Index theorem on T^2/Z_N orbifolds with magnetic flux

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Index theorem on T^2/Z_N orbifolds with magnetic flux

(一様磁場のかかったT^2/Z_Nオービフォールド模型における指数定理)

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Abstract

The Standard Model (SM) achieved significant success in 2012 with the discovery of the Higgs boson. However, many mysteries remain, including the generation number problem. The problem is that the quarks and leptons have been observed to exist in three copies with the same quantum numbers except for their masses. However, the theoretical origin of three-generation is unknown. In the context of higher-dimensional theories, the four-dimensional theory is derived as an effective field theory (EFT) of the higher-dimensional theory. In the extra-dimensional model, the number of chiral zero modes corresponds to the number of generations. The degeneracy of the chiral zero modes is caused by topological quantities in the extra dimension. Consequently, multiple generations can be obtained through the degeneracy of chiral zero modes. In this dissertation, we will focus on the $T^2/\mathbb{Z}_N$ $(N = 2, 3, 4, 6)$ orbifolds with magnetic flux background. This model has been extensively investigated in various studies as a potential solution to the mysteries of the Standard Model, such as the three-generation, flavor structures. The Atiyah-Singer index theorem serves as a crucial tool for understanding generation structure. While typically applied to smooth manifolds, the $T^2/\mathbb{Z}_N$ orbifolds, having singularities, appear extra nontrivial contributions. We attempt to derive the Atiyah-Singer index theorem on the magnetized $T^2/\mathbb{Z}_N$ orbifolds.

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Contents

Chapter 1

Introduction

1.1 Mysteries in the Standard Model

The Standard Model (SM) is a four-dimensional $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge theory with the quarks, leptons, gauge bosons, and Higgs boson. Its significant triumph came in 2012 with the identification of the Higgs boson. Nevertheless, several mysteries persist within the Standard Model. We outline some of these mysteries below:

- **Generation number**:
  Quarks and leptons are observed to exist in three copies, each sharing identical quantum numbers except for their masses. These distinct sets are commonly referred to as generations. The Kobayashi-Masukawa theory posits that having more than or equal to three generations is necessary to manifest experimentally confirmed CP-violation. However, within the Standard Model, the theoretical underpinning for precisely why there are three generations remains unexplained.

- **Mass hierarchy of fermions**:
  Quarks and leptons have huge mass differences among the generations. For instance, the mass ratios in the up-type quark sector are given by $m_u : m_c : m_t = 0.00001 : 0.07 : 1$. Since fermions get masses through the Higgs mechanism, it implies that Yukawa coupling takes on exponentially different values. In the SM, where Yukawa couplings are treated as free parameters, the theory cannot explain why these coupling constants have exponential differences.

- **Flavor mixing and the CP-violating phase**:
  The flavor mixing and the CP-violating phase are induced through the weak interaction. The strength of mixing and the CP-violating are described by the three mixing angles and CP-violating phase in the Cabibbo-Kobayashi-Masukawa (CKM) matrix. Although experimental observations have provided values for these parameters, their theoretical origin remains elusive within the framework of the Standard Model.

In addition to these, there are many unexplained mysteries in the Standard Model, such as the strong CP problem, the origin of neutrino masses, dark matter, and dark energy.

Hence, we anticipate the existence of a more fundamental theory than the Standard Model that would provide answers to these mysteries. Unraveling such a theory beyond the Standard Model stands as a crucial objective in particle physics.

1.2 Extra dimensional model

One of the most promising candidates beyond the Standard Model which can explain these mysteries is the higher-dimensional theories with extra dimensions. In addition to the four dimensions of space-time, there are extra spatial dimensions in higher-dimensional theories, and the structure of the extra dimension affects matters such as quarks and leptons in the four dimensions.

1.2.1 Simple example

We consider the simplest extra-dimensional model, $\mathcal{M}^4 \times S^1$, encompassing a 4D Minkowski spacetime and an additional 1D extra dimension in the form of a circle with a radius R. Let $x^\mu (\mu = 0, 1, 2, 3)$ denote the 4D Minkowski coordinates and y the extra dimension coordinates. Due to the circular nature of the extra dimension, a 5D scalar field Φ satisfies the boundary condition

$$\Phi(x, y + 2\pi R) = \Phi(x, y). \tag{1.2.1}$$

It can be expanded

$$\Phi(x, y) = \sum_{n=-\infty}^{\infty} \phi_n(x) \frac{e^{in\frac{y}{R}}}{\sqrt{2\pi R}} \equiv \sum_{n=-\infty}^{\infty} \phi_n(x) f_n(y) \tag{1.2.2}$$

$f_n(y)$ $(n \in \mathbb{Z})$ satisfies orthonormality condition

$$\int_0^{2\pi R} dy f_m^*(y) f_n(y) = \delta_{m,n}. \tag{1.2.3}$$

The action for a 5D scalar field is given by

$$S[\Phi] = \int d^4x \int_0^{2\pi R} dy \left\{\partial_I \Phi^*(x, y) \partial^I \Phi(x, y)\right\} \tag{1.2.4}$$

where $I = 0, 1, 2, 3, 4$. Substituting (1.2.2) into (1.2.4)

$$\begin{aligned}
S[\Phi] &= \int d^4x \int_0^{2\pi R} dy \{\partial_\mu \Phi^*(x,y)\partial^\mu \Phi(x,y) + \partial_4 \Phi^*(x,y)\partial^4 \Phi(x,y)\} \\
&= \int d^4x \int_0^{2\pi R} dy \{\partial_\mu \Phi^*(x,y)\partial^\mu \Phi(x,y) - |\partial_4 \Phi(x,y)|^2\} \\
&= \int d^4x \int_0^{2\pi R} dy \{\sum_n \frac{1}{2\pi R}\partial_\mu \phi_n^*(x)\partial^\mu \phi_n(x) - \sum_{n,m} \phi_n^*(x)\phi_m(x)\frac{nm}{R^2}\frac{e^{-in\frac{y}{R}}}{\sqrt{2\pi R}}\frac{e^{im\frac{y}{R}}}{\sqrt{2\pi R}}\} \\
&= \int d^4x \sum_n \{\partial_\mu \phi_n^*(x)\partial^\mu \phi_n(x) - \frac{nm}{R^2}\delta_{n,m}\phi_n^*(x)\phi_m(x)\} \\
&= \int d^4x \sum_n \{\partial_\mu \phi_n^*(x)\partial^\mu \phi_n(x) - \frac{n^2}{R^2}\phi_n^*(x)\phi_n(x)\} \\
&= \int d^4x \sum_n \{\partial_\mu \phi_n^*(x)\partial^\mu \phi_n(x) - m_n^2\phi_n^*(x)\phi_n(x)\}. \qquad (1.2.5)
\end{aligned}$$

In the fourth equality, we use the orthonormality condition (1.2.4). The 4D mass of $\phi_n(x)$ is given by

$$m_n = \frac{|n|}{R} \quad (n \in \mathbb{Z}). \qquad (1.2.6)$$

The mode functions of massive particles $\phi_n(x)$ are called Kaluza-Klein (KK) modes. On the other hand, the mode functions of massless particles are called zero modes. According to (1.2.6), the masses of KK particles are generally inversely proportional to the size of the extra dimension. As a consequence, massive particles are generally not observable in experiments. Therefore, only zero modes are observable, and these zero modes are identified as the particles within the Standard Model.

1.2.2 Extra dimensional model and the mysteries in the SM

Let's delve into the exploration of how the mysteries within the SM can be addressed through the lens of the extra-dimensional model. Various studies have been conducted to provide explanations for challenges such as the generation problem of quarks and leptons [1–13], mass hierarchy [2, 14–24], and CP violation [23, 25–28]. In this subsection, we will outline the fundamental concepts and approaches to tackling these issues.

- **Generation number**:
  As discussed in the preceding subsection, the SM particles correspond to chiral zero modes in the extra-dimensional model. To achieve three generations, chiral zero modes must exhibit a threefold degeneracy. Unfortunately, in the straightforward model $\mathcal{M}^4 \times S^1$, chiral zero modes do not appear. The topology of the extra dimension plays a crucial role in determining whether the degeneracy of chiral zero modes arises, and the Atiyah-Singer index theorem serves as a powerful tool for comprehending the structure of generation numbers. This thesis centers on exploring this specific aspect.

- **Mass hierarchy of fermions, flavor mixing, and the CP-violating phase** :
  The mass hierarchies of fermions are explained by overlap integrals KK modes. As an example, let us consider the 5D Yukawa interaction for a 5D scalar field Φ and Dirac fields Ψ and Ψ'

  $$S_{\text{Yukawa}} = g_{5\text{D}} \int d^4x \int dy \Phi(x,y) \bar{\Psi}(x,y) \bar{\Psi}'(x,y) + (h.c.), \tag{1.2.7}$$

  where $g_{5\text{D}}$ is a 5D Yukawa coupling constant. Ψ and Ψ' can be decomposed as

  $$\Phi(x,y) = \psi_L^{(0)}(x) f^{(0)}(y) + (\text{massive terms}), \tag{1.2.8}$$
  $$\Phi'(x,y) = \psi_R'^{(0)}(x) g^{(0)}(y) + (\text{massive terms}), \tag{1.2.9}$$

  where $\psi_{L/R}^{(0)}(x)$ denote 4D massless chiral fermions, and $f^{(0)}(y)$ and $g^{(0)}(y)$ are zero modes. Upon dimensionality reduction from (1.2.7), we obtain the following 4D mass term

  $$S_{\text{Yukawa}} = \int d^4x m_{4\text{D}} \bar{\psi}_L^{(0)}(x) \psi_R'^{(0)}(x) + (h.c.) + (\text{massive terms}), \tag{1.2.10}$$

  where 4D mass $m_{4\text{D}}$ is given by

  $$m_{4\text{D}} = g_{5\text{D}} \int dy \langle \Phi(y) \rangle f^{(0)\dagger}(y) g^{(0)}(y). \tag{1.2.11}$$

  Consequently, in scenarios where the scalar vacuum expectation value (VEV) and the KK modes are localized at specific positions, such as with Gaussian functions, substantial mass differences among fermions can be achieved without the need for fine-tuning of $g_{5\text{D}}$.

  In cases where chiral zero modes exhibit degeneracy, overlaps between mode functions of different generations occur. These overlap integrals introduce off-diagonal components in mass matrices, resulting in flavor mixing. By diagonalizing these mass matrices, we obtain the CKM matrix. Moreover, if the mode functions are genuine complex functions, the CP-violating phase can also appear in the CKM matrix.

1.3 Outline

As we have seen in the above, the characteristics of the mode functions in the extra dimension play a crucial role in shaping an effective 4D theory. In this dissertation, we will focus on the generation number problem. Whether the chiral zero modes are degenerate or not is deeply related to the topological quantity of the extra dimension. The Atiyah-Singer index theorem [29] is a powerful tool to obtain the number of chiral fermions, and states that the index of a Dirac operator $\not{D}$

$$\text{Ind}(i\not{D}) \equiv n_+ - n_- \tag{1.3.1}$$

is topologically invariant. Here, $n_\pm$ are the numbers of ± chiral zero modes for the Dirac operator. As an example, the Atiyah-Singer index theorem on the torus T^2 with magnetic flux is known as [30, 31]

$$n_+ - n_- = \frac{q}{2\pi}\int_{T^2} F = M. \tag{1.3.2}$$

Here, M denotes the flux quanta in the torus compactification. The number of chiral zero modes is decided by the flux quanta. In the torus model, the only scenario that yields three-generation is when $M = 3$.

In this dissertation, we will focus on the $T^2/\mathbb{Z}_N$ ($N = 2, 3, 4, 6$) orbifolds with magnetic flux background. This model has been extensively investigated in various studies as a potential solution to the mysteries of the Standard Model, such as the three-generation [1, 32–34], flavor structures [26, 27, 35], and some applications to physics beyond the Standard Model [36, 37].

The number of chiral zero modes on the magnetized orbifold $T^2/\mathbb{Z}_N$ ($N = 2, 3, 4, 6$) has been explored in [32, 33, 38]. However, a list of chiral zero-mode numbers on the orbifolds has not been completed, due to its complicated dependence on the flux quanta M, the Scherk-Schwarz (SS) twist phase (α_1, α_2), and the $\mathbb{Z}_N$ eigenvalue η under the $\mathbb{Z}_N$ rotation. Moreover, unlike the index theorem on the torus, a simple formula for the number of zero modes on the orbifolds has not been known.

In the previous paper [1], a complete list of the number of chiral zero modes was found and the index $n_+ - n_-$ of the models was given by a simple formula

$$n_+ - n_- = \frac{M - V_+}{N} + 1, \tag{1.3.3}$$

where V_+ is the sum of the winding numbers for + chirality modes at the fixed points of the $T^2/\mathbb{Z}_N$ ($N = 2, 3, 4, 6$) orbifolds. We call this formula the "zero-mode counting formula". However, it is not clear whether this formula can be regarded as the index theorem on the $T^2/\mathbb{Z}_N$ orbifolds with magnetic flux. This is because in Ref. [1] $n_+ - n_-$ and $\frac{M-V_+}{N} + 1$ have been computed separately, and verified the equality of them by comparing the values for each case of M, the $\mathbb{Z}_N$ eigenvalue, the Scherk-Schwarz (SS) twist phase (α_1, α_2) and N.

In the paper [39], the index theorem on the $T^2/\mathbb{Z}_N$ orbifolds without magnetic flux was derived as

$$n_+ - n_- = -\frac{V_+}{N} + 1 = \frac{1}{2N}(-V_+ + V_-), \tag{1.3.4}$$

where V_- is the sum of the winding numbers for − chirality modes at the fixed points with a relation $V_+ + V_- = 2N$. In this work, the terms $-\frac{V_+}{N} + 1$ were derived using the trace formula without magnetic flux. Given the absence of magnetic flux, there is no contribution from the bulk, and the index is solely determined by the contribution at the fixed points on the $T^2/\mathbb{Z}_N$ orbifolds.

In Eq.(3.2.22), M/N can be considered as the flux contribution of the bulk as in Eq.(1.3.2). The additional contribution at the fixed points, $-\frac{V_+}{N} + 1$, is consistent with Eq.(1.3.4), which was derived as the index theorem. Therefore, Eq.(3.2.22) is expected to be derived as the index theorem. In this dissertation, in order to clarify the relationship between Eq.(3.2.22) and the index theorem, we will check whether the right-hand side of Eq.(3.2.22) is derived as the index theorem. This dissertation is based on [40].

This dissertation is organized as follows: In Chapter 2, we briefly review the magnetized torus model. In Chapter 3, we construct the mode functions on the $T^2/\mathbb{Z}_N$ $(N = 2, 3, 4, 6)$ orbifolds with magnetic flux. In Chapter 4, we construct the blow-up manifolds of $T^2/\mathbb{Z}_N$ $(N = 2, 3, 4, 6)$ orbifolds to apply the Atiyah-Singer index theorem directly. Then, we derive the AS index theorem on the blow-up manifolds and reinterpret the zero-mode counting formula. In Chapter 5, to derive the AS index theorem on the magnetized $T^2/\mathbb{Z}_N$ $(N = 2, 3, 4, 6)$ orbifolds, we evaluate the trace formula by using the complete set of the mode functions on the $T^2/\mathbb{Z}_N$ orbifolds. Chapter 6 is devoted to summary and discussion.

Chapter 2

Magnetized torus model

2.1 Mode functions on T^2

In this section, we succinctly review mode functions on the two-dimensional (2d) torus T^2 in the presence of a magnetic flux background [14, 32, 33].

2.1.1 Setup

We consider a six-dimensional (6d) abelian gauge theory compactified on T^2. By use of the complex coordinate $z = y_1 + y_2\tau$, where $\boldsymbol{y} = (y_1, y_2)$ $(0 \leq y_1, y_2 < 1)$ is the oblique coordinate, the torus T^2 is defined by the identification

$$z \sim z + 1 \sim z + \tau \qquad (\, \tau \in \mathbb{C},\ \ \mathrm{Im}\tau > 0\,) \tag{2.1.1}$$

under torus lattice shifts.

The non-zero magnetic flux f on T^2 is acquired through the integration over T^2 of the field strength F as $f = \int_{T^2} F$.

$$F(z) = \frac{if}{2\mathrm{Im}\tau} dz \wedge d\bar{z}. \tag{2.1.2}$$

For $F = dA$, the (1-form) vector potential is given by

$$A(z) = \frac{f}{2\mathrm{Im}\tau}\mathrm{Im}(\bar{z}dz). \tag{2.1.3}$$

The torus lattice shift of the vector potential corresponds to the following gauge transformation

$$A(z+1) = A(z) + d\Lambda_1(z), \tag{2.1.4}$$

$$A(z+\tau) = A(z) + d\Lambda_2(z), \tag{2.1.5}$$

where $\Lambda_1(z)$ and $\Lambda_2(z)$ are gauge parameters given by

$$\Lambda_1(z) = \frac{f}{2\mathrm{Im}\tau}\mathrm{Im}z, \qquad \Lambda_2(z) = \frac{f}{2\mathrm{Im}\tau}\mathrm{Im}\bar{\tau}z. \tag{2.1.6}$$

We consider a 6d Weyl fermion in the presence of a magnetic flux background:

$$\mathcal{L}_{6\mathrm{d}} = i\bar{\Psi}\Gamma^K D_K\Psi, \qquad \Gamma_7\Psi = \Psi, \tag{2.1.7}$$

where $K(= 0, 1, 2, 3, 5, 6)$ denotes the 6d spacetime index and $D_K = \partial_K - iqA_K$ is the covariant derivative. Γ^K denote 6d Gamma matrices and Γ_7 is the 6d chiral operator. The 6d Gamma matrices are taken as

$$\{\Gamma^K, \Gamma^L\} = -2\eta^{KL} \qquad (K, L = 0, 1, 2, 3, 5, 6), \tag{2.1.8}$$

$$\eta^{KL} = \mathrm{diag}\,(-1, 1, 1, 1, 1, 1), \tag{2.1.9}$$

$$\Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix} \qquad (\mu = 0, 1, 2, 3), \tag{2.1.10}$$

$$\Gamma^5 = \begin{pmatrix} 0 & i\gamma_5 \\ i\gamma_5 & 0 \end{pmatrix}, \qquad \Gamma^6 = \begin{pmatrix} 0 & \gamma_5 \\ -\gamma_5 & 0 \end{pmatrix}, \qquad \Gamma_7 = \begin{pmatrix} \gamma_5 & 0 \\ 0 & -\gamma_5 \end{pmatrix}. \tag{2.1.11}$$

The 6d Weyl fermion $\Psi(x, z)$ can be decomposed into 4d Weyl right/left-handed fermions $\psi^{(4)}_{\mathrm{R/L}}(x)$ as

$$\Psi(x, z) = \sum_{n,j}\{\psi^{(4)}_{\mathrm{R},n,j}(x) \otimes \psi^{(2)}_{+,n,j}(z) + \psi^{(4)}_{\mathrm{L},n,j}(x) \otimes \psi^{(2)}_{-,n,j}(z)\}, \tag{2.1.12}$$

where x^μ $(\mu = 0, 1, 2, 3)$ denotes the 4d Minkowski coordinate. The 4d Weyl fermions $\psi^{(4)}_{\mathrm{R/L},n,j}(x)$ satisfy $\gamma_5\psi^{(4)}_{\mathrm{R/L},n,j}(x) = \pm\psi^{(4)}_{\mathrm{R/L},n,j}(x)$. The 2d Weyl fermions $\psi^{(2)}_{\pm,n,j}(z)$ are expressed as

$$\psi^{(2)}_{+,n,j}(z) = \begin{pmatrix} \psi^j_{+,n}(z) \\ 0 \end{pmatrix}, \qquad \psi^{(2)}_{-,n,j}(z) = \begin{pmatrix} 0 \\ \psi^j_{-,n}(z) \end{pmatrix}, \tag{2.1.13}$$

where n and j label the Landau level and the degeneracy of mode functions on each level, respectively.

Due to the existence of the vector potential, the 2d Weyl fermions have to obey the pseudo periodic boundary conditions

$$\psi^j_{\pm,n}(z + 1) = U_1(z)\psi^j_{\pm,n}(z), \qquad \psi^j_{\pm,n}(z + \tau) = U_2(z)\psi^j_{\pm,n}(z) \tag{2.1.14}$$

with

$$U_i(z) = e^{iq\Lambda_i(z)}e^{2\pi i\alpha_i} \quad (i = 1, 2), \tag{2.1.15}$$

where α_i $(i = 1, 2)$ corresponds to the Scherk-Schwarz twist phase. The important point here is that in order to be invariant under gauge transformations, the flux must be quantized. To see this, we examine gauge transformations for two-dimensional chiral fermions in the next two paths.

- Path1 $z \to z+1 \to z+1+\tau$:

$$\begin{aligned}\psi^j_{\pm,n}(z+1+\tau) &= U_2(z+1)\psi^j_{\pm,n}(z+1)\\ &= U_2(z+1)U_1(z)\psi^j_{\pm,n}(z)\end{aligned} \tag{2.1.16}$$

- Path2 $z \to z+\tau \to z+\tau+1$:

$$\begin{aligned}\psi^j_{\pm,n}(z+\tau+1) &= U_1(z+\tau)\psi^j_{\pm,n}(z+\tau)\\ &= U_1(z+\tau)U_2(z)\psi^j_{\pm,n}(z)\end{aligned} \tag{2.1.17}$$

The results of Path 1 and Path 2 should be equal. Therefore,

$$\begin{gathered}U_2(z+1)U_1(z) = U_1(z+\tau)U_2(z)\\ e^{iq\Lambda_2(z+1)}e^{2\pi i\alpha_2}e^{iq\Lambda_1(z)}e^{2\pi i\alpha_1} \overset{(2.1.15)}{=} e^{iq\Lambda_1(z+\tau)}e^{2\pi i\alpha_1}e^{iq\Lambda_2(z)}e^{2\pi i\alpha_2}\\ \Lambda_2(z+1)-\Lambda_2(z) = \Lambda_1(z+\tau)-\Lambda_1(z) \quad \mathrm{mod}\frac{2\pi}{q}\\ \frac{f}{2\,\mathrm{Im}\,\tau}\mathrm{Im}\,(\bar\tau(z+1)) - \frac{f}{2\,\mathrm{Im}\,\tau}\mathrm{Im}\,(\bar\tau z) \overset{(2.1.6)}{=} \frac{f}{2\,\mathrm{Im}\,\tau}\mathrm{Im}\,(z+\tau) - \frac{f}{2\,\mathrm{Im}\,\tau}\mathrm{Im}\,z \quad \mathrm{mod}\frac{2\pi}{q}\\ \frac{f}{2\,\mathrm{Im}\,\tau}\mathrm{Im}\,(\bar\tau) = \frac{f}{2\,\mathrm{Im}\,\tau}\mathrm{Im}\,\tau \quad \mathrm{mod}\frac{2\pi}{q}\\ \Longrightarrow f = \frac{2\pi M}{q} \quad M \in \mathbb{Z}\end{gathered} \tag{2.1.18}$$

The consistency condition that the 2d Weyl fermions can be well defined on the torus leads to the magnetic flux quantization

$$\frac{qf}{2\pi} \equiv M \in \mathbb{Z}. \tag{2.1.19}$$

We derive the equations that the mode functions $\psi^j_{\pm,n}(z)$ should satisfy. The 6d Dirac equation is

$$i\Gamma^K D_K \Psi = 0. \tag{2.1.20}$$

Substituting Eqs.(2.1.11) and (2.1.12) into Eq.(2.1.20), we get

$$\begin{aligned}i\Gamma^K D_K\Psi &= 0\\ i\Gamma^\mu D_\mu\Psi + i\Gamma^5 D_5\Psi + i\Gamma^6 D_6\Psi &= 0\\ \sum_{n,j}\Big(i\gamma^\mu\partial_\mu\psi^{(4)}_{\mathrm{R},n,j}(x)\otimes\psi^{(2)}_{+,n,j}(z) + i\gamma^\mu\partial_\mu\psi^{(4)}_{\mathrm{L},n,j}(x)\otimes\psi^{(2)}_{-,n,j}(z)&\\ +i\gamma_5\psi^{(4)}_{\mathrm{R},n,j}(x)\otimes i\sigma_1(\partial_5 - iqA_5)\psi^{(2)}_{+,n,j}(z) + i\gamma_5\psi^{(4)}_{\mathrm{L},n,j}(x)\otimes i\sigma_1(\partial_5 - iqA_5)\psi^{(2)}_{-,n,j}(z)&\\ +i\gamma_5\psi^{(4)}_{\mathrm{R},n,j}(x)\otimes i\sigma_2(\partial_6 - iqA_6)\psi^{(2)}_{+,n,j}(z) + i\gamma_5\psi^{(4)}_{\mathrm{L},n,j}(x)\otimes i\sigma_2(\partial_6 - iqA_6)\psi^{(2)}_{-,n,j}(z)\Big) &= 0.\end{aligned} \tag{2.1.21}$$

We impose to satisfy the 4d Dirac equation

$$i\gamma^{\mu}\partial_{\mu}\psi^{(4)}_{\mathrm{L/R},n,j}(x)=m_n\psi^{(4)}_{\mathrm{R/L},n,j}(x). \tag{2.1.22}$$

From this, further calculations can be performed for Eq.(2.1.21)

$$\begin{aligned}&\sum_{n,j}\Bigg[\psi^{(4)}_{\mathrm{R},n,j}(x)\otimes\left\{m_n\begin{pmatrix}0\\ \psi^j_{-,n}(z)\end{pmatrix}-\begin{pmatrix}0 & (\partial_5-iqA_5)-i(\partial_6-iqA_6)\\ (\partial_5-iqA_5+i(\partial_6-iqA_6) & 0\end{pmatrix}\begin{pmatrix}\psi^j_{+,n}(z)\\ 0\end{pmatrix}\right\}\\ &\quad+\psi^{(4)}_{\mathrm{L},n,j}(x)\otimes\left\{m_n\begin{pmatrix}\psi^j_{+,n}(z)\\ 0\end{pmatrix}+\begin{pmatrix}0 & (\partial_5-iqA_5)-i(\partial_6-iqA_6)\\ (\partial_5-iqA_5+i(\partial_6-iqA_6) & 0\end{pmatrix}\begin{pmatrix}0\\ \psi^j_{-,n}(z)\end{pmatrix}\right\}\Bigg]=0\end{aligned}$$

$$\begin{aligned}&\sum_{n,j}\Bigg[\psi^{(4)}_{\mathrm{R},n,j}(x)\otimes\left\{m_n\begin{pmatrix}0\\ \psi^j_{-,n}(z)\end{pmatrix}-\begin{pmatrix}0 & 2(\partial_z-\frac{\pi M}{2\mathrm{Im}\tau}\bar{z})\\ 2(\partial_{\bar{z}}+\frac{\pi M}{2\mathrm{Im}\tau}z) & 0\end{pmatrix}\begin{pmatrix}\psi^j_{+,n}(z)\\ 0\end{pmatrix}\right\}\\ &\quad+\psi^{(4)}_{\mathrm{L},n,j}(x)\otimes\left\{m_n\begin{pmatrix}\psi^j_{+,n}(z)\\ 0\end{pmatrix}+\begin{pmatrix}0 & 2(\partial_z-\frac{\pi M}{2\mathrm{Im}\tau}\bar{z})\\ 2(\partial_{\bar{z}}+\frac{\pi M}{2\mathrm{Im}\tau}z) & 0\end{pmatrix}\begin{pmatrix}0\\ \psi^j_{-,n}(z)\end{pmatrix}\right\}\Bigg]=0,\end{aligned} \tag{2.1.23}$$

where we use as following relations

$$\partial_z=\frac{\partial_5-i\partial_6}{2}, \tag{2.1.24}$$

$$\partial_{\bar{z}}=\frac{\partial_5+i\partial_6}{2}, \tag{2.1.25}$$

$$A_5=-\frac{f}{2\mathrm{Im}\tau}x_6, \tag{2.1.26}$$

$$A_6=\frac{f}{2\mathrm{Im}\tau}x_5, \tag{2.1.27}$$

$$A_5+iA_6=i\frac{f}{2\mathrm{Im}\tau}(x_5+ix_6)=i\frac{f}{2\mathrm{Im}\tau}z=i\frac{\pi M}{q\mathrm{Im}\tau}z, \tag{2.1.28}$$

$$A_5-iA_6=-i\frac{f}{2\mathrm{Im}\tau}(x_5-ix_6)=-i\frac{f}{2\mathrm{Im}\tau}\bar{z}=-i\frac{\pi M}{q\mathrm{Im}\tau}\bar{z}. \tag{2.1.29}$$

Therefore, the mode functions $\psi^j_{\pm,n}(z)$ are required to satisfy the equations

$$-2D_z\psi^j_{-,n}(z)=m_n\psi^j_{+,n}(z), \tag{2.1.30}$$

$$2D_{\bar{z}}\psi^j_{+,n}(z)=m_n\psi^j_{-,n}(z), \tag{2.1.31}$$

where

$$D_z\equiv\partial_z-\frac{\pi M}{2\mathrm{Im}\tau}\bar{z},\qquad D_{\bar{z}}\equiv\partial_{\bar{z}}+\frac{\pi M}{2\mathrm{Im}\tau}z \tag{2.1.32}$$

with the orthonormality condition

$$\int_{T^2}dzd\bar{z}\,[\psi^j_{\pm,n}(z)]^*\psi^k_{\pm,m}(z)=\delta^{jk}\delta_{nm}. \tag{2.1.33}$$

Our interest is how many zero mode solutions exist for $m_n=0$. In the next subsection, we discuss zero modes on T^2.

