Since $H^0(C, L^{-1} \otimes \Omega^1_C) \simeq H^1(C, L)^* = \{0\}$, $(\beta_i)_i$ is uniquely determined.

When $\lambda = 0$, the cocycle $(b_{ij}\gamma_j)_{i,j}$ is zero because $\alpha_i \in \Omega^1_C(U_i)$. Conversely, assume that $(b_{ij}\gamma_j)_{i,j} = 0$. Then there exists $\tilde{\alpha}_i \in \Omega^1_C(U_i)$ for each i such that $\alpha_i - \alpha_j = b_{ij}\gamma_j = \tilde{\alpha}_i - \tilde{\alpha}_j$. The cocycle $(\alpha_i - \tilde{\alpha}_i)_i$ defines a global logarithmic 1-form on C. By the residue theorem, we have

$$\sum_{i=1}^{n} \lambda \nu_{i,0} = 0.$$

By assumption, we obtain $\lambda = 0$.

For a point $([\gamma], [b]) \in (\mathbb{P}H^0(C, L \otimes \Omega^1_C(D)) \times V_0) \setminus \Sigma$, there exist a unique complex number λ and a unique $\lambda\boldsymbol{\nu}$-parabolic λ-connection (E, ∇, l_*) such that $\mathrm{tr}\nabla = \lambda\nabla_L$, $\varphi_\nabla = \gamma$, and (E, l_*) is the quasi-parabolic bundle corresponding to b. Then $\lambda \neq 0$ and $(E, \lambda^{-1}\nabla, l_*)$ is a $\boldsymbol{\nu}$-parabolic connection with the determinant (L, ∇_L) whose image by App $\times$ Bun is $([\gamma], [b])$. If a $\boldsymbol{\nu}$-parabolic connection (E, ∇', l_*) satisfies $\mathrm{tr}\nabla' = \nabla_L$ and $\varphi_{\nabla'} \in [\gamma]$, then there is a unique complex number μ such that $\varphi_{\nabla'} = \mu\lambda^{-1}\gamma$. A $\mu\boldsymbol{\nu}$-parabolic μ-connection $(E, \mu\lambda^{-1}\nabla, l_*)$ satisfies $\mathrm{tr}(\mu\lambda^{-1}\nabla) = \mu\nabla_L$ and $\varphi_{\mu\lambda^{-1}\nabla} = \mu\lambda^{-1}\gamma$, so we have $\mu = 1$ and $\nabla' = \lambda^{-1}\nabla$ by the uniqueness. Therefore, the morphism

$$\mathrm{App} \times \mathrm{Bun} \colon M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))^0 \longrightarrow (\mathbb{P}H^0(C, L \otimes \Omega^1_C(D)) \times V_0) \setminus \Sigma$$

is bijective. By Zariski's main theorem (for example, see Chapter 3, §9, Proposition 1 in [Mu]), App $\times$ Bun is an isomorphism. □

The following proposition is the same as Proposition 4.6 in [LS] and follows by using the same argument of the proof.

Proposition 5.1.8. Suppose that $\sum_{i=1}^{n} \nu_{i,0} = 0$. Then $M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))^0$ is isomorphic to the total space of the cotangent bundle T^*V_0 and the map Bun: $M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))^0 \to V_0$ corresponds to the natural projection $T^*V_0 \to V_0$. Moreover, the section $\nabla_0 \colon V_0 \to M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))^0$ corresponding to the zero section $V_0 \to T^*V_0$ is given by those reducible connections preserving the destabilizing subbundle $\mathcal{O}_C$.

5.1.4 Another proof of Theorem 5.1.6

We will show App $\times$ Bun is a birational map in another way. First, we show the existence of a parabolic connection over a given parabolic bundle. The following lemma is an analogy of Lemma 2.5 in [FL].

Lemma 5.1.9. Suppose that $\sum_{i=1}^{n} w_i < 1$. Then for each $(E, l_*) \in P^{\boldsymbol{\alpha}}(L)$, there is a $\boldsymbol{\nu}$-parabolic connection (E, ∇, l_*) such that $\mathrm{tr}\nabla \simeq \nabla_L$.

Proof. Let $\{U_i\}_i$ be an open covering of C and ∇'_i be a logarithmic connection on U_i satisfying $(\mathrm{res}_{t_k}(\nabla'_i) - \nu_{k,1}\mathrm{id})(l_{k,1}) = 0, (\mathrm{res}_{t_k}(\nabla'_i) - \nu_{k,0}\mathrm{id})(E|_{t_i}) \subset l_{k,1}$ at each $t_k \in U_i$ and $\mathrm{tr}\nabla'_i = \nabla_L|_{U_i}$. We define sheaves $\mathcal{E}_0$ and $\mathcal{E}_1$ on C by

$$\begin{aligned} \mathcal{E}^0 &:= \{s \in \mathcal{E}nd(E) \mid \mathrm{tr}(s) = 0 \text{ and } s_{t_i}(l_{i,1}) \subset l_{i,1} \text{ for any } i\}, \\ \mathcal{E}^1 &:= \{s \in \mathcal{E}nd(E) \otimes \Omega^1_C(D) \mid \mathrm{tr}(s) = 0 \text{ and } \mathrm{res}_{t_i}(s)(l_{i,j}) \subset l_{i,j+1} \text{ for any } i, j\}. \end{aligned}$$

Then the isomorphism $\mathcal{E}^1 \simeq (\mathcal{E}^0)^\vee \otimes \Omega^1_C$ holds. Differences $\nabla'_i - \nabla'_j$ define the cocycle

$$(\nabla'_i - \nabla'_j)_{i,j} \in H^1(C, \mathcal{E}^1).$$

By Serre duality and the simplicity of E, we obtain

$$H^1(C, \mathcal{E}^1) \simeq H^0(C, \mathcal{E}^0)^* = \{0\}.$$

Hence, there exists $\Phi_i \in \mathcal{E}^1(U_i)$ for each i such that $\nabla'_i - \nabla'_j = \Phi_i - \Phi_j$. Set $\nabla_i = \nabla'_i - \Phi_i$. Then $(\nabla_i)_i$ defines a $\boldsymbol{\nu}$-parabolic connection ∇ over (E, l_*) satisfying $\mathrm{tr}\nabla \simeq \nabla_L$. □

For a quasi-parabolic bundle $(E, l_*) \in V_0$, let us fix a $\boldsymbol{\nu}$-parabolic connection $(E, \nabla, l_*) \in \mathrm{Bun}^{-1}((E, l_*))$. Let $(E, \nabla', l_*) \in \mathrm{Bun}^{-1}((E, l_*))$ be another $\boldsymbol{\nu}$-parabolic connection. Then $\nabla' - \nabla$ is a global section of $\mathcal{E}^1$ which is the sheaf defined in the proof of Lemma 5.1.9. Therefore, we have the isomorphism $\mathrm{Bun}^{-1}((E, l_*)) \simeq \nabla + H^0(C, \mathcal{E}^1)$.

For a section $\Theta \in H^0(\mathcal{E}nd(E) \otimes \Omega^1_C(D))$, we define the section $\varphi_\Theta \in H^0(C, L \otimes \Omega^1_C(D))$ by the composite

$$\mathcal{O}_C \hookrightarrow E \xrightarrow{\Theta} E \otimes \Omega^1_C(D) \to E/\mathcal{O}_C \otimes \Omega^1_C(D) \simeq L \otimes \Omega^1_C(D)$$

and define the map

$$\varphi\colon H^0(C, \mathcal{E}nd(E)\otimes\Omega^1_C(D)) \longrightarrow H^0(C, L\otimes\Omega^1(D))$$

by $\varphi(\Theta)=\varphi_\Theta$. It is clearly linear. Let us define the sheaf $\mathcal{F}^1$ by

$$\mathcal{F}^1=\{s\in\mathcal{E}nd(E)\otimes\Omega^1_C(D)\mid \mathrm{res}_{t_i}(s)(l_{i,j})\subset l_{i,j+1}\text{ for all } i,j\}.$$

Assume that $\Theta\in H^0(C,\mathcal{F}^1)$ satisfies $\varphi_\Theta=0$, that is, $\Theta(\mathcal{O}_C)\subset\mathcal{O}_C\otimes\Omega^1_C(D)$. By definitions of V_0 and $\mathcal{E}^1$, we obtain $\mathrm{res}_{t_i}(\Theta)(\mathcal{O}_C|_{t_i})\subset\mathcal{O}_C|_{t_i}\cap l_{i,1}=\{0\}$ for any i. Hence, we have $\Theta(\mathcal{O}_C)\subset\mathcal{O}_C\otimes\Omega_C$, that is, $\Theta|_{\mathcal{O}_C}$ is a global section of Ω^1_C.

