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Mode recombination formula and nonanalytic term in an effective potential at finite temperature on a compactified space

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We develop a new formula called a mode recombination formula, and we can recast the effective potential at finite temperature in one-loop approximation for fermion and scalar fields on the D -dimensional spacetime, $S^1_\tau \times R^{D-(p+1)} \times \prod_{i=1}^p S^1_i$ into a convenient form for discussing nonanalytic terms, which cannot be written in the form of any positive integer power of the field-dependent mass squared, in the effective potential. The formula holds irrespective of whether the field is a fermion or a scalar and of boundary conditions for spatial S^1_i directions and clarifies the importance of zero modes in the Matsubara and Kaluza-Klein modes for the existence of the nonanalytic terms. The effective potential is drastically simplified further to obtain the nonanalytic terms in easier and more transparent way. In addition to reproducing previous results, we find that there exists no nonanalytic term for the fermion field with arbitrary boundary condition for the spatial S^1_i direction, which is also the case for the scalar field with the antiperiodic boundary condition for the spatial direction.

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I. INTRODUCTION

Quantum field theory at finite temperature has provided a fundamental theoretical framework in high energy physics. In particular, the effective potential at finite temperature is a crucial tool to investigate physical phenomena involved with the order of the phase transition and its strength.

In the pioneering work by the Dolan-Jackiw [1], they found that there exists a nonanalytic term, which cannot be written in the form of any positive integer power of field-dependent mass squared, in the effective potential at finite temperature for a real scalar field. The nonanalytic term obtained by them has three-halves power of the mass squared, and it turns out to play an essential role to trigger the first order phase transition [2,3]. Moreover, the magnitude of the term is related with the strength of the first order phase transition, which, in turn, put certain constraint on physical quantity such as the Higgs mass, for example, in the study of electroweak baryogenesis [4]. Hence, the nonanalytic term in the effective potential is an important quantity that must be studied in detail.

Quantum field theory with compactified dimensions has been one of the attractive approaches for physics beyond the standard model. Orbifold compactification provides a framework for gauge-Higgs unification, where the Higgs field is unified into higher dimensional gauge fields [5,6]. The order of the finite temperature phase transition in the gauge-Higgs unification has been studied in [7,8], and the first order phase transition actually occurs due to the nonanalytic term in the effective potential. It has been also shown that the quantum field theory at finite temperature with compactified dimensions can possess rich phase structures [9,10]. Compactified dimensions also provide the theoretical framework for studying quantum field theory itself. For instance, from a point of view of dimensional reduction [11,12], models with several numbers of S^1 have been investigated.

In the previous paper [13], we studied all the possible nonanalytic terms in the effective potential at finite temperature in one-loop approximation for a real scalar field on the D -dimensional spacetime, $S^1_\tau \times R^{D-(p+1)} \times \prod_{i=1}^p S^1_i$, where the $S^1_\tau, R^{D-(p+1)}, S^1_i$ stand for the Euclidean time direction, the $D - (p + 1)$ dimensional flat Euclidean space, the spatial compactified direction, respectively. The effective potential contains the modified Bessel function of the second kind accompanied with multiple mode summations with respect to the winding mode associated with each S^1 . By using the integral representation for the modified Bessel function given by the inverse Mellin

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transformation [14,15] and the analytical extension for the mode summation [16], we recast the effective potential into the integral form on the complex plane¹ and performed the residue integration in order to obtain the nonanalytic terms. We found that the only nonanalytic power comes from a term $M^{D-(p+1)}$ which is not analytic in M^2 when the dimension of the flat Euclidian space, $D - (p + 1)$ is odd. We obtained the coefficient of this term, but other non-analytic terms proportional to $\log M$ were not studied.

In this paper, we develop a new formula called a mode recombination formula, which plays a central role for discussions in the present paper. The effective potential can be recast into the convenient form for studying the nonanalytic terms. The formula also clarifies that only the zero mode in the Kaluza-Klein mode associated with each S^1 is crucial for the existence of the nonanalytic terms in the case of the scalar field satisfying the periodic boundary condition for the spatial $S^1_i (i = 1, \dots, p)$ direction. Then, the effective potential relevant for the nonanalytic terms can be simplified further drastically and is given in terms of the contribution of the single mode summation with respect to the winding mode associated with each S^1 . This is quite different from the previous paper, where it includes the contribution of the multiple mode summations. The integral form for the simplified effective potential on the complex plane is easy to perform the residue integration in order to obtain the nonanalytic terms. We reproduce the previous results in easier and more transparent way.