2.1.2 Zero mode functions on T^2

It follows from Eqs.(2.1.30) and (2.1.31) that the zero mode functions $\psi^j_{\pm,0}(z)$ should satisfy

$$D_{\bar{z}}\psi^j_{+,0}(z) = \left(\partial_{\bar{z}} + \frac{\pi M}{2\mathrm{Im}\tau}z\right)\psi^j_{+,0}(z) = 0, \tag{2.1.34}$$

$$D_{z}\psi^j_{-,0}(z) = \left(\partial_{z} - \frac{\pi M}{2\mathrm{Im}\tau}\bar{z}\right)\psi^j_{-,0}(z) = 0. \tag{2.1.35}$$

In the case of $M > 0$, the zero mode solutions that satisfy Eq.(2.1.34) and the boundary conditions (2.1.14) are found to be

$$\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,0}(z) = \mathcal{N}_{T^2}\, e^{i\pi M z\,\mathrm{Im}\,z/\mathrm{Im}\,\tau}\,\vartheta\begin{bmatrix}\frac{j+\alpha_1}{M}\\ -\alpha_2\end{bmatrix}(Mz, M\tau) = e^{-\frac{\pi M}{2\mathrm{Im}\tau}|z|^2} g^{(j+\alpha_1,\alpha_2)}(z) \qquad (j = 0, 1, \cdots, M-1), \tag{2.1.36}$$

$$g^{(j+\alpha_1,\alpha_2)}(z) = \mathcal{N}_{T^2}\, e^{\frac{\pi M}{2\mathrm{Im}\tau}z^2}\, e^{2\pi i\frac{j+\alpha_1}{M}\alpha_2}\,\vartheta\begin{bmatrix}\frac{j+\alpha_1}{M}\\ -\alpha_2\end{bmatrix}(Mz, M\tau). \tag{2.1.37}$$

where $j = 0, 1, \cdots, M-1$. $\mathcal{N}_{T^2}$ is the normalization constant and is given by[1]

$$\mathcal{N}_{T^2} = \left(\frac{2M}{\mathrm{Im}\tau}\right)^{\frac{1}{4}}. \tag{2.1.39}$$

ϑ is the theta function defined by

$$\vartheta\begin{bmatrix}a\\ b\end{bmatrix}(z,\tau) = \sum_{l=-\infty}^{\infty} e^{i\pi(a+l)^2\tau} e^{2\pi i(a+l)(z+b)}. \tag{2.1.40}$$

Let us confirm that Eq.(2.1.36) is the solution to Eq.(2.1.34) below.

$$\begin{aligned}\left(\partial_{\bar{z}} + \frac{\pi M}{2\mathrm{Im}\tau}z\right)\psi^j_{+,0}(z) &\overset{(2.1.36)}{=} \left(\partial_{\bar{z}} + \frac{\pi M}{2\mathrm{Im}\tau}z\right)e^{-\frac{\pi M}{2\mathrm{Im}\tau}|z|^2} g^{(j+\alpha_1,\alpha_2)}(z)\\ &= \left(-\frac{\pi M}{2\mathrm{Im}\tau}z + \frac{\pi M}{2\mathrm{Im}\tau}z\right)e^{-\frac{\pi M}{2\mathrm{Im}\tau}|z|^2} g^{(j+\alpha_1,\alpha_2)}(z)\\ &= 0.\end{aligned} \tag{2.1.41}$$

---

[1] In [14], the normalization factor is

$$\mathcal{N}_{T^2} = \left(\frac{2M\mathrm{Im}\tau}{\mathcal{A}^2}\right)^{\frac{1}{4}}, \tag{2.1.38}$$

where $\mathcal{A} = \mathrm{Im}\tau$ in this paper.

Next, verify that the boundary conditions (2.1.14) are satisfied

$$\begin{aligned}
\psi^{j}_{+,0}(z+1) &= \mathcal{N}_{T^2}\, e^{i\pi M(z+1)\,\mathrm{Im}\,(z+1)/\mathrm{Im}\,\tau} \sum_{l=-\infty}^{\infty} e^{i\pi\left(\frac{j+\alpha_1}{M}+l\right)^2 M\tau} e^{2\pi i\left(\frac{j+\alpha_1}{M}+l\right)(M(z+1)-\alpha_2)} \\
&= e^{i\pi M\,\mathrm{Im}\,(z+1)/\mathrm{Im}\,\tau} \sum_{l=-\infty}^{\infty} e^{2\pi i\left(\frac{j+\alpha_1}{M}+l\right)M} \psi^{j}_{+,0}(z) \\
&\overset{j,M,l\in\mathbb{Z}}{=} e^{\left(i\pi M\frac{\mathrm{Im}\,z}{\mathrm{Im}\,\tau}+2\pi i\alpha_1\right)} \psi^{j}_{+,0}(z) \\
&= e^{iq\Lambda_1(z)} e^{2\pi i\alpha_1} \psi^{j}_{+,0}(z), \qquad (2.1.42) \\
\psi^{j}_{+,0}(z+\tau) &= \mathcal{N}_{T^2}\, e^{i\pi M(z+\tau)\,\mathrm{Im}\,(z+\tau)/\mathrm{Im}\,\tau} \sum_{l=-\infty}^{\infty} e^{i\pi\left(\frac{j+\alpha_1}{M}+l\right)^2 M\tau} e^{2\pi i\left(\frac{j+\alpha_1}{M}+l\right)(M(z+\tau)-\alpha_2)} \\
&= \mathcal{N}_{T^2}\, e^{(i\pi Mz+i\pi M\tau\,\mathrm{Im}\,(z+\tau)/\mathrm{Im}\,\tau+i\pi Mz\,\mathrm{Im}\,(z)/\mathrm{Im}\,\tau)} \sum_{l=-\infty}^{\infty} e^{i\pi\left(\frac{j+\alpha_1}{M}+l+1\right)^2 M\tau} e^{2\pi i\left(\frac{j+\alpha_1}{M}+l+1\right)(Mz-\alpha_2)} e^{-i\pi M\tau-2\pi i(Mz-\alpha_2)} \\
&= e^{(i\pi M\tau\,\mathrm{Im}\,z/\mathrm{Im}\,\tau-i\pi Mz+2\pi i\alpha_2)} \psi^{j}_{+,0}(z) \\
&= e^{\left(i\frac{\pi M}{\mathrm{Im}\,\tau}\mathrm{Im}\,(z\bar{\tau})+2\pi i\alpha_2\right)} \psi^{j}_{+,0}(z) \\
&= e^{iq\Lambda_2(z)} e^{2\pi i\alpha_2} \psi^{j}_{+,0}(z). \qquad (2.1.43)
\end{aligned}$$

We note that there is no normalizable solution to Eq.(2.1.35) for $M > 0$. We restrict our considerations to $M > 0$ in this paper, although we can analyze the case of $M < 0$ in a similar way. We emphasize that the number of the zero modes on T^2 with magnetic flux is $|M|$. Therefore, we obtain $|M|$-generation 4d chiral fermions. At the end of this subsection, we derive the normalization constant (2.1.39).

$$\begin{aligned}
&\int_{T^2} dzd\bar{z}\, [\psi^{j}_{\pm,0}(z)]^* \psi^{k}_{\pm,0}(z) \\
&= \int_{T^2} dzd\bar{z} \left(\mathcal{N}_{T^2}\, e^{i\pi Mz\,\mathrm{Im}\,z/\mathrm{Im}\,\tau}\, \vartheta\begin{bmatrix} \frac{j+\alpha_1}{M} \\ -\alpha_2 \end{bmatrix}(Mz, M\tau) \right)^* \mathcal{N}_{T^2}\, e^{i\pi Mz\,\mathrm{Im}\,z/\mathrm{Im}\,\tau}\, \vartheta\begin{bmatrix} \frac{k+\alpha_1}{M} \\ -\alpha_2 \end{bmatrix}(Mz, M\tau) \\
&= \int_{T^2} dzd\bar{z}\, |\mathcal{N}_{T^2}|^2 e^{-2\pi M\,(\mathrm{Im}\,z)^2/\mathrm{Im}\,\tau} \\
&\quad \times \sum_{l,l'=-\infty}^{\infty} \exp\left[i\pi M \left\{ \left(\frac{j+\alpha_1}{M}+l\right)^2 \tau - \left(\frac{k+\alpha_1}{M}+l'\right)^2 \bar{\tau} \right\} \right. \\
&\quad \left. + 2i\pi M \left\{ \left(\frac{j+\alpha_1}{M}+l\right) z - \left(\frac{k+\alpha_1}{M}+l'\right) \bar{z} \right\} - 2i\pi\alpha_2 \left\{ \left(\frac{j+\alpha_1}{M}+l\right) - \left(\frac{k+\alpha_1}{M}+l'\right) \right\} \right].
\end{aligned} \qquad (2.1.44)$$

To perform further calculations, rewrite Eq.(2.1.44) in the oblique coordinates (y_1, y_2) $(z = y_1 + y_2\tau)$.

$$
\begin{aligned}
&\int_{T^2} dz d\bar{z}\, [\psi^j_{\pm,0}(z)]^* \psi^k_{\pm,0}(z) \\
&= \mathrm{Im}\,\tau \int_0^1 dy_1 \int_0^1 dy_2 |\mathcal{N}_{T^2}|^2 e^{-2\pi M\,(\mathrm{Im}\,(\tau y_2))^2/\mathrm{Im}\,\tau} \\
&\quad \times \sum_{l,l'=-\infty}^{\infty} \exp\left[i\pi M \left\{ \left(\frac{j+\alpha_1}{M} + l\right)^2 \tau - \left(\frac{k+\alpha_1}{M} + l'\right)^2 \bar{\tau} \right\}\right. \\
&\quad \left. + 2i\pi M \left\{ \left(\frac{j+\alpha_1}{M} + l\right)(y_1 + y_2\tau) - \left(\frac{k+\alpha_1}{M} + l'\right)(y_1 + y_2\bar{\tau}) \right\} - 2i\pi\alpha_2 \left\{ \left(\frac{j+\alpha_1}{M} + l\right) - \left(\frac{k+\alpha_1}{M} + l'\right) \right\} \right] \\
&= \delta_{j,k}\delta_{l,l'} \mathrm{Im}\,\tau \int_0^1 dy_2 |\mathcal{N}_{T^2}|^2 e^{-2\pi M\,(\mathrm{Im}\,\tau) y_2^2} \sum_{l=-\infty}^{\infty} \exp\left[-2\pi M \left(\frac{j+\alpha_1}{M} + l\right)^2 \mathrm{Im}\,\tau - 4\pi M \left(\frac{j+\alpha_1}{M} + l\right) \mathrm{Im}\,\tau y_2 \right] \\
&= \delta_{j,k} \mathrm{Im}\,\tau \int_0^1 dy_2 |\mathcal{N}_{T^2}|^2 \sum_{l=-\infty}^{\infty} \exp\left[-2\pi M \mathrm{Im}\,\tau \left(y_2 + \frac{j+\alpha_1}{M} + l \right)^2 \right] \\
&\overset{y_2+l=y_2'}{=} \delta_{j,k} |\mathcal{N}_{T^2}|^2 \mathrm{Im}\,\tau \sum_{l=-\infty}^{\infty} \int_l^{l+1} dy_2' \exp\left[-2\pi M \mathrm{Im}\,\tau \left(y_2' + \frac{j+\alpha_1}{M} \right)^2 \right] \\
&= \delta_{j,k} |\mathcal{N}_{T^2}|^2 \mathrm{Im}\,\tau \int_{-\infty}^{\infty} dy_2' \exp\left[-2\pi M \mathrm{Im}\,\tau \left(y_2' + \frac{j+\alpha_1}{M} \right)^2 \right] \\
&= \delta_{j,k} |\mathcal{N}_{T^2}|^2 \sqrt{\frac{\mathrm{Im}\,\tau}{2M}},
\end{aligned}
\tag{2.1.45}
$$

In the second equality, we integrated for y_1. Eq.(2.1.39) has been obtained.

2.1.3 Kaluza-Klein mode functions on T^2

In this subsection, we discuss Kaluza-Klein mode functions on T^2, which will be used in the proof of the index theorem on the 2d toroidal orbifolds $T^2/\mathbb{Z}_N$ $(N = 2, 3, 4, 6)$.

From Eqs.(2.1.30) and (2.1.31), the mode functions $\psi^j_{\pm,n}(z)$ obey

$$
\begin{pmatrix} -4D_z D_{\bar{z}} & 0 \\ 0 & -4D_{\bar{z}} D_z \end{pmatrix} \begin{pmatrix} \psi^j_{+,n}(z) \\ \psi^j_{-,n}(z) \end{pmatrix} = m_n^2 \begin{pmatrix} \psi^j_{+,n}(z) \\ \psi^j_{-,n}(z) \end{pmatrix}. \tag{2.1.46}
$$

Here, we define the following operator

$$
\Delta \equiv -2(D_z D_{\bar{z}} + D_{\bar{z}} D_z) \tag{2.1.47}
$$

Δ and D_z, $D_{\bar{z}}$ satisfy the following relations

$$
[\Delta, D_z] = \frac{4\pi M}{\mathrm{Im}\tau} D_z, \tag{2.1.48}
$$

$$
[\Delta, D_{\bar{z}}] = -\frac{4\pi M}{\mathrm{Im}\tau} D_{\bar{z}}, \tag{2.1.49}
$$

where we use $[D_z, D_{\bar{z}}] = \pi M/\mathrm{Im}\tau$. These can be used to construct the excitation modes using the creation and annihilation operators in a manner similar to the discussion of the harmonic oscillator in quantum mechanics. The eigenvalue equation (2.1.46) can easily be solved by introducing the annihilation and the creation operators as

$$\hat{a}_+ \equiv i\sqrt{\frac{\mathrm{Im}\tau}{\pi M}} D_{\bar{z}} = i\sqrt{\frac{\mathrm{Im}\tau}{\pi M}}\left(\partial_{\bar{z}} + \frac{\pi M}{2\mathrm{Im}\tau} z\right), \tag{2.1.50}$$

$$\hat{a}_+^\dagger \equiv i\sqrt{\frac{\mathrm{Im}\tau}{\pi M}} D_z = i\sqrt{\frac{\mathrm{Im}\tau}{\pi M}}\left(\partial_z - \frac{\pi M}{2\mathrm{Im}\tau}\bar{z}\right) \tag{2.1.51}$$

with

$$[\hat{a}_+, \hat{a}_+^\dagger] = 1. \tag{2.1.52}$$

Δ is described by $\hat{a}_+, \hat{a}_+^\dagger$ as

$$\Delta = \frac{4\pi M}{\mathrm{Im}\tau}\left(\hat{a}_+^\dagger \hat{a}_+ + \frac{1}{2}\right) \tag{2.1.53}$$

In terms of the creation operator $\hat{a}_+^\dagger$ and the zero mode functions $\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,0}(z)$, the massive mode functions $\psi^{(j+\alpha_1,\alpha_2)}_{T^2\pm,n}(z)$ can be constructed as

$$\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(z) \equiv \frac{1}{\sqrt{n!}}(\hat{a}_+^\dagger)^n \psi^{(j+\alpha_1,\alpha_2)}_{T^2+,0}(z), \tag{2.1.54}$$

$$\begin{aligned}\psi^{(j+\alpha_1,\alpha_2)}_{T^2-,n}(z) &= \frac{2}{m_n} D_{\bar{z}} \psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(z) \\ &= -i\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n-1}(z) \qquad (n = 1, 2, \cdots).\end{aligned} \tag{2.1.55}$$

Here, in the first equality of Eq.(2.1.55), we have used the relation (2.1.31) and also used the mass eigenvalue

$$m_n = \sqrt{\frac{4\pi M}{\mathrm{Im}\tau} n} \qquad (n = 0, 1, 2, \cdots). \tag{2.1.56}$$

It should be emphasized that there is no zero mode function for $\psi_{-,0}$ with $M > 0$, but there exist the non-zero mode functions for both $\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(z)$ and $\psi^{(j+\alpha_1,\alpha_2)}_{T^2-,n}(z)$ ($n = 1, 2, \cdots$ and $j = 0, 1, \cdots, |M| - 1$). The mass spectrum of $\psi^{(j+\alpha_1,\alpha_2)}_{T^2\pm,n}(z)$ is illustrated in Figure 2.1.

2.1.4 The Atiyah-Singer index theorem on magnetized torus

The AS index theorem on the torus with magnetic flux is

$$n_+ - n_- = \frac{q}{2\pi}\int_{T^2} F = M, \tag{2.1.57}$$

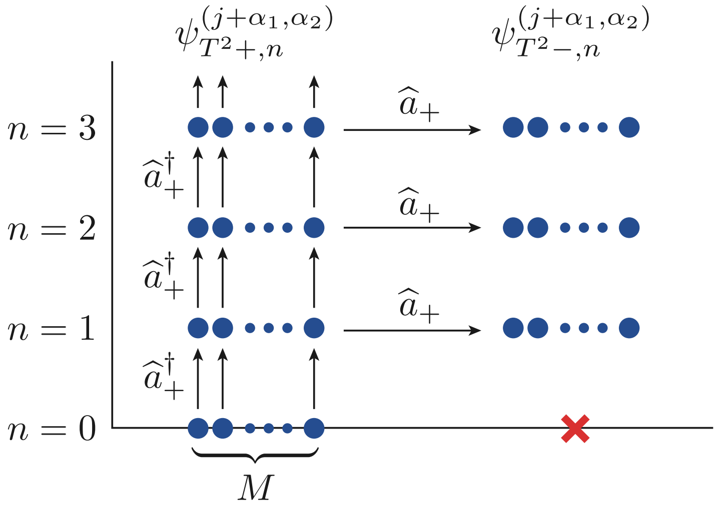

Figure 2.1: The mass spectrum of $\psi^{(j+\alpha_1,\alpha_2)}_{T^2\pm,n}(z)$ for $M > 0$ is presented. There exist M degenerate zero mode solutions $\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,0}(z)$. The zero modes and their Kaluza-Klein modes are visually represented by blue-filled circles. The arrows mean that the $(n+1)$th modes $\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n+1}(z)$ can be obtained by acting the creation operator $\hat{a}_+^\dagger$ on nth modes $\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(z)$, and that the nth modes $\psi^{(j+\alpha_1,\alpha_2)}_{T^2-,n}(z)$ can be obtained by employing the annihilation operator $\hat{a}_+$ on the nth modes $\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(z)$.

where $n_\pm$ are the numbers of ± chiral zero modes. Refer to the appendix A for a detailed proof of Eq. (2.1.57). The index theorem (2.1.57) shows that the number of the independent chiral zero modes is determined by the magnetic flux quantization number M on the magnetized torus. In other words, the generation number in this model is precisely M. We emphasize that the index $n_+ - n_-$ is contingent solely upon the flux.

Chapter 3

Magnetized $T^2/\mathbb{Z}_N$ orbifold model

3.1 Mode functions on $T^2/\mathbb{Z}_N$ $(N = 2, 3, 4, 6)$ orbifolds

In this section, we construct mode functions on the $T^2/\mathbb{Z}_N$ orbifolds with magnetic flux. The $T^2/\mathbb{Z}_N$ orbifold is defined by the torus identification (2.1.1) and the identification under the $\mathbb{Z}_N$ rotation

$$z \sim \omega z. \tag{3.1.1}$$

where $\omega = e^{i2\pi/N} \in \mathbb{Z}_N$. First, let's consider the possible N of orbifolds. In order for the identification of the orbifold (3.1.1) to be consistent with the identification of the torus (2.1.1), the lattice points must remain lattice points even after a $\mathbb{Z}_N$ rotation. Since $z = 1$ is a lattice point, the point obtained by rotating it by $\mathbb{Z}_N$, $z = \omega$, must also be a lattice point. Similarly, the point obtained by $\mathbb{Z}_N$ inverse rotation, $z = \omega^{-1}$, must also be a lattice point. Additionally, the sum of lattice points must be a lattice point, so

$$z = \omega + \omega^{-1} = e^{i2\pi/N} + e^{-i2\pi/N} = 2\cos\left(\frac{2\pi}{N}\right) = n \qquad (n \in \mathbb{Z}). \tag{3.1.2}$$

Since $-1 \leq \cos(2\pi/N) \leq 1$, we get $n = -2, -1, 0, 1, 2$. Thus, $N = 2, 3, 4, 6$. Furthermore, for $N = 3, 4, 6$, it must be that $\omega = \tau$. This is because, for $N = 3, 4, 6$, lattice point $z = \omega$ is not real, so it must coincide with lattice points, $z = \tau$. Therefore, we only consider the $T^2/\mathbb{Z}_N$ $(N = 2, 3, 4, 6)$ orbifolds.

3.1.1 Zero mode functions on $T^2/\mathbb{Z}_2$

In this subsection, we derive $\mathbb{Z}_2$ zero mode eigenstates on the $T^2/\mathbb{Z}_2$ orbifold. The $T^2/\mathbb{Z}_2$ orbifold is defined by the torus identification (2.1.1) and an additional $\mathbb{Z}_2$ one

$$z \sim -z. \tag{3.1.3}$$

In the case of $T^2/\mathbb{Z}_2$, there is no restriction on τ except for $\mathrm{Im}\tau > 0$. We often use $\omega = e^{i\pi} = -1$ for the $T^2/\mathbb{Z}_2$ orbifold. To maintain consistency with the $\mathbb{Z}_2$ orbifold identification, the SS phase (α_1, α_2) must be quantized as [32]

$$(\alpha_1, \alpha_2) = (0,0), (1/2,0), (0,1/2), (1/2,1/2). \tag{3.1.4}$$

The $\mathbb{Z}_2$ zero mode eigenstates $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,0}(z)_\eta$ on $T^2/\mathbb{Z}_2$ satisfy

$$\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,0}(-z)_\eta = \eta\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,0}(z)_\eta, \tag{3.1.5}$$

where $\eta = \pm 1$ are $\mathbb{Z}_2$ eigenvalues. In terms of the zero mode functions $\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,0}(z)$ on T^2, the $\mathbb{Z}_2$ eigenstates $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,0}(z)_\eta$ on $T^2/\mathbb{Z}_2$ can be constructed as

$$\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,0}(z)_{\pm 1} \equiv \mathcal{N}^{(j+\alpha_1,\alpha_2)}_{+,\pm 1}[\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,0}(z) \pm \psi^{(j+\alpha_1,\alpha_2)}_{T^2+,0}(-z)], \tag{3.1.6}$$

where $j = 0, 1, \cdots, M-1$, and $\mathcal{N}^{(j+\alpha_1,\alpha_2)}_{+,\pm 1}$ are normalization constants.

It should be noticed that all of the $\mathbb{Z}_2$ eigenstates (3.1.6) are not linearly independent. According to Eq.(2.1.37) and properties of the theta function, the zero modes $\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,0}$ satisfy

$$\begin{aligned}\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,0}(-z) &= \psi^{(M-(j+\alpha_1),-\alpha_2)}_{T^2+,0}(z) \\ &= e^{-4\pi i\frac{j+\alpha_1}{M}\alpha_2}\psi^{(M-(j+\alpha_1),\alpha_2)}_{T^2+,0}(z).\end{aligned} \tag{3.1.7}$$

Therefore, we get

$$\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,0}(z)_{\pm 1} = \mathcal{N}^{(j+\alpha_1,\alpha_2)}_{+,\pm 1}[\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,0}(z) \pm e^{-4\pi i\frac{j+\alpha_1}{M}\alpha_2}\psi^{(M-(j+\alpha_1),\alpha_2)}_{T^2+,0}(z)]. \tag{3.1.8}$$

Consequently, the linearly independent $\mathbb{Z}_2$ eigenstates depend on M, (α_1, α_2) and η, and are explicitly detailed in Table 3.1. The normalization constants $\mathcal{N}^{(j+\alpha_1,\alpha_2)}_{+,\eta}$ for the $\mathbb{Z}_2$ zero mode eigenstates are determined by the orthonormality condition

$$\int_{T^2/\mathbb{Z}_2} dzd\bar{z}\ [\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,0}(z)_\eta]^*\psi^{(k+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,0}(z)_\eta = \delta^{jk}, \tag{3.1.9}$$

and turn out to depend on j as well as M, (α_1, α_2) and η, as illustrated in Table 3.2. As we will observe in the next section, the set of the independent $\mathbb{Z}_2$ eigenstates and the values of the normalization constants $\mathcal{N}^{(j+\alpha_1,\alpha_2)}_{+,\eta}$ play a crucial role in deriving the index theorem on $T^2/\mathbb{Z}_2$.

3.1.2 $\mathbb{Z}_2$ eigen mode functions and mass spectrum

The $\mathbb{Z}_2$ eigen mode functions $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2\pm,n}(z)_\eta$ satisfy

$$\begin{pmatrix}-4D_zD_{\bar{z}} & 0 \\ 0 & -4D_{\bar{z}}D_z\end{pmatrix}\begin{pmatrix}\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,n}(z)_\eta \\ \psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2-,n}(z)_\eta\end{pmatrix} = m_n^2\begin{pmatrix}\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,n}(z)_\eta \\ \psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2-,n}(z)_\eta\end{pmatrix} \tag{3.1.10}$$

(α_1, α_2)	M	$\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2,+,0}(z)_{+1}$	$\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2,+,0}(z)_{-1}$
$(0,0)$	even	$\frac{M}{2}+1 \quad (j=0,1,\cdots,\frac{M}{2})$	$\frac{M}{2}-1 \quad (j=1,2,\cdots,\frac{M}{2}-1)$
$(0,0)$	odd	$\frac{M+1}{2} \quad (j=0,1,\cdots,\frac{M-1}{2})$	$\frac{M-1}{2} \quad (j=1,2,\cdots,\frac{M-1}{2})$
$(\frac{1}{2},0)$	even	$\frac{M}{2} \quad (j=0,1,\cdots,\frac{M}{2}-1)$	$\frac{M}{2} \quad (j=0,1,\cdots,\frac{M}{2}-1)$
$(\frac{1}{2},0)$	odd	$\frac{M+1}{2} \quad (j=0,1,\cdots,\frac{M-1}{2})$	$\frac{M-1}{2} \quad (j=0,1,\cdots,\frac{M-3}{2})$
$(0,\frac{1}{2})$	even	$\frac{M}{2} \quad (j=0,1,\cdots,\frac{M}{2}-1)$	$\frac{M}{2} \quad (j=1,2,\cdots,\frac{M}{2})$
$(0,\frac{1}{2})$	odd	$\frac{M+1}{2} \quad (j=0,1,\cdots,\frac{M-1}{2})$	$\frac{M-1}{2} \quad (j=1,2,\cdots,\frac{M-1}{2})$
$(\frac{1}{2},\frac{1}{2})$	even	$\frac{M}{2} \quad (j=0,1,\cdots,\frac{M}{2}-1)$	$\frac{M}{2} \quad (j=0,1,\cdots,\frac{M}{2}-1)$
$(\frac{1}{2},\frac{1}{2})$	odd	$\frac{M-1}{2} \quad (j=0,1,\cdots,\frac{M-3}{2})$	$\frac{M+1}{2} \quad (j=0,1,\cdots,\frac{M-1}{2})$

Table 3.1: The number of zero modes on the $T^2/\mathbb{Z}_2$ orbifold.

(α_1, α_2)	M	$\mathcal{N}^{(j)}_{+,+1}$	$\mathcal{N}^{(j)}_{+,-1}$
$(0,0)$	even	$\frac{1}{\sqrt{2}}\,(j=0,\frac{M}{2}), \quad 1\,(j=\text{others})$	$1\,(j=1,2,\cdots,\frac{M}{2}-1)$
$(0,0)$	odd	$\frac{1}{\sqrt{2}}\,(j=0), \quad 1\,(j=\text{others})$	$1\,(j=1,2,\cdots,\frac{M-1}{2})$
$(\frac{1}{2},0)$	even	$1\,(j=0,1,\cdots,\frac{M}{2}-1)$	$1\,(j=0,1,\cdots,\frac{M}{2}-1)$
$(\frac{1}{2},0)$	odd	$\frac{1}{\sqrt{2}}\,(j=\frac{M-1}{2}), \quad 1\,(j=\text{others})$	$1\,(j=0,1,\cdots,\frac{M-3}{2})$
$(0,\frac{1}{2})$	even	$\frac{1}{\sqrt{2}}\,(j=0), \quad 1\,(j=\text{others})$	$\frac{1}{\sqrt{2}}\,(j=\frac{M}{2}), \quad 1\,(j=\text{others})$
$(0,\frac{1}{2})$	odd	$\frac{1}{\sqrt{2}}\,(j=0), \quad 1\,(j=\text{others})$	$1\,(j=1,2,\cdots,\frac{M-1}{2})$
$(\frac{1}{2},\frac{1}{2})$	even	$1\,(j=0,1,\cdots,\frac{M}{2}-1)$	$1\,(j=0,1,\cdots,\frac{M}{2}-1)$
$(\frac{1}{2},\frac{1}{2})$	odd	$1\,(j=0,1,\cdots,\frac{M-3}{2})$	$\frac{1}{\sqrt{2}}\,(j=\frac{M-1}{2}), \quad 1\,(j=\text{others})$

Table 3.2: The normalization constants of zero modes on the $T^2/\mathbb{Z}_2$ orbifold.

with

$$\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,n}(-z)_\eta = \eta\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,n}(z)_\eta\,, \tag{3.1.11}$$

$$\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2-,n}(-z)_\eta = -\eta\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2-,n}(z)_\eta\,. \tag{3.1.12}$$

We note that if the $\mathbb{Z}_2$ eigenvalue of $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,n}(z)_\eta$ is η, then that of $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2-,n}(z)_\eta$ must be $\omega\eta = -\eta$. The additional factor $\omega = -1$ arises from a rotation matrix acting on 2d spinors, and it is also understood from the relation (2.1.31) with the property $\hat{a}_+(-z) = -\hat{a}_+(z)$.

Regarding the zero modes $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,0}(z)_\eta$ on the $T^2/\mathbb{Z}_2$ orbifold, the $\mathbb{Z}_2$ massive mode func-

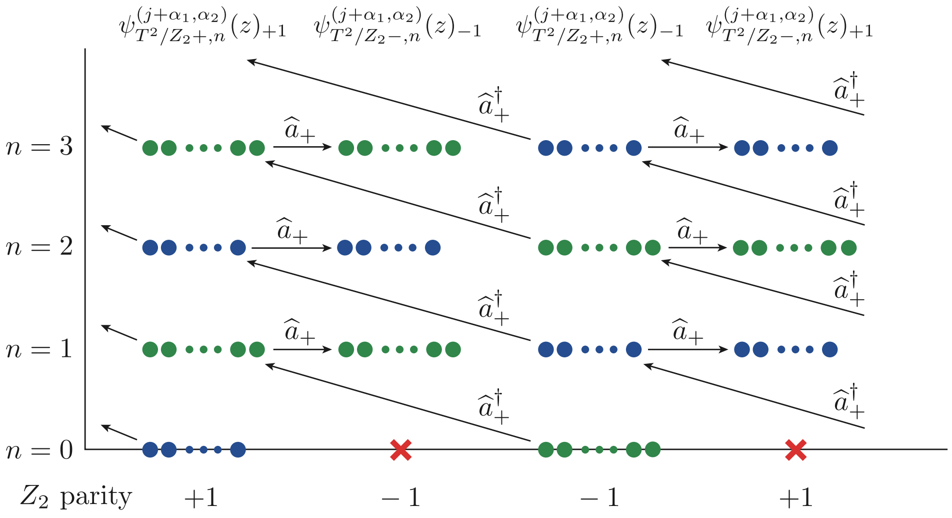

Figure 3.1: The mass spectrum of $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2\pm,n}(z)_\eta$ for $M > 0$. The blue (green)-filled circles correspond to zero modes with $\mathbb{Z}_2$ parity $+1$ (-1) and their Kaluza-Klein modes. The arrows mean that $\hat{a}_+^\dagger$ operates on the nth modes $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,n}(z)_\eta$ and creates the $(n+1)$th modes the next modes $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,n+1}(z)_{-\eta}$ with the opposite $\mathbb{Z}_N$ eigenvalue, and also that $\hat{a}_+$ operates on the nth modes $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,n}(z)_\eta$ and creates the nth modes $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2-,n}(z)_{-\eta}$.

tions $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2\pm,n}(z)_\eta$ $(n = 1, 2, \cdots)$ can be constructed as

$$\begin{aligned}
\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,n}(z)_\eta &\equiv \frac{1}{\sqrt{n!}}[\hat{a}_+^\dagger(z)]^n \psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,0}(z)_{\omega^n\eta} \\
&= \mathcal{N}^{(j+\alpha_1,\alpha_2)}_{+,\omega^n\eta} \sum_{l=0}^{1} \eta^{-l} \psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(\omega^l z), \qquad (3.1.13)\\
\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2-,n}(z)_\eta &= \frac{2}{m_n} D_{\bar{z}} \psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,n}(z)_\eta \\
&= -i\mathcal{N}^{(j+\alpha_1,\alpha_2)}_{+,\omega^n\eta} \sum_{l=0}^{1} (\omega\eta)^{-l} \psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n-1}(\omega^l z), \qquad (3.1.14)
\end{aligned}$$

where $\omega = -1$ and $\eta = \pm 1$. We notice that the number of the degeneracy of $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2\pm,n}(z)_\eta$ with a fixed η depends on n, in general, because that of $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,0}(z)_{\omega^n\eta}$ does from Table 3.1. The mass spectrum of $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2\pm,n}(z)_\eta$ is illustrated in Figure 3.1.