Lemma 5.1.10. The linear map $H^0(C,\mathcal{F}^1)\cap\operatorname{Ker}\varphi\to H^0(C,\Omega^1_C)$, $\Theta\mapsto\Theta|_{\mathcal{O}_C}$ is an isomorphism.

Proof. For $\mu\in H^0(C,\Omega^1_C)$, we define $\Theta=\mathrm{id}_E\otimes\mu$. Then we have $\Theta\in H^0(C,\mathcal{F}^1)\cap\operatorname{Ker}\varphi$ and $\Theta|_{\mathcal{O}_C}=\mu$. The linear map is hence surjective. We show that the map is injective. If $\Theta\in H^0(C,\mathcal{F}^1)\cap\operatorname{Ker}\varphi$ satisfies $\Theta|_{\mathcal{O}_C}=0$, then Θ induces the homomorphism $\hat{\Theta}\colon L\simeq E/\mathcal{O}_C\to E\otimes\Omega^1_C(D)$. $\mathrm{res}_{t_i}(\Theta)=0$ implies $\mathrm{res}_{t_i}(\hat{\Theta})=0$, so we obtain $\hat{\Theta}(L)\subset E\otimes\Omega^1_C$. Since $\operatorname{rank}E=2$, we have isomorphisms $E^\vee\simeq E\otimes(\det E)^{-1}\simeq E\otimes L^{-1}$. By this isomorphism and Serre duality,

$$\mathrm{Hom}(L,E\otimes\Omega^1_C)\simeq H^0(C,L^{-1}\otimes E\otimes\Omega^1_C)\simeq H^0(C,E^\vee\otimes\Omega^1_C)\simeq H^1(C,E)^*=\{0\}$$

Hence we obtain $\hat{\Theta}=0$ and this implies $\Theta=0$. □

Proof. (Another proof the second assertion of Theorem 5.1.6)

We show that for each $(E,l_*)\in V_0$, the morphism

$$\mathrm{App}\colon \mathrm{Bun}^{-1}((E,l_*))\longrightarrow\mathbb{P}H^0(C,L\otimes\Omega^1_C(D))$$

is injective.

Let us fix a $\boldsymbol{\nu}$-parabolic connection $(E,\nabla,l_*)\in\mathrm{Bun}^{-1}((E,l_*))$. If there exists $\Theta\in H^0(C,\mathcal{E}^1)$ such that $\varphi_\nabla=\varphi_\Theta$, then $\nabla-\Theta$ is a $\boldsymbol{\nu}$-parabolic connection and $\varphi_{\nabla-\Theta}=0$. It is a contradiction. Thus, we have

$$\{\varphi_\Theta\mid\Theta\in H^0(C,\mathcal{E}^1)\}\cap\mathbb{C}\varphi_\nabla=\{0\}.$$

Hence, we only need to show that the linear map $\varphi\colon H^0(C,\mathcal{E}^1)\to H^0(C,L\otimes\Omega^1(D))$ is injective. Suppose that a section $\Theta\in H^0(C,\mathcal{E}^1)$ satisfies $\varphi_\Theta=0$. By the proof of Lemma 5.1.10, there is a section $\mu\in H^0(C,\Omega^1_C)$ such that $\Theta=\mathrm{id}_E\otimes\mu$. Since $\operatorname{tr}\Theta=0$, we get $\mu=0$ and this means $\Theta=0$. □

5.1.5 Lagrangian fibrations

Recall the canonical symplectic structure on $M^{\boldsymbol{\alpha}}(\boldsymbol{\nu},(L,\nabla_L))$ (see section 6 in [IIS1] and section 7 in [In] for more detail). Take a point $x=(E,\nabla,l_*)\in M^{\boldsymbol{\alpha}}(\boldsymbol{\nu},(L,\nabla_L))$. Let $\mathcal{E}^\bullet$ be the complex of sheaves defined by

$$\mathcal{E}^0\longrightarrow\mathcal{E}^1,\ s\longmapsto\nabla\circ s-s\circ\nabla,$$

where $\mathcal{E}^0$ and $\mathcal{E}^1$ are sheaves defined in Lemma 5.1.9. Then there exists the canonical isomorphism between the tangent space $T_xM^{\boldsymbol{\alpha}}(\boldsymbol{\nu},(L,\nabla_L))$ and the hypercohomology group $\mathbf{H}^1(\mathcal{E}^\bullet)$. Take an open covering $\{U_i\}_i$ of C. In Čech cohomology an element of $\mathbf{H}^1(\mathcal{E}^\bullet)$ can be written by the form $\{(B_{ij}),(\Phi_i)\}$, where $(B_{ij})_{i,j}\in C^1(\mathcal{E}^0)$, $(\Phi_i)_i\in C^0(\mathcal{E}^1)$ and $(\nabla B_{ij}-B_{ij}\nabla)_{i,j}=(\Phi_j-\Phi_i)_{i,j}$ in $C^1(\mathcal{E}^1)$. The canonical symplectic form Ω on $M^{\boldsymbol{\alpha}}(\boldsymbol{\nu},(L,\nabla_L))$ is defined by

$$\begin{aligned}\Omega_x\colon \mathbf{H}^1(\mathcal{E}^\bullet)\otimes\mathbf{H}^1(\mathcal{E}^\bullet)&\longrightarrow\mathbf{H}^2(\mathcal{O}_C\xrightarrow{d}\Omega^1_C)\cong\mathbb{C}\\ (\{(B_{ij}),(\Phi_i)\},\{(B'_{ij}),(\Phi'_i)\})&\longmapsto(\{\operatorname{tr}(B_{ij}\circ B'_{jk})\},-\{(\operatorname{tr}(B_{ij}\circ\Phi'_j)-\operatorname{tr}(\Phi_i\circ B'_{ij}))\})\end{aligned}$$

at each x. We can see that the homomorphisms $H^0(C,\mathcal{E}^1)\to\mathbf{H}^1(C,\mathcal{E}^\bullet)$ and $\mathbf{H}^1(C,\mathcal{E}^\bullet)\to H^1(C,\mathcal{E}^0)$ defined by $(\Phi_i)_i\mapsto\{0,(\Phi_i)_i\}$ and $\{(B_{ij})_{i,j},(\Phi_i)_i\}\mapsto(B_{ij})_{i,j}$, respectively, give an exact sequence

$$H^0(C,\mathcal{E}^0)\longrightarrow H^0(C,\mathcal{E}^1)\longrightarrow\mathbf{H}^1(C,\mathcal{E}^\bullet)\longrightarrow H^1(C,\mathcal{E}^0)\longrightarrow H^1(C,\mathcal{E}^1).$$

When $(E,\nabla,l_*)\in M^{\boldsymbol{\alpha}}(\boldsymbol{\nu},(L,\nabla_L))^0$, we have $H^1(C,\mathcal{E}^1)\simeq H^0(C,\mathcal{E}^0)^*=\{0\}$. We note that each element in $H^1(C,\mathcal{E}^0)$ gives a deformation of (E,l_*).

Proposition 5.1.11. $\mathrm{App}\colon M^{\boldsymbol{\alpha}}(\boldsymbol{\nu},(L,\nabla_L))^0\to|L\otimes\Omega^1_C(D)|$ and $\mathrm{Bun}\colon M^{\boldsymbol{\alpha}}(\boldsymbol{\nu},(L,\nabla_L))^0\to V_0$ are Lagrangian fibrations.