3.2 Zero mode functions on $T^2/\mathbb{Z}_N$ $(N = 3, 4, 6)$

The $T^2/\mathbb{Z}_N$ $(N = 3, 4, 6)$ orbifolds are defined by the identification

$$z \sim \omega z \qquad (\omega = e^{i2\pi/N}) \qquad (3.2.1)$$

in addition to the torus identification (2.1.1). The SS phase (α_1, α_2) must be quantized as

$$\alpha = \alpha_1 = \alpha_2 = \begin{cases} 0, 1/3, 2/3 & (M = \text{even}) \\ 1/6, 3/6, 5/6 & (M = \text{odd}) \end{cases} \quad \text{for } T^2/\mathbb{Z}_3, \tag{3.2.2}$$

$$\alpha = \alpha_1 = \alpha_2 = 0, 1/2 \qquad \text{for } T^2/\mathbb{Z}_4, \tag{3.2.3}$$

$$\alpha = \alpha_1 = \alpha_2 = \begin{cases} 0 & (M = \text{even}) \\ 1/2 & (M = \text{odd}) \end{cases} \quad \text{for } T^2/\mathbb{Z}_6. \tag{3.2.4}$$

The $\mathbb{Z}_N$ eigenstates $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N\pm,n}(z)_\eta$ satisfy the following eigenvalue equations

$$\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(\omega z)_\eta = \eta\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z)_\eta, \tag{3.2.5}$$

$$\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N-,n}(\omega z)_\eta = \omega\eta\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N-,n}(z)_\eta, \tag{3.2.6}$$

where $\eta = \omega^l$ $(l = 0, 1, \cdots, N-1)$ is the $\mathbb{Z}_N$ eigenvalue with $\omega = e^{i2\pi/N}$. In terms of the zero mode functions $\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,0}(z)$ on the torus T^2, the $\mathbb{Z}_N$ zero mode functions $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,0}(z)_\eta$ on the $T^2/\mathbb{Z}_N$ orbifolds can be constructed as

$$\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,0}(z)_\eta \equiv \mathcal{N}^{(j+\alpha_1,\alpha_2)}_{+,\eta} \sum_{l=0}^{N-1} \eta^{-l}\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,0}(\omega^l z) = e^{-\frac{\pi M}{2\mathrm{Im}\tau}|z|^2} h^{(j+\alpha_1,\alpha_2)}_{+,\eta}(z), \tag{3.2.7}$$

$$h^{(j+\alpha_1,\alpha_2)}_{+,\eta}(z) = \mathcal{N}^{(j+\alpha_1,\alpha_2)}_{+,\eta} \sum_{l=0}^{N-1} \eta^{-l} g^{(j+\alpha_1,\alpha_2)}(\omega^l z). \tag{3.2.8}$$

In this context, $h^{(j+\alpha_1,\alpha_2)}_{+,\eta}(z)$ represents the holomorphic function of z.

3.2.1 Fixed points and the winding numbers

A significant characteristic of the $T^2/\mathbb{Z}_N$ orbifolds is the existence of the fixed points z_I^{fp} defined by

$$z_I^{\mathrm{fp}} = \omega z_I^{\mathrm{fp}} + m + n\tau \qquad \text{for } {}^\exists m, n \in \mathbb{Z}. \tag{3.2.9}$$

The $\mathbb{Z}_N$ fixed points on the $T^2/\mathbb{Z}_N$ orbifolds are given by

$$z_I^{\mathrm{fp}} = \begin{cases} 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} & \text{on } T^2/\mathbb{Z}_2, \\ 0, \frac{2+\tau}{3}, \frac{1+2\tau}{3} & \text{on } T^2/\mathbb{Z}_3, \\ 0, \frac{1+\tau}{2} & \text{on } T^2/\mathbb{Z}_4, \\ 0 & \text{on } T^2/\mathbb{Z}_6, \end{cases} \tag{3.2.10}$$

and the respective values of (m, n) in Eq.(3.2.9) are

$$(m, n) = \begin{cases} (0,0), (1,0), (0,1), (1,1) & \text{on } T^2/\mathbb{Z}_2, \\ (0,0), (1,0), (1,1) & \text{on } T^2/\mathbb{Z}_3, \\ (0,0), (1,0) & \text{on } T^2/\mathbb{Z}_4, \\ (0,0) & \text{on } T^2/\mathbb{Z}_6. \end{cases} \tag{3.2.11}$$

Note that there are additional fixed points for $N = 4, 6$, since the $\mathbb{Z}_4$ ($\mathbb{Z}_6$) group includes $\mathbb{Z}_2$ ($\mathbb{Z}_2$ and $\mathbb{Z}_3$) as its subgroup. Although they are not invariant under the $\mathbb{Z}_4$ ($\mathbb{Z}_6$) transformation, they are invariant under the $\mathbb{Z}_2$ ($\mathbb{Z}_2$ and $\mathbb{Z}_3$) transformation up to the torus shifts. The additional fixed points are identified as

$$\mathbb{Z}_2 \text{ fixed points}: \quad z_I^{\text{fp}} = \frac{1}{2}, \frac{\tau}{2} \qquad \text{on } T^2/\mathbb{Z}_4, \tag{3.2.12}$$

$$\mathbb{Z}_3 \text{ fixed points}: \quad z_I^{\text{fp}} = \frac{1+\tau}{3}, \frac{2+2\tau}{3} \qquad \text{on } T^2/\mathbb{Z}_6, \tag{3.2.13}$$

$$\mathbb{Z}_2 \text{ fixed points}: \quad z_I^{\text{fp}} = \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \qquad \text{on } T^2/\mathbb{Z}_6. \tag{3.2.14}$$

It is important to emphasize that the fixed points depicted in Figure 3.2 are singular points on the $T^2/\mathbb{Z}_N$ orbifolds. The existence of these fixed points is indeed the decisive difference from smooth manifolds like the torus.

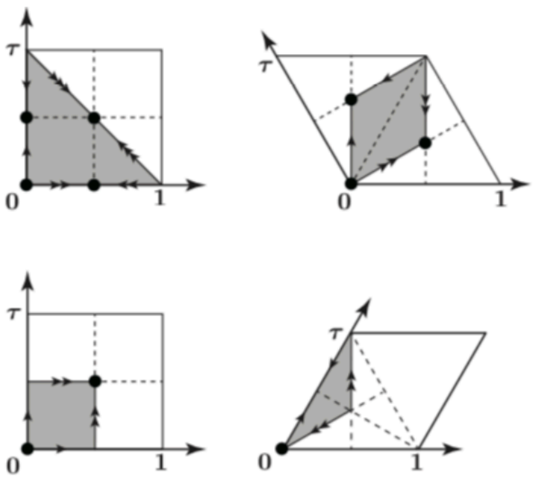

Figure 3.2: The fixed points of T^2/Z_N $(N = 2, 3, 4, 6)$ orbifolds

We introduce winding numbers at the fixed points on the $T^2/\mathbb{Z}_N$ orbifolds. Let us define the winding number for the $\mathbb{Z}_N$ eigenstate $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z)_\eta$ at a fixed point z_p^f as

$$\chi_p(\eta, (\alpha_1, \alpha_2), M) \equiv \frac{1}{2\pi i} \oint_{C_p} d\boldsymbol{l} \cdot \nabla \log(\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z)_\eta). \tag{3.2.15}$$

Here, C_p denotes a sufficiently small circle encircled anticlockwise around the fixed point $z = z_p^f$. The line integral along the contour C_p gives a winding number, indicating how many times $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z)_\eta$ wraps around the origin. It should be noted that the winding numbers at fixed points are determined only by the $\mathbb{Z}_N$ transformation (3.2.5) and the boundary conditions (2.1.14).

We define the winding numbers $\chi_{\pm p}$ around the $\mathbb{Z}_N$ fixed points z_p^f as[1]

$$\chi_{+p} \equiv \chi_p(\eta, (\alpha_1, \alpha_2), M), \tag{3.2.17}$$
$$\chi_{-p} \equiv -\chi_p(\omega\eta, (\alpha_1, \alpha_2), M) + N. \tag{3.2.18}$$

Let us examine the winding numbers $\chi_{\pm p}$ around the fixed point z_p^f. From Eqs.(2.1.14) and (3.2.5), we can show that $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z)_\eta$ satisfies the relation

$$\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z_p^f + \omega Z)_\eta = e^{i(mq\Lambda_1(\omega Z)+nq\Lambda_2(\omega Z))}\eta e^{im(q\Lambda_1(z_p^f)+2\pi\alpha_1)+in(q\Lambda_2(z_p^f)+2\pi\alpha_2)+i\pi Mmn} \times \psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z_p^f + Z)_\eta, \tag{3.2.19}$$

Here, $Z = z - z_p^f$, and (m, n) are determined by Eq. (3.2.9), with the values of (m, n) at each fixed point indicated by Eq.(3.2.11). It follows that by taking the limit as $Z \to 0$, the winding numbers $\chi_{\pm p}$ around the $\mathbb{Z}_N$ fixed point z_p^f are found to be

$$\chi_{+p} = N\left(\frac{mq}{2\pi}\Lambda_1(z_p^f) + m\alpha_1 + \frac{nq}{2\pi}\Lambda_2(z_p^f) + n\alpha_2 + \frac{1}{2}Mmn\right) + l \qquad \text{mod } N, \tag{3.2.20}$$
$$\chi_{-p} = -N\left(\frac{mq}{2\pi}\Lambda_1(z_p^f) + m\alpha_1 + \frac{nq}{2\pi}\Lambda_2(z_p^f) + n\alpha_2 + \frac{1}{2}Mmn\right) - l - 1 + N \qquad \text{mod } N, \tag{3.2.21}$$

where $\eta = \omega^l$ $(l = 0, 1, \cdots, N-1)$. Since the winding numbers $\chi_{\pm p}$ can be obtained from Eq.(3.2.19) up to mod N around the $\mathbb{Z}_N$ fixed point z_p^f, we restrict the values of $\chi_{\pm p}$ to $0, 1, \cdots, N-1$.

As seen here, $T^2/\mathbb{Z}_N$ orbifolds have the fixed points, and around these fixed points, winding numbers can be defined. These are properties ”added” due to the presence of singular points, and as investigated in previous paper [1], the number of chiral zero modes on the $T^2/\mathbb{Z}_N$ $(N = 2, 3, 4, 6)$ orbifolds was found and the index was given by

$$n_+ - n_- = \frac{M - V_+}{N} + 1. \tag{3.2.22}$$

Here, V_+ is the sum of the winding numbers for + chirality modes at the fixed points of the $T^2/\mathbb{Z}_N$ $(N = 2, 3, 4, 6)$ orbifolds. We call this formula the “zero-mode counting formula”. As can be seen from this, compared to the index (2.1.57) of a smooth manifold like the torus, additional contributions from winding numbers emerge. It is not, however, clear whether this formula can be regarded as the index theorem on the $T^2/\mathbb{Z}_N$ orbifolds with magnetic flux.

---

[1] We may define χ_{-p} as [39]

$$\chi_{-p} \equiv \chi_p(\bar{\omega}\bar{\eta}, (-\alpha_1, -\alpha_2), -M). \tag{3.2.16}$$

It is interesting to note that Eqs.(3.2.17) and (3.2.18) turn out to lead to the same results, as verified by explicit computations.

This is because in Ref. [1] $n_+ - n_-$ and $\frac{M-V_+}{N} + 1$ have been computed separately, and have been verified by comparing the values for each case of M, the $\mathbb{Z}_N$ eigenvalue, the Scherk-Schwarz (SS) twist phase (α_1, α_2) and N. One of our primary objectives is to prove this formula as the AS index theorem.

Chapter 4

Index theorem on magnetized blow-up manifold of $T^2/\mathbb{Z}_N$ orbifold

In this section, we review our paper [11]. The difficulty in proving the Atiyah-Singer index theorem on magnetized $T^2/\mathbb{Z}_N$ orbifolds arises from the presence of singular points. Therefore, we apply the Atiyah-Singer index theorem to the blow-up manifold of $T^2/\mathbb{Z}_N$ orbifold, which is constructed by removing the fixed points.

4.1 Blow-up manifold of magnetized $T^2/\mathbb{Z}_N$ orbifold

We construct the blow-up manifolds of the magnetized $T^2/\mathbb{Z}_N$ orbifolds by removing the singular points. To achieve this, we cut out the singularities of the magnetized $T^2/\mathbb{Z}_N$ orbifolds and attach smooth manifolds (parts of S^2) to them, as illustrated in Figure 4.1. The smooth manifolds without singularities are referred to as the blow-up manifolds of the $T^2/\mathbb{Z}_N$ orbifolds. Then, we can directly apply the AS index theorem to the blow-up manifolds. We then need to connect mode functions on the orbifolds with those on parts of S^2 while preserving the orbifold information. Through this analysis, it turns out that winding numbers of mode functions on the $T^2/\mathbb{Z}_N$ orbifolds are related to localized flux and localized curvature.

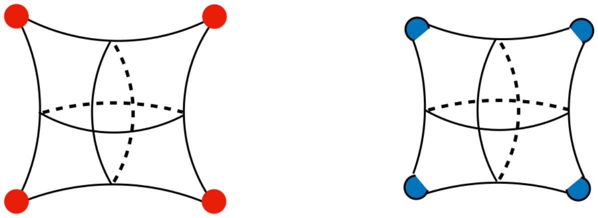

Figure 4.1: The left figure shows $T^2/\mathbb{Z}_2$ orbifold, with the red points indicating the fixed points of $T^2/\mathbb{Z}_2$ orbifold. By cutting around the fixed points and embedding the parts of S^2 as caps, we can construct the blow-up manifold, as illustrated in the right figure.

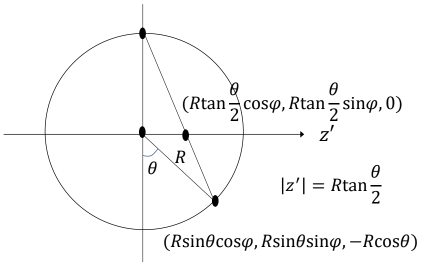

Figure 4.2: The cross section of S^2 with the radius R is depicted. We project a point of S^2 with the 3d coordinate, $(R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, -R\cos\theta)$, from the north pole of S^2, onto the complex plane passing through the center of S^2. The 3d coordinate of this point on the complex plane is $(R\tan\frac{\theta}{2}\cos\varphi, R\tan\frac{\theta}{2}\sin\varphi, 0)$, where (R, θ, φ) are spherical coordinate parameters. We define the complex coordinate on the complex plane $\mathbb{CP}^1$, z', such that $z' = |z'|e^{i\varphi} = R\tan\frac{\theta}{2}e^{i\varphi}$ at the point with the 3d coordinate $(R\tan\frac{\theta}{2}\cos\varphi, R\tan\frac{\theta}{2}\sin\varphi, 0)$. Then, we represent the coordinate of a point on S^2 by the complex coordinate of the projected point on $\mathbb{CP}^1$, which is z'.

4.1.1 Magnetized S^2

In this subsection, we review zero mode functions on S^2 with magnetic flux [41]. Let z' be the complex coordinate on $S^2 \simeq \mathbb{CP}^1$ defined by projecting a point of S^2 into the complex plane passing through the center of S^2 from the north pole of S^2, as illustrated in Figure 4.2. The radius of S^2 is taken to be R.

The magnetic flux on S^2 is quantized as

$$\frac{1}{2\pi}\int_{S^2} F' = M', \tag{4.1.1}$$

where M' is an integer. The field strength is

$$\frac{F'}{2\pi} = \frac{i}{2\pi}\frac{R^2 M'}{(R^2+|z'|^2)^2}dz' \wedge d\bar{z}'. \tag{4.1.2}$$

The gauge potentials on S^2 are given by

$$A_{\bar{z}'} = \frac{i}{2}\frac{M'}{R^2+|z'|^2}z', \qquad A_{z'} = -\frac{i}{2}\frac{M'}{R^2+|z'|^2}\bar{z}'. \tag{4.1.3}$$

The mode functions on the magnetized S^2 satisfy the Dirac equations

$$\frac{R^2+|z'|^2}{R}i\left(\partial_{\bar{z}'} + i\frac{1}{2}\omega_{\bar{z}'} - iA_{\bar{z}'}\right)\psi_{S^2,+,n}(z') = m_n\psi_{S^2,-,n}(z'), \tag{4.1.4}$$

$$\frac{R^2+|z'|^2}{R}i\left(\partial_{z'} - i\frac{1}{2}\omega_{z'} - iA_{z'}\right)\psi_{S^2,-,n}(z') = m_n\psi_{S^2,+,n}(z') \tag{4.1.5}$$

with

$$\omega_{\bar{z}'} = \frac{i}{2}\frac{2}{R^2+|z'|^2}z', \qquad \omega_{z'} = -\frac{i}{2}\frac{2}{R^2+|z'|^2}\bar{z}', \tag{4.1.6}$$

where $\omega_{\bar{z}'}$ and $\omega_{z'}$ represent the spin connections arising from the non-vanishing curvature on S^2:

$$\frac{1}{2\pi}\int_{S^2} R' = \chi(S^2) = 2. \tag{4.1.7}$$

Here, R' is the curvature on S^2, and χ is the Euler characteristic. Note that the spin connections (4.1.6) can be obtained by replacing the flux M' in the gauge potentials (4.1.3) by the Euler characteristic $\chi(S^2) = 2$.

The positive chirality zero mode solutions of Eq.(4.1.4) with $m_n = 0$ are expressed as

$$\psi_{S^2,+,0}(z') = \frac{f_+(z')}{(R^2+|z'|^2)^{\frac{M'-1}{2}}}, \tag{4.1.8}$$

where $f_+(z')$ is a holomorphic function of z'. These solutions are normalizable and well-defined on S^2 only if $M' > 0$ and $f_+(z')$ is represented as a $(M'-1)$th order polynomial. This implies that the number of the independent solutions is M'. On the other hand, normalizable and well-defined negative chirality zero modes on S^2 are obtained similarly, but only if $M' < 0$. In this case, an anti-holomorphic function $f_-(\bar{z}')$ is expressed as a $(|M'|-1)$th order polynomial.

The above results are consistent with the AS index theorem on the magnetized S^2, i.e.

$$n_+ - n_- = \frac{1}{2\pi}\int_{S^2} F' = M'. \tag{4.1.9}$$

The number of the chiral zero modes is indeed determined by the flux quantization number M'. It is crucial to emphasize that in the magnetized S^2 model, although both the flux and the curvature exist, only the flux contributes to the AS index theorem.

4.1.2 The relation between winding number, localized flux, and localized curvature at the fixed point

In order to directly apply the AS index theorem, it is necessary for manifolds to be smooth without singularities. Since the $T^2/\mathbb{Z}_N$ orbifolds have the singularities at the fixed points, we replace cones around the fixed points with parts of S^2 to remove the singularities (see Figure 4.1). It is crucial to construct the smooth blow-up manifolds without losing the orbifold information. Specifically, preserving information on the winding numbers of the mode functions at the fixed points during the blow-up process is of utmost importance. To achieve this, we use a singular gauge transformation to connect the mode functions on $T^2/\mathbb{Z}_N$ to those on S^2, as we will see later.

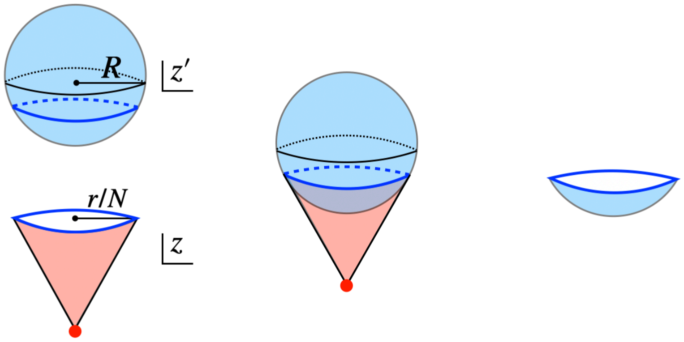

Figure 4.3: When cutting around the fixed point on the orbifold with a radius r, a cone (depicted as a red cone in this figure) with a base of radius r/N is obtained. The magnetized S^2 (shown as a blue ball in this figure) is embedded in this cone, aligning it with the connection line represented by the blue line. The $(N-1)/2N$-part of S^2 with the radius $R = r/\sqrt{N^2-1}$ is embedded by this operation. Here, z and z' denote the coordinates of $T^2/\mathbb{Z}_N$ and S^2, respectively. The coordinates at the connection points are $z = re^{i\varphi/N}$ and $z' = re^{i\varphi}/(N+1)$, respectively.

First, we construct the blow-up manifolds by cutting around the singularities of the $T^2/\mathbb{Z}_N$ orbifolds and replacing them with parts of the magnetized S^2, as illustrated in Figures 4.1 and 4.3. This replacement should be performed for each fixed point on the orbifolds. The construction method for blow-up manifolds is discussed in detail in [42], and specific relations of coordinates at the connection points are omitted here. It is necessary to smoothly connect the zero modes (3.2.7) and (4.1.8) on the connection line. However, there is an obstacle. If zero modes on the $T^2/\mathbb{Z}_N$ orbifold have non-zero winding numbers, they cannot be connected to zero modes on S^2 because the boundary conditions for $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+,0}(z)$ around the fixed point differ from those of $\psi_{S^2,+,0}(z')$. $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+}(z)$ obey the boundary condition (3.2.19). On the other hand, $\psi_{S^2,+,0}(z')$ have no phase. To resolve the above problem, we utilize singular gauge transformations to remove non-zero winding numbers from mode functions on $T^2/\mathbb{Z}_N$, as will be discussed below.

We perform a singular gauge transformation $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+}(z) \rightarrow \tilde{\psi}^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+}(z)$ such that $\tilde{\psi}^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+}(z)$ has no winding number:

$$\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+}(\omega z) = \omega^l \psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+}(z) \quad \rightarrow \quad \tilde{\psi}^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+}(\omega z) = \tilde{\psi}^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+}(z). \tag{4.1.10}$$

Here, we have considered the case where the fixed point is $z^f_p = 0$. It's important to note that the following analysis can be applied to other fixed points by the following replacement:

$$z \rightarrow Z, \tag{4.1.11}$$

$$l \rightarrow \chi_{+p}. \tag{4.1.12}$$

The singular gauge transformation is defined by

$$A \;\rightarrow\; \tilde{A}(z) \equiv A(z) + \delta A(z), \tag{4.1.13}$$

$$\delta A(z) = \delta A_z dz + \delta A_{\bar{z}} d\bar{z} = iU_{\xi^F} dU_{\xi^F}^{-1} \simeq -i\frac{\xi^F}{2}\frac{1}{z}dz + i\frac{\xi^F}{2}\frac{1}{\bar{z}}d\bar{z}, \tag{4.1.14}$$

with

$$U_{\xi^F}(z) = \left(\frac{g_1(z)}{(g_1(z))^*}\right)^{\frac{\xi^F}{2}} \simeq \left(\frac{g_1'(0)z}{(g_1'(0)z)^*}\right)^{\frac{\xi^F}{2}}, \tag{4.1.15}$$

where $g_1(z)$ denotes a specific holomorphic function with $\mathbb{Z}_N$ eigenvalue 1. The detailed form of $g_1(z)$ is discussed in [24]. The rightest-hand sides of Eqs.(4.1.14) and (4.1.15) are approximate expressions near $z = 0$. Under the singular gauge transformation, the field strength is modified as

$$\frac{F}{2\pi} \;\rightarrow\; \frac{\tilde{F}}{2\pi} \equiv \frac{F}{2\pi} + \frac{\delta F}{2\pi}, \tag{4.1.16}$$

$$\frac{\delta F}{2\pi} = i\xi^F \delta(z)\delta(\bar{z}) dz \wedge d\bar{z}. \tag{4.1.17}$$

We note that from Eq.(4.1.16), ξ^F/N can be regarded as a localized flux at the fixed point $z = 0$.

We further need to consider a singular gauge transformation for the spin connection in a manner similar to the gauge potentials to remove winding numbers of both $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+}(z)$ and $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,-}(z)$. This transformation is defined by

$$\omega \;\rightarrow\; \tilde{\omega} = \omega + \delta\omega = \delta\omega, \quad (\omega = 0), \tag{4.1.18}$$

$$\delta\omega = iU_{\xi^R} dU_{\xi^R}^{-1} \overset{z\simeq 0}{\simeq} -i\frac{\xi^R}{2}\frac{1}{z}dz + i\frac{\xi^R}{2}\frac{1}{\bar{z}}d\bar{z}, \tag{4.1.19}$$

with

$$U_{\xi^R}(z) = \left(\frac{g_1(z)}{(g_1(z))^*}\right)^{\frac{\xi^R}{2}} \overset{z\simeq 0}{\simeq} \left(\frac{g_1'(0)z}{(g_1'(0)z)^*}\right)^{\frac{\xi^R}{2}}. \tag{4.1.20}$$

We also note that ξ^R/N can be regarded as a localized curvature at the fixed point. ξ^R/N is determined by the deficit angle as

$$2\pi\frac{\xi^R}{N} = 2\pi - \frac{2\pi}{N} = 2\pi\frac{N-1}{N}, \tag{4.1.21}$$

at the $\mathbb{Z}_N$ fixed point.

From Eqs.(4.1.15) and (4.1.20), the mode functions are transformed under the singular gauge transformation as

$$\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+}(z) \;\rightarrow\; \tilde{\psi}^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+}(z) = U_{\xi^F}(z)U^{-1/2}_{\xi^R}(z)\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+}(z), \tag{4.1.22}$$

$$\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,-}(z) \;\rightarrow\; \tilde{\psi}^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,-}(z) = U_{\xi^F}(z)U^{1/2}_{\xi^R}(z)\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,-}(z). \tag{4.1.23}$$

Then, Eqs.(3.2.5) and (3.2.6) are modified such as

$$\tilde{\psi}^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+}(\omega z) = \omega^{\xi^F-\frac{\xi^R}{2}+l}\tilde{\psi}^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+}(z), \tag{4.1.24}$$

$$\tilde{\psi}^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,-}(\omega z) = \omega^{\xi^F+\frac{\xi^R}{2}+l+1}\tilde{\psi}^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,-}(z). \tag{4.1.25}$$

Note that the contributions of the localized curvature ξ^R act with opposite signs on the chirality positive and negative mode functions.

We arrive at the conditions to obtain mode functions with vanishing winding numbers as

$$\xi^F = \frac{N-1}{2} - l + \ell N \qquad \text{for} \quad {}^\forall \ell \in \mathbb{Z}. \tag{4.1.26}$$

Here, we used $\xi^R = N-1$ for the $\mathbb{Z}_N$ fixed point. It is interesting to point out that a new degree of freedom ℓ appears. It comes from mod N property of Eqs.(4.1.24) and (4.1.25). For $z^f_p \neq 0$, the same argument can be applied by replacing l with the winding number χ_{+p} at the fixed point z^f_p, yielding the following relationship:

$$\xi^F_p = \frac{N-1}{2} - \chi_{+p} + \ell_p N \qquad \text{for} \quad {}^\forall \ell_p \in \mathbb{Z}. \tag{4.1.27}$$

Eq.(4.1.27) implies that the winding number χ_{+p} can be expressed in terms of the localized flux ξ^F_p and the localized curvature $\xi^R_p = N-1$. In other words, the singular gauge transformations (4.1.22) and (4.1.23) replace the information of the winding number on the orbifolds with the localized flux and localized curvature at the fixed points of $T^2/\mathbb{Z}_N$. This operation is expected to connect mode functions on $T^2/\mathbb{Z}_N$ with those on S^2 without losing the orbifold information. In the next subsection, we will see that it allows us to reinterpret the index formula (3.2.22).