Proof. Take a point $x = (E, \nabla, l_*) \in M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))^0$ and put $[\gamma] = \mathrm{App}(x)$ and $[b] = \mathrm{Bun}(x)$, where $\gamma = (\gamma_i)_i \in H^0(C, L \otimes \Omega^1_C(D))$ and $b = (b_{ij})_{i,j} \in H^1(C, L^{-1}(-D))$ are nonzero elements. Then a transition matrix M_{ij} of E and a connection matrix A_i of ∇ have the form

$$M_{ij} = \begin{pmatrix} 1 & b_{ij} \\ 0 & c_{ij} \end{pmatrix},\ A_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix},$$

respectively. The natural homomorphism

$$T_x\mathrm{App}^{-1}([\gamma]) \oplus T_x\mathrm{Bun}^{-1}([b]) \longrightarrow T_x M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L)) \cong \mathbf{H}^1(C, \mathcal{E}^\bullet)$$

is an isomorphism. Since any element in $T_x\mathrm{Bun}^{-1}([b])$ does not deform (E, l_*), we have $T_x\mathrm{Bun}^{-1}([b]) \subset H^0(C, \mathcal{E}^1)$. So $\Omega|_{\mathrm{Bun}^{-1}([b])} = 0$ and $T_x\mathrm{App}^{-1}([\gamma]) \to H^1(C, \mathcal{E}^0)$ is an isomorphism. Take $\{(B_{ij})_{ij}, (\Phi_i)_i\} \in \mathbf{H}^1(C, \mathcal{E}^\bullet)$. Since the homomorphism

$$T_{[b]}\mathbb{P}H^1(C, L^{-1}(-D)) \cong H^1(C, L^{-1}(-D))/[b] \to H^1(C, \mathcal{E}^0) \cong T_{(E,l_*)}P^{\boldsymbol{\alpha}}(L),\ (g_{ij})_{i,j} \mapsto \left(\begin{pmatrix} 0 & g_{ij} \\ 0 & 0 \end{pmatrix}\right)_{i,j}$$

is an isomorphism, B_{ij} and Φ_i can be written by the form

$$B_{ij} = \begin{pmatrix} 0 & g_{ij} \\ 0 & 0 \end{pmatrix},\ \Phi_i = \begin{pmatrix} \zeta_i & \eta_i \\ \theta_i & -\zeta_i \end{pmatrix},$$

where $\zeta_i, \eta_i, \in \Omega^1_C(U_i)$ and $\theta_i \in \Omega^1_C(D)(U_i)$. We note that $(b_{ij}\gamma_j)_{i,j}$ is a nonzero cocycle in $H^1(C, \Omega^1_C)$ (see Step 2 in the proof of Theorem 5.1.6). So we have $H^1(C, L^{-1}(-D)) = [b] \oplus \mathrm{Ker}\,\langle \gamma,\ \rangle$, where $\langle\ ,\ \rangle$ is the natural pairing

$$\langle\ ,\ \rangle \colon H^0(C, L \otimes \Omega^1_C(D)) \times H^1(C, L^{-1}(-D)) \longrightarrow H^1(C, \Omega^1_C).$$

Since $b = 0$ in $H^1(C, \mathcal{E}^0)$, the composite

$$\mathrm{Ker}\,\langle \gamma,\ \rangle \to H^1(C, L^{-1}(-D)) \to H^1(C, \mathcal{E}^0)$$

becomes an isomorphism. So we may assume that $(g_{ij})_{i,j} \in \mathrm{Ker}\,\langle \gamma,\ \rangle$. The condition $\nabla B_{ij} - B_{ij}\nabla = dB_{ij} + A_iB_{ij} - B_{ij}A_j = M_{ij}\Phi_j - \Phi_i M_{ij}$ is equivalent to

$$\begin{cases} -g_{ij}\gamma_j = \zeta_j - \zeta_i + b_{ij}\theta_j \\ dg_{ij} + \alpha_i g_{ij} - g_{ij}\delta_j = \eta_j - \eta_i c_{ij} - b_{ij}(\zeta_i + \zeta_j) \\ c_{ij}\theta_j - \theta_i = 0. \end{cases}$$

So $\theta = (\theta_i)_i$ defines a global section of $L \otimes \Omega^1_C(D)$ and $(b_{ij}\theta_j)_{i,j}$ is zero in $H^1(C, \Omega^1_C)$. Assume that $\{(B_{ij})_{ij}, (\Phi_i)_i\} \in T_x\mathrm{App}^{-1}([\gamma])$. Then θ is an element of $[\gamma]$, and so θ must be zero. Hence we have

$$\Omega_x(\{(B_{ij})_{ij}, (\Phi_i)_i\}, \{(B'_{ij})_{ij}, (\Phi'_i)_i\}) = 0$$

for any $\{(B_{ij})_{ij}, (\Phi_i)_i\}, \{(B'_{ij})_{ij}, (\Phi'_i)_i\} \in T_x\mathrm{App}^{-1}([\gamma])$, which means that $\Omega|_{\mathrm{App}^{-1}([\gamma])} = 0$. □

5.2 Rank 3 case

This section is devoted to the relation between the moduli space of parabolic bundles and parabolic logarithmic connections of rank three on the projective line with three points.

5.2.1 The moduli space of w-stable parabolic bundles

In this subsection, we determine w-stable parabolic bundles on $\mathbb{P}^1$ of rank 3 and degree -2, and investigate the moduli space and the wall-crossing behavior. Let us fix $\boldsymbol{t} \in T_3$.

We assume that

$$\alpha_{1,3} - \alpha_{1,2} = \alpha_{1,2} - \alpha_{1,1} = \alpha_{2,3} - \alpha_{2,2} = \alpha_{2,2} - \alpha_{2,1} = \alpha_{3,3} - \alpha_{3,2} = \alpha_{3,2} - \alpha_{3,1} =: w.$$

Then we have $0 < w < 1/2$. We consider the case of $\deg E = -2$. Take a nonzero subbundle $F \subsetneq E$. If $\operatorname{rank} F = 2$, then the inequality (2.1) is equivalent to

$$-4 - 3\deg F + \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_{i,j}(2 - 3d_{i,j}(F)) > 0, \tag{5.9}$$

and we have

$$\sum_{j=1}^{3} \alpha_{i,j}(2 - 3d_{i,j}(F)) = \begin{cases} -3w & F|_{t_i} = l_{i,1} \\ 0 & F|_{t_i} \neq l_{i,1}, F|_{t_i} \supset l_{i,2} \\ 3w & F|_{t_i} \not\supset l_{i,2}. \end{cases}$$

In the case of $\operatorname{rank} F = 1$, (2.1) is equivalent to

$$-2 - 3\deg F + \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_{i,j}(1 - 3d_{i,j}(F)) > 0, \tag{5.10}$$

and we have

$$\sum_{j=1}^{3} \alpha_{i,j}(1 - 3d_{i,j}(F)) = \begin{cases} 3w & F|_{t_i} \not\subset l_{i,1} \\ 0 & F|_{t_i} \subset l_{i,1}, F|_{t_i} \neq l_{i,2} \\ -3w & F|_{t_i} = l_{i,2}. \end{cases}$$

The stability condition is determined by w under the assumption, so we call the special case of the $\boldsymbol{\alpha}$-stability the w-stability.

Let (E, l_*) be a w-stable parabolic bundle with $\deg E = -2$. The vector bundle E can be written by the form $\mathcal{O}_{\mathbb{P}^1}(m_1) \oplus \mathcal{O}_{\mathbb{P}^1}(m_2) \oplus \mathcal{O}_{\mathbb{P}^1}(m_3)$, where $m_1 \geq m_2 \geq m_3$ and $m_1 + m_2 + m_3 = -2$. Suppose that $m_1 \geq 1$. Since $w < 1/2$, we have

$$-2 - 3\deg \mathcal{O}_{\mathbb{P}^1}(m_1) + \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_{i,j}(1 - 3d_{i,j}(\mathcal{O}_{\mathbb{P}^1}(m_1))) \leq -5 + 9w < 0.$$

So E is isomorphic to $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Suppose that $\mathcal{O}_{\mathbb{P}^1}|_{t_i} = l_{i,2}$ for some i. Then we have

$$-2 - 3\deg \mathcal{O}_{\mathbb{P}^1} + \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_{i,j}(1 - 3d_{i,j}(\mathcal{O}_{\mathbb{P}^1})) \leq -2 + 3w < 0.$$

So $\mathcal{O}_{\mathbb{P}^1}|_{t_i} \neq l_{i,2}$ for any i. Let l'_i be the image of $l_{i,2}$ by the quotient $E|_{t_i} \to (E/\mathcal{O}_{\mathbb{P}^1})|_{t_i}$. Since $\mathcal{O}_{\mathbb{P}^1}|_{t_i} \neq l_{i,2}$, l'_i is not zero for any i. For a parabolic structure $l'_* = \{l'_i\}_{1\leq i\leq 3}$ on $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, put

$$n(l'_*) := \max_{\mathcal{O}_{\mathbb{P}^1}(-1)\cong F \subset \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}} \#\{i \mid F|_{t_i} = l'_i\}.$$

A parabolic bundle $(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, l'_*)$ with $n(l'_*) = 1$ and 3 is unique up to isomorphism, respectively. When $n(l'_*) = 2$, there are three isomorphism classes of such parabolic bundccles, that is, those isomorphism classes are determined by the pair of numbers $1 \leq i < j \leq 3$. Let $(*)$ be the following condition;

- $(*)$ There is no subbundle $F \subset E$ such that $F \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, l_{i,2} \subset F|_{t_i}$ and $F|_{t_j} = l_{j,1}$ for some i and any $j \neq i$.

Proposition 5.2.1. Let $P^w(-2) := P^w_{(\mathbb{P}^1,\boldsymbol{t})}(3, -2)$.

(1) If $0 < w < 2/9, 4/9 < w < 1/2$, then $P^w(-2) = \emptyset$.

(2) If $2/9 < w < 1/3$, then a w-stable parabolic bundle (E, l_*) fits into a nonsplit exact sequence

$$0 \longrightarrow (\mathcal{O}_{\mathbb{P}^1}, \emptyset) \longrightarrow (E, l_*) \longrightarrow (\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, l'_*) \longrightarrow 0, \tag{5.11}$$

where $n(l'_*) = 1$. In particular, $P^w(-2)$ is isomorphic to $\mathbb{P}^1$.

(3) If $1/3 < w < 4/9$, then a w-stable parabolic bundle (E, l_*) is either type of the following:

(i) $E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\#\{i \mid \mathcal{O}_{\mathbb{P}^1}|_{t_i} \subset l_{i,1}\} = 0$, $n(l'_*) = 1$, and the condition $(*)$ holds.

(ii) $E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\#\{i \mid \mathcal{O}_{\mathbb{P}^1}|_{t_i} \subset l_{i,1}\} = 1$, $n(l'_*) = 1$, and the condition $(*)$ holds.

In particular, $P^w(-2)$ is isomorphic to $\mathbb{P}^1$.

Proof. Assume that $w < 2/9$. Then we have

$$-2 - 3\deg \mathcal{O}_{\mathbb{P}^1} + \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_{i,j}(1 - 3d_{i,j}(\mathcal{O}_{\mathbb{P}^1}))) \leq -2 + 9w < 0,$$

which means that $P^w(-2) = \emptyset$.

Assume that $2/9 < w < 1/3$. If $\mathcal{O}_{\mathbb{P}^1}|_{t_i} \subset l_{i,1}$ for some i, then we have

$$-2 - 3\deg \mathcal{O}_{\mathbb{P}^1} + \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_{i,j}(1 - 3d_{i,j}(\mathcal{O}_{\mathbb{P}^1})) \leq -2 + 6w < 0.$$

So $\mathcal{O}_{\mathbb{P}^1}|_{t_i} \not\subset l_{i,1}$ for any i. Hence (E, l_*) fits into an exact sequence

$$0 \longrightarrow (\mathcal{O}_{\mathbb{P}^1}, \emptyset) \longrightarrow (E, l_*) \longrightarrow (\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, l'_* = \{l'_i\}_{1\leq i\leq 3}) \longrightarrow 0. \tag{5.12}$$

If (5.12) splits, that is, there exists a subbundle F such that $F \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ and $F|_{t_i} = l_{i,1}$ for all i, then we have

$$-4 - 3\deg F + \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_{i,j}(2 - 3d_{i,j}(F)) = 2 - 9w < 0.$$

So (5.12) does not split. Suppose that $n(l'_*) \geq 2$. Then we can take a subbundle $F \subset E$ satisfying $F \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ and $F|_{t_i} = l_{i,2}, F|_{t_j} = l_{j,2}$ for some $1 \leq i < j \leq 3$ and we have

$$-4 - 3\deg(\mathcal{O}_{\mathbb{P}^1} \oplus F) + \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_{i,j}(2 - 3d_{i,j}((\mathcal{O}_{\mathbb{P}^1} \oplus F))) \leq -1 + 3w < 0.$$

Hence $n(l'_*) = 1$ and we have

$$P^w(-2) \cong \mathbb{P}\mathrm{Ext}^1((\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, l'_*), (\mathcal{O}_{\mathbb{P}^1}, \emptyset)) \cong \mathbb{P}H^1((\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2})(-D)) \cong \mathbb{P}^1.$$

Assume that $1/3 < w < 1/2$. If $n(l'_*) \geq 2$, then we can take a subbundle $F \subset E$ satisfying $F \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ and $F|_{t_i} = l_{i,2}, F|_{t_j} = l_{j,2}$ for some $1 \leq i < j \leq 3$, and we have

$$-2 - 3\deg F + \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_{i,j}(1 - 3d_{i,j}(F)) \leq 1 - 3w < 0.$$

So $n(l'_*) = 1$. In this case, we can take a unique subbundle $F \subset E$ such that $F \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ and $F|_{t_i} = l_{i,2}$ for any i, and we have

$$-2 - 3\deg F + \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_{i,j}(1 - 3d_{i,j}(F)) = 4 - 9w.$$

So $P^w(-2) = \emptyset$ if $w > 4/9$. Assume that $1/3 < w < 4/9$. Suppose that $\#\{i \mid \mathcal{O}_{\mathbb{P}^1}|_{t_i} \subset l_{i,1}\} \geq 2$. Then we have

$$-2 - 3\deg \mathcal{O}_{\mathbb{P}^1} + \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_{i,j}(1 - 3d_{i,j}(\mathcal{O}_{\mathbb{P}^1})) \leq -2 + 3w < 0.$$

So $\#\{i \mid \mathcal{O}_{\mathbb{P}^1}|_{t_i} \subset l_{i,1}\} \leq 1$. We consider the case $\mathcal{O}_{\mathbb{P}^1}|_{t_i} \not\subset l_{i,1}$ for any i. Then we can take a unique subbundle $F_{ij} \subset E$ such that $F_{ij} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, F_{ij}|_{t_i} = l_{i,1}$ and $F_{ij}|_{t_j} = l_{j,1}$ for each $1 \leq i < j \leq 3$. If $l_{m,2} \subset F_{ij}|_{t_m}$ for $m \neq i, j$, then we have

$$-4 - 3\deg F + \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_{i,j}(2 - 3d_{i,j}(F)) = 2 - 6w < 0.$$

So such a parabolic bundle becomes w-unstable, which is a contradiction. We can see that such a parabolic bundle $p_{ij} \in \mathbb{P}\mathrm{Ext}^1((\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, l'_*), (\mathcal{O}_{\mathbb{P}^1}, \emptyset))$ is unique for each $1 \le i < j \le 3$. Next we consider the case $\mathcal{O}_{\mathbb{P}^1}|_{t_i} \not\subseteq l_{m,1}$ for some m. Let i, j be different elements of $\{1,2,3\} \setminus \{m\}$. Then we can take a unique subbundle $F_{ij} \subset E$ such that $F_{ij} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, F_{ij}|_{t_i} = l_{i,1}$ and $F_{ij}|_{t_j} = l_{j,1}$. In the same reason of the above, we have $l_{m,2} \not\subseteq F|_{t_m}$. We can see that such a parabolic bundle p_m is unique up to isomorphism. Therefore we have

$$P^w(-2) \cong (\mathbb{P}\mathrm{Ext}^1((\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, l'_*), (\mathcal{O}_{\mathbb{P}^1}, \emptyset)) \setminus \{p_{12}, p_{13}, p_{23}\}) \sqcup \{p_1, p_2, p_3\} \cong \mathbb{P}^1.$$