4.1.3 Flux condition

Now, we can explore zero-mode functions on the blow-up manifolds of magnetized $T^2/\mathbb{Z}_N$. The mode functions on the blow-up regions (parts of S^2 regions) are those on S^2 in Eq.(4.1.8), $\psi_{S^2,+,0}(z')$, while those on the bulk region (remaining region of $T^2/\mathbb{Z}_N$ by cutting out regions around fixed points) are those on $T^2/\mathbb{Z}_N$ in Eq.(4.1.22), $\tilde{\psi}^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+,0}(z)$, with the localized curvature (4.1.21) and the localized flux (4.1.26). In particular, the $\tilde{\psi}^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+}(z)$ near $z=0$ is approximated as

$$\tilde{\psi}^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+,0}(z) \simeq |z|^{l-\ell N}e^{-\frac{\pi M}{2\mathrm{Im}\tau}|z|^2}|g'_1(0)|^{l-\ell N}\tilde{h}^j_+(z), \tag{4.1.28}$$

where $\tilde{h}^j_+(z)$ denotes the holomorphic function. Then, we should connect mode functions on bulk regions (4.1.28) and those on the blow-up regions (4.1.8) smoothly at the junction points. That is, $\tilde{\psi}^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+,0}(z)$ at $z = re^{i\varphi/N}$ and $\psi_{S^2,+,0}(z')$ at $z' = \frac{r}{N+1}e^{i\varphi}$ should satisfy the following junction conditions:

$$\tilde{\psi}^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+,0}(z)\Big|_{z=re^{i\varphi/N}} = \psi_{S^2,+,0}(z')\Big|_{z'=\frac{r}{N+1}e^{i\varphi}}, \tag{4.1.29}$$

$$\frac{1}{e^{-i\varphi/N}}\frac{d\tilde{\psi}^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}^l_N,+,0}(z)}{dz}\Big|_{z=re^{i\varphi/N}} = \frac{1}{\frac{N+1}{N}e^{-i\varphi}}\frac{d\psi_{S^2,+,0}(z')}{dz'}\Big|_{z'=\frac{r}{N+1}e^{i\varphi}}. \tag{4.1.30}$$

The detailed analysis is discussed in Ref. [24]. In particular, from the non-holomorphic parts of Eqs.(4.1.29) and (4.1.30), we derive the following relation:

$$\frac{\pi r^2}{N\mathrm{Im}\tau}M + \frac{N-1}{2N} - \frac{l}{N} + \ell = \frac{N-1}{2N}M'. \tag{4.1.31}$$

For $z^f_p \neq 0$, by replacing l with the winding number χ_{+p} we obtain

$$\frac{\pi r_p^2}{N\mathrm{Im}\tau}M + \frac{N-1}{2N} - \frac{\chi_{+p}}{N} + \ell_p = \frac{N-1}{2N}M'_p. \tag{4.1.32}$$

By using the relation (4.1.27) (or (4.1.26)), Eq.(4.1.32) can be expressed as

$$\frac{\pi r_p^2}{N\mathrm{Im}\tau}M + \frac{\xi^F_p}{N} = \frac{N-1}{2N}M'_p. \tag{4.1.33}$$

Now, we can easily understand the physical meaning of this relationship. The left-hand side represents the flux, including the localized flux on the cutout area around the fixed point of $T^2/\mathbb{Z}_N$, while the right-hand side represents the flux on the embedded area of S^2. Thus, it means that the magnetic flux is not modified under the blow-up process. This is crucial in deriving the AS index theorem, as we will see in the next section. In particular, in the orbifold limit $r_I \to 0$, Eq.(4.1.33) is expressed as

$$\frac{\xi^F_p}{N} = \frac{N-1}{2N}M'_p\Big|_{r_p=0}, \tag{4.1.34}$$

which shows that the flux on the embedded area of S^2 (right-hand side of Eq.(4.1.34)) corresponds to the localized flux on the orbifold fixed point (left-hand side of Eq.(4.1.34)).

We also notice that the curvature is not modified under the blow-up process:

$$\frac{\xi^R_p}{N} = \frac{N-1}{N} = \frac{N-1}{2N} \times 2, \tag{4.1.35}$$

where the left-hand side represents the curvature, which corresponds to the deficit angle at the fixed point on $T^2/\mathbb{Z}_N$, while the right-hand side represents the curvature in the embedded area of S^2.

4.2 Index theorem on the blow-up manifold

Our purpose is to establish the AS index theorem on the $T^2/\mathbb{Z}_N$ orbifolds with magnetic flux background. Due to the presence of singularities on the $T^2/\mathbb{Z}_N$ orbifolds, the AS index theorem cannot be applied directly to the orbifold models. Our strategy is to replace the $T^2/\mathbb{Z}_N$ orbifolds with the blow-up manifolds without singularities and apply the AS index theorem to them.

4.2.1 Index theorem on the blow-up manifold

The AS index theorem on the blow-up manifolds can be obtained as

$$
\begin{aligned}
n_+ - n_- &= \int_{\text{blow-up manifold}} \frac{F}{2\pi} && (4.2.1)\\
&= \int_{T^2/\mathbb{Z}_N\,\text{bulk}} \frac{F}{2\pi} + \sum_p \int_{\frac{N-1}{2N}\times S^2} \frac{F'}{2\pi} && (4.2.2)\\
&= \left(\frac{M}{N} - \sum_p \frac{\pi r_p^2}{N\mathrm{Im}\tau} M\right) + \sum_p \frac{N-1}{2N} M'_p(r_p) && (4.2.3)\\
&= \left(\frac{M}{N} - \sum_p \frac{\pi r_p^2}{N\mathrm{Im}\tau} M\right) + \sum_p \left(\frac{\pi r_p^2}{N\mathrm{Im}\tau} M + \frac{\xi_p^F}{N}\right) && (4.2.4)\\
&= \frac{M}{N} + \sum_p \frac{\xi_p^F}{N}. && (4.2.5)
\end{aligned}
$$

Regarding the above equations, there are several points to note. In Eq.(4.2.1), it is emphasized that the index $n_+ - n_-$ on the blow-up manifolds depends solely on the flux and not on the curvature. This is due to the general property that the AS index theorem on a two-dimensional compact manifold only involves the contribution of the flux. For the first term of Eq.(4.2.2) (and Eq.(4.2.3)), the $T^2/\mathbb{Z}_N$ bulk refers to the region of the $T^2/\mathbb{Z}_N$ orbifold from which the areas near the fixed points are removed. The second term of Eq.(4.2.2) (and Eq.(4.2.3)) represents the contribution of the magnetic flux on the embedded area of S^2 replacing each fixed point. The sum over p is taken for the fixed points of the $T^2/\mathbb{Z}_N$ orbifolds. For Eq.(4.2.4), the relation (4.1.33) is utilized.

For the final result (4.2.5), it is crucial that the AS index theorem on the blow-up manifolds does not depend on the blow-up radius r_p, as expected. In other words, the result of the AS index theorem holds even in the orbifold limit $r_p \to 0$:

$$
n_+ - n_- = \int_{T^2/\mathbb{Z}_N} \frac{\tilde{F}}{2\pi} = \frac{M}{N} + \sum_p \frac{\xi_p^F}{N}. \tag{4.2.6}
$$

Here, $\tilde{F}$ is defined in Eq.(4.1.16), and this term arises from the limit of the right-hand side of Eq.(4.2.2) as follows: in the $r_p \to 0$ ($R \to 0$) limit, the second term of Eq.(4.2.2) with the field strength (4.1.2) can be expressed as

$$\int_{\frac{N-1}{2N}\times S^2} iM'_p\delta(z')\delta(\bar{z}')dz'\wedge d\bar{z}', \tag{4.2.7}$$

(see Appendix A in Ref. [43]), and it corresponds to Eq.(4.1.17) by considering Eq.(4.1.34). Thus, the first term of the rightest-hand side of Eq.(4.2.6) represents the contribution of the homogeneous magnetic flux on the $T^2/\mathbb{Z}_N$ orbifolds, originating from the first term of (4.1.16), while the second term represents the sum of localized fluxes at each fixed point, derived from the second term of (4.1.16). Therefore, Eq.(4.2.6) represents the AS index theorem on the $T^2/\mathbb{Z}_N$ orbifold, and the index can be determined solely by the contribution of the flux.

From Eq.(4.1.27), the localized flux ξ^F_p is determined by the localized curvature $\xi^R_p = N-1$ and the winding number χ_{+p} at the fixed points. The winding numbers at the fixed points are investigated in [1], and we can derive the values of the localized flux. We can verify that the number of chiral zero modes, computed using the zero-mode counting formula in Ref. [1], are completely consistent with the relation (4.2.6). The results are summarized in Table 4.1–4.5.

flux	parity	twist	localized flux				index	(3.2.22)
M	η	(α_1,α_2)	$\frac{\xi^F_1}{N}$	$\frac{\xi^F_2}{N}$	$\frac{\xi^F_3}{N}$	$\frac{\xi^F_4}{N}$	$\frac{M}{N}+\sum_{p=1}^{4}\frac{\xi^F_p}{N}$	$\frac{M-V_+}{N}+1$
$2m+1$	$+1$	$(0,0)$	$1/4$	$1/4$	$1/4$	$-1/4$	$(M+1)/2$	$(M+1)/2$
		$(\frac{1}{2},0)$	$1/4$	$-1/4$	$1/4$	$1/4$	$(M+1)/2$	$(M+1)/2$
		$(0,\frac{1}{2})$	$1/4$	$1/4$	$-1/4$	$1/4$	$(M+1)/2$	$(M+1)/2$
		$(\frac{1}{2},\frac{1}{2})$	$1/4$	$-1/4$	$-1/4$	$-1/4$	$(M-1)/2$	$(M-1)/2$
	-1	$(0,0)$	$-1/4$	$-1/4$	$-1/4$	$1/4$	$(M-1)/2$	$(M-1)/2$
		$(\frac{1}{2},0)$	$-1/4$	$1/4$	$-1/4$	$-1/4$	$(M-1)/2$	$(M-1)/2$
		$(0,\frac{1}{2})$	$-1/4$	$-1/4$	$1/4$	$-1/4$	$(M-1)/2$	$(M-1)/2$
		$(\frac{1}{2},\frac{1}{2})$	$-1/4$	$1/4$	$1/4$	$1/4$	$(M+1)/2$	$(M+1)/2$
$2m+2$	$+1$	$(0,0)$	$1/4$	$1/4$	$1/4$	$1/4$	$M/2+1$	$M/2+1$
		$(\frac{1}{2},0)$	$1/4$	$-1/4$	$1/4$	$-1/4$	$M/2$	$M/2$
		$(0,\frac{1}{2})$	$1/4$	$1/4$	$-1/4$	$-1/4$	$M/2$	$M/2$
		$(\frac{1}{2},\frac{1}{2})$	$1/4$	$-1/4$	$-1/4$	$1/4$	$M/2$	$M/2$
	-1	$(0,0)$	$-1/4$	$-1/4$	$-1/4$	$-1/4$	$M/2-1$	$M/2-1$
		$(\frac{1}{2},0)$	$-1/4$	$1/4$	$-1/4$	$1/4$	$M/2$	$M/2$
		$(0,\frac{1}{2})$	$-1/4$	$-1/4$	$1/4$	$1/4$	$M/2$	$M/2$
		$(\frac{1}{2},\frac{1}{2})$	$-1/4$	$1/4$	$1/4$	$-1/4$	$M/2$	$M/2$

Table 4.1: The values of localized fluxes at fixed points and index ($\ell = 0$) on $T^2/\mathbb{Z}_2$.

4.2.2 Reinterpretation of the zero-mode counting formula

Now, we can reinterpret the zero-mode counting formula (3.2.22). Utilizing the relation (4.1.27), the AS index theorem (4.2.5) can be expressed in terms of the winding numbers χ_{+p}

flux	parity	twist	localized flux			index	(3.2.22)
M	η	α	$\frac{\xi_1^F}{N}$	$\frac{\xi_2^F}{N}$	$\frac{\xi_3^F}{N}$	$\frac{M}{N}+\sum_{p=1}^{3}\frac{\xi_p^F}{N}$	$\frac{M-V_+}{N}+1$
$6m+1$	1	$1/6$	$1/3$	0	$1/3$	$(M+2)/3$	$(M+2)/3$
		$1/2$	$1/3$	$-1/3$	$-1/3$	$(M-1)/3$	$(M-1)/3$
		$5/6$	$1/3$	$1/3$	0	$(M+2)/3$	$(M+2)/3$
	ω	$1/6$	0	$-1/3$	0	$(M-1)/3$	$(M-1)/3$
		$1/2$	0	$1/3$	$1/3$	$(M+2)/3$	$(M+2)/3$
		$5/6$	0	0	$-1/3$	$(M-1)/3$	$(M-1)/3$
	ω^2	$1/6$	$-1/3$	$1/3$	$-1/3$	$(M-1)/3$	$(M-1)/3$
		$1/2$	$-1/3$	0	0	$(M-1)/3$	$(M-1)/3$
		$5/6$	$-1/3$	$-1/3$	$1/3$	$(M-1)/3$	$(M-1)/3$
$6m+2$	1	0	$1/3$	0	0	$(M+1)/3$	$(M+1)/3$
		$1/3$	$1/3$	$-1/3$	$1/3$	$(M+1)/3$	$(M+1)/3$
		$2/3$	$1/3$	$1/3$	$-1/3$	$(M+1)/3$	$(M+1)/3$
	ω	0	0	$-1/3$	$-1/3$	$(M-2)/3$	$(M-2)/3$
		$1/3$	0	$1/3$	0	$(M+1)/3$	$(M+1)/3$
		$2/3$	0	0	$1/3$	$(M+1)/3$	$(M+1)/3$
	ω^2	0	$-1/3$	$1/3$	$1/3$	$(M+1)/3$	$(M+1)/3$
		$1/3$	$-1/3$	0	$-1/3$	$(M-2)/3$	$(M-2)/3$
		$2/3$	$-1/3$	$-1/3$	0	$(M-2)/3$	$(M-2)/3$
$6m+3$	1	$1/6$	$1/3$	$-1/3$	0	$M/3$	$M/3$
		$1/2$	$1/3$	$1/3$	$1/3$	$M/3+1$	$M/3+1$
		$5/6$	$1/3$	0	$-1/3$	$M/3$	$M/3$
	ω	$1/6$	0	$1/3$	$-1/3$	$M/3$	$M/3$
		$1/2$	0	0	0	$M/3$	$M/3$
		$5/6$	0	$-1/3$	$1/3$	$M/3$	$M/3$
	ω^2	$1/6$	$-1/3$	0	$1/3$	$M/3$	$M/3$
		$1/2$	$-1/3$	$-1/3$	$-1/3$	$M/3-1$	$M/3-1$
		$5/6$	$-1/3$	$1/3$	0	$M/3$	$M/3$

Table 4.2: The values of localized fluxes at fixed points and index ($\ell = 0$) on $T^2/\mathbb{Z}_3$.

as

$$n_+ - n_- = \frac{M}{N} + \sum_p \left(\frac{-\chi_{+p}}{N} + \frac{N-1}{2N} + \ell_p \right) = \frac{M-V_+}{N} + 1 + \sum_p \ell_p. \qquad (4.2.8)$$

flux	parity	twist	localized flux			index	(3.2.22)
M	η	α	$\frac{\xi_1^F}{N}$	$\frac{\xi_2^F}{N}$	$\frac{\xi_3^F}{N}$	$\frac{M}{N}+\sum_{p=1}^{3}\frac{\xi_p^F}{N}$	$\frac{M-V_+}{N}+1$
$6m+4$	1	0	$1/3$	$-1/3$	$-1/3$	$(M-1)/3$	$(M-1)/3$
		$1/3$	$1/3$	$1/3$	0	$(M+2)/3$	$(M+2)/3$
		$2/3$	$1/3$	0	$1/3$	$(M+2)/3$	$(M+2)/3$
	ω	0	0	$1/3$	$1/3$	$(M+2)/3$	$(M+2)/3$
		$1/3$	0	0	$-1/3$	$(M-1)/3$	$(M-1)/3$
		$2/3$	0	$-1/3$	0	$(M-1)/3$	$(M-1)/3$
	ω^2	0	$-1/3$	0	0	$(M-1)/3$	$(M-1)/3$
		$1/3$	$-1/3$	$-1/3$	$1/3$	$(M-1)/3$	$(M-1)/3$
		$2/3$	$-1/3$	$1/3$	$-1/3$	$(M-1)/3$	$(M-1)/3$
$6m+5$	1	$1/6$	$1/3$	$1/3$	$-1/3$	$(M+1)/3$	$(M+1)/3$
		$1/2$	$1/3$	0	0	$(M+1)/3$	$(M+1)/3$
		$5/6$	$1/3$	$-1/3$	$1/3$	$(M+1)/3$	$(M+1)/3$
	ω	$1/6$	0	0	$1/3$	$(M+1)/3$	$(M+1)/3$
		$1/2$	0	$-1/3$	$-1/3$	$(M-2)/3$	$(M-2)/3$
		$5/6$	0	$1/3$	0	$(M+1)/3$	$(M+1)/3$
	ω^2	$1/6$	$-1/3$	$-1/3$	0	$(M-2)/3$	$(M-2)/3$
		$1/2$	$-1/3$	$1/3$	$1/3$	$(M+1)/3$	$(M+1)/3$
		$5/6$	$-1/3$	0	$-1/3$	$(M-2)/3$	$(M-2)/3$
$6m+6$	1	0	$1/3$	$1/3$	$1/3$	$M/3+1$	$M/3+1$
		$1/3$	$1/3$	0	$-1/3$	$M/3$	$M/3$
		$2/3$	$1/3$	$-1/3$	0	$M/3$	$M/3$
	ω	0	0	0	0	$M/3$	$M/3$
		$1/3$	0	$-1/3$	$1/3$	$M/3$	$M/3$
		$2/3$	0	$1/3$	$-1/3$	$M/3$	$M/3$
	ω^2	0	$-1/3$	$-1/3$	$-1/3$	$M/3-1$	$M/3-1$
		$1/3$	$-1/3$	$1/3$	0	$M/3$	$M/3$
		$2/3$	$-1/3$	0	$1/3$	$M/3$	$M/3$

Table 4.3: The values of localized fluxes at fixed points and index ($\ell = 0$) on $T^2/\mathbb{Z}_3$.

Here, we employed the relation

$$\sum_p \frac{N-1}{2N} = \frac{1}{2}\sum_p \frac{\xi_p^R}{N} = 1, \tag{4.2.9}$$

at the last equality. It can be confirmed through the following steps:

$$T^2/\mathbb{Z}_2 \ : \ \sum_p \frac{\xi_p^R}{4} = 4 \times \frac{1}{4} = 1 \qquad \left(z_p^f = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \right), \tag{4.2.10}$$

$$T^2/\mathbb{Z}_3 \ : \ \sum_p \frac{\xi_p^R}{6} = 3 \times \frac{2}{6} = 1 \qquad \left(z_p^f = 0, \frac{2+\tau}{3}, \frac{1+2\tau}{3} \right), \tag{4.2.11}$$

$$T^2/\mathbb{Z}_4 \ : \ \sum_{p_{\mathbb{Z}_4}} \frac{\xi_{p_{\mathbb{Z}_4}}^R}{8} + \frac{1}{2} \sum_{p_{\mathbb{Z}_2}} \frac{\xi_{p_{\mathbb{Z}_2}}^R}{4} = 2 \times \frac{3}{8} + \frac{1}{2} \times 2 \times \frac{1}{4} = 1 \left(z_{p_{\mathbb{Z}_4}}^f = 0, \frac{1+\tau}{2}, \ z_{p_{\mathbb{Z}_2}}^f = \frac{1}{2}, \frac{\tau}{2} \right), \tag{4.2.12}$$

$$T^2/\mathbb{Z}_6 \ : \ \sum_{p_{\mathbb{Z}_6}} \frac{\xi_{p_{\mathbb{Z}_6}}^R}{12} + \frac{1}{2} \sum_{p_{\mathbb{Z}_3}} \frac{\xi_{p_{\mathbb{Z}_3}}^R}{6} + \frac{1}{3} \sum_{p_{\mathbb{Z}_2}} \frac{\xi_{p_{\mathbb{Z}_2}}^R}{4} = \frac{5}{12} + \frac{2}{6} + \frac{1}{4} = 1$$
$$\left(z_{p_{\mathbb{Z}_6}}^f = 0, \quad z_{p_{\mathbb{Z}_3}}^f = \frac{1+\tau}{3}, \frac{2+2\tau}{3}, \quad z_{p_{\mathbb{Z}_2}}^f = \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} \right). \tag{4.2.13}$$

Note that $\mathbb{Z}_4$ and $\mathbb{Z}_6$ have subgroups, and their contributions must include the fixed points associated with these subgroups.

Thus, the zero-mode counting formula (3.2.22) can be derived from Eq.(5.1.4) by taking $\ell_p = 0$. The previously mysterious physical meaning of the term +1 in Eq.(3.2.22) is now clarified. The factor +1 represents the contribution of the sum of localized curvatures at fixed points. When attempting to express the index theorem in terms of winding numbers, the presence of +1 becomes necessary to subtract the contribution of localized curvature, as winding numbers encompass both localized flux and localized curvature contributions (refer to Eq. (4.1.27)). This analysis elucidates that the zero-mode counting formula solely incorporates the contribution of flux.

An interesting observation in our analysis is the presence of a new degree of freedom ℓ_p. The AS index theorem implies that additional zero modes can appear. A comprehensive discussion on this aspect will be provided in [24].

flux	parity	twist	localized flux				index	(3.2.22)
M	η	α	$\frac{\xi_1^F}{N}$	$\frac{\xi_2^F}{N}$	$\frac{\xi_3^F}{N}$	$\frac{\xi_4^F}{N}$	$\frac{M}{N}+\sum_{p=1}^{4}\frac{\xi_p^F}{N}$	$\frac{M-V_+}{N}+1$
$4m+1$	1	0	3/8	1/8	1/8	1/8	$(M+3)/4$	$(M+3)/4$
		1/2	3/8	−3/8	−1/8	−1/8	$(M-1)/4$	$(M-1)/4$
	i	0	1/8	−1/8	−1/8	−1/8	$(M-1)/4$	$(M-1)/4$
		1/2	1/8	3/8	1/8	1/8	$(M+3)/4$	$(M+3)/4$
	−1	0	−1/8	−3/8	1/8	1/8	$(M-1)/4$	$(M-1)/4$
		1/2	−1/8	1/8	−1/8	−1/8	$(M-1)/4$	$(M-1)/4$
	$-i$	0	−3/8	3/8	−1/8	−1/8	$(M-1)/4$	$(M-1)/4$
		1/2	−3/8	−1/8	1/8	1/8	$(M-1)/4$	$(M-1)/4$
$4m+2$	1	0	3/8	−1/8	1/8	1/8	$(M+2)/4$	$(M+2)/4$
		1/2	3/8	3/8	−1/8	−1/8	$(M+2)/4$	$(M+2)/4$
	i	0	1/8	−3/8	−1/8	−1/8	$(M-2)/4$	$(M-2)/4$
		1/2	1/8	1/8	1/8	1/8	$(M+2)/4$	$(M+2)/4$
	−1	0	−1/8	3/8	1/8	1/8	$(M+2)/4$	$(M+2)/4$
		1/2	−1/8	−1/8	−1/8	−1/8	$(M-2)/4$	$(M-2)/4$
	$-i$	0	−3/8	1/8	−1/8	−1/8	$(M-2)/4$	$(M-2)/4$
		1/2	−3/8	−3/8	1/8	1/8	$(M-2)/4$	$(M-2)/4$
$4m+3$	1	0	3/8	−3/8	1/8	1/8	$(M+1)/4$	$(M+1)/4$
		1/2	3/8	1/8	−1/8	−1/8	$(M+1)/4$	$(M+1)/4$
	i	0	1/8	3/8	−1/8	−1/8	$(M+1)/4$	$(M+1)/4$
		1/2	1/8	−1/8	1/8	1/8	$(M+1)/4$	$(M+1)/4$
	−1	0	−1/8	1/8	1/8	1/8	$(M+1)/4$	$(M+1)/4$
		1/2	−1/8	−3/8	−1/8	−1/8	$(M-3)/4$	$(M-3)/4$
	$-i$	0	−3/8	−1/8	−1/8	−1/8	$(M-3)/4$	$(M-3)/4$
		1/2	−3/8	3/8	1/8	1/8	$(M+1)/4$	$(M+1)/4$
$4m+4$	1	0	3/8	3/8	1/8	1/8	$M/4+1$	$M/4+1$
		1/2	3/8	−1/8	−1/8	−1/8	$M/4$	$M/4$
	i	0	1/8	1/8	−1/8	−1/8	$M/4$	$M/4$
		1/2	1/8	−3/8	1/8	1/8	$M/4$	$M/4$
	−1	0	−1/8	−1/8	1/8	1/8	$M/4$	$M/4$
		1/2	−1/8	3/8	−1/8	−1/8	$M/4$	$M/4$
	$-i$	0	−3/8	−3/8	−1/8	−1/8	$M/4-1$	$M/4-1$
		1/2	−3/8	1/8	1/8	1/8	$M/4$	$M/4$

Table 4.4: The values of localized fluxes at fixed points and index ($\ell = 0$) on $T^2/\mathbb{Z}_4$.

flux	parity	twist	localized flux						index	(3.2.22)
M	η	α	$\frac{\xi_1^F}{N}$	$\frac{\xi_2^F}{N}$	$\frac{\xi_3^F}{N}$	$\frac{\xi_4^F}{N}$	$\frac{\xi_5^F}{N}$	$\frac{\xi_6^F}{N}$	$\frac{M}{N}+\sum_{p=1}^{6}\frac{\xi_p^F}{N}$	$\frac{M-V_+}{N}+1$
$6m+1$	1	$1/2$	$5/12$	$-2/12$	$-2/12$	$-1/12$	$-1/12$	$-1/12$	$(M-1)/6$	$(M-1)/6$
	ω	$1/2$	$3/12$	$2/12$	$2/12$	$1/12$	$1/12$	$1/12$	$(M+5)/6$	$(M+5)/6$
	ω^2	$1/2$	$1/12$	0	0	$-1/12$	$-1/12$	$-1/12$	$(M-1)/6$	$(M-1)/6$
	ω^3	$1/2$	$-1/12$	$-2/12$	$-2/12$	$1/12$	$1/12$	$1/12$	$(M-1)/6$	$(M-1)/6$
	ω^4	$1/2$	$-3/12$	$2/12$	$2/12$	$-1/12$	$-1/12$	$-1/12$	$(M-1)/6$	$(M-1)/6$
	ω^5	$1/2$	$-5/12$	0	0	$1/12$	$1/12$	$1/12$	$(M-1)/6$	$(M-1)/6$
$6m+2$	1	0	$5/12$	0	0	$1/12$	$1/12$	$1/12$	$(M+4)/6$	$(M+4)/6$
	ω	0	$3/12$	$-2/12$	$-2/12$	$-1/12$	$-1/12$	$-1/12$	$(M-2)/6$	$(M-2)/6$
	ω^2	0	$1/12$	$2/12$	$2/12$	$1/12$	$1/12$	$1/12$	$(M+4)/6$	$(M+4)/6$
	ω^3	0	$-1/12$	0	0	$-1/12$	$-1/12$	$-1/12$	$(M-2)/6$	$(M-2)/6$
	ω^4	0	$-3/12$	$-2/12$	$-2/12$	$1/12$	$1/12$	$1/12$	$(M-2)/6$	$(M-2)/6$
	ω^5	0	$-5/12$	$2/12$	$2/12$	$-1/12$	$-1/12$	$-1/12$	$(M-2)/6$	$(M-2)/6$
$6m+3$	1	$1/2$	$5/12$	$2/12$	$2/12$	$-1/12$	$-1/12$	$-1/12$	$(M+3)/6$	$(M+3)/6$
	ω	$1/2$	$3/12$	0	0	$1/12$	$1/12$	$1/12$	$(M+3)/6$	$(M+3)/6$
	ω^2	$1/2$	$1/12$	$-2/12$	$-2/12$	$-1/12$	$-1/12$	$-1/12$	$(M-3)/6$	$(M-3)/6$
	ω^3	$1/2$	$-1/12$	$2/12$	$2/12$	$1/12$	$1/12$	$1/12$	$(M+3)/6$	$(M+3)/6$
	ω^4	$1/2$	$-3/12$	0	0	$-1/12$	$-1/12$	$-1/12$	$(M-3)/6$	$(M-3)/6$
	ω^5	$1/2$	$-5/12$	$-2/12$	$-2/12$	$1/12$	$1/12$	$1/12$	$(M-3)/6$	$(M-3)/6$
$6m+4$	1	0	$5/12$	$-2/12$	$-2/12$	$1/12$	$1/12$	$1/12$	$(M+2)/6$	$(M+2)/6$
	ω	0	$3/12$	$2/12$	$2/12$	$-1/12$	$-1/12$	$-1/12$	$(M+2)/6$	$(M+2)/6$
	ω^2	0	$1/12$	0	0	$1/12$	$1/12$	$1/12$	$(M+2)/6$	$(M+2)/6$
	ω^3	0	$-1/12$	$-2/12$	$-2/12$	$-1/12$	$-1/12$	$-1/12$	$(M-4)/6$	$(M-4)/6$
	ω^4	0	$-3/12$	$2/12$	$2/12$	$1/12$	$1/12$	$1/12$	$(M+2)/6$	$(M+2)/6$
	ω^5	0	$-5/12$	0	0	$-1/12$	$-1/12$	$-1/12$	$(M-4)/6$	$(M-4)/6$
$6m+5$	1	$1/2$	$5/12$	0	0	$-1/12$	$-1/12$	$-1/12$	$(M+1)/6$	$(M+1)/6$
	ω	$1/2$	$3/12$	$-2/12$	$-2/12$	$1/12$	$1/12$	$1/12$	$(M+1)/6$	$(M+1)/6$
	ω^2	$1/2$	$1/12$	$2/12$	$2/12$	$-1/12$	$-1/12$	$-1/12$	$(M+1)/6$	$(M+1)/6$
	ω^3	$1/2$	$-1/12$	0	0	$1/12$	$1/12$	$1/12$	$(M+1)/6$	$(M+1)/6$
	ω^4	$1/2$	$-3/12$	$-2/12$	$-2/12$	$-1/12$	$-1/12$	$-1/12$	$(M-5)/6$	$(M-5)/6$
	ω^5	$1/2$	$-5/12$	$2/12$	$2/12$	$1/12$	$1/12$	$1/12$	$(M+1)/6$	$(M+1)/6$
$6m+6$	1	0	$5/12$	$2/12$	$2/12$	$1/12$	$1/12$	$1/12$	$M/6+1$	$M/6+1$
	ω	0	$3/12$	0	0	$-1/12$	$-1/12$	$-1/12$	$M/6$	$M/6$
	ω^2	0	$1/12$	$-2/12$	$-2/12$	$1/12$	$1/12$	$1/12$	$M/6$	$M/6$
	ω^3	0	$-1/12$	$2/12$	$2/12$	$-1/12$	$-1/12$	$-1/12$	$M/6$	$M/6$
	ω^4	0	$-3/12$	0	0	$1/12$	$1/12$	$1/12$	$M/6$	$M/6$
	ω^5	0	$-5/12$	$-2/12$	$-2/12$	$-1/12$	$-1/12$	$-1/12$	$M/6-1$	$M/6-1$

Table 4.5: The values of localized fluxes at fixed points and index ($\ell = 0$) on $T^2/\mathbb{Z}_6$.

Chapter 5

Index theorem on magnetized $T^2/\mathbb{Z}_N$ orbifold model

In the previous section, the AS index theorem was applied directly by removing singularities and constructing smooth blow-up manifolds of $T^2/\mathbb{Z}_N$ orbifolds. In this chapter, we derive the AS index theorem on magnetized $T^2/\mathbb{Z}_N$ orbifolds by explicitly calculating the index $\mathrm{Ind}(i\not{D})_\eta$ using the trace formula without blow-up [40].