□

As the above proof shows, p_{12}, p_{13}, p_{23} become w-unstable and p_1, p_2, p_3 become w-stable when w is across $1/3$. Let us investigate this in detail. Assume that $2/9 < w < 1/3$. In this case, a w-stable parabolic bundle (E, l_*) fits into a nonsplit exact sequence (5.11). Then we can take nonzero homomorphisms $s_1, s_2 \colon \mathcal{O}_{\mathbb{P}^1}(-1) \to E$ satisfying $l_{1,2} = (\mathrm{Im}\, s_1)|_{t_1}$, $l_{2,2} = (\mathrm{Im}\, s_2)|_{t_2}$, $0 \neq (\mathrm{Im}\, s_1)|_{t_2} \subset l_{2,1}$, $0 \neq (\mathrm{Im}\, s_2)|_{t_1} \subset l_{1,1}$. Let e_1, e_2 be local basis corresponding to s_1, s_2, respectively, and e_0 be the nonzero section of $\mathcal{O}_{\mathbb{P}^1} \subset E$. Let us denote $ae_0 + be_1 + ce_2$ by the matrix ${}^t(a\ b\ c)$. Since $n(l'_*) = 1$, we can wright l_* by the form

$$l_{1,2} = \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix},\ l_{1,1} = \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix},\quad l_{2,2} = \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix},\ l_{2,1} = \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$l_{3,2} = \mathbb{C}\begin{pmatrix}a+b\\1\\1\end{pmatrix},\ l_{3,1} = \mathbb{C}\begin{pmatrix}a\\1\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}b\\0\\1\end{pmatrix},$$

where $a, b \in \mathbb{C}$. The exact sequence (5.11) splits if and only if $(a,b) = (0,0)$, and parabolic bundles defined by $(a,b), (a',b')$ are isomorphic to each other if and only if $(a,b), (a',b')$ are the same up to scalar multiplicities. In this way, we also prove that $P^w(-2) \cong \mathbb{P}^1$. The parabolic bundles p_{12}, p_{13}, p_{23} in the proof of Proposition 5.2.1 correspond to the case $a+b=0, b=0, a=0$, respectively. Let us fix $a \neq 0$ and put $\mu = a + b$. Let $\tilde{l}_*$ be the parabolic structure defined by

$$\tilde{l}_{1,2} = \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix},\ \tilde{l}_{1,1} = \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix},\quad \tilde{l}_{2,2} = \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix},\ \tilde{l}_{2,1} = \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\tilde{l}_{3,2} = \mathbb{C}\begin{pmatrix}1\\1\\1\end{pmatrix},\ \tilde{l}_{3,1} = \mathbb{C}\begin{pmatrix}1\\\frac{\mu}{a}\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

When $\mu \neq 0$, the homomorphism defined by the matrix

$$\begin{pmatrix}\mu & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

is an isomorphism from $(E, \tilde{l}_*)$ to (E, l_*). When $\mu = 0$, $(E, \tilde{l}_*)$ and (E, l_*) are parabolic bundles corresponding to p_3 and p_{12} in the proof of Proposition 5.2.1, respectively. So p_3 and p_{12} are infinitesimally close to each other. In the same way, we can see that p_1, p_2 are infinitesimally close to p_{23}, p_{13}, respectively.

5.2.2 The moduli space of λ-connections

In this subsection, we consider the compactification of the moduli space of parabolic connections by using λ-connections. Let $M_3^w(\boldsymbol{t}, \boldsymbol{\nu})$ be the moduli space of rank 3 w-stable $\boldsymbol{\nu}$-parabolic logarithmic connection on $(\mathbb{P}^1, \boldsymbol{t})$. Let $\overline{M_3^w(\boldsymbol{t}, \boldsymbol{\nu})^0}$ be the moduli space of $\lambda\boldsymbol{\nu}$-parabolic λ-connections over $(\mathbb{P}^1, \boldsymbol{t})$ whose underlying parabolic bundle is w-stable, that is,

$$\overline{M_3^w(\boldsymbol{t}, \boldsymbol{\nu})^0} := \{(\lambda, E, \nabla, l_*) \mid (E, l_*) \in P^w(-2)\} / \sim .$$

Here two objects $(\lambda_1, E_1, \nabla_1, (l_1)_*), (\lambda_2, E_2, \nabla_2, (l_2)_*)$ are equivalent if there exists an isomorphism $\sigma\colon (E_1, (l_1)_*) \to (E_2, (l_2)_*)$ and $\mu \in \mathbb{C}^*$ such that the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\nabla_1} & E_1 \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \\ \sigma \downarrow & & \downarrow \sigma\otimes\mathrm{id} \\ E_2 & \xrightarrow{\mu\nabla_2} & E_2 \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \end{array}$$

commutes. The locus defined by $\lambda = 0$ on $\overline{M_3^w(\boldsymbol{t},\boldsymbol{\nu})^0}$ is isomorphic to the projectivization $\mathbb{P}T^*P^w(-2)$ of the cotangent bundle of $P^w(-2)$. By definition,

$$M_3^w(\boldsymbol{t},\boldsymbol{\nu})^0 := \{\lambda \neq 0\} = \overline{M_3^w(\boldsymbol{t},\boldsymbol{\nu})^0} \setminus \mathbb{P}T^*P^w(-2)$$

is just the moduli space of $\boldsymbol{\nu}$-parabolic connections whose underlying parabolic bundle is w-stable. The following result when $\nu_{1,0} + \nu_{2,0} + \nu_{3,0} = 0$ is a version of Proposition 4.6 in [LS] in the present setting.

Theorem 5.2.2. Assume that $2/9 < w < 1/3$. Then we have

$$\overline{M_3^w(\boldsymbol{t},\boldsymbol{\nu})^0} \cong \begin{cases} \mathbb{P}^1 \times \mathbb{P}^1 & \nu_{1,0} + \nu_{2,0} + \nu_{3,0} \neq 0 \\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) & \nu_{1,0} + \nu_{2,0} + \nu_{3,0} = 0. \end{cases}$$

Proof. Let $U_0 := \mathbb{C}$ and $U_\infty := \mathbb{C}$. For $a \in U_0$ and $b \in U_\infty$, let us define a parabolic structure $(l_a)_*$ and $(l_b)_*$ on $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ by

$$(l_a)_{1,2} = (l_b)_{1,2} = \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix},\ (l_a)_{1,1} = (l_b)_{1,1} = \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix},$$

$$(l_a)_{2,2} = (l_b)_{2,2} = \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix},\ (l_a)_{2,1} = (l_b)_{2,1} = \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix},$$

$$(l_a)_{3,2} = \mathbb{C}\begin{pmatrix}a+1\\1\\1\end{pmatrix},\ (l_a)_{3,1} = \mathbb{C}\begin{pmatrix}a\\1\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}1\\0\\1\end{pmatrix}, (l_b)_{3,2} = \mathbb{C}\begin{pmatrix}1+b\\1\\1\end{pmatrix},\ (l_b)_{3,1} = \mathbb{C}\begin{pmatrix}1\\1\\0\end{pmatrix} + \mathbb{C}\begin{pmatrix}b\\0\\1\end{pmatrix}.$$

Then (U_0, a) and (U_∞, b) define coordinates on $P^w(-2)$, and we have $a = 1/b$ when $a, b \neq 0$. Put

$$\begin{aligned} c_{11}(z) &= \nu_{2,0}(t_2 - t_3)(z - t_1) + \nu_{1,0}(t_1 - t_3)(z - t_2), \\ c_{22}(z) &= \nu_{2,1}(t_2 - t_3)(z - t_1) + \nu_{1,2}(t_1 - t_3)(z - t_2), \\ c_{33}(z) &= \nu_{2,2}(t_2 - t_3)(z - t_1) + \nu_{1,1}(t_1 - t_3)(z - t_2), \end{aligned}$$

$$c^0_{12}(a) = a(1+\nu_{1,0}+\nu_{2,0}-\nu_{1,2}-\nu_{2,1})+(1-(\nu_{1,2}+\nu_{2,1}+\nu_{3,1})),\ c^\infty_{12}(b) = (1-\nu_{1,2}-\nu_{2,1}-\nu_{3,0})+b((\nu_{1,1}+\nu_{2,2}+\nu_{3,2})-1),$$

$$c^0_{13}(a) = a((\nu_{1,2}+\nu_{2,1}+\nu_{3,2})-1)+(1-(\nu_{1,1}+\nu_{2,2}+\nu_{3,0})),\ c^\infty_{13}(b) = (1-\nu_{1,1}-\nu_{2,2}-\nu_{3,1})+b(1+\nu_{1,0}+\nu_{2,0}-\nu_{1,1}-\nu_{2,2}),$$

$$c^0_{31} = c^\infty_{21} = -(\nu_{1,0} + \nu_{2,0} + \nu_{3,0}),$$

$$c^0_{23} = (\nu_{1,2} + \nu_{2,1} + \nu_{3,2}) - 1,\ c^\infty_{23}(b) = (\nu_{1,2} + \nu_{2,1} + \nu_{3,2}) - 1 + (1 + b)(\nu_{1,0} + \nu_{2,0} + \nu_{3,0}),$$

$$c^0_{32}(a) = (\nu_{1,1} + \nu_{2,2} + \nu_{3,2}) - 1 + (a + 1)(\nu_{1,0} + \nu_{2,0} + \nu_{3,0}),\ c^\infty_{32} = (\nu_{1,1} + \nu_{2,2} + \nu_{3,2}) - 1,$$

$$\nabla_0(a) := d + \begin{pmatrix} c_{11}(z) & c^0_{12}(a)(z-t_1)(z-t_2) & c^0_{13}(a)(z-t_1)(z-t_2) \\ 0 & (z-t_1)(z-t_2) + c_{22}(z) & c^0_{23}(t_3-t_1)(z-t_2) \\ c^0_{31}h'(t_3) & c^0_{32}(a)(t_3-t_2)(z-t_1) & (z-t_1)(z-t_2) + c_{33}(z) \end{pmatrix} \frac{dz}{h(z)},$$

$$\Phi_0(a) := \begin{pmatrix} 0 & a(a+1)(z-t_1)(z-t_2) & -a(a+1)(z-t_1)(z-t_2) \\ h'(t_3) & 0 & -(a+1)(t_3-t_1)(z-t_2) \\ -ah'(t_3) & a(a+1)(t_3-t_2)(z-t_1) & 0 \end{pmatrix} \frac{dz}{h(z)},$$

$$\nabla_\infty(b) := d + \begin{pmatrix} c_{11}(z) & c^\infty_{12}(b)(z-t_1)(z-t_2) & c^\infty_{13}(b)(z-t_1)(z-t_2) \\ c^\infty_{21}h'(t_3) & (z-t_1)(z-t_2) + c_{22}(z) & c^\infty_{23}(b)(t_3-t_1)(z-t_2) \\ 0 & c^\infty_{32}(t_3-t_2)(z-t_1) & (z-t_1)(z-t_2) + c_{33}(z) \end{pmatrix} \frac{dz}{h(z)},$$

$$\Phi_\infty(b) := \begin{pmatrix} 0 & b(1+b)(z-t_1)(z-t_2) & -b(1+b)(z-t_1)(z-t_2) \\ bh'(t_3) & 0 & -b(1+b)(t_3-t_1)(z-t_2) \\ -h'(t_3) & (1+b)(t_3-t_2)(z-t_1) & 0 \end{pmatrix} \frac{dz}{h(z)}.$$

Then we have

$$\mathrm{Bun}^{-1}(U_0) \cong \mathbb{P}(\mathbb{C}\nabla_0 \oplus \mathbb{C}\Phi_0),\ \mathrm{Bun}^{-1}(U_\infty) \cong \mathbb{P}(\mathbb{C}\nabla_\infty \oplus \mathbb{C}\Phi_\infty),$$

where $\mathrm{Bun}\colon \overline{M_3^w(\boldsymbol{t},\boldsymbol{\nu})^0} \to P^w(-2)$ is the forgetful map. We can see that

$$\nabla_\infty = \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (\nabla_0 - (\nu_{1,0}+\nu_{2,0}+\nu_{3,0})a^{-1}\Phi_0) \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \Phi_\infty = \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (a^{-2}\Phi_0) \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and so we have

$$(\nabla_\infty, \Phi_\infty) \cong (\nabla_0, \Phi_0) \begin{pmatrix} 1 & 0 \\ -(\nu_{1,0}+\nu_{2,0}+\nu_{3,0})a^{-1} & a^{-2} \end{pmatrix}.$$

Hence we obtain the theorem. □

Let us consider the relation between the moduli space of $\boldsymbol{\nu}$-parabolic ϕ-connections $\overline{M_3^{\boldsymbol{\alpha}}(\boldsymbol{t},\boldsymbol{\nu})}$ and the moduli space of $\lambda\boldsymbol{\nu}$-parabolic λ-connections $\overline{M_3^w(\boldsymbol{t},\boldsymbol{\nu})^0}$. We assume that $\nu_{i,0} \neq \nu_{i,1} \neq \nu_{i,2} \neq \nu_{i,0}$ for each i for simplicity. Let $\varphi\colon \widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu}) \to \mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1})$, $\varphi'\colon \overline{M_3^{\boldsymbol{\alpha}}(\boldsymbol{t},\boldsymbol{\nu})} \to \mathbb{P}^2$ and $\rho\colon \mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1}) \to \mathbb{P}^2$ be the morphism defined in Section 4 (see the diagram (4.28) in the proof of Theorem 4.1.1). Let $D_i \subset \mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1})$ be the fiber over t_i and $\hat{D}_i$ be the strict transform of D_i under φ. Let $H_i = \rho(D_i)$ and $\hat{H}_i$ be the strict transform of H_i under φ'. Let D_0 be the section of $\mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1})$ over $\mathbb{P}^1$ defined by the injection $\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \hookrightarrow \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1}$. Let $b_{i,j} \in \mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1})$ be the point defined in the Subsection 4.7 and put $c_{i,j} = \rho(b_{i,j}) \in \mathbb{P}^2$. We can see that three points $c_{1,i}, c_{2,j}, c_{3,k}$ are on the same line if and only if $\nu_{1,i} + \nu_{2,j} + \nu_{3,k} = 1$, and six points $c_{1,i_1}, c_{1,i_2}, c_{2,j_1}, c_{2,j_2}, c_{3,k_1}, c_{3,k_2}$ are on the same conic if and only if $\nu_{1,i_1} + \nu_{1,i_2} + \nu_{2,j_1} + \nu_{2,j_2} + \nu_{3,k_1} + \nu_{3,k_2} = 2$.

The following proposition follows from the proof of Proposition 4.7.1 and Proposition 4.7.3.

Proposition 5.2.3. Assume that $0 < \alpha_{i,j} \ll 1$ and $\nu_{i,0} \neq \nu_{i,1} \neq \nu_{i,2} \neq \nu_{i,0}$ for each i. Take $(E, \nabla, l_*) \in M_3^{\boldsymbol{\alpha}}(\boldsymbol{t},\boldsymbol{\nu})$. Then the type of (E, l_*) is one of the following:

- (i) $E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\#\{i \mid \mathcal{O}_{\mathbb{P}^1}|_{t_i} \subset l_1^{(i)}\} = 0$, $n(l'_*) = 1$, and the condition $(*)$ holds.
- (i)′ $E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\#\{i \mid \mathcal{O}_{\mathbb{P}^1}|_{t_i} \subset l_1^{(i)}\} = 0$, $n(l'_*) = 1$, and the condition $(*)$ does not hold.
- (ii) $E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\#\{i \mid \mathcal{O}_{\mathbb{P}^1}|_{t_i} \subset l_1^{(i)}\} = 1$, $n(l'_*) = 1$, and the condition $(*)$ holds.
- (iii) $E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\#\{i \mid \mathcal{O}_{\mathbb{P}^1}|_{t_i} \subset l_1^{(i)}\} = 0$, $n(l'_*) \geq 2$, and the condition $(*)$ holds.