5.1 Index theorem on $T^2/\mathbb{Z}_2$ orbifold with magnetic flux

We derive the index on the $T^2/\mathbb{Z}_2$ orbifold with magnetic flux by using the trace formula

$$\mathrm{Ind}(i\not{D})_\eta = \lim_{\rho\to\infty} \mathrm{tr}[\sigma_3 e^{\not{D}^2/\rho^2}]_\eta, \tag{5.1.1}$$

where the subscript η signifies that the trace on the right-hand side of Eq.(5.1.1) is restricted to the functional space spanned by the mode functions $\{\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2\pm,n}(z)_\eta\}$. The regularization factor $e^{\not{D}^2/\rho^2}$ is introduced in the trace of Eq.(5.1.1) to make the trace of σ_3 well defined. Utilizing the relation $\not{D}^2 = \sigma^a\sigma^b D_a D_b = D^2 - \frac{iq}{4}[\sigma^a,\sigma^b]F_{ab}$, where $D^2 = 4D_zD_{\bar{z}} - 2\pi M/\mathrm{Im}\tau$ and $\sigma^a(a=1,2)$ is the Pauli matrix, we expand Eq.(5.1.1) around D^2 as follows:

$$\mathrm{Ind}(i\not{D})_\eta = \lim_{\rho\to\infty}\left\{\mathrm{tr}[\sigma_3 e^{D^2/\rho^2}] + \frac{1}{\rho^2}\mathrm{tr}[\sigma_3(-\tfrac{iq}{4})[\sigma^a,\sigma^b]F_{ab}e^{D^2/\rho^2}] + O(\rho^{-4})\right\}. \tag{5.1.2}$$

The second term of Eq.(5.1.2) can be evaluated using the Fujikawa method [44, 45] and is given by $M/2$, where the factor $1/2$ arises from the fact that the area of the $T^2/\mathbb{Z}_2$ orbifold is $1/2$ of that of the torus T^2. The proof is given in Appendix B.1. The third term of $O(\rho^{-4})$ will vanish in the limit of $\rho\to\infty$.

The first term of Eq.(5.1.2) is typically neglected because $\mathrm{tr}[\sigma_3]$ is expected to vanish with $\rho\to\infty$. However, this is not the case since the $T^2/\mathbb{Z}_2$ orbifold has singular points, which correspond to the fixed points, i.e. $z = 0, 1/2, \tau/2$ and $(1+\tau)/2$. Therefore, we cannot directly apply the Atiyah-Singer index theorem for the $T^2/\mathbb{Z}_2$ orbifold. One of our main purposes is to show that the first term of Eq.(5.1.2) does not vanish and yields the desired result as the index theorem, as explained below.

Using the complete set of the mode functions $\{\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,n}(z)_\eta \;\; (n=0,1,\cdots)\}$ and $\{\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2-,n}(z)_\eta$ $(n=1,2,\cdots)\}$, the first term of Eq.(5.1.2) can be expressed as

$$\begin{aligned}
&\lim_{\rho\to\infty} \mathrm{tr}[\sigma_3 e^{D^2/\rho^2}]_\eta \\
&= \lim_{\rho\to\infty} \int_{T^2/\mathbb{Z}_2} dz d\bar{z} \lim_{z'\to z} e^{D^2(z)/\rho^2} \left\{ \sum_{n=0}^{\infty} \sum_j \psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,n}(z)_\eta [\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2+,n}(z')_\eta]^* \right. \\
&\qquad\qquad \left. - \sum_{n=1}^{\infty} \sum_j \psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2-,n}(z)_\eta [\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_2-,n}(z')_\eta]^* \right\}, \qquad (5.1.3)
\end{aligned}$$

where j denotes the label of the degeneracy of the linearly independent $\mathbb{Z}_2$ eigenfunctions, as given in Table 3.1. It is interesting to note that each of the first and the second terms in Eq.(5.1.3) is divergent, but their combination in Eq.(5.1.3) is finite. Indeed, Eq.(5.1.3) can be evaluated as

$$\lim_{\rho\to\infty} \mathrm{tr}[\sigma_3 e^{D^2/\rho^2}]_\eta = \frac{1}{2} \int_{T^2} dz d\bar{z} \sum_{n=0}^{\infty} \sum_{j=0}^{M-1} \eta(1-\omega) \psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(z) [\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(\omega z)]^*. \qquad (5.1.4)$$

We note that there appear the mode functions defined on T^2 (but not on $T^2/\mathbb{Z}_2$) as well as the integration over the area of T^2 in Eq.(5.1.4).

The proof of Eq.(5.1.4) is lengthy because we need to verify the expression (5.1.4) for every case of M = even/odd, $(\alpha_1,\alpha_2) = (0,0), (1/2,0), (0,1/2), (1/2,1/2)$ and $\eta = \pm 1$, separately. We show Eq.(5.1.4) only for the case of $(\alpha_1,\alpha_2) = (0,0)$ with M = even/odd and $\eta = \pm 1$.

$\underline{(\alpha_1,\alpha_2) = (0,0),\ M = 2m+1,\ \eta = +1}$

Let us first consider the case of $(\alpha_1,\alpha_2) = (0,0)$, $M = 2m+1$ and $\eta = +1$. We start with

$$I_{\eta=+1} \equiv \sum_{n=0}^{\infty} \sum_j \psi^{(j,0)}_{T^2/\mathbb{Z}_2+,n}(z)_{+1} [\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,n}(z)_{+1}]^* - \sum_{n=1}^{\infty} \sum_j \psi^{(j,0)}_{T^2/\mathbb{Z}_2-,n}(z)_{+1} [\psi^{(j,0)}_{T^2/\mathbb{Z}_2-,n}(z)_{+1}]^*. \quad (5.1.5)$$

From Table 3.1 and Eqs.(3.1.13) and (3.1.14), the label j for $\psi^{(j,0)}_{T^2/\mathbb{Z}_2\pm,n}(z)_{+1}$ runs from 0 (1) to $\frac{M-1}{2}$ $(\frac{M-1}{2})$ for n = even (odd). Thus, $I_{\eta=+1}$ can be represented as

$$\begin{aligned}
I_{\eta=+1} =& \sum_{k=0}^{\infty}\sum_{j=0}^{\frac{M-1}{2}} \psi^{(j,0)}_{T^2/\mathbb{Z}_2+,2k}(z)_{+1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,2k}(z)_{+1}]^* \\
&+ \sum_{k=0}^{\infty}\sum_{j=1}^{\frac{M-1}{2}} \psi^{(j,0)}_{T^2/\mathbb{Z}_2+,2k+1}(z)_{+1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,2k+1}(z)_{+1}]^* \\
&- \sum_{k=0}^{\infty}\sum_{j=0}^{\frac{M-1}{2}} \psi^{(j,0)}_{T^2/\mathbb{Z}_2-,2k+2}(z)_{+1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2-,2k+2}(z)_{+1}]^* \\
&- \sum_{k=0}^{\infty}\sum_{j=1}^{\frac{M-1}{2}} \psi^{(j,0)}_{T^2/\mathbb{Z}_2-,2k+1}(z)_{+1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2-,2k+1}(z)_{+1}]^*. \qquad (5.1.6)
\end{aligned}$$

An important observation is that we can add $j = 0$ terms in the second and the forth terms of Eq.(5.1.6) because

$$\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,0}(z)_{-1} = \mathcal{N}^{(j,0)}_{+,-1}\Big[\psi^{(j,0)}_{T^2+,0}(z) - \psi^{(j,0)}_{T^2+,0}(-z)\Big] \qquad (5.1.7)$$

identically vanishes for $j = 0$. Then, taking $\mathcal{N}^{(0,0)}_{+,-1} = 1/\sqrt{2}$, we can rewrite Eq.(5.1.6) as follows:

$$\begin{aligned}
I_{\eta=+1} &= \sum_{n=0}^{\infty}\sum_{j=0}^{\frac{M-1}{2}} \psi^{(j,0)}_{T^2/\mathbb{Z}_2+,n}(z)_{+1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,n}(z)_{+1}]^* - \sum_{n=1}^{\infty}\sum_{j=0}^{\frac{M-1}{2}} \psi^{(j,0)}_{T^2/\mathbb{Z}_2-,n}(z)_{+1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2-,n}(z)_{+1}]^* \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{\frac{M-1}{2}}\sum_{l=0}^{1}\sum_{l'=0}^{1}(1-\omega^{-l+l'})|\mathcal{N}^{(j,0)}_{+,+1}|^2\psi^{(j,0)}_{T^2+,n}(\omega^l z)[\psi^{(j,0)}_{T^2+,n}(\omega^{l'}z)]^* \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{\frac{M-1}{2}}(1-\omega)|\mathcal{N}^{(j,0)}_{+,+1}|^2\Big\{\psi^{(j,0)}_{T^2+,n}(z)[\psi^{(j,0)}_{T^2+,n}(\omega z)]^* + \psi^{(j,0)}_{T^2+,n}(\omega z)[\psi^{(j,0)}_{T^2+,n}(z)]^*\Big\}, \qquad (5.1.8)
\end{aligned}$$

where we have used the relations (3.1.13), (3.1.14) and $\mathcal{N}^{(j,0)}_{+,+1} = \mathcal{N}^{(j,0)}_{+,-1}$ for $j = 0, 1, \cdots, \frac{M-1}{2}$ in the second equality, and $\omega = -1$ in the last equality.

Using the relation (3.1.7) and $\psi^{(M,0)}_{T^2+,n}(z) = \psi^{(0,0)}_{T^2+,n}(z)$, we can further rewrite Eq.(5.1.8) as

$$\begin{aligned}
I_{\eta=+1} =& \sum_{n=0}^{\infty}(1-\omega)\Big\{2|\mathcal{N}^{(0,0)}_{+,+1}|^2\psi^{(0,0)}_{T^2+,n}(z)[\psi^{(0,0)}_{T^2+,n}(\omega z)]^* \\
&+ \sum_{j=1}^{\frac{M-1}{2}}|\mathcal{N}^{(j,0)}_{+,+1}|^2(\psi^{(j,0)}_{T^2+,n}(z)[\psi^{(j,0)}_{T^2+,n}(\omega z)]^* + \psi^{(M-j,0)}_{T^2+,n}(z)[\psi^{(M-j,0)}_{T^2+,n}(\omega z)]^*)\Big\} \\
=& \sum_{n=0}^{\infty}\sum_{j=0}^{M-1}(1-\omega)\psi^{(j,0)}_{T^2+,n}(z)[\psi^{(j,0)}_{T^2+,n}(\omega z)]^*, \qquad (5.1.9)
\end{aligned}$$

where we have used $\mathcal{N}^{(0,0)}_{+,+1} = 1/\sqrt{2}$ and $\mathcal{N}^{(j,0)}_{+,+1} = 1$ ($j = 1, 2, \cdots, \frac{M-1}{2}$) for M = odd from Table 3.2.

Taking the limit of $z' \to z$ and $\rho \to \infty$ in Eq.(5.1.3) and then substituting Eq.(5.1.9) into Eq.(5.1.3), we finally arrive at

$$\lim_{\rho\to\infty} \mathrm{tr}[\sigma_3 e^{D^2/\rho^2}]_{\eta=+1} = \frac{1}{2}\int_{T^2} dzd\bar{z} \sum_{n=0}^{\infty}\sum_{j=0}^{M-1}(1-\omega)\psi^{(j,0)}_{T^2+,n}(z)[\psi^{(j,0)}_{T^2+,n}(\omega z)]^*, \tag{5.1.10}$$

where we have used the fact that $\int_{T^2/\mathbb{Z}_2} dzd\bar{z}$ can be replaced by $\frac{1}{2}\int_{T^2} dzd\bar{z}$ in the final expression. Thus, we have succeeded in verifying Eq.(5.1.4) for $(\alpha_1, \alpha_2) = (0, 0)$, M = odd and $\eta = +1$.

<u>$(\alpha_1, \alpha_2) = (0, 0),\ M = 2m + 1,\ \eta = -1$</u>

We next discuss the case of $(\alpha_1, \alpha_2) = (0, 0)$, $M = 2m + 1$ and $\eta = -1$. Let us consider

$$I_{\eta=-1} \equiv \sum_{n=0}^{\infty}\sum_{j}\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,n}(z)_{-1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,n}(z)_{-1}]^* - \sum_{n=1}^{\infty}\sum_{j}\psi^{(j,0)}_{T^2/\mathbb{Z}_2-,n}(z)_{-1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2-,n}(z)_{-1}]^*. \tag{5.1.11}$$

From Table 3.1 and Eqs.(3.1.13) and (3.1.14), the label j for $\psi^{(j,0)}_{T^2/\mathbb{Z}_2\pm,n}(z)_{-1}$ runs from 1 (0) to $\frac{M-1}{2}$ ($\frac{M-1}{2}$) for n = even (odd). Thus, $I_{\eta=-1}$ can be expressed as

$$\begin{aligned} I_{\eta=-1} = &\sum_{k=0}^{\infty}\sum_{j=1}^{\frac{M-1}{2}}\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,2k}(z)_{-1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,2k}(z)_{-1}]^* \\ &+\sum_{k=0}^{\infty}\sum_{j=0}^{\frac{M-1}{2}}\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,2k+1}(z)_{-1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,2k+1}(z)_{-1}]^* \\ &-\sum_{k=0}^{\infty}\sum_{j=1}^{\frac{M-1}{2}}\psi^{(j,0)}_{T^2/\mathbb{Z}_2-,2k+2}(z)_{-1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2-,2k+2}(z)_{-1}]^* \\ &-\sum_{k=0}^{\infty}\sum_{j=0}^{\frac{M-1}{2}}\psi^{(j,0)}_{T^2/\mathbb{Z}_2-,2k+1}(z)_{-1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2-,2k+1}(z)_{-1}]^*. \end{aligned} \tag{5.1.12}$$

Since $\psi^{(j,0)}_{T^2/\mathbb{Z}_2\pm,2k}(z)_{-1} = 0$ for $j = 0$, we can add $j = 0$ terms in the first and the third terms of Eq.(5.1.12). Then, we find

$$\begin{aligned}
I_{\eta=-1} &= \sum_{n=0}^{\infty}\sum_{j=0}^{\frac{M-1}{2}} \psi^{(j,0)}_{T^2/\mathbb{Z}_2+,n}(z)_{-1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,n}(z)_{-1}]^* - \sum_{n=1}^{\infty}\sum_{j=0}^{\frac{M-1}{2}} \psi^{(j,0)}_{T^2/\mathbb{Z}_2-,n}(z)_{-1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2-,n}(z)_{-1}]^* \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{\frac{M-1}{2}}\sum_{l=0}^{1}\sum_{l'=0}^{1} \eta^{-l+l'}(1-\omega^{-l+l'})|\mathcal{N}^{(j,0)}_{+,+1}|^2 \psi^{(j,0)}_{T^2+,n}(\omega^l z)[\psi^{(j,0)}_{T^2+,n}(\omega^{l'} z)]^* \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{\frac{M-1}{2}} \eta(1-\omega)|\mathcal{N}^{(j,0)}_{+,+1}|^2 \Big\{\psi^{(j,0)}_{T^2+,n}(z)[\psi^{(j,0)}_{T^2+,n}(\omega z)]^* + \psi^{(j,0)}_{T^2+,n}(\omega z)[\psi^{(j,0)}_{T^2+,n}(z)]^*\Big\}, \qquad (5.1.13)
\end{aligned}$$

where we have used the relations (3.1.13), (3.1.14) and $\mathcal{N}^{(j,0)}_{+,+1} = \mathcal{N}^{(j,0)}_{+,-1}$ for $j = 0, 1, \cdots, \frac{M-1}{2}$ in the second equality, and $\eta = \omega = -1$ in the third equality.

Using the relation (3.1.7) and $\psi^{(M,0)}_{T^2+,n}(z) = \psi^{(0,0)}_{T^2+,n}(z)$, we can further rewrite Eq.(5.1.13) as

$$\begin{aligned}
I_{\eta=-1} &= \sum_{n=0}^{\infty} \eta(1-\omega)\Big\{2|\mathcal{N}^{(0,0)}_{+,+1}|^2 \psi^{(0,0)}_{T^2+,n}(z)[\psi^{(0,0)}_{T^2+,n}(\omega z)]^* \\
&\quad + \sum_{j=1}^{\frac{M-1}{2}} |\mathcal{N}^{(j,0)}_{+,+1}|^2 (\psi^{(j,0)}_{T^2+,n}(z)[\psi^{(j,0)}_{T^2+,n}(\omega z)]^* + \psi^{(M-j,0)}_{T^2+,n}(z)[\psi^{(M-j,0)}_{T^2+,n}(\omega z)]^*)\Big\} \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{M-1} \eta(1-\omega)\psi^{(j,0)}_{T^2+,n}(z)[\psi^{(j,0)}_{T^2+,n}(\omega z)]^*, \qquad (5.1.14)
\end{aligned}$$

where we have used $\mathcal{N}^{(0,0)}_{+,+1} = 1/\sqrt{2}$ and $\mathcal{N}^{(j,0)}_{+,+1} = 1$ $(j = 1, 2, \cdots, \frac{M-1}{2})$ for M = odd from Table 3.2.

Taking the limit of $z' \to z$ and $\rho \to \infty$ in Eq.(5.1.3) and then substituting Eq.(5.1.14) into Eq.(5.1.3), we find that Eq.(5.1.4) holds for $(\alpha_1, \alpha_2) = (0, 0)$, M = odd and $\eta = -1$. We have succeeded in verifying Eq.(5.1.4) for the case of $(\alpha_1, \alpha_2) = (0, 0)$, M = odd and $\eta = \pm 1$.

$(\alpha_1, \alpha_2) = (0, 0)$, $M = 2m + 2$, $\eta = +1$

Let us consider the case of $(\alpha_1, \alpha_2) = (0, 0)$, $M = 2m + 2$ and $\eta = +1$.

$$I_{\eta=+1} \equiv \sum_{n=0}^{\infty}\sum_{j} \psi^{(j,0)}_{T^2/\mathbb{Z}_2+,n}(z)_{+1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,n}(z)_{+1}]^* - \sum_{n=1}^{\infty}\sum_{j} \psi^{(j,0)}_{T^2/\mathbb{Z}_2-,n}(z)_{+1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2-,n}(z)_{+1}]^*. \qquad (5.1.15)$$

From Table 3.1 and Eqs.(3.1.13) and (3.1.14), the label j for $\psi^{(j,0)}_{T^2/\mathbb{Z}_2\pm,n}(z)_{+1}$ runs from 0 (1) to $\frac{M}{2}$ ($\frac{M}{2}-1$) for n = even (odd). Thus, $I_{\eta=+1}$ can be represented as

$$\begin{aligned}
I_{\eta=+1} =& \sum_{k=0}^{\infty}\sum_{j=0}^{\frac{M}{2}} \psi^{(j,0)}_{T^2/\mathbb{Z}_2+,2k}(z)_{+1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,2k}(z)_{+1}]^* \\
&+ \sum_{k=0}^{\infty}\sum_{j=1}^{\frac{M}{2}-1} \psi^{(j,0)}_{T^2/\mathbb{Z}_2+,2k+1}(z)_{+1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,2k+1}(z)_{+1}]^* \\
&- \sum_{k=0}^{\infty}\sum_{j=0}^{\frac{M}{2}} \psi^{(j,0)}_{T^2/\mathbb{Z}_2-,2k+2}(z)_{+1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2-,2k+2}(z)_{+1}]^* \\
&- \sum_{k=0}^{\infty}\sum_{j=1}^{\frac{M}{2}-1} \psi^{(j,0)}_{T^2/\mathbb{Z}_2-,2k+1}(z)_{+1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2-,2k+1}(z)_{+1}]^*. \qquad (5.1.16)
\end{aligned}$$

Since $\psi^{(j,0)}_{T^2/\mathbb{Z}_2\pm,2k+1}(z)_{-1} = 0$ for $j = 0, \frac{M}{2}$, we can add $j = 0, \frac{M}{2}$ terms in the second and the forth terms of Eq.(5.1.16). Then, we find

$$\begin{aligned}
I_{\eta=+1} &= \sum_{n=0}^{\infty}\sum_{j=0}^{\frac{M}{2}} \psi^{(j,0)}_{T^2/\mathbb{Z}_2+,n}(z)_{+1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,n}(z)_{+1}]^* - \sum_{n=1}^{\infty}\sum_{j=0}^{\frac{M}{2}} \psi^{(j,0)}_{T^2/\mathbb{Z}_2-,n}(z)_{+1}[\psi^{(j,0)}_{T^2/\mathbb{Z}_2-,n}(z)_{+1}]^* \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{\frac{M}{2}}\sum_{l=0}^{1}\sum_{l'=0}^{1}(1-\omega^{-l+l'})|\mathcal{N}^{(j,0)}_{+,+1}|^2\psi^{(j,0)}_{T^2+,n}(\omega^l z)[\psi^{(j,0)}_{T^2+,n}(\omega^{l'}z)]^* \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{\frac{M}{2}}(1-\omega)|\mathcal{N}^{(j,0)}_{+,+1}|^2\Big\{\psi^{(j,0)}_{T^2+,n}(z)[\psi^{(j,0)}_{T^2+,n}(\omega z)]^* + \psi^{(j,0)}_{T^2+,n}(\omega z)[\psi^{(j,0)}_{T^2+,n}(z)]^*\Big\}, \qquad (5.1.17)
\end{aligned}$$

where we have used the relations (3.1.13), (3.1.14) and $\mathcal{N}^{(j,0)}_{+,+1} = \mathcal{N}^{(j,0)}_{+,-1}$ for $j = 0, 1, \cdots, \frac{M}{2}$ in the second equality, and $\omega = -1$ in the last equality.

Using the relation (3.1.7) and $\psi^{(M,0)}_{T^2+,n}(z) = \psi^{(0,0)}_{T^2+,n}(z)$, we can further rewrite Eq.(5.1.17) as

$$\begin{aligned}
I_{\eta=+1} =& \sum_{n=0}^{\infty}(1-\omega)\Big\{2|\mathcal{N}^{(0,0)}_{+,+1}|^2\psi^{(0,0)}_{T^2+,n}(z)[\psi^{(0,0)}_{T^2+,n}(\omega z)]^* + 2|\mathcal{N}^{(M/2,0)}_{+,+1}|^2\psi^{(M/2,0)}_{T^2+,n}(z)[\psi^{(M/2,0)}_{T^2+,n}(\omega z)]^* \\
&+ \sum_{j=1}^{\frac{M}{2}-1}|\mathcal{N}^{(j,0)}_{+,+1}|^2(\psi^{(j,0)}_{T^2+,n}(z)[\psi^{(j,0)}_{T^2+,n}(\omega z)]^* + \psi^{(M-j,0)}_{T^2+,n}(z)[\psi^{(M-j,0)}_{T^2+,n}(\omega z)]^*)\Big\} \\
=& \sum_{n=0}^{\infty}\sum_{j=0}^{M-1}(1-\omega)\psi^{(j,0)}_{T^2+,n}(z)[\psi^{(j,0)}_{T^2+,n}(\omega z)]^*, \qquad (5.1.18)
\end{aligned}$$

where we have used $\mathcal{N}^{(0,0)}_{+,+1} = \mathcal{N}^{(M/2,0)}_{+,+1} = 1/\sqrt{2}$ and $\mathcal{N}^{(j,0)}_{+,+1} = 1$ ($j = 1, 2, \cdots, \frac{M}{2}-1$) for M = even from Table 3.2.

Taking the limit of $z' \to z$ and $\rho \to \infty$ in Eq.(5.1.3) and then substituting Eq.(5.1.18) into Eq.(5.1.3), we find that Eq.(5.1.4) holds for $(\alpha_1, \alpha_2) = (0, 0),\ M = \text{even}$ and $\eta = +1$.

$(\alpha_1, \alpha_2) = (0, 0),\ M = 2m + 2,\ \eta = -1$

We next discuss the case of $(\alpha_1, \alpha_2) = (0, 0),\ M = 2m + 2$ and $\eta = -1$. Let us consider

$$I_{\eta=-1} \equiv \sum_{n=0}^{\infty} \sum_{j} \psi^{(j,0)}_{T^2/\mathbb{Z}_2+,n}(z)_{-1} [\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,n}(z)_{-1}]^* - \sum_{n=1}^{\infty} \sum_{j} \psi^{(j,0)}_{T^2/\mathbb{Z}_2-,n}(z)_{-1} [\psi^{(j,0)}_{T^2/\mathbb{Z}_2-,n}(z)_{-1}]^*. \tag{5.1.19}$$

From Table 3.1 and Eqs.(3.1.13) and (3.1.14), the label j for $\psi^{(j,0)}_{T^2/\mathbb{Z}_2\pm,n}(z)_{-1}$ runs from 1 (0) to $\frac{M}{2} - 1$ ($\frac{M}{2}$) for n = even (odd). Thus, $I_{\eta=-1}$ can be expressed as

$$\begin{aligned} I_{\eta=-1} = & \sum_{k=0}^{\infty} \sum_{j=1}^{\frac{M}{2}-1} \psi^{(j,0)}_{T^2/\mathbb{Z}_2+,2k}(z)_{-1} [\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,2k}(z)_{-1}]^* \\ & + \sum_{k=0}^{\infty} \sum_{j=0}^{\frac{M}{2}} \psi^{(j,0)}_{T^2/\mathbb{Z}_2+,2k+1}(z)_{-1} [\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,2k+1}(z)_{-1}]^* \\ & - \sum_{k=0}^{\infty} \sum_{j=1}^{\frac{M}{2}-1} \psi^{(j,0)}_{T^2/\mathbb{Z}_2-,2k+2}(z)_{-1} [\psi^{(j,0)}_{T^2/\mathbb{Z}_2-,2k+2}(z)_{-1}]^* \\ & - \sum_{k=0}^{\infty} \sum_{j=0}^{\frac{M}{2}} \psi^{(j,0)}_{T^2/\mathbb{Z}_2-,2k+1}(z)_{-1} [\psi^{(j,0)}_{T^2/\mathbb{Z}_2-,2k+1}(z)_{-1}]^*. \end{aligned} \tag{5.1.20}$$

Since $\psi^{(j,0)}_{T^2/\mathbb{Z}_2\pm,2k}(z)_{-1} = 0$ for $j = 0, \frac{M}{2}$, we can add $j = 0, \frac{M}{2}$ terms in the first and the third terms of Eq.(5.1.20). Then, we find

$$\begin{aligned} I_{\eta=-1} &= \sum_{n=0}^{\infty} \sum_{j=0}^{\frac{M}{2}} \psi^{(j,0)}_{T^2/\mathbb{Z}_2+,n}(z)_{-1} [\psi^{(j,0)}_{T^2/\mathbb{Z}_2+,n}(z)_{-1}]^* - \sum_{n=1}^{\infty} \sum_{j=0}^{\frac{M}{2}} \psi^{(j,0)}_{T^2/\mathbb{Z}_2-,n}(z)_{-1} [\psi^{(j,0)}_{T^2/\mathbb{Z}_2-,n}(z)_{-1}]^* \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{\frac{M}{2}} \sum_{l=0}^{1} \sum_{l'=0}^{1} \eta^{-l+l'} (1-\omega^{-l+l'}) |\mathcal{N}^{(j,0)}_{+,+1}|^2 \psi^{(j,0)}_{T^2+,n}(\omega^l z) [\psi^{(j,0)}_{T^2+,n}(\omega^{l'} z)]^* \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{\frac{M}{2}} \eta(1-\omega) |\mathcal{N}^{(j,0)}_{+,+1}|^2 \Big\{ \psi^{(j,0)}_{T^2+,n}(z) [\psi^{(j,0)}_{T^2+,n}(\omega z)]^* + \psi^{(j,0)}_{T^2+,n}(\omega z) [\psi^{(j,0)}_{T^2+,n}(z)]^* \Big\}, \end{aligned} \tag{5.1.21}$$

where we have used the relations (3.1.13), (3.1.14) and $\mathcal{N}^{(j,0)}_{+,+1} = \mathcal{N}^{(j,0)}_{+,-1}$ for $j = 0, 1, \cdots, \frac{M}{2}$ in the second equality, and $\eta = \omega = -1$ in the third equality.

Using the relation (3.1.7) and $\psi^{(M,0)}_{T^2+,n}(z) = \psi^{(0,0)}_{T^2+,n}(z)$, we can further rewrite Eq.(5.1.21) as

$$\begin{aligned}
I_{\eta=-1} &= \sum_{n=0}^{\infty} \eta(1-\omega)\Big\{2|\mathcal{N}^{(0,0)}_{+,+1}|^2\psi^{(0,0)}_{T^2+,n}(z)[\psi^{(0,0)}_{T^2+,n}(\omega z)]^* + 2|\mathcal{N}^{(M/2,0)}_{+,+1}|^2\psi^{(M/2,0)}_{T^2+,n}(z)[\psi^{(M/2,0)}_{T^2+,n}(\omega z)]^* \\
&\quad + \sum_{j=1}^{\frac{M}{2}-1}|\mathcal{N}^{(j,0)}_{+,+1}|^2(\psi^{(j,0)}_{T^2+,n}(z)[\psi^{(j,0)}_{T^2+,n}(\omega z)]^* + \psi^{(M-j,0)}_{T^2+,n}(z)[\psi^{(M-j,0)}_{T^2+,n}(\omega z)]^*)\Big\} \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{M-1} \eta(1-\omega)\psi^{(j,0)}_{T^2+,n}(z)[\psi^{(j,0)}_{T^2+,n}(\omega z)]^*, \qquad (5.1.22)
\end{aligned}$$

where we have used $\mathcal{N}^{(0,0)}_{+,+1} = \mathcal{N}^{(M/2,0)}_{+,+1} = 1/\sqrt{2}$ and $\mathcal{N}^{(j,0)}_{+,+1} = 1$ $(j = 1, 2, \cdots, \frac{M}{2} - 1)$ for $M =$ even from Table 3.2.

Taking the limit of $z' \to z$ and $\rho \to \infty$ in Eq.(5.1.3) and then substituting Eq.(5.1.22) into Eq.(5.1.3), we find that Eq.(5.1.4) holds for $(\alpha_1, \alpha_2) = (0, 0)$, $M =$ even and $\eta = -1$. We have succeeded in verifying Eq.(5.1.4) for the case of $(\alpha_1, \alpha_2) = (0, 0)$, $M =$ even and $\eta = \pm 1$. We can similarly show that Eq.(5.1.4) holds for other cases of (α_1, α_2), M and η.