For (E, l_*) whose type is (iii), $n(l'_*) = 3$ when $\nu_{1,2}+\nu_{2,2}+\nu_{3,2} = 1$ and $n(l'_*) = 2$ when $\nu_{1,2}+\nu_{2,2}+\nu_{3,2} \neq 1$

Assume that $\boldsymbol{\nu}$ satisfies the condition

$$\nu_{1,2} + \nu_{2,2} + \nu_{3,2} \neq 1 \tag{5.13}$$

and

$$\nu_{1,j_1} + \nu_{2,2} + \nu_{3,2} \neq 1,\ \nu_{1,2} + \nu_{2,j_2} + \nu_{3,2} \neq 1,\ \nu_{1,2} + \nu_{2,2} + \nu_{3,j_3} \neq 1 \tag{5.14}$$

for any $j_1, j_2, j_3 = 0, 1$. When $2/9 < w < 1/3$, $P^w(-2)$ consists of parabolic bundles of the type (i) and (i)′. We can obtain $\overline{M_3^w(\boldsymbol{t},\boldsymbol{\nu})^0}$ from $\widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$ by the following three steps.

Step 1: contract the locus consisting of the type (ii) and (iii). We have

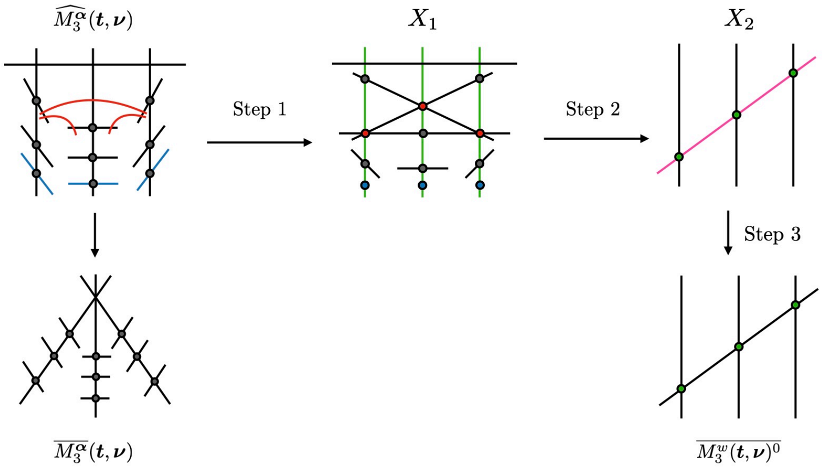
$$\{(E,\nabla,l_*) \in M_3^{\boldsymbol{\alpha}}(\boldsymbol{t},\boldsymbol{\nu}) \mid \text{the type of } (E,l_*) \text{ is (ii)}\} = (\varphi^{-1}(b_{1,0}) \setminus D_1) \cup (\varphi^{-1}(b_{2,0}) \setminus D_2) \cup (\varphi^{-1}(b_{3,0}) \setminus D_3).$$

By Proposition 4.7.3, $\varphi^{-1}(b_{i,j})$ is a (-1)-curve. From (4.25), the closure of the set

$$\{(E,\nabla,l_*) \in M_3^{\boldsymbol{\alpha}}(\boldsymbol{t},\boldsymbol{\nu}) \mid l'_i \text{ and } l'_j \text{ lie on some subbundle } \mathcal{O}_{\mathbb{P}^1}(-1) \cong F' \subset \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\}$$

on $\overline{M_3^{\boldsymbol{\alpha}}(\boldsymbol{t},\boldsymbol{\nu})}$ is the closure of the locus defined by

$$(h'(t_i)(\nu_{i,2} - \mathrm{res}_{t_i}(\tfrac{dz}{z-t_3})) - p)(t_j - q) - (h'(t_j)(\nu_{j,2} - \mathrm{res}_{t_j}(\tfrac{dz}{z-t_3})) - p)(t_i - q) = 0,$$



where (q,p) is the coordinate defined in the proof of Proposition 4.7.1, which is just the strict transform $\hat{L}_{ij} \subset \overline{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$ of the line $L_{ij} \subset \mathbb{P}^2$ passing through $c_{i,2}$ and $c_{j,2}$ under φ'. Since any $c_{m,n}$ for $(m,n) \neq (i,2),(j,2)$ is not on $L_{i,j}$ from the condition (5.13) and (5.14), the intersection number of $\hat{L}_{ij}$ is -1. By contracting $\varphi^{-1}(b_{1,0}), \varphi^{-1}(b_{2,0}), \varphi^{-1}(b_{3,0})$ and the inverse images of $\hat{L}_{12}, \hat{L}_{23}, \hat{L}_{13}$ under PC, we obtain a morphism $\rho_1 \colon \widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu}) \to X_1$, where X_1 is a smooth projective surface.

Step 2: contract the locus defined by rank $\phi = 2$. Since $\varphi \colon \widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu}) \to \mathbb{P}(\Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \oplus \mathcal{O}_{\mathbb{P}^1})$ is the blow-up at 9 points $\{b_{i,j}\}_{0\leq j\leq 2}^{1\leq i\leq 3}$, $\hat{D}_i$ is a (-3)-curve for each i. $\hat{H}_i$ intersects with $\varphi^{-1}(c_{i,0})$ and $\hat{L}_{jm}$ $(j,m \neq i)$ at one point, respectively. So the image $\rho_1(\hat{D}_i) \subset X_1$ is a (-1)-curve. Contracting $\hat{D}_1, \hat{D}_2, \hat{D}_3$, we obtain a morphism $\rho_2 \colon X_1 \to X_2$. When $\nu_{1,0} + \nu_{2,0} + \nu_{3,0} = 0$, there exists a conic $C \subset \mathbb{P}^2$ passing through six points $c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}, c_{3,1}, c_{3,2}$. Let $\hat{C} \subset \widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$ be the strict transform of C under $\rho \circ \varphi = \varphi' \circ \mathrm{PC}$. Then $\rho_1(\hat{C}) \cong \rho_2(\rho_1(\hat{C}))$ is a projective line and intersects with $\rho_2(\rho_1(\varphi^{-1}(b_{i,1})))$ for each $i = 1,2,3$. So X_2 is isomorphic to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$. Since C does not intersect with $\varphi'^{-1}(c_{i,0})$, and C intersects with each $\hat{H}_i$ and $\hat{L}_{mn}$ at two points, we have $\rho_2(\rho_1(\hat{C}))^2 = \rho_1(\hat{C})^2 = \hat{C}^2 = -2$. $\rho_2(\rho_1(\hat{C}))$ is the unique section whose intersection number is -2. When $\nu_{1,0}+\nu_{2,0}+\nu_{3,0} \neq 0$, there is no projective line contained in X_2 which intersects with $\rho_2(\rho_1(\varphi^{-1}(b_{i,1})))$ for each $i = 1,2,3$. So X_2 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Step 3: change D_0 to $\mathbb{P}T^*P^w(-2)$. D_0 and $\mathbb{P}T^*P^w(-2)$ are infinitesimally close to each other. A $\boldsymbol{\nu}$-parabolic connection

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \nabla = d + \begin{pmatrix} 0 & a_{12}(z) & a_{13}(z) \\ 1 & (z-t_1)(z-t_2) - p & 0 \\ 0 & z-q & (z-t_1)(z-t_2)+p \end{pmatrix} \frac{dz}{h(z)}$$

whose apparent singularity q is not t_1, t_2 and t_3 has the limits

$$\begin{pmatrix} p^{-2} & 0 & 0 \\ 0 & p^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} (\phi,\nabla) \begin{pmatrix} p^{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p^{-1} \end{pmatrix} \xrightarrow{p\to\infty} \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & g(z) \\ 1 & 0 & 0 \\ 0 & z-q & 1 \end{pmatrix} \frac{dz}{h(z)} \right), \tag{5.15}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p^2 \end{pmatrix} (\phi,\nabla) \begin{pmatrix} p^{-1} & 0 & 0 \\ 0 & p^{-2} & 0 \\ 0 & 0 & p^{-3} \end{pmatrix} \xrightarrow{p\to\infty} \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & g(z) \\ 1 & -1 & 0 \\ 0 & z-q & 1 \end{pmatrix} \frac{dz}{h(z)} \right), \tag{5.16}$$

where $g(z) = \sum_{i=1}^{3} \frac{1}{(q-t_i)h'(t_i)} \prod_{j\neq i}(z-t_j)$. Put

$$C(q;z) := \begin{pmatrix} \frac{(t_3-t_1)h'(t_3)}{(t_2-t_1)(q-t_1)(q-t_3)} & \frac{(t_3-t_2)(z+q-t_1-t_2)}{(t_1-t_2)(q-t_2)} & \frac{(t_3-t_1)(z+q-t_1-t_2)}{(t_2-t_1)(q-t_1)} \\ 0 & \frac{t_3-t_2}{t_1-t_2} & \frac{t_3-t_1}{t_2-t_1} \\ 0 & \frac{(t_3-t_2)(q-t_1)}{t_1-t_2} & \frac{(t_3-t_1)(q-t_2)}{t_2-t_1} \end{pmatrix},$$

$$C_1(q;z) := \begin{pmatrix} -(q-t_2)(q-t_3) & 0 & z+q-t_2-t_3 \\ 0 & -(q-t_2)(q-t_3) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$C_2(q;z) := \begin{pmatrix} -(q-t_2)^{-1}(q-t_3)^{-1} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & q-t_1 \end{pmatrix}.$$