Since $\{\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(z)$ $(n = 0, 1, \cdots;\ j = 0, 1, \cdots, M-1)\}$ forms the complete set of the mode functions on the torus T^2, Eq.(5.1.4) can be expressed as

$$\lim_{\rho\to\infty} \mathrm{tr}[\sigma_3 e^{D^2/\rho^2}]_\eta = \frac{1}{2}\int_{T^2} dz d\bar{z}\, \eta(1-\omega)\delta^2_{T^2}(z, \omega z), \qquad (5.1.23)$$

where

$$\delta^2_{T^2}(z, w) \equiv \sum_{n=0}^{\infty}\sum_{j=0}^{M-1} \psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(z)[\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(w)]^*. \qquad (5.1.24)$$

To proceed further, we need the explicit representation of the specific delta function $\delta^2_{T^2}(z, w)$ on the torus T^2. The specific delta function $\delta^2_{T^2}(z, w)$, defined by Eq.(5.1.24) should satisfy the relations

$$\delta^2_{T^2}(z+1, w) = e^{iq\Lambda_1(z)+i2\pi\alpha_1}\delta^2_{T^2}(z, w), \qquad (5.1.25)$$

$$\delta^2_{T^2}(z+\tau, w) = e^{iq\Lambda_2(z)+i2\pi\alpha_2}\delta^2_{T^2}(z, w), \qquad (5.1.26)$$

$$\delta^2_{T^2}(z, w+1) = \delta^2_{T^2}(z, w)e^{-iq\Lambda_1(w)-i2\pi\alpha_1}, \qquad (5.1.27)$$

$$\delta^2_{T^2}(z, w+\tau) = \delta^2_{T^2}(z, w)e^{-iq\Lambda_2(w)-i2\pi\alpha_2}, \qquad (5.1.28)$$

$$\int_{T^2} dw d\bar{w}\, \delta^2_{T^2}(z, w)\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(w) = \psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(z), \qquad (5.1.29)$$

$$\int_{T^2} dz d\bar{z}\, [\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(z)]^*\delta^2_{T^2}(z, w) = [\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(w)]^*. \qquad (5.1.30)$$

It is found that the specific delta function $\delta^2_{T^2}(z, w)$, which satisfies Eqs.(5.1.25)–(5.1.30), can be represented in terms of the standard delta function $\delta^2(z)$ as follows:

$$\delta^2_{T^2}(z, w) = \sum_{m,n\in\mathbb{Z}} e^{i\theta_{m,n}(z)}\delta^2(z - w - m - n\tau), \qquad (5.1.31)$$

where

$$\theta_{m,n}(z) = mq\Lambda_1(z) + 2\pi m\alpha_1 + nq\Lambda_2(z) + 2\pi n\alpha_2 + \pi Mmn. \tag{5.1.32}$$

The non-trivial phase $\theta_{m,n}(z)$ plays a crucial role in obtaining the correct index and is directly connected to winding numbers at fixed points on the orbifolds, as we will see in the Appendix **??**.

By utilizing the expression (5.1.31) of the delta function with the nontrivial phase (5.1.32), we can rewrite Eq.(5.1.23) as

$$\begin{aligned}\lim_{\rho\to\infty} \mathrm{tr}[\sigma_3 e^{D^2/\rho^2}]_\eta &= \int_{T^2} dz d\bar{z} \sum_{m,n\in\mathbb{Z}} \eta e^{i\theta_{m,n}(z)} \delta^2(z - \omega z - m - n\tau) \\ &= \frac{1}{4}\int_{T^2} dz d\bar{z} \sum_{m,n\in\mathbb{Z}} \eta e^{i\theta_{m,n}(z)} \delta^2(z - (m + n\tau)/2),\end{aligned} \tag{5.1.33}$$

where we have used $\omega = -1$.

By taking the fundamental domain of the torus T^2 to be $-\varepsilon \le y_1, y_2 < 1 - \varepsilon$ with a small positive number ε instead of $0 \le y_1, y_2 < 1$, the values of (m, n), which remain in the summation of Eq.(5.1.33) after the z-integration, are given by

$$(m, n) = (0, 0), (1, 0), (0, 1), (1, 1). \tag{5.1.34}$$

Then, we get

$$\lim_{\rho\to\infty} \mathrm{tr}[\sigma_3 e^{D^2/\rho^2}]_\eta = \frac{1}{2\times 2}\int_{T^2} dz d\bar{z} \sum_{p=1}^{4} W_p \delta^2(z - z_p^f), \tag{5.1.35}$$

where z_p^f $(p = 1, 2, 3, 4)$ are given by

$$z_1^f = 0, \quad z_2^f = 1/2, \quad z_3^f = \tau/2, \quad z_4^f = (1 + \tau)/2, \tag{5.1.36}$$

which are nothing but the fixed points on the $T^2/\mathbb{Z}_2$ orbifold. The coefficients W_p in Eq.(5.1.35) are

$$W_1 = \eta e^{i\theta_{m,n}(z)}\Big|_{z=z_1^f=0}^{(m,n)=(0,0)}, \tag{5.1.37}$$

$$W_2 = \eta e^{i\theta_{m,n}(z)}\Big|_{z=z_2^f=1/2}^{(m,n)=(1,0)}, \tag{5.1.38}$$

$$W_3 = \eta e^{i\theta_{m,n}(z)}\Big|_{z=z_3^f=\tau/2}^{(m,n)=(0,1)}, \tag{5.1.39}$$

$$W_4 = \eta e^{i\theta_{m,n}(z)}\Big|_{z=z_4^f=(1+\tau)/2}^{(m,n)=(1,1)}. \tag{5.1.40}$$

The explicit values of W_p $(p = 1, 2, 3, 4)$ are summarized in Table 5.1.

We have succeeded in deriving the index of the $T^2/\mathbb{Z}_2$ orbifold, and the results shown in Table 5.1 are found to be consistent with the number of the zero modes in Table 3.1. The physical significance of the coefficient W_p is not, however, clear at this moment. In Section 5.3, we will clarify that W_p is related to the winding numbers at the fixed point z_p^f on the $T^2/\mathbb{Z}_2$ orbifold.

flux	parity	twist	coefficients of the delta functions				sum of coefficients	index
M	η	(α_1,α_2)	W_1	W_2	W_3	W_4	$\sum_{p=1}^{4} W_p$	$M/2+\sum_{p=1}^{4} W_p/4$
$2m+1$	$+1$	$(0,0)$	$+1$	$+1$	$+1$	-1	$+2$	$(M+1)/2$
		$(\frac{1}{2},0)$	$+1$	-1	$+1$	$+1$	$+2$	$(M+1)/2$
		$(0,\frac{1}{2})$	$+1$	$+1$	-1	$+1$	$+2$	$(M+1)/2$
		$(\frac{1}{2},\frac{1}{2})$	$+1$	-1	-1	-1	-2	$(M-1)/2$
	-1	$(0,0)$	-1	-1	-1	$+1$	-2	$(M-1)/2$
		$(\frac{1}{2},0)$	-1	$+1$	-1	-1	-2	$(M-1)/2$
		$(0,\frac{1}{2})$	-1	-1	$+1$	-1	-2	$(M-1)/2$
		$(\frac{1}{2},\frac{1}{2})$	-1	$+1$	$+1$	$+1$	$+2$	$(M+1)/2$
$2m+2$	$+1$	$(0,0)$	$+1$	$+1$	$+1$	$+1$	$+4$	$M/2+1$
		$(\frac{1}{2},0)$	$+1$	-1	$+1$	-1	0	$M/2$
		$(0,\frac{1}{2})$	$+1$	$+1$	-1	-1	0	$M/2$
		$(\frac{1}{2},\frac{1}{2})$	$+1$	-1	-1	$+1$	0	$M/2$
	-1	$(0,0)$	-1	-1	-1	-1	-4	$M/2-1$
		$(\frac{1}{2},0)$	-1	$+1$	-1	$+1$	0	$M/2$
		$(0,\frac{1}{2})$	-1	-1	$+1$	$+1$	0	$M/2$
		$(\frac{1}{2},\frac{1}{2})$	-1	$+1$	$+1$	-1	0	$M/2$

Table 5.1: The coefficients W_p of the delta functions in Eq.(5.1.35) and the index for the $T^2/\mathbb{Z}_2$ orbifold.

5.2 Index theorem on $T^2/\mathbb{Z}_N$ $(N = 3,4,6)$ orbifolds with magnetic flux

The situation becomes more challenging for $T^2/\mathbb{Z}_N$ $(N = 3,4,6)$ orbifolds. The difficulty arises from the fact that the set of functions $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,0}(z)_\eta$ for $j = 0,1,\ldots,M-1$ with a fixed η are not always guaranteed to be linearly independent. Although we have obtained the complete set of the linearly independent $\mathbb{Z}_2$ eigenfunctions on the $T^2/\mathbb{Z}_2$ orbifold in Section 3.1.1, it is highly nontrivial to construct complete sets of $\mathbb{Z}_N$ eigenfunctions on the $T^2/\mathbb{Z}_N$ $(N = 3,4,6)$ orbifolds except for some small M [32]. Indeed, the complete sets of $\mathbb{Z}_N$ eigenfunctions on the $T^2/\mathbb{Z}_N$ $(N = 3,4,6)$ orbifolds for general M are unknown, so that we cannot follow the analysis given in the case of $T^2/\mathbb{Z}_2$ to derive the index theorem for $T^2/\mathbb{Z}_N$ $(N = 3,4,6)$.

To evaluate $\lim_{\rho\to\infty}\mathrm{tr}[\sigma_3 e^{D^2/\rho^2}]_\eta$ for the $T^2/\mathbb{Z}_N$ $(N = 3,4,6)$ orbifolds, we generalize the expression (5.1.4) for $T^2/\mathbb{Z}_2$ to

$$\lim_{\rho\to\infty}\mathrm{tr}[\sigma_3 e^{D^2/\rho^2}]_\eta = \frac{1}{N}\int_{T^2} dz d\bar{z}\sum_{n=0}^{\infty}\sum_{j=0}^{M-1}\sum_{k=1}^{N-1}\eta^k(1-\omega^k)\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(z)[\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(\omega^k z)]^*. \quad (5.2.1)$$

This formula is true for the case of $M = 0$, as shown in [39]. We can also verify Eq.(5.2.1) for some small M by explicitly constructing complete sets of $\mathbb{Z}_N$ eigenfunctions on the $T^2/\mathbb{Z}_N$ $(N = 3,4,6)$ orbifolds (see the Appendix B.3). Unfortunately, proving the relation

(5.2.1) for arbitrary M has not been achieved at this point. The proof of Eq.(5.2.1) is left for future work. In the following, we will show that the relation (5.2.1) leads to the correct results.

By using of the completeness relation (5.1.24) with the representation (5.1.31), Eq.(5.2.1) can be expressed as

$$\lim_{\rho\to\infty} \mathrm{tr}[\sigma_3 e^{D^2/\rho^2}]_\eta = \frac{1}{N}\int_{T^2} dz d\bar{z} \sum_{k=1}^{N-1} \sum_{m_k,n_k\in\mathbb{Z}} \eta^k (1-\omega^k) e^{i\theta_{m_k,n_k}(z)} \delta^2(z-\omega^k z - m_k - n_k\tau). \quad (5.2.2)$$

We evaluate Eq.(5.2.2) for $N = 3, 4$ and 6, separately.

5.2.1 $T^2/\mathbb{Z}_3$

In the case of $N = 3$, we have

$$\begin{aligned} &\lim_{\rho\to\infty} \mathrm{tr}[\sigma_3 e^{D^2/\rho^2}]_\eta \\ &= \frac{1}{3}\int_{T^2} dz d\bar{z} \sum_{k=1}^{2} \sum_{m_k,n_k\in\mathbb{Z}} \eta^k (1-\omega^k) e^{i\theta_{m_k,n_k}(z)} \delta^2(z-\omega^k z - m_k - n_k\tau). \end{aligned} \quad (5.2.3)$$

Since we have taken the fundamental domain of the torus T^2 to be $-\varepsilon \leq y_1, y_2 < 1-\varepsilon$, the values of (m_1, n_1) and (m_2, n_2), which remain in the summations of Eq.(5.2.3) after the z-integration, are given by

$$(m_1, n_1) = (0,0), (1,0), (1,1), \quad (5.2.4)$$
$$(m_2, n_2) = (0,0), (1,1), (0,1). \quad (5.2.5)$$

Then, it follows that Eq.(5.2.3) reduces to

$$\lim_{\rho\to\infty} \mathrm{tr}[\sigma_3 e^{D^2/\rho^2}]_\eta = \frac{1}{2\times 3}\int_{T^2} dz d\bar{z} \sum_{p=1}^{3} W_p \delta^2(z - z_p^f). \quad (5.2.6)$$

Here, we have used the relations

$$\delta^2(z-\omega z - m_1 - n_1\tau) = \frac{1}{3}\delta^2(z - (2m_1 - n_1)/3 - (m_1+n_1)\tau/3), \quad (5.2.7)$$
$$\delta^2(z-\omega^2 z - m_2 - n_2\tau) = \frac{1}{3}\delta^2(z - (m_2 + n_2)/3 - (-m_2+2n_2)\tau/3). \quad (5.2.8)$$

The z_p^f ($p = 1, 2, 3$) are the fixed points of the $T^2/\mathbb{Z}_3$ orbifold, i.e.

$$z_1^f = 0, \quad z_2^f = \frac{1}{3}(2+\tau), \quad z_3^f = \frac{1}{3}(1+2\tau) \quad (5.2.9)$$

and the coefficients W_p $(p = 1, 2, 3)$ are given by

$$W_1 = \frac{2}{3}\{\eta(1-\omega)e^{i\theta_{m_1,n_1}(z)}\Big|_{z=z_1^f}^{(m_1,n_1)=(0,0)} + \eta^2(1-\omega^2)e^{i\theta_{m_2,n_2}(z)}\Big|_{z=z_1^f}^{(m_2,n_2)=(0,0)}\}, \tag{5.2.10}$$

$$W_2 = \frac{2}{3}\{\eta(1-\omega)e^{i\theta_{m_1,n_1}(z)}\Big|_{z=z_2^f}^{(m_1,n_1)=(1,0)} + \eta^2(1-\omega^2)e^{i\theta_{m_2,n_2}(z)}\Big|_{z=z_2^f}^{(m_2,n_2)=(1,1)}\}, \tag{5.2.11}$$

$$W_3 = \frac{2}{3}\{\eta(1-\omega)e^{i\theta_{m_1,n_1}(z)}\Big|_{z=z_3^f}^{(m_1,n_1)=(1,1)} + \eta^2(1-\omega^2)e^{i\theta_{m_2,n_2}(z)}\Big|_{z=z_3^f}^{(m_2,n_2)=(0,1)}\}. \tag{5.2.12}$$

The explicit values of W_p $(p = 1, 2, 3)$ are summarized in Table 5.2 and 5.3.

flux	parity	twist	coefficients of the delta functions			sum of coefficients	index
M	η	α	W_1	W_2	W_3	$\sum_{p=1}^{3} W_p$	$M/3 + \sum_{p=1}^{3} W_p/6$
$6m+1$	1	1/6	+2	0	+2	+4	$(M+2)/3$
		1/2	+2	−2	−2	−2	$(M-1)/3$
		5/6	+2	+2	0	+4	$(M+2)/3$
	ω	1/6	0	−2	0	−2	$(M-1)/3$
		1/2	0	+2	+2	+4	$(M+2)/3$
		5/6	0	0	−2	−2	$(M-1)/3$
	ω^2	1/6	−2	+2	−2	−2	$(M-1)/3$
		1/2	−2	0	0	−2	$(M-1)/3$
		5/6	−2	−2	+2	−2	$(M-1)/3$
$6m+2$	1	0	+2	0	0	+2	$(M+1)/3$
		1/3	+2	−2	+2	+2	$(M+1)/3$
		2/3	+2	+2	−2	+2	$(M+1)/3$
	ω	0	0	−2	−2	−4	$(M-2)/3$
		1/3	0	+2	0	+2	$(M+1)/3$
		2/3	0	0	+2	+2	$(M+1)/3$
	ω^2	0	−2	+2	+2	+2	$(M+1)/3$
		1/3	−2	0	−2	−4	$(M-2)/3$
		2/3	−2	−2	0	−4	$(M-2)/3$
$6m+3$	1	1/6	+2	−2	0	0	$M/3$
		1/2	+2	+2	+2	+6	$M/3+1$
		5/6	+2	0	−2	0	$M/3$
	ω	1/6	0	+2	−2	0	$M/3$
		1/2	0	0	0	0	$M/3$
		5/6	0	−2	+2	0	$M/3$
	ω^2	1/6	−2	0	+2	0	$M/3$
		1/2	−2	−2	−2	−6	$M/3-1$
		5/6	−2	+2	0	0	$M/3$

Table 5.2: The coefficients W_p of the delta functions in Eqs.(5.2.10)-(5.2.12) and the index for the $T^2/\mathbb{Z}_3$ orbifold.

flux	parity	twist	coefficients of the delta functions			sum of coefficients	index
M	η	α	W_1	W_2	W_3	$\sum_{p=1}^{3} W_p$	$M/3 + \sum_{p=1}^{3} W_p/6$
$6m+4$	1	0	$+2$	-2	-2	-2	$(M-1)/3$
		$1/3$	$+2$	$+2$	0	$+4$	$(M+2)/3$
		$2/3$	$+2$	0	$+2$	$+4$	$(M+2)/3$
	ω	0	0	$+2$	$+2$	$+4$	$(M+2)/3$
		$1/3$	0	0	-2	-2	$(M-1)/3$
		$2/3$	0	-2	0	-2	$(M-1)/3$
	ω^2	0	-2	0	0	-2	$(M-1)/3$
		$1/3$	-2	-2	$+2$	-2	$(M-1)/3$
		$2/3$	-2	$+2$	-2	-2	$(M-1)/3$
$6m+5$	1	$1/6$	$+2$	$+2$	-2	$+2$	$(M+1)/3$
		$1/2$	$+2$	0	0	$+2$	$(M+1)/3$
		$5/6$	$+2$	-2	$+2$	$+2$	$(M+1)/3$
	ω	$1/6$	0	0	$+2$	$+2$	$(M+1)/3$
		$1/2$	0	-2	-2	-4	$(M-2)/3$
		$5/6$	0	$+2$	0	$+2$	$(M+1)/3$
	ω^2	$1/6$	-2	-2	0	-4	$(M-2)/3$
		$1/2$	-2	$+2$	$+2$	$+2$	$(M+1)/3$
		$5/6$	-2	0	-2	-4	$(M-2)/3$
$6m+6$	1	0	$+2$	$+2$	$+2$	$+6$	$M/3+1$
		$1/3$	$+2$	0	-2	0	$M/3$
		$2/3$	$+2$	-2	0	0	$M/3$
	ω	0	0	0	0	0	$M/3$
		$1/3$	0	-2	$+2$	0	$M/3$
		$2/3$	0	$+2$	-2	0	$M/3$
	ω^2	0	-2	-2	-2	-6	$M/3-1$
		$1/3$	-2	$+2$	0	0	$M/3$
		$2/3$	-2	0	$+2$	0	$M/3$

Table 5.3: The coefficients W_p of the delta functions in Eqs.(5.2.10)-(5.2.12) and the index for the $T^2/\mathbb{Z}_3$ orbifold.

5.2.2 $T^2/\mathbb{Z}_4$

In the case of $N = 4$, we have

$$\begin{aligned} &\lim_{\rho\to\infty} \mathrm{tr}[\sigma_3 e^{D^2/\rho^2}]_\eta \\ &= \frac{1}{4}\int_{T^2} dz d\bar{z} \sum_{k=1}^{3} \sum_{m_k, n_k \in \mathbb{Z}} \eta^k (1-\omega^k) e^{i\theta_{m_k,n_k}(z)} \delta^2(z - \omega^k z - m_k - n_k \tau). \end{aligned} \tag{5.2.13}$$

The values of (m_1, n_1), (m_2, n_2) and (m_3, n_3), which remain in the summations of Eq.(5.2.13) after the z-integration, are found to be

$$(m_1, n_1) = (0, 0), (1, 0), \tag{5.2.14}$$

$$(m_2, n_2) = (0, 0), (1, 0), (0, 1), (1, 1), \tag{5.2.15}$$

$$(m_3, n_3) = (0, 0), (0, 1). \tag{5.2.16}$$

Then, it follows that Eq.(5.2.13) can be written into the form

$$\lim_{\rho\to\infty} \mathrm{tr}[\sigma_3 e^{D^2/\rho^2}]_\eta = \frac{1}{2\times 4}\int_{T^2} dz d\bar{z} \sum_{p=1}^{4} W_p \delta^2(z - z_p{}^f), \tag{5.2.17}$$

where we have used the relations

$$\delta^2(z - \omega z - m_1 - n_1\tau) = \frac{1}{2}\delta^2(z - (m_1 - n_1)/2 - (m_1 + n_1)\tau/2), \tag{5.2.18}$$

$$\delta^2(z - \omega^2 z - m_2 - n_2\tau) = \frac{1}{4}\delta^2(z - m_2/2 - n_2\tau/2), \tag{5.2.19}$$

$$\delta^2(z - \omega^3 z - m_3 - n_3\tau) = \frac{1}{2}\delta^2(z - (m_3 + n_3)/2 - (-m_3 + n_3)\tau/2). \tag{5.2.20}$$

The z_1^f and z_2^f are the $\mathbb{Z}_4$ fixed points under the $\mathbb{Z}_4$ identification $z \sim \omega z$ ($\omega = e^{i2\pi/4} = i$), i.e.

$$\mathbb{Z}_4 \text{ fixed point} : \ z_1^f = 0, \quad z_2^f = \frac{1}{2}(1+\tau). \tag{5.2.21}$$

Since the $\mathbb{Z}_4$ group includes $\mathbb{Z}_2$ as its subgroup, there are additionally two "$\mathbb{Z}_2$ fixed points"[1] given by

$$\text{“}\mathbb{Z}_2 \text{ fixed point”} : \ z_3^f = \frac{1}{2}, \quad z_4^f = \frac{\tau}{2}. \tag{5.2.23}$$

The coefficients W_p $(p = 1, 2, 3, 4)$ in Eq.(5.2.17) are

$$\begin{aligned} W_1 = {} & \eta(1-\omega)e^{i\theta_{m_1,n_1}(z)}\Big|_{z=z_1^f}^{(m_1,n_1)=(0,0)} + \frac{1}{2}\eta^2(1-\omega^2)e^{i\theta_{m_2,n_2}(z)}\Big|_{z=z_1^f}^{(m_2,n_2)=(0,0)} \\ & + \eta^3(1-\omega^3)e^{i\theta_{m_3,n_3}(z)}\Big|_{z=z_1^f}^{(m_3,n_3)=(0,0)}, \end{aligned} \tag{5.2.24}$$

$$\begin{aligned} W_2 = {} & \eta(1-\omega)e^{i\theta_{m_1,n_1}(z)}\Big|_{z=z_2^f}^{(m_1,n_1)=(1,0)} + \frac{1}{2}\eta^2(1-\omega^2)e^{i\theta_{m_2,n_2}(z)}\Big|_{z=z_2^f}^{(m_2,n_2)=(1,1)} \\ & + \eta^3(1-\omega^3)e^{i\theta_{m_3,n_3}(z)}\Big|_{z=z_2^f}^{(m_3,n_3)=(0,1)}, \end{aligned} \tag{5.2.25}$$

$$W_3 = \frac{1}{2}\eta^2(1-\omega^2)e^{i\theta_{m_2,n_2}(z)}\Big|_{z=z_3^f}^{(m_2,n_2)=(1,0)}, \tag{5.2.26}$$

$$W_4 = \frac{1}{2}\eta^2(1-\omega^2)e^{i\theta_{m_2,n_2}(z)}\Big|_{z=z_4^f}^{(m_2,n_2)=(0,1)}. \tag{5.2.27}$$

---

[1] "$\mathbb{Z}_2$ fixed points" are defined by

$$z_I^f = \omega^2 z_I^f + m + n\tau \qquad \text{for} \quad {}^\exists m, n \in \mathbb{Z}. \tag{5.2.22}$$

The explicit values of W_p $(p = 1, 2, 3, 4)$ are summarized in Table 5.4.

flux	parity	twist	coefficients of the delta functions				sum of coefficients	index
M	η	α	W_1	W_2	W_3	W_4	$\sum_{p=1}^{4} W_p$	$M/4 + \sum_{p=1}^{4} W_p/8$
$4m+1$	1	0	$+3$	$+1$	$+1$	$+1$	$+6$	$(M+3)/4$
		$1/2$	$+3$	-3	-1	-1	-2	$(M-1)/4$
	i	0	$+1$	-1	-1	-1	-2	$(M-1)/4$
		$1/2$	$+1$	$+3$	$+1$	$+1$	$+6$	$(M+3)/4$
	-1	0	-1	-3	$+1$	$+1$	-2	$(M-1)/4$
		$1/2$	-1	$+1$	-1	-1	-2	$(M-1)/4$
	$-i$	0	-3	$+3$	-1	-1	-2	$(M-1)/4$
		$1/2$	-3	-1	$+1$	$+1$	-2	$(M-1)/4$
$4m+2$	1	0	$+3$	-1	$+1$	$+1$	$+4$	$(M+2)/4$
		$1/2$	$+3$	$+3$	-1	-1	$+4$	$(M+2)/4$
	i	0	$+1$	-3	-1	-1	-4	$(M-2)/4$
		$1/2$	$+1$	$+1$	$+1$	$+1$	$+4$	$(M+2)/4$
	-1	0	-1	$+3$	$+1$	$+1$	$+4$	$(M+2)/4$
		$1/2$	-1	-1	-1	-1	-4	$(M-2)/4$
	$-i$	0	-3	$+1$	-1	-1	-4	$(M-2)/4$
		$1/2$	-3	-3	$+1$	$+1$	-4	$(M-2)/4$
$4m+3$	1	0	$+3$	-3	$+1$	$+1$	$+2$	$(M+1)/4$
		$1/2$	$+3$	$+1$	-1	-1	$+2$	$(M+1)/4$
	i	0	$+1$	$+3$	-1	-1	$+2$	$(M+1)/4$
		$1/2$	$+1$	-1	$+1$	$+1$	$+2$	$(M+1)/4$
	-1	0	-1	$+1$	$+1$	$+1$	$+2$	$(M+1)/4$
		$1/2$	-1	-3	-1	-1	-6	$(M-3)/4$
	$-i$	0	-3	-1	-1	-1	-6	$(M-3)/4$
		$1/2$	-3	$+3$	$+1$	$+1$	$+2$	$(M+1)/4$
$4m+4$	1	0	$+3$	$+3$	$+1$	$+1$	$+8$	$M/4+1$
		$1/2$	$+3$	-1	-1	-1	0	$M/4$
	i	0	$+1$	$+1$	-1	-1	0	$M/4$
		$1/2$	$+1$	-3	$+1$	$+1$	0	$M/4$
	-1	0	-1	-1	$+1$	$+1$	0	$M/4$
		$1/2$	-1	$+3$	-1	-1	0	$M/4$
	$-i$	0	-3	-3	-1	-1	-8	$M/4-1$
		$1/2$	-3	$+1$	$+1$	$+1$	0	$M/4$

Table 5.4: The coefficients W_p of the delta functions in Eqs.(5.2.24)-(5.2.27) and the index for the $T^2/\mathbb{Z}_4$ orbifold.

5.2.3 $T^2/\mathbb{Z}_6$

In the case of $N = 6$, we obtain

$$\lim_{\rho\to\infty} \mathrm{tr}[\sigma_3 e^{D^2/\rho^2}]_\eta$$
$$= \frac{1}{6}\int_{T^2} dz d\bar{z} \sum_{k=1}^{5} \sum_{m_k,n_k\in\mathbb{Z}} \eta^k (1-\omega^k) e^{i\theta_{m_k,n_k}(z)} \delta^2(z-\omega^k z - m_k - n_k\tau). \tag{5.2.28}$$

The values of (m_1, n_1),$\cdots$, (m_5, n_5), which remain in the summations of Eq.(5.2.28) after the z-integration, are given by

$$(m_1, n_1) = (0,0), \tag{5.2.29}$$
$$(m_2, n_2) = (0,0), (1,0), (2,0), \tag{5.2.30}$$
$$(m_3, n_3) = (0,0), (1,0), (0,1)(1,1), \tag{5.2.31}$$
$$(m_4, n_4) = (0,0), (0,1), (0,2), \tag{5.2.32}$$
$$(m_5, n_5) = (0,0). \tag{5.2.33}$$

Then, we can rewrite Eq.(5.2.28) into the form

$$\lim_{\rho\to\infty} \mathrm{tr}[\sigma_3 e^{D^2/\rho^2}]_\eta = \frac{1}{2\times 6}\int_{T^2} dz d\bar{z} \sum_{p=1}^{6} W_p \delta^2(z - z_p^f), \tag{5.2.34}$$

where we have used the relations

$$\delta^2(z-\omega z - m_1 - n_1\tau) = \delta^2(z + n_1 - (m_1+n_1)\tau), \tag{5.2.35}$$
$$\delta^2(z-\omega^2 z - m_2 - n_2\tau) = \frac{1}{3}\delta^2(z - (m_2-n_2)/3 - (m_2+2n_2)\tau/3), \tag{5.2.36}$$
$$\delta^2(z-\omega^3 z - m_3 - n_3\tau) = \frac{1}{4}\delta^2(z - m_3/2 - n_3\tau/2), \tag{5.2.37}$$
$$\delta^2(z-\omega^4 z - m_4 - n_4\tau) = \frac{1}{3}\delta^2(z - (2m_4+n_4)/3 - (-m_4+n_4)\tau/3), \tag{5.2.38}$$
$$\delta^2(z-\omega^5 z - m_5 - n_5\tau) = \delta^2(z - (m_5+n_5) + m_5\tau). \tag{5.2.39}$$

Note that there is one $\mathbb{Z}_6$ fixed point under the $\mathbb{Z}_6$ identification $z \sim \omega z$, i.e.

$$\mathbb{Z}_6 \text{ fixed point} : \ z_1^f = 0. \tag{5.2.40}$$

Since the $\mathbb{Z}_6$ group includes $\mathbb{Z}_3$ and $\mathbb{Z}_2$ as its subgroups, there are additionally two "$\mathbb{Z}_3$ fixed points" and three "$\mathbb{Z}_2$ fixed points":

$$\text{“}\mathbb{Z}_3 \text{ fixed point”} : \quad z_2^f = \frac{1}{3}(1+\tau), \quad z_3^f = \frac{2}{3}(1+\tau), \tag{5.2.41}$$
$$\text{“}\mathbb{Z}_2 \text{ fixed point”} : \quad z_4^f = \frac{1}{2}, \quad z_5^f = \frac{\tau}{2}, \quad z_6^f = \frac{1}{2}(1+\tau). \tag{5.2.42}$$

The coefficients W_p $(p = 1, 2, \cdots, 6)$ are given by

$$\begin{aligned} W_1 = &\, 2\eta(1-\omega)e^{i\theta_{m_1,n_1}(z)}\Big|_{z=z_1^f}^{(m_1,n_1)=(0,0)} + \frac{2}{3}\eta^2(1-\omega^2)e^{i\theta_{m_2,n_2}(z)}\Big|_{z=z_1^f}^{(m_2,n_2)=(0,0)} \\ &+ \frac{1}{2}\eta^3(1-\omega^3)e^{i\theta_{m_3,n_3}(z)}\Big|_{z=z_1^f}^{(m_3,n_3)=(0,0)} + \frac{2}{3}\eta^4(1-\omega^4)e^{i\theta_{m_4,n_4}(z)}\Big|_{z=z_1^f}^{(m_4,n_4)=(0,0)} \\ &+ 2\eta^5(1-\omega^5)e^{i\theta_{m_5,n_5}(z)}\Big|_{z=z_1^f}^{(m_5,n_5)=(0,0)}, \end{aligned} \tag{5.2.43}$$

$$W_2 = \frac{2}{3}\eta^2(1-\omega^2)e^{i\theta_{m_2,n_2}(z)}\Big|_{z=z_2^f}^{(m_2,n_2)=(1,0)} + \frac{2}{3}\eta^4(1-\omega^4)e^{i\theta_{m_4,n_4}(z)}\Big|_{z=z_2^f}^{(m_4,n_4)=(0,1)}, \tag{5.2.44}$$

$$W_3 = \frac{2}{3}\eta^2(1-\omega^2)e^{i\theta_{m_2,n_2}(z)}\Big|_{z=z_3^f}^{(m_2,n_2)=(2,0)} + \frac{2}{3}\eta^4(1-\omega^4)e^{i\theta_{m_4,n_4}(z)}\Big|_{z=z_3^f}^{(m_4,n_4)=(0,2)}, \tag{5.2.45}$$

$$W_4 = \frac{1}{2}\eta^3(1-\omega^3)e^{i\theta_{m_3,n_3}(z)}\Big|_{z=z_4^f}^{(m_3,n_3)=(1,0)}, \tag{5.2.46}$$

$$W_5 = \frac{1}{2}\eta^3(1-\omega^3)e^{i\theta_{m_3,n_3}(z)}\Big|_{z=z_5^f}^{(m_3,n_3)=(0,1)}, \tag{5.2.47}$$

$$W_6 = \frac{1}{2}\eta^3(1-\omega^3)e^{i\theta_{m_3,n_3}(z)}\Big|_{z=z_6^f}^{(m_3,n_3)=(1,1)}. \tag{5.2.48}$$

The explicit values of W_p $(p = 1, 2, \cdots, 6)$ are summarized in Table 5.5.

5.3 Winding numbers at fixed points on $T^2/\mathbb{Z}_N$ orbifolds

In this section, we clarify the relation between the winding numbers and the coefficients W_p in front of the delta functions in Eqs.(5.1.35), (5.2.6), (5.2.17) and (5.2.34).

A crucial observation is that the coefficient W_p can be represented, in terms of the winding numbers $\chi_{\pm p}$ given in Eq.(3.2.20) and (3.2.21), as[2]

$$W_p = -2\chi_{+p} + N - 1 = -\chi_{+p} + \chi_{-p}. \tag{5.3.1}$$

We derive the formula (5.3.1) for the $T^2/\mathbb{Z}_N$ $(N = 2, 3, 4, 6)$ orbifolds.

5.3.1 $T^2/\mathbb{Z}_2$

From Eqs.(5.1.33) and (5.1.35), we have

$$\begin{aligned} W_p &= \eta e^{i\theta_{m,n}(z_p^f)} \\ &= \eta e^{imq\Lambda_1(z_p^f)+i2\pi m\alpha_1+inq\Lambda_2(z_p^f)+i2\pi n\alpha_2+i\pi Mmn} \\ &= \omega^{\chi_{+p}} \qquad (p = 1, 2, 3, 4), \end{aligned} \tag{5.3.2}$$

---

[2] Note that in Eqs.(3.2.17)– (3.2.21) and (5.3.1), N should take the value 2 for the $\mathbb{Z}_2$ fixed points on the $T^2/\mathbb{Z}_4$ and $T^2/\mathbb{Z}_6$ orbifolds, and 3 for the $\mathbb{Z}_3$ fixed points on the $T^2/\mathbb{Z}_6$ orbifold.

flux	parity	twist	coefficients of the delta functions						sum of coefficients	index
M	η	α	W_1	W_2	W_3	W_4	W_5	W_6	$\sum_{p=1}^{6} W_p$	$M/6 + \sum_{p=1}^{6} W_p/12$
$6m+1$	1	$1/2$	$+5$	-2	-2	-1	-1	-1	-2	$(M-1)/6$
	ω	$1/2$	$+3$	$+2$	$+2$	$+1$	$+1$	$+1$	$+10$	$(M+5)/6$
	ω^2	$1/2$	$+1$	0	0	-1	-1	-1	-2	$(M-1)/6$
	ω^3	$1/2$	-1	-2	-2	$+1$	$+1$	$+1$	-2	$(M-1)/6$
	ω^4	$1/2$	-3	$+2$	$+2$	-1	-1	-1	-2	$(M-1)/6$
	ω^5	$1/2$	-5	0	0	$+1$	$+1$	$+1$	-2	$(M-1)/6$
$6m+2$	1	0	$+5$	0	0	$+1$	$+1$	$+1$	$+8$	$(M+4)/6$
	ω	0	$+3$	-2	-2	-1	-1	-1	-4	$(M-2)/6$
	ω^2	0	$+1$	$+2$	$+2$	$+1$	$+1$	$+1$	$+8$	$(M+4)/6$
	ω^3	0	-1	0	0	-1	-1	-1	-4	$(M-2)/6$
	ω^4	0	-3	-2	-2	$+1$	$+1$	$+1$	-4	$(M-2)/6$
	ω^5	0	-5	$+2$	$+2$	-1	-1	-1	-4	$(M-2)/6$
$6m+3$	1	$1/2$	$+5$	$+2$	$+2$	-1	-1	-1	$+6$	$(M+3)/6$
	ω	$1/2$	$+3$	0	0	$+1$	$+1$	$+1$	$+6$	$(M+3)/6$
	ω^2	$1/2$	$+1$	-2	-2	-1	-1	-1	-6	$(M-3)/6$
	ω^3	$1/2$	-1	$+2$	$+2$	$+1$	$+1$	$+1$	$+6$	$(M+3)/6$
	ω^4	$1/2$	-3	0	0	-1	-1	-1	-6	$(M-3)/6$
	ω^5	$1/2$	-5	-2	-2	$+1$	$+1$	$+1$	-6	$(M-3)/6$
$6m+4$	1	0	$+5$	-2	-2	$+1$	$+1$	$+1$	$+4$	$(M+2)/6$
	ω	0	$+3$	$+2$	$+2$	-1	-1	-1	$+4$	$(M+2)/6$
	ω^2	0	$+1$	0	0	$+1$	$+1$	$+1$	$+4$	$(M+2)/6$
	ω^3	0	-1	-2	-2	-1	-1	-1	-8	$(M-4)/6$
	ω^4	0	-3	$+2$	$+2$	$+1$	$+1$	$+1$	$+4$	$(M+2)/6$
	ω^5	0	-5	0	0	-1	-1	-1	-8	$(M-4)/6$
$6m+5$	1	$1/2$	$+5$	0	0	-1	-1	-1	$+2$	$(M+1)/6$
	ω	$1/2$	$+3$	-2	-2	$+1$	$+1$	$+1$	$+2$	$(M+1)/6$
	ω^2	$1/2$	$+1$	$+2$	$+2$	-1	-1	-1	$+2$	$(M+1)/6$
	ω^3	$1/2$	-1	0	0	$+1$	$+1$	$+1$	$+2$	$(M+1)/6$
	ω^4	$1/2$	-3	-2	-2	-1	-1	-1	-10	$(M-5)/6$
	ω^5	$1/2$	-5	$+2$	$+2$	$+1$	$+1$	$+1$	$+2$	$(M+1)/6$
$6m+6$	1	0	$+5$	$+2$	$+2$	$+1$	$+1$	$+1$	$+12$	$M/6+1$
	ω	0	$+3$	0	0	-1	-1	-1	0	$M/6$
	ω^2	0	$+1$	-2	-2	$+1$	$+1$	$+1$	0	$M/6$
	ω^3	0	-1	$+2$	$+2$	-1	-1	-1	0	$M/6$
	ω^4	0	-3	0	0	$+1$	$+1$	$+1$	0	$M/6$
	ω^5	0	-5	-2	-2	-1	-1	-1	-12	$M/6-1$

Table 5.5: The coefficients W_p of the delta functions in Eqs.(5.2.43)-(5.2.48) and the index for the $T^2/\mathbb{Z}_6$ orbifold .

where we have used $\eta = \omega^l$ $(l = 0, 1)$ and Eqs.(5.1.32) and (3.2.20). Since χ_{+p} is taken to be 0 or 1, the following identity holds:

$$\omega^{\chi_{+p}} = -2\chi_{+p} + 1 \qquad (\omega = -1). \tag{5.3.3}$$

Thus, we obtain

$$W_p = -2\chi_{+p} + 1, \tag{5.3.4}$$

which is the relation (5.3.1) for $T^2/\mathbb{Z}_2$.

5.3.2 $T^2/\mathbb{Z}_3$

From Eqs.(5.2.10)–(5.2.12), we find

$$\begin{aligned} W_p &= \frac{2}{3}\{\eta(1-\omega)e^{i\theta_{m_1,n_1}(z_p^f)} + \eta^2(1-\omega^2)e^{i\theta_{m_2,n_2}(z_p^f)}\} \\ &= \frac{2}{3}\{\eta(1-\omega)e^{i\theta_{m_1,n_1}(z_p^f)} + \eta^2(1-\omega^2)e^{i2\theta_{m_1,n_1}(z_p^f)}\} \\ &= \frac{2}{3}\{(1-\omega)\omega^{\chi_{+p}} + (1-\omega^2)\omega^{2\chi_{+p}}\}, \end{aligned} \tag{5.3.5}$$

where (m_k, n_k) are determined from the fixed point equations $z_p^f = \omega^k z_p^f + m_k + n_k\tau$ $(k = 1, 2)$. At the second equality of Eq.(5.3.5), we have used the relation

$$e^{i\theta_{m_2,n_2}(z_p^f)} = e^{i2\theta_{m_1,n_1}(z_p^f)}. \tag{5.3.6}$$

The proof of Eq.(5.3.6) will be given in the subsection B.2.

For $\chi_{+p} = 0, 1, 2$, the following identity holds

$$\omega^{\chi_{+p}} = 1 + \left(\frac{5}{2}\omega - 1\right)\chi_{+p} - \frac{3}{2}\omega\chi_{+p}^2 \tag{5.3.7}$$

with $\omega = e^{i2\pi/3}$. Inserting Eq.(5.3.7) into Eq.(5.3.5) and using $\chi_{+p}(\chi_{+p} - 1)(\chi_{+p} - 2) = 0$, we have

$$W_p = -2\chi_{+p} + 2. \tag{5.3.8}$$

5.3.3 $T^2/\mathbb{Z}_4$

We first note that the $T^2/\mathbb{Z}_4$ orbifold has two $\mathbb{Z}_4$ fixed points ($z_1^f = 0$, $z_2^f = (1+\tau)/2$) and two $\mathbb{Z}_2$ fixed points ($z_3^f = 1/2$, $z_4^f = \tau/2$). For the $\mathbb{Z}_4$ fixed points z_p^f $(p = 1, 2)$, from Eqs.(5.2.24) and (5.2.25) we obtain

$$\begin{aligned} W_p &= \eta(1-\omega)e^{i\theta_{m_1,n_1}(z_p^f)} + \frac{1}{2}\eta^2(1-\omega^2)e^{i\theta_{m_2,n_2}(z_p^f)} + \eta^3(1-\omega^3)e^{i\theta_{m_3,n_3}(z_p^f)} \\ &= \eta(1-\omega)e^{i\theta_{m_1,n_1}(z_p^f)} + \frac{1}{2}\eta^2(1-\omega^2)e^{i2\theta_{m_1,n_1}(z_p^f)} + \eta^3(1-\omega^3)e^{i3\theta_{m_1,n_1}(z_p^f)} \\ &= (1-\omega)\omega^{\chi_{+p}} + \frac{1}{2}(1-\omega^2)\omega^{2\chi_{+p}} + (1-\omega^3)\omega^{3\chi_{+p}}, \end{aligned} \tag{5.3.9}$$

where we have used

$$e^{i\theta_{m_k,n_k}(z_p^f)} = e^{ik\theta_{m_1,n_1}(z_p^f)} \qquad (k = 2, 3), \tag{5.3.10}$$

and (m_k, n_k) are solutions to the fixed point equations $z_p^f = \omega^k z_p^f + m_k + n_k\tau$ $(k = 1, 2, 3)$.

Using the identities

$$\omega^{\chi_{+p}} = 1 + \left(-\frac{1}{3} + \frac{8}{3}\omega\right)\chi_{+p} - (1 + 2\omega)\chi_{+p}^2 + \left(\frac{1}{3} + \frac{1}{3}\omega\right)\chi_{+p}^3, \tag{5.3.11}$$

$$\chi_{+p}(\chi_{+p} - 1)(\chi_{+p} - 2)(\chi_{+p} - 3) = 0, \tag{5.3.12}$$

for $\chi_{+p} = 0, 1, 2, 3$ with $\omega = e^{i2\pi/4} = i$, we can show that the right-hand side of Eq.(5.3.9) reduces to

$$W_p = -2\chi_{+p} + 3 \qquad \text{for} \quad p = 1, 2. \tag{5.3.13}$$

For the $\mathbb{Z}_2$ fixed point $z_3^f = 1/2$ with $z_3^f = \omega^2 z_3^f + m_2 + n_2\tau$, from Eq.(5.2.26) we have

$$\begin{aligned} W_3 &= \frac{1}{2}\eta^2(1 - \omega^2)e^{i\theta_{m_2,n_2}(z_3^f)} \\ &= (\omega^2)^{\chi_{+3}} \\ &= -2\chi_{+3} + 1, \end{aligned} \tag{5.3.14}$$

where we have used $\omega^2 = -1$, $\eta^2 = (\omega^2)^l$ $(l = 0, 1)$ and Eq.(3.2.20) with $N = 2$. In the last equality of Eq.(5.3.14), we have used the analysis in the subsection 5.3.1. A similar discussion holds for another $\mathbb{Z}_2$ fixed point $z_4^f = \tau/2$, and we find

$$W_4 = -2\chi_{+4} + 1. \tag{5.3.15}$$

5.3.4 $T^2/\mathbb{Z}_6$

We first note that the $T^2/\mathbb{Z}_6$ orbifold contains

$$\begin{aligned} &\text{one } \mathbb{Z}_6 \text{ fixed point} : z_1^f = 0, \\ &\text{two } \mathbb{Z}_3 \text{ fixed points} : z_2^f = (1 + \tau)/3,\ z_3^f = 2(1 + \tau)/3, \\ &\text{three } \mathbb{Z}_2 \text{ fixed points} : z_4^f = 1/2\ z_5^f = \tau/2\ z_6^f = (1 + \tau)/2. \end{aligned} \tag{5.3.16}$$

For the $\mathbb{Z}_6$ fixed point z_1^f, from (5.2.43) we have

$$\begin{aligned} W_1 &= 2\eta(1 - \omega)e^{i\theta_{m_1,n_1}(z_1^f)} + \frac{2}{3}\eta^2(1 - \omega^2)e^{i\theta_{m_2,n_2}(z_1^f)} + \frac{1}{2}\eta^3(1 - \omega^3)e^{i\theta_{m_3,n_3}(z_1^f)} \\ &\qquad + \frac{2}{3}\eta^4(1 - \omega^4)e^{i\theta_{m_4,n_4}(z_1^f)} + 2\eta^5(1 - \omega^5)e^{i\theta_{m_5,n_5}(z_1^f)} \\ &= 2(1 - \omega)\omega^{\chi_{+1}} + \frac{2}{3}(1 - \omega^2)\omega^{2\chi_{+1}} + \frac{1}{2}(1 - \omega^3)\omega^{3\chi_{+1}} \\ &\qquad + \frac{2}{3}(1 - \omega^4)\omega^{4\chi_{+1}} + 2(1 - \omega^5)\omega^{5\chi_{+1}}, \end{aligned} \tag{5.3.17}$$

we have used the relations

$$e^{i\theta_{m_k,n_k}(z_1^f)} = e^{ik\theta_{m_1,n_1}(z_1^f)} \qquad (k = 1,2,3,4,5), \tag{5.3.18}$$

and $\eta e^{i\theta_{m_1,n_1}(z_1^f)} = \omega^{\chi_{+1}}$.

Using the identities

$$\begin{aligned} \omega^{\chi_{+1}} &= 1 + \frac{1}{60}(-25+63\omega)\chi_{+1} + \frac{1}{24}(-23+9\omega)\chi_{+1}^2 \\ &\quad + \frac{1}{24}(10-13\omega)\chi_{+1}^3 + \frac{1}{24}(-1+3\omega)\chi_{+1}^4 + \frac{1}{120}(-\omega)\chi_{+1}^5, \qquad (5.3.19) \\ &\chi_{+1}(\chi_{+1}-1)(\chi_{+1}-2)(\chi_{+1}-3)(\chi_{+1}-4)(\chi_{+1}-5) = 0 \qquad (5.3.20) \end{aligned}$$

with $\chi_{+1} = 0, 1, \cdots, 5$ and $\omega = e^{i2\pi/6}$, we can show that Eq.(5.3.17) takes the form

$$W_1 = -2\chi_{+1} + 5. \tag{5.3.21}$$

For the $\mathbb{Z}_3$ fixed point $z_2^f = (1+\tau)/3$ with $z_2^f = \omega^2 z_2^f + m_2 + n_2\tau$ and $z_2^f = \omega^4 z_2^f + m_4 + n_4\tau$, from Eq.(5.2.44) we have

$$\begin{aligned} W_2 &= \frac{2}{3}\eta^2(1-\omega^2)e^{i\theta_{m_2,n_2}(z_2^f)} + \frac{2}{3}\eta^4(1-\omega^4)e^{i\theta_{m_4,n_4}(z_2^f)} \\ &= \frac{2}{3}(1-\omega^2)(\omega^2)^{\chi_{+2}} + \frac{2}{3}(1-\omega^4)(\omega^2)^{2\chi_{+2}} \\ &= -2\chi_{+2} + 2, \end{aligned} \tag{5.3.22}$$

where we have used Eq.(3.2.20) with $N = 3$ and $\eta^2 = (\omega^2)^l$ $(l = 0,1,2)$. In the last equality of Eq.(5.3.22), we followed the analysis in the subsection 5.3.2. A similar analysis for another $\mathbb{Z}_3$ fixed point $z_3^f = 2(1+\tau)/3$ shows

$$W_3 = -2\chi_{+3} + 2. \tag{5.3.23}$$

For the $\mathbb{Z}_2$ fixed point $z_4^f = 1/2$, $z_5^f = \tau/2$ and $z_6^f = (1+\tau)/2$ with $z_p^f = \omega^3 z_p^f + m_3 + n_3\tau$ $(p = 4,5,6)$, from Eqs.(5.2.46)–(5.2.48) we have

$$\begin{aligned} W_p &= \frac{1}{2}\eta^3(1-\omega^3)e^{i\theta_{m_3,n_3}(z_p^f)} \\ &= \frac{1}{2}(1-\omega^3)(\omega^3)^{\chi_{+p}} \\ &= -2\chi_{+p} + 1 \qquad (p = 4,5,6), \end{aligned} \tag{5.3.24}$$

where we have used Eq.(3.2.20) with $N = 2$ and $\eta^3 = (\omega^3)^l$ $(l = 0,1)$. In the last equality of Eq.(5.3.24), we followed the analysis in the subsection 5.3.1. From the above, the formula (5.3.1) was derived.

By employing the relation (5.3.1), we can express $\lim_{\rho\to\infty} \mathrm{tr}[\sigma_3 e^{D^2/\rho^2}]_\eta$ into the form

$$\lim_{\rho\to\infty} \mathrm{tr}[\sigma_3 e^{D^2/\rho^2}]_\eta = \frac{1}{2N}\int_{T^2} dz d\bar{z} \sum_p W_p \delta^2(z - z_p^f) = \frac{1}{2N}(-V_+ + V_-), \tag{5.3.25}$$

where $V_\pm$ are the sums of the winding numbers $\chi_{\pm p}$ around the fixed points z_p^f, i.e.

$$V_\pm = \sum_p \chi_{\pm p}. \tag{5.3.26}$$

Therefore, our conclusion is that the index formula on the $T^2/\mathbb{Z}_N$ $(N = 2, 3, 4, 6)$ orbifold with magnetic flux is given by

$$n_+ - n_- = \frac{M}{N} - \frac{V_+ - V_-}{2N}. \tag{5.3.27}$$

Utilizing the relation $\chi_{-p} = -\chi_{+p} + N - 1$ for the $\mathbb{Z}_N$ fixed point z_p^f, the formula (5.3.27) can be rewritten as

$$n_+ - n_- = \frac{M - V_+}{N} + 1, \tag{5.3.28}$$

which is the relation found in Ref. [1]. We confirm the zero mode counting formula (3.2.22) as an expression of the AS index theorem.

Chapter 6

Summary and discussion

In this dissertation, we have derived the Atiyah-Singer (AS) index theorem on the $T^2/\mathbb{Z}_N$ orbifolds with magnetic flux.

In Chapter 1, we introduced the mysteries of the Standard Model and extra-dimensional models as compelling candidates beyond the Standard Model. Especially, the magnetized $T^2/\mathbb{Z}_N$ orbifolds are strong candidates for models beyond the Standard Model, encompassing features such as the three-generation and flavor structures. This dissertation has centered on deriving the AS index theorem to clarify the generation structure in the magnetized $T^2/\mathbb{Z}_N$ orbifold models.

In Chapter 2, we have briefly reviewed mode functions on the two-dimensional torus with magnetic flux background. Then, in Chapter 3, we constructed mode functions on the $T^2/\mathbb{Z}_N$ orbifolds with magnetic flux, emphasizing fixed points and winding numbers as crucial features of orbifolds. In our previous paper [1], we have got the zero-mode counting formula, which provides the numbers of the chiral zero modes on $T^2/\mathbb{Z}_N$ orbifolds. However, It was unclear whether the formula could be regarded as the index theorem because the equality between the left-hand side and the right-hand side of Eq.(3.2.22) was merely verified in Ref. [1]. Moreover, the appearance of the sum of winding numbers V_+ and the physical meaning of the factor $+1$ in the formula (3.2.22) were not obvious.

To confirm the zero-mode counting formula (3.2.22) as the index theorem and also to reveal the physical meanings of the right-hand side of the formula (3.2.22), we have constructed the blow-up manifolds without singularities from the $T^2/\mathbb{Z}_N$ orbifolds in Chapter 4. This involved cutting out around the singularities of the $T^2/\mathbb{Z}_N$ orbifolds and attaching smooth manifolds (parts of S^2) to them. In the process, two crucial conditions, (4.1.27) and (4.1.34), emerged. The first condition comes from modifying the boundary conditions of mode functions on the $T^2/\mathbb{Z}_N$ orbifolds through an appropriate singular gauge transformation. This is needed to connect mode functions on $T^2/\mathbb{Z}_N$ and those on S^2. This condition means that the contributions of the winding numbers can be expressed in terms of localized flux and localized curvature at each fixed point. The second condition arises from the junction conditions of mode functions on $T^2/\mathbb{Z}_N$ and those on S^2. This condition implies that the magnetic flux (as well as the curvature) is not modified under the blow-up process. These conditions are crucial for deriving the AS index theorem on the $T^2/\mathbb{Z}_N$ orbifolds. We have applied the AS index theorem to the blow-up manifolds of the $T^2/\mathbb{Z}_N$ orbifolds, and the numbers of chiral zero modes are determined solely by the magnetic flux on the blow-up manifolds. Importantly, the total flux remains unchanged under the blow-up process, even in the orbifold limit $r_p \to 0$. The AS index theorem on $T^2/\mathbb{Z}_N$ orbifolds with magnetic flux background is expressed by Eq.(4.2.6), indicating that the index is influenced by the contribution of homogeneous magnetic flux M and localized fluxes ξ_p^F at the fixed points. We have verified that the number of chiral zero modes obtained from the zero-mode counting formula (3.2.22) in [1] is completely consistent with Eq.(4.2.6). Additionally, the zero-mode counting formula can be reinterpreted from the perspective of the blow-up manifolds. The factor +1 in the formula (3.2.22) is identified as the contribution of the localized curvature at the fixed points, necessary to remove the contribution of localized curvature from winding numbers since the latter includes contributions from both localized flux and localized curvature. Notably, the AS index theorem on two-dimensional manifolds requires only information about fluxes. Remarkably, a new degree of freedom ℓ in Eq.(5.1.4), which emerges from the indeterminacy of mod N, suggests that there are new $|\ell|$ number of chiral zero modes. These new chiral zero modes will be discussed in detail in Ref. [24]. It will be important to extend this blow-up analysis for the AS index theorem on higher dimensional toroidal orbifolds such as $T^4/\mathbb{Z}_N$ and $T^6/\mathbb{Z}_N$.[1] We would study them elsewhere.

In Chapter 5, we have derived the index

$$n_+ - n_- = \frac{M}{N} + \frac{1}{2N}(-V_+ + V_-) = \frac{M - V_+}{N} + 1 \tag{6.1.1}$$

on the $T^2/\mathbb{Z}_N$ orbifolds with magnetic flux by using the trace formula without blow-up. As a result, we have clarified where the distinctions between smooth manifolds and orbifold models specifically arise, and why the winding number comes into play. The first term M/N of Eq.(6.1.1) can be evaluated using the Fujikawa method, where the factor $1/N$ arises from the fact that the area of the $T^2/\mathbb{Z}_N$ orbifold is $1/N$ of that of the torus T^2. The primary focus of this chapter is to derive the nontrivial terms of $(-V_+ + V_-)/2N$ (or $-V_+/N + 1$) in Eq.(6.1.1) from the trace formula.

To derive the index $n_+ - n_-$, we have used the trace formula (5.1.1) in this paper. By constructing the complete set of the mode functions on $T^2/\mathbb{Z}_2$ in terms of those on T^2, we successfully evaluated the first term on the right-hand side of Eq.(5.1.2) for the $T^2/\mathbb{Z}_2$ orbifold. Subsequently, we have found that the first term of Eq.(5.1.2) is determined solely by the information at the fixed points on $T^2/\mathbb{Z}_2$, as shown in Eq.(5.1.35). Furthermore, we have established that the coefficients W_p of the delta functions $\delta^2(z - z_p^f)$ are related to the winding numbers $-\chi_{+p} + \chi_{-p}$ (or $-2\chi_{+p} + 1$) at the fixed points z_p^f (see Eq.(5.3.1)). Finally, we derived the index formula presented in Eq.(5.3.27) (or Eq.(5.3.28)).

Although it was not possible to construct complete sets of mode functions on the $T^2/\mathbb{Z}_N$ ($N = 3, 4, 6$) orbifolds for arbitrary M, we were able to get the desired result (5.3.27) starting from the expression (5.2.1). As a result, we confirmed the zero-mode counting formula (3.2.22) given in Ref. [1] as an expression of the Atiyah-Singer index theorem. It is interesting to note that M/N and $-V_+/N + 1$ (or $(-V_+ + V_-)/(2N)$) are not always integer-valued,

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[1] The higher dimensional orbifold models with bulk magnetic fluxes were studied [46–50].

but the sum of them $(M - V_+)/N + 1$ (or $(M/N + (-V_+ + V_-)/(2N))$ becomes an integer in any case. Therefore, the combination is nontrivial from the perspective of the index theorem.

Several tasks remain to be addressed. The first one is to verify the relation (5.2.1) for arbitrary M. To this end, one might try to construct complete sets of mode functions on the $T^2/\mathbb{Z}_N$ $(N = 3, 4, 6)$ orbifolds, but it seems to be hard since the $\mathbb{Z}_N$ transformation of the mode functions on T^2 is quite complicated. Creative approaches are needed to overcome this difficulty.

An interesting extension of this work involves exploring higher dimensions. In the two-dimensional case, we observed the winding number as a topological invariant in the index theorem. For higher dimensions, we anticipate that other topological objects besides the winding number contribute to it. The extension to higher dimensions should be researched in the future.

Appendix A

Index theorem on magnetized T^2

In this appendix, we derive the Atiyah-Singer index theorem on magnetized T^2:

$$\mathrm{Ind}(i\not{D}) = \frac{1}{2\pi}\int_{T^2} F = M. \tag{A.0.1}$$

Where F is the field strength and M denotes the flux quanta.

$$i\not{D} \equiv i\sum_{a=1}^{2} \gamma_a(\partial_a - iA_a), \tag{A.0.2}$$

where γ_a $(a = 1, 2)$ denotes 2d Gamma matrices.

$$\begin{aligned} \{\gamma_a, \gamma_b\} &= 2\delta_{ab} \\ (\gamma_a)^\dagger &= \gamma_a \end{aligned} \tag{A.0.3}$$

2d chiral fermion $\psi_\pm$ satisfy

$$\gamma_3\psi_\pm = \pm\psi_\pm \tag{A.0.4}$$

The chirality operator γ_3 is given by

$$\begin{gathered} \gamma_3 \equiv -i\gamma_1\gamma_2 \\ \gamma_3\gamma_a = -\gamma_a\gamma_3, \quad (\gamma_3)^2 = I_2, \quad (\gamma_3)^\dagger = \gamma_3 \end{gathered} \tag{A.0.5}$$

Now we take $\gamma_3 = \sigma_3$.