Then we have

$$C_1(q;z)\left(\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&-1&g(z)\\1&0&0\\0&z-q&1\end{pmatrix}\frac{dz}{h(z)}\right)C_2(q;z) = \left(\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&(z-t_2)(z-t_3)&0\\1&0&0\\0&z-q&z-t_1\end{pmatrix}\frac{dz}{h(z)}\right),$$

and

$$C(q;z)^{-1}\begin{pmatrix}0&-1&g(z)\\1&-1&0\\0&z-q&1\end{pmatrix}\frac{dz}{h(z)}C(q;z) = \frac{(t_3-t_1)(q-t_2)}{h'(t_2)(q-t_1)(q-t_3)}\Phi_0(-\tfrac{(t_3-t_2)(q-t_1)}{(t_3-t_1)(q-t_2)}).$$

So a $\boldsymbol{\nu}$-parabolic ϕ-connection with rank $\phi = 1$ and a parabolic Higgs bundle is infinitesimally closed to each other. In the case of $q = t_1, t_2, t_3$, we can also see it by using (4.27) and (4.29). Therefore we can obtain $\overline{M_3^w(\boldsymbol{t},\boldsymbol{\nu})^0}$ from $\widehat{M_3^{\boldsymbol{\alpha}}}(\boldsymbol{t},\boldsymbol{\nu})$.

5.2.3 Parabolic bundles and the apparent singularities

We fix $2/9 < w < 1/3$. Let $V_0 \subset P^w(-2)$ be the subset consisting of parabolic bundles of the type (i). The set V_0 is the set of $P^w(-2)$ minus 3 points by Proposition 5.2.1. Let $(E, l_*) \in V_0$ and ∇ be a $\lambda\boldsymbol{\nu}$-logarithmic λ-connection on (E, l_*). Assume that $\nu_{1,0} + \nu_{2,0} + \nu_{3,0} \neq 0$. Then there exists a unique filtration $E =: F_0 \supset F_1 \supset F_2 \supset 0$ such that $F_2 \cong \mathcal{O}_{\mathbb{P}^1}, F_1 \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, and $\nabla(F_2) \subset F_1 \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$. We define the apparent singularity $\mathrm{App}(E, \nabla, l_*)$ by the zero of the nonzero homomorphism

$$\mathcal{O}_{\mathbb{P}^1}(-1) \cong F_1/F_2 \xrightarrow{\nabla} (E/F_1) \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t})) \cong \mathcal{O}_{\mathbb{P}^1}.$$

When $\lambda \neq 0$, this definition is the same of the definition in Subsection 4.3.

Remark 5.2.4. Assume that $(E, l_*) \in P^w(-2) \setminus V_0$. Then for any parabolic connection ∇ over (E, l_*), there exists a unique filtration $E = F_0 \supset F_1 \supset F_2 \supset 0$ such that $F_2 \cong \mathcal{O}_{\mathbb{P}^1}, F_1 \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, and $\nabla(F_2) \subset F_1 \otimes \Omega^1_{\mathbb{P}^1}(D(\boldsymbol{t}))$. However, we can see that for a parabolic Higgs field Φ over (E, l_*), such filtration is not unique. So we can not define the apparent map App over $\overline{M_3^w(\boldsymbol{t},\boldsymbol{\nu})^0}$.

The following is a version of Theorem 4.3 in [LS] in the present setting.

Proposition 5.2.5. We fix $2/9 < w < 1/3$ and assume that $\nu_{1,0} + \nu_{2,0} + \nu_{3,0} \neq 0$. Then the morphism

$$\mathrm{App} \times \mathrm{Bun} \colon \mathrm{Bun}^{-1}(V_0) \longrightarrow \mathbb{P}^1 \times V_0$$

is finite and its generic fiber consists of three points.

Proof. Consider fibers of App $\times$ Bun. We have

$$(\mu\nabla_0 + \lambda\Phi_0)\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\mu c_{11}(z)\\ \lambda h'(t_3)\\ (\mu c^0_{31} - \lambda a)h'(t_3)\end{pmatrix}\frac{dz}{h(z)}.$$

So F_1 is generated by the sections ${}^t(1,0,0)$ and ${}^t(0, \lambda, (\mu c^0_{31} - \lambda a))$. Since

$$(\mu\nabla_0+\lambda\Phi_0)\begin{pmatrix}0\\ \lambda\\ \mu c^0_{31}-\lambda a\end{pmatrix} = \begin{pmatrix} * \\ \mu\lambda((z-t_1)(z-t_2)+c_{22}(z)) + (\mu c^0_{31}-\lambda a)(\mu c^0_{23} - \lambda(a+1))(t_3-t_1)(z-t_2) \\ \lambda(\mu c^0_{32}(a) + \lambda a(a+1))(t_3-t_2)(z-t_1) + \mu(\mu c^0_{31}-\lambda a)((z-t_1)(z-t_2)+c_{33}(z))\end{pmatrix},$$

the apparent singularity of $\mu\nabla_0 + \lambda\Phi_0$ is the zero of the polynomial

$$\begin{aligned}&\lambda\{\lambda(\mu c^0_{32}(a) + \lambda a(a+1))(t_3-t_2)(z-t_1) + \mu(\mu c^0_{31} - \lambda a)((z-t_1)(z-t_2) + c_{33}(z))\}\\ &-(\mu c^0_{31} - \lambda a)\{\mu\lambda((z-t_1)(z-t_2)+c_{22}(z)) + (\mu c^0_{31} - \lambda a)(\mu c^0_{23} - \lambda(a+1))(t_3-t_1)(z-t_2)\}\\ =&f_1(a;\mu,\lambda)(z-t_1) + f_2(a;\mu,\lambda)(z-t_2),\end{aligned}$$

where

$$\begin{aligned}
f_1(a;\mu,\lambda) =&(t_3-t_2)\{a(a+1)\lambda^3+(c^0_{32}(a)+(\nu_{2,2}-\nu_{2,1})a)\lambda^2\mu-(\nu_{2,2}-\nu_{2,1})c^0_{31}\mu^2\lambda\},\\
f_2(a;\mu,\lambda) =&(t_3-t_1)\{a^2(a+1)\lambda^3-((\nu_{1,2}-\nu_{1,1})a+2a(a+1)c^0_{31}+a^2c^0_{32}(a))\lambda^2\mu\\
&+((\nu_{1,2}-\nu_{1,1})c^0_{31}+2ac^0_{31}c^0_{23}+(a+1)(c^0_{31})^2)\lambda\mu^2-(c^0_{31})^2c^0_{23}\mu^3\}.
\end{aligned}$$

Hence $\mathrm{App}\colon \mathrm{Bun}^{-1}((E,(l_a)_*))\cong\mathbb{P}(\mathbb{C}\nabla_0(a)\oplus\mathbb{C}\Phi_0(a))\to\mathbb{P}^1$ is defined by

$$\mathrm{App}(\mu\nabla_0+\lambda\Phi_0)=(f_1(a;\mu,\lambda)+f_2(a;\mu,\lambda):t_1f_1(a;\mu,\lambda)+t_2f_2(a;\mu,\lambda)),$$

which implies that a generic fiber consists of three points. Since $\mathrm{App}\times\mathrm{Bun}$ is proper, $\mathrm{App}\times\mathrm{Bun}$ is finite. □

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