Index is defined by

$$\mathrm{Ind}(i\not{D}) \equiv N_{+,0} - N_{-,0}, \tag{A.0.6}$$

where $N_{\pm,0}$ are the numbers of ± chiral zero modes for the Dirac operator. The eigenvalue equation is given by

$$-(\not{D})^2\psi_{\pm,n}(y) = m_n^2\psi_{\pm,n}(y), \tag{A.0.7}$$

We obtain the relation between $\psi_{+,n}$ and $\psi_{-,n}$ as

$$\psi_{-,n} = \frac{1}{m_n} i\not{D}\psi_{+,n} \tag{A.0.8}$$

$\psi_{+,n}$ and $\psi_{-,n}$ satisfy the eigenvalue equation Eq.(A.0.7) and (A.0.8). Therefore, the number of eigenstates $\psi_{+,n}$ and $\psi_{-,n}$ with eigenvalue $m_n \neq 0$ are

$$N_{+,n} = N_{-,n} \qquad (m_n^2 \neq 0), \tag{A.0.9}$$

where $N_{\pm,n}$ are the numbers of ± chiral modes satisfied (A.0.7). When $m_n = 0$

$$\begin{aligned} -\not{D}^2\psi_{+,0} = 0 \quad &\Leftrightarrow \quad i\not{D}\psi_{+,0} = 0 \\ -\not{D}^2\psi_{-,0} = 0 \quad &\Leftrightarrow \quad i\not{D}\psi_{-,0} = 0 \end{aligned} \tag{A.0.10}$$

The index (A.0.6) can be expanded as follows

$$\mathrm{Ind}(i\not{D}) = \sum_n (N_{+,n} - N_{-,n}). \tag{A.0.11}$$

We need add the regularization function $e^{-\frac{m_n^2}{\Lambda^2}}$.

$$\mathrm{Ind}(i\not{D}) \equiv \lim_{\Lambda\to\infty} \sum_n (N_{+,n} - N_{-,n}) e^{-\frac{m_n^2}{\Lambda^2}} \tag{A.0.12}$$

The set of eigenstates

$$\{\psi_{\lambda,n}^{(j)}(y);\ n = 0, 1, 2, \cdots;\ \lambda = \pm 1;\ j = 1, 2, \cdots, N_{\lambda,n}\} \tag{A.0.13}$$

satisfies

$$\begin{aligned} -\not{D}^2\psi_{\lambda,n}^{(j)}(y) &= m_n^2\psi_{\lambda,n}^{(j)}(y), \\ \gamma_3\psi_{\lambda,n}^{(j)}(y) &= \lambda\psi_{\lambda,n}^{(j)}(y), \end{aligned} \tag{A.0.14}$$

where λ is the label of chirality and j is the label of degeneracy. $\psi_{\lambda,n}^{(j)}$ satisfy the orthonormality condition

$$\int_{T^2} d^2y (\psi_{\lambda,n}^{(j)}(y))^\dagger \psi_{\lambda',n'}^{(j')}(y) = \delta_{n,n'}\delta_{\lambda,\lambda'}\delta_{j,j'}. \tag{A.0.15}$$

$\{\psi_{\lambda,n}^{(j)}(y)\}$ is the complete set

$$\sum_n \sum_{\lambda=\pm 1} \sum_{j=1}^{N_{\lambda,n}} \psi_{\lambda,n}^{(j)}(x)(\psi_{\lambda,n}^{(j)}(y))^\dagger = \delta^2(x-y)_{T^2}. \tag{A.0.16}$$

We expand Eq.(A.0.12) by the complete set

$$\begin{aligned}
\mathrm{Ind}(i\not{D}) &= \lim_{\Lambda\to\infty}\sum_n (N_{+,n}-N_{-,n})e^{-\frac{m_n^2}{\Lambda^2}} \\
&\overset{(A.0.15)}{=} \lim_{\Lambda\to\infty}\int_{T^2} d^2y \sum_n \{\sum_{j=1}^{N_{+,n}} (\psi_{+,n}^{(j)}(y))^\dagger \psi_{+,n}^{(j)}(y) - \sum_{j=1}^{N_{-,n}} (\psi_{-,n}^{(j)}(y))^\dagger \psi_{-,n}^{(j)}(y)\} e^{-\frac{m_n^2}{\Lambda^2}} \\
&\overset{(A.0.14)}{=} \lim_{\Lambda\to\infty}\int_{T^2} d^2y \sum_n \sum_{\lambda=\pm1} \sum_{j=1}^{N_{+,n}} (\psi_{\lambda,n}^{(j)}(y))^\dagger \gamma_3 e^{\frac{\not{D}^2}{\Lambda^2}} \psi_{\lambda,n}^{(j)}(y) \\
&= \lim_{\Lambda\to\infty}\int_{T^2} d^2y \sum_n \sum_{\lambda=\pm1} \sum_{j=1}^{N_{+,n}} \lim_{x\to y} \mathrm{tr}[\gamma_3 (e^{\frac{\not{D}^2}{\Lambda^2}} \psi_{\lambda,n}^{(j)}(y))(\psi_{\lambda,n}^{(j)}(x))^\dagger] \\
&\overset{(A.0.16)}{=} \lim_{\Lambda\to\infty}\lim_{x\to y}\int_{T^2} d^2y \mathrm{tr}[\gamma_3 e^{\frac{\not{D}^2}{\Lambda^2}} \delta^2(x-y)_{T^2}]
\end{aligned} \tag{A.0.17}$$

In the second equality from the last, we use the relation$\vec{A}^\dagger\vec{B} = \mathrm{tr}[\vec{B}\vec{A}^\dagger]$. Here, we rewrite the delta function as

$$\delta^2(x-y)_{T^2} \quad \longrightarrow \quad \frac{1}{L^2}\sum_{l\in\mathbb{Z}} e^{-ik_l(x-y)}, \tag{A.0.18}$$

where L is the size of T^2 and wave vector k_l is given by

$$k_l = \left(\frac{2\pi}{L}l_1, \frac{2\pi}{L}l_2\right) \qquad (l_1, l_2 \in \mathbb{Z}). \tag{A.0.19}$$

We show the proof of Eq.(A.0.18). An arbitrary function $f(y)$ of T^2 can be expanded the Fourier series

$$f(y) = \sum_{l\in\mathbb{Z}} f_l e^{ik_l y}. \tag{A.0.20}$$

The orthonormality condition is

$$\int_{T^2} d^2y e^{ik_l y} = L^2\delta_{l,0} \tag{A.0.21}$$

where $\int_{T^2} d^2y = L^2$.

$$\int_{T^2} d^2y f(y) e^{-ik_l y} = \int_{T^2} d^2y \sum_{l\in\mathbb{Z}} f_l e^{ik_l y} e^{-ik_l y} = f_l L^2 \tag{A.0.22}$$

Substituting Eq.(A.0.22) into Eq.(A.0.20)

$$f(y) = \sum_{l'\in\mathbb{Z}} \frac{1}{L^2}\int_{T^2} d^2x f(x) e^{-ik_l(x-y)}$$

$$\Longrightarrow \delta^2(x-y)_{T^2} = \sum_{l'\in\mathbb{Z}} \frac{1}{L^2} e^{-ik_l(x-y)}. \tag{A.0.23}$$

To further the calculation, we consider

$$\not{D}^2(y) = (D_a(y))^2 I_2 - \frac{i}{4}[\gamma_a, \gamma_b] F_{ab}(y), \tag{A.0.24}$$

where $F_{ab} = \partial_a A_b(y) - \partial_b A_a(y)$.

$$\begin{aligned} e^{ik_l(x-y)} D_a(y) e^{-ik_l(x-y)} &= e^{ik_l(x-y)}(\partial_a - iA_a(y)) e^{-ik_l(x-y)} \\ &= ik_{la} + D_a(y) \end{aligned} \tag{A.0.25}$$

Eq.(A.0.24) is obtained by the following calculation

$$\begin{aligned} \not{D}^2 &= (\gamma_a D_a)(\gamma_b D_b) \\ &= \gamma_a \gamma_b D_a D_b \\ &= \frac{1}{2}\{\gamma_a, \gamma_b\} D_a D_b + \frac{1}{2}[\gamma_a, \gamma_b] D_a D_b \\ &\overset{(A.0.3)}{=} (D_a)^2 I_2 + \frac{1}{2}[\gamma_a, \gamma_b][D_a, D_b] \\ &= (D_a)^2 I_2 + \frac{1}{2}[\gamma_a, \gamma_b][\partial_a - iA_a, \partial_b - iA_b] \\ &= (D_a)^2 I_2 - \frac{i}{4}[\gamma_a, \gamma_b] F_{ab}. \end{aligned} \tag{A.0.26}$$

From this, we get

$$\begin{aligned} \mathrm{Ind}(i\not{D}) &= \lim_{\Lambda\to\infty} \lim_{x\to y} \int_{T^2} d^2y \mathrm{tr}[\gamma_3 e^{\frac{\not{D}^2}{\Lambda^2}} \delta^2(x-y)_{T^2}] \\ &\overset{(A.0.18),(A.0.24)}{=} \lim_{\Lambda\to\infty} \lim_{x\to y} \int_{T^2} d^2y \frac{1}{L^2} \sum_{l\in\mathbb{Z}} \mathrm{tr}[\gamma_3 \exp\{\frac{1}{\Lambda^2}((D_a)^2 - \frac{i}{4}[\gamma_a, \gamma_b] F_{ab})\} e^{-ik_l(x-y)}] \\ &= \lim_{\Lambda\to\infty} \lim_{x\to y} \int_{T^2} d^2y \frac{1}{L^2} \sum_{l\in\mathbb{Z}} \mathrm{tr}[\gamma_3 e^{-ik_l(x-y)} e^{ik_l(x-y)} \exp\{\frac{1}{\Lambda^2}((D_a)^2 - \frac{i}{4}[\gamma_a, \gamma_b] F_{ab})\} e^{-ik_l(x-y)}] \\ &\overset{(A.0.25)}{=} \lim_{\Lambda\to\infty} \lim_{x\to y} \int_{T^2} d^2y \frac{1}{L^2} \sum_{l\in\mathbb{Z}} \mathrm{tr}[\gamma_3 e^{-ik_l(x-y)} \exp\{\frac{1}{\Lambda^2}((ik_{la} + D_a)^2 - \frac{i}{4}[\gamma_a, \gamma_b] F_{ab})\}] \\ &= \lim_{\Lambda\to\infty} \int_{T^2} d^2y \frac{1}{L^2} \sum_{l\in\mathbb{Z}} \mathrm{tr}[\gamma_3 \exp\{\frac{1}{\Lambda^2}((ik_{la} + D_a)^2 - \frac{i}{4}[\gamma_a, \gamma_b] F_{ab})\}] \end{aligned} \tag{A.0.27}$$

We expand around $\Lambda \to \infty$

$$\begin{aligned} \exp\{\frac{1}{\Lambda^2}((ik_{la} + D_a)^2 - \frac{i}{4}[\gamma_a, \gamma_b] F_{ab})\} &= \exp\{\frac{1}{\Lambda^2}(-k_l^2 + 2ik_l D + D^2 - \frac{i}{4}[\gamma_a, \gamma_b] F_{ab})\} \\ &= e^{-\frac{k_l^2}{\Lambda^2}} \exp\{\frac{1}{\Lambda^2}(2ik_l D + D^2 - \frac{i}{4}[\gamma_a, \gamma_b] F_{ab})\} \end{aligned}$$

$$= e^{-\frac{k_l^2}{\Lambda^2}}\{1 + \frac{1}{\Lambda^2}(2ik_l D + D^2 - \frac{i}{4}[\gamma_a, \gamma_b]F_{ab}) + O(\frac{1}{\Lambda^3})\}. \tag{A.0.28}$$

We use the following relations

$$\begin{aligned} \mathrm{tr}[\gamma_3] &= 0, \\ \mathrm{tr}[\gamma_3\gamma_a\gamma_b] &= 2i\varepsilon_{ab}, \end{aligned} \tag{A.0.29}$$

where $\varepsilon_{12} = \varepsilon_{21} = 1$. Substituting Eq.(A.0.28) and (A.0.29) into Eq.(A.0.27), we get

$$\begin{aligned} \mathrm{Ind}(i\not{D}) &= \lim_{\Lambda\to\infty}\int_{T^2} d^2y\frac{1}{L^2}\sum_{l\in\mathbb{Z}} \mathrm{tr}[\gamma_3 e^{-\frac{k_l^2}{\Lambda^2}}\{1 + \frac{1}{\Lambda^2}(2ik_l D + D^2 - \frac{i}{4}[\gamma_a, \gamma_b]F_{ab}) + O(\frac{1}{\Lambda^3})\}] \\ &= \lim_{\Lambda\to\infty}\int_{T^2} d^2y\frac{1}{L^2}\sum_{l\in\mathbb{Z}} e^{-\frac{k_l^2}{\Lambda^2}}\{\frac{1}{\Lambda^2}(-\frac{i}{4})(2i\varepsilon_{ab}F_{ab} - 2i\varepsilon_{ba}F_{ab}) + O(\frac{1}{\Lambda^3})\} \\ &= \lim_{\Lambda\to\infty}\int_{T^2} d^2y\frac{1}{L^2}\sum_{l\in\mathbb{Z}} e^{-\frac{k_l^2}{\Lambda^2}}\{\frac{1}{\Lambda^2}\varepsilon_{ab}F_{ab} + O(\frac{1}{\Lambda^3})\}. \end{aligned} \tag{A.0.30}$$

Here, we use the Poisson summation formula

$$\sum_{l=-\infty}^{\infty} e^{-(\frac{l+\alpha}{R})^2 t} = R\sqrt{\frac{\pi}{t}}\sum_{m=-\infty}^{\infty} e^{-\frac{(\pi m R)^2}{t} + i2\pi m\alpha} \tag{A.0.31}$$

The exponential term of Eq.(A.0.30) is expressed as

$$\begin{aligned} \sum_{l\in\mathbb{Z}} e^{-\frac{k_l^2}{\Lambda^2}} &\equiv \sum_{l_1\in\mathbb{Z}}\sum_{l_2\in\mathbb{Z}} \exp\{-\tfrac{1}{\Lambda^2}[(\tfrac{2\pi l_1}{L})^2 + (\tfrac{2\pi l_2}{L})^2]\} \\ &= \left(\sum_{l\in\mathbb{Z}} \exp\left(-\frac{l^2}{\frac{\Lambda L}{2\pi}}\right)\right)^2. \end{aligned} \tag{A.0.32}$$

By using Eq.(A.0.31) and (A.0.32), we get

$$\begin{aligned} \lim_{\Lambda\to\infty}\frac{1}{\Lambda^2 L^2}\sum_{l\in\mathbb{Z}} e^{-\frac{k_l^2}{\Lambda^2}} &= \lim_{\Lambda\to\infty}\frac{1}{\Lambda^2 L^2}\left(\frac{\Lambda L}{2\pi}\sqrt{\pi}\sum_{m\in\mathbb{Z}}\exp\left[-\left(\frac{\pi m\Lambda L}{2\pi}\right)^2\right]\right)^2 \\ &= \frac{1}{4\pi} \end{aligned} \tag{A.0.33}$$

Finally, we get

$$\begin{aligned} \mathrm{Ind}(i\not{D}) &= \lim_{\Lambda\to\infty}\int_{T^2} d^2y\frac{1}{L^2}\sum_{l\in\mathbb{Z}} e^{-\frac{k_l^2}{\Lambda^2}}\{\frac{1}{\Lambda^2}\varepsilon_{ab}F_{ab} + O(\frac{1}{\Lambda^3})\} \\ &= \int_{T^2} d^2y\frac{1}{4\pi}\varepsilon_{ab}F_{ab} \end{aligned}$$

$$
\begin{aligned}
&= \int_{T^2} d^2y \frac{1}{4\pi} 2B \\
&= \frac{1}{2\pi} \int_{T^2} d^2yB \\
&= \frac{1}{2\pi} \int_{T^2} F. \qquad \text{(A.0.34)}
\end{aligned}
$$

Appendix B

The proof

B.1 Evaluation of the second term in Eq.(5.1.2)

In this Appendix, we evaluate the second term of Eq.(5.1.2) for the $T^2/\mathbb{Z}_N$ $(N = 2, 3, 4, 6)$ orbifolds, and show

$$\lim_{\rho\to\infty}\left\{\tfrac{1}{\rho^2}\mathrm{tr}[\sigma_3(-\tfrac{iq}{4})[\sigma^a,\sigma^b]F_{ab}e^{D^2/\rho^2}]\right\} = \frac{M}{N}. \tag{B.1.1}$$

Using the relation $[\sigma^a, \sigma^b] = 2i\varepsilon^{abc}\sigma^c$, the left-hand side of (B.1.1) is expressed as

$$\lim_{\rho\to\infty}\left\{\tfrac{1}{\rho^2}\mathrm{tr}[\sigma_3(-\tfrac{iq}{4})[\sigma^a,\sigma^b]F_{ab}e^{D^2/\rho^2}]\right\} = \lim_{\rho\to\infty}\left\{\tfrac{qF_{12}}{\rho^2}\mathrm{tr}[I_2e^{D^2/\rho^2}]\right\} = \lim_{\rho\to\infty}\left\{\tfrac{2\pi M}{(\mathrm{Im}\tau)\rho^2}\mathrm{tr}[I_2e^{D^2/\rho^2}]\right\}, \tag{B.1.2}$$

where we have used $(\sigma_3)^2 = I_2$ in the first equality and Eqs.(1.3.2), (2.1.19) in the second equality. By using the complete set of the mode functions $\{\,\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z)_\eta\ \ (n = 0, 1, \cdots)\,\}$ and $\{\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N-,n}(z)_\eta\ (n = 1, 2, \cdots)\}$, Eq.(B.1.2) can be expressed as

$$\begin{aligned}\lim_{\rho\to\infty}\left\{\tfrac{2\pi M}{(\mathrm{Im}\tau)\rho^2}\mathrm{tr}[I_2e^{D^2/\rho^2}]\right\} &= \lim_{\rho\to\infty}\tfrac{2\pi M}{(\mathrm{Im}\tau)\rho^2}\int_{T^2/\mathbb{Z}_N}dzd\bar{z}\lim_{z'\to z}e^{D^2(z)/\rho^2}\\ &\times\left\{\sum_{n=0}^{\infty}\sum_{j}\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z)_\eta[\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z')_\eta]^* + \sum_{n=1}^{\infty}\sum_{j}\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N-,n}(z)_\eta[\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N-,n}(z')_\eta]^*\right\}.\end{aligned} \tag{B.1.3}$$

We can evaluate Eq.(B.1.3) in the same way as the calculation of the first term, and we get

$$\begin{aligned}&\lim_{\rho\to\infty}\left\{\tfrac{2\pi M}{(\mathrm{Im}\tau)\rho^2}\mathrm{tr}[I_2e^{D^2/\rho^2}]\right\}\\ &= \lim_{\rho\to\infty}\frac{2\pi M}{(\mathrm{Im}\tau)\rho^2}\frac{1}{N}\int_{T^2}dzd\bar{z}\lim_{z'\to z}e^{D^2(z)/\rho^2}\sum_{n=0}^{\infty}\sum_{j=0}^{M-1}\sum_{l=0}^{N-1}\eta^l(1+\omega^l)\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(z)[\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(\omega^lz')]^*\end{aligned}$$

$$= \lim_{\rho\to\infty} \frac{2\pi M}{(\mathrm{Im}\tau)\rho^2} \frac{1}{N} \int_{T^2} dz d\bar{z} \lim_{z'\to z} e^{D^2(z)/\rho^2} \sum_{l=0}^{N-1} \eta^l (1+\omega^l) \sum_{m_l,n_l\in\mathbb{Z}} e^{i\theta_{m_l,n_l}(\omega^l z')} \delta^2(z-\omega^l z' - m_l - n_l\tau). \tag{B.1.4}$$

Here, we have used the assumption which has been used in the evaluation of the first term for $N = 3, 4, 6$ in Section 5.2. Further calculations show that

$$\begin{aligned}
&\lim_{\rho\to\infty} \left\{ \tfrac{2\pi M}{(\mathrm{Im}\tau)\rho^2} \mathrm{tr}[I_2 e^{D^2/\rho^2}] \right\} \\
&= \lim_{\rho\to\infty} \frac{2\pi M}{N(\mathrm{Im}\tau)\rho^2} \int_{T^2} dz d\bar{z} \int \frac{d^2k}{(2\pi)^2} \lim_{z'\to z} \sum_{l=0}^{N-1} \eta^l (1+\omega^l) \\
&\qquad \times \sum_{m_l,n_l\in\mathbb{Z}} e^{i\theta_{m_l,n_l}(\omega^l z')} e^{-ik\cdot(-z+\omega^l z' + m_l + n_l\tau)} e^{-ik\cdot z} e^{D^2(z)/\rho^2} e^{ik\cdot z} \\
&= \lim_{\rho\to\infty} \frac{2\pi M}{N(\mathrm{Im}\tau)} \int_{T^2} dz d\bar{z} \int \frac{d^2k}{(2\pi)^2} \sum_{l=0}^{N-1} \eta^l (1+\omega^l) \\
&\qquad \times \sum_{m_l,n_l\in\mathbb{Z}} e^{i\theta_{m_l,n_l}(\omega^l z)} e^{-ik\cdot(-z+\omega^l z + m_l + n_l\tau)\rho} \exp\left[\left(ik_a + \tfrac{D_a}{\rho} \right) \left(ik^a + \tfrac{D^a}{\rho} \right) \right],
\end{aligned} \tag{B.1.5}$$

where we have used the relation $e^{-ik\cdot z} e^{D^2(z)/\rho^2} e^{ik\cdot z} = \exp[\frac{1}{\rho^2}(ik_a + D_a)(ik^a + D^a)]$ and replaced $k \to \rho k$ in the second equality. From the Riemann-Lebesgue lemma[1], only the contribution of $l = m_0 = n_0 = 0$ remains. Therefore, we get

$$\begin{aligned}
\lim_{\rho\to\infty} \left\{ \tfrac{2\pi M}{(\mathrm{Im}\tau)\rho^2} \mathrm{tr}[I_2 e^{D^2/\rho^2}] \right\} &= \frac{2\pi M}{N(\mathrm{Im}\tau)} 2 \int_{T^2} dz d\bar{z} \lim_{\rho\to\infty} \int \frac{d^2k}{(2\pi)^2} \exp\left[\left(ik_a + \tfrac{D_a}{\rho} \right) \left(ik^a + \tfrac{D^a}{\rho} \right) \right] \\
&= \frac{2\pi M}{N(\mathrm{Im}\tau)} 2\mathrm{Im}\tau \frac{\pi}{(2\pi)^2} \\
&= \frac{M}{N}.
\end{aligned} \tag{B.1.7}$$

B.2 Proof of Eqs.(5.3.6), (5.3.10) and (5.3.18)

In this Appendix, we prove Eqs.(5.3.6), (5.3.10) and (5.3.18). To this end, we start with the relation (3.2.19)

$$\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z^f_p + \omega Z)_\eta = e^{i(m_1 q\Lambda_1(\omega Z) + n_1 q \Lambda_2(\omega Z))} e^{i\theta_{m_1,n_1}(z^f_p)} \eta\, \psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z^f_p + Z)_\eta, \tag{B.2.1}$$

---

[1] The Riemann-Lebesgue lemma is

$$\lim_{|X|\to\infty} \int_{-\infty}^{\infty} d^D k\, f(k)\, e^{-ik\cdot X} = 0, \tag{B.1.6}$$

where D is the dimension and $f(k)$ is an integrable function. Now we set $X = (-z + \omega^l z' + m + n\tau)\rho$.

where m_1 and n_1 are defined through the fixed point equation

$$z_p^f = \omega z_p^f + m_1 + n_1 \tau \tag{B.2.2}$$

and we have used the definition (5.1.32) of $\theta_{m,n}(z)$.

In the following, we first show the relation

$$e^{i\theta_{m_2,n_2}(z_p^f)} = e^{i2\theta_{m_1,n_1}(z_p^f)}, \tag{B.2.3}$$

where (m_1, n_1) and (m_2, n_2) are related to z_p^f as

$$z_p^f = \omega z_p^f + m_1 + n_1 \tau, \tag{B.2.4}$$
$$z_p^f = \omega^2 z_p^f + m_2 + n_2 \tau, \tag{B.2.5}$$

by computing $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z_p^f + \omega^2 Z)_\eta$ in two ways.

We first evaluate $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z_p^f + \omega^2 Z)_\eta$, by use of the formula (B.2.1) twice, as follows:

$$\begin{aligned}
\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z_p^f + \omega^2 Z)_\eta &= \psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z_p^f + \omega(\omega Z))_\eta \\
&= e^{im_1 q\Lambda_1(\omega^2 Z)+in_1 q\Lambda_2(\omega^2 Z)} e^{i\theta_{m_1,n_1}(z_p^f)} \eta \psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z_p^f + \omega Z)_\eta \\
&= e^{im_1 q\Lambda_1(\omega^2 Z)+in_1 q\Lambda_2(\omega^2 Z)} e^{im_1 q\Lambda_1(\omega Z)+in_1 q\Lambda_2(\omega Z)} \\
&\quad \times e^{i2\theta_{m_1,n_1}(z_p^f)} \eta^2 \psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z_p^f + Z)_\eta.
\end{aligned} \tag{B.2.6}$$

Next, we evaluate $\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z_p^f + \omega^2 Z)_\eta$ with Eq.(B.2.5) as follows:

$$\begin{aligned}
\psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z_p^f + \omega^2 Z)_\eta &= \psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(\omega^2 z_p^f + m_2 + n_2\tau + \omega^2 Z)_\eta \\
&= e^{im_2 q\Lambda_1(\omega^2 Z)+in_2 q\Lambda_2(\omega^2 Z)} e^{i\theta_{m_2,n_2}(z_p^f)} \eta^2 \psi^{(j+\alpha_1,\alpha_2)}_{T^2/\mathbb{Z}_N+,n}(z_p^f + Z)_\eta,
\end{aligned} \tag{B.2.7}$$

where we have used the boundary conditions (2.1.14) and the $\mathbb{Z}_N$ transformation (3.2.5). Equating Eq.(B.2.6) with Eq.(B.2.7) and taking the limit of $Z \to 0$, we have

$$e^{i\theta_{m_2,n_2}(z_p^f)} = e^{i2\theta_{m_1,n_1}(z_p^f)}. \tag{B.2.8}$$

Similarly, for the fixed point z_p^f which satisfies the fixed point equations

$$z_p^f = \omega z_p^f + m_1 + n_1 \tau, \tag{B.2.9}$$
$$z_p^f = \omega^k z_p^f + m_k + n_k \tau, \tag{B.2.10}$$

we can show

$$e^{i\theta_{m_k,n_k}(z_p^f)} = e^{ik\theta_{m_1,n_1}(z_p^f)}. \tag{B.2.11}$$

B.3 Verification of Eq.(5.2.1) for the case of $M = 1, (\alpha_1, \alpha_2) = (\frac{1}{2}, \frac{1}{2})$ and $\eta = \omega$

In this Appendix, we verify Eq.(5.2.1) for the case of $M = 1$, $(\alpha_1, \alpha_2) = (\frac{1}{2}, \frac{1}{2})$ and $\eta = \omega$ as an example. In this case, we have the zero modes $\psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N+,0}(z)_\omega$. We start with

$$I_\omega \equiv \sum_{n=0}^{\infty} \psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N+,n}(z)_\omega [\psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N+,n}(z)_\omega]^* - \sum_{n=1}^{\infty} \psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N-,n}(z)_\omega [\psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N-,n}(z)_\omega]^*. \tag{B.3.1}$$

I_ω can be represented as

$$\begin{aligned}
I_\omega = &\sum_{k=0}^{\infty} \psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N+,Nk}(z)_\omega [\psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N+,Nk}(z)_\omega]^* \\
&+ \sum_{k=0}^{\infty} \psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N+,Nk+1}(z)_\omega [\psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N+,Nk+1}(z)_\omega]^* \\
&+ \cdots + \sum_{k=0}^{\infty} \psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N+,Nk+N-1}(z)_\omega [\psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N+,Nk+N-1}(z)_\omega]^* \\
&- \sum_{k=0}^{\infty} \psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N-,Nk+1}(z)_\omega [\psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N-,Nk+1}(z)_\omega]^* \\
&- \cdots - \sum_{k=0}^{\infty} \psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N-,Nk+N}(z)_\omega [\psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N-,Nk+N}(z)_\omega]^*.
\end{aligned} \tag{B.3.2}$$

Since $\psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N\pm,p}(z)_\omega = 0$ for $p = Nk+1, \cdots, Nk+N-1$, we can take $\mathcal{N}^{(\frac{1}{2},\frac{1}{2})}_{+,1} = \mathcal{N}^{(\frac{1}{2},\frac{1}{2})}_{+,\omega} = \cdots = \mathcal{N}^{(\frac{1}{2},\frac{1}{2})}_{+,\omega^{(N-1)}}$. Then, we find

$$\begin{aligned}
I_\omega &= \sum_{n=0}^{\infty} \psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N+,n}(z)_\omega [\psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N+,n}(z)_\omega]^* - \sum_{n=1}^{\infty} \psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N-,n}(z)_\omega [\psi^{(\frac{1}{2},\frac{1}{2})}_{T^2/\mathbb{Z}_N-,n}(z)_\omega]^* \\
&= \sum_{n=0}^{\infty} \sum_{l=0}^{N-1} \sum_{l'=0}^{N-1} \eta^{-l+l'} (1 - \omega^{-l+l'}) |\mathcal{N}^{(\frac{1}{2},\frac{1}{2})}_{+,\omega}|^2 \psi^{(\frac{1}{2},\frac{1}{2})}_{T^2+,n}(\omega^l z) [\psi^{(\frac{1}{2},\frac{1}{2})}_{T^2+,n}(\omega^{l'} z)]^* \\
&= \sum_{n=0}^{\infty} \sum_{l=0}^{N-1} \eta^l (1 - \omega^l) |\mathcal{N}^{(\frac{1}{2},\frac{1}{2})}_{+,\omega}|^2 \Big\{ \psi^{(\frac{1}{2},\frac{1}{2})}_{T^2+,n}(z) [\psi^{(\frac{1}{2},\frac{1}{2})}_{T^2+,n}(\omega^l z)]^* + \psi^{(\frac{1}{2},\frac{1}{2})}_{T^2+,n}(\omega z) [\psi^{(\frac{1}{2},\frac{1}{2})}_{T^2+,n}(\omega^{(l+1)} z)]^* \\
&\qquad + \cdots + \psi^{(\frac{1}{2},\frac{1}{2})}_{T^2+,n}(\omega^{(N-1)} z) [\psi^{(\frac{1}{2},\frac{1}{2})}_{T^2+,n}(\omega^{(l+N-1)} z)]^* \Big\},
\end{aligned} \tag{B.3.3}$$

Using the relation $\psi^{(\frac{1}{2},\frac{1}{2})}_{T^2+,n}(\omega z) = \omega\bar{\omega}^n \psi^{(\frac{1}{2},\frac{1}{2})}_{T^2+,n}(z)$, discussed in [32], we can further rewrite Eq.(B.3.3) as

$$\begin{aligned} I_\omega &= \sum_{n=0}^{\infty} \sum_{l=0}^{N-1} \eta^l (1-\omega^l) N |\mathcal{N}^{(\frac{1}{2},\frac{1}{2})}_{+,\omega}|^2 \psi^{(\frac{1}{2},\frac{1}{2})}_{T^2+,n}(z) [\psi^{(\frac{1}{2},\frac{1}{2})}_{T^2+,n}(\omega^l z)]^* \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^{N-1} \eta^l (1-\omega^l) \psi^{(\frac{1}{2},\frac{1}{2})}_{T^2+,n}(z) [\psi^{(\frac{1}{2},\frac{1}{2})}_{T^2+,n}(\omega^l z)]^*, \end{aligned} \tag{B.3.4}$$

where we have used $\mathcal{N}^{(\frac{1}{2},\frac{1}{2})}_{+,\omega} = 1/\sqrt{N}$. We have succeeded in verifying Eq.(5.2.1) for the case of $M = 1$ and $(\alpha_1, \alpha_2) = (\frac{1}{2}, \frac{1}{2})$. In general, however, the relation $\psi^{(j+\alpha_1,\alpha_2)}_{T^2+,n}(\omega z) = \sum_{k=0}^{M-1} D_{jk} \psi^{(\frac{1}{2},\frac{1}{2})}_{T^2+,n}(z)$ discussed in [33] is quite complicated and seems to be difficult to prove in arbitrary cases.

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