The mode recombination formula holds irrespective of whether the field is a fermion or a scalar and of the boundary condition for the spatial $S^1_i (i = 1, \dots, p)$ direction. The formula also provides a convenient form for studying the nonanalytic terms in the effective potential for the case of the fermion field with arbitrary boundary condition for the S^1_i direction. The zero mode for the Euclidean time direction is removed due to the antiperiodic boundary condition followed from the quantum statistics for the fermion. This changes the pole structure of the analytical extension for the mode summation with respect to the Matsubara mode compared with that of the scalar case. We find that there is no nonanalytic term for the case of the fermion. This immediately implies that the effective potential does not possess the nonanalytic term for the case of the real scalar field satisfying the antiperiodic boundary condition for at least one spatial S^1_i direction.

This paper is organized as follows. We present the setup in the next section. We prove the mode recombination formula and present the convenient form of the effective potential for studying the nonanalytic terms in each case of the fermion and the scalar in the Sec. III. We reproduce the same result as the one in the previous paper in easier and more transparent way in the Sec. IV. We also study the

nonanalytic terms for the case of fermion with arbitrary boundary condition in the Sec. V and of the real scalar with the antiperiodic boundary condition in the Sec. VI. The final section is devoted to conclusions and discussions which also include the case for a higher dimensional gauge field.

II. SETUP

Let us first present the setup for our discussions. We study nonanalytic terms in the effective potential at finite temperature in one-loop approximation for a real scalar (fermion) field on the D -dimensional spacetime, $S^1_\tau \times R^{D-(p+1)} \times \prod_{i=1}^p S^1_i$. We employ the Euclidean time formulation for finite temperature quantum field theory and then the Euclidean time direction, whose coordinate is denoted by τ , is compactified on S^1_τ . The spatial p directions are compactified on the p numbers of S^1 and their coordinates are $y^i (i = 1, \dots, p)$. We denote the circumference of each S^1_i as $L_i (i = 0, 1, \dots, p)$ and L_0 stands for the inverse temperature T^{-1} . The $R^{D-(p+1)}$ is the $D - (p + 1)$ dimensional flat Euclidean space whose coordinates are $x^k (k = 1, \dots, D - (p + 1))$.

The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}(\partial_N \phi)^2 - \frac{m_s^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \bar{\psi}(i\Gamma_N \partial_N + m_f)\psi + g\phi\bar{\psi}\psi, \quad (2.1)$$

where the N stands for $N = (\tau, k, i)$, and $\phi(\psi)$ is the scalar (fermion) field whose bulk mass is $m_s(m_f)$. The g is the Yukawa coupling.

One needs to specify the boundary conditions for the S^1_τ and $S^1_i (i = 1, \dots, p)$ directions. For a given field $\Phi(\tau, x^k, y^i)$, the boundary condition for the S^1_τ direction is specified by

$$\Phi(\tau + L_0, x^k, y^i) = e^{2\pi i \eta_0} \Phi(\tau, x^k, y^i). \quad (2.2)$$

The parameter η_0 is definitely determined by quantum statistics to be 0 (periodic) for the scalar field or to be $\frac{1}{2}$ (antiperiodic) for the fermion field. On the other hand, the boundary condition for the $S^1_i (i = 1, \dots, p)$ direction is parametrized by

$$\Phi(\tau, x^k, y^i + L_i) = e^{2\pi i \eta_i} \Phi(\tau, x^k, y^i). \quad (2.3)$$

The parameter η_i can take 0 or $\frac{1}{2}$ for the real scalar field and can be arbitrary for the fermion field.

We employ the standard prescription to calculate the effective potential at finite temperature in one-loop approximation. Let us quickly review the calculations given in the previous paper [13]. For those who are familiar with it, readers can directly go to the next section. By taking up the

¹The studies for dimensional reduction based on the integral form have been carried out in Refs. [11,12].

quadratic terms in the shifted Lagrangian around the constant field φ for the scalar field ϕ in Eq. (2.1), one needs to evaluate

$$V_{\text{eff}} = (-1)^f \mathcal{N} \frac{1}{2} \left(\prod_{i=0}^p \frac{1}{L_i} \sum_{n_i=-\infty}^{\infty} \right) \int \frac{d^{D-(p+1)} p_E}{(2\pi)^{D-(p+1)}} \log \left[p_E^2 + \left(\frac{2\pi}{L_0} \right)^2 (n_0 + \eta_0)^2 + \sum_{i=1}^p \left(\frac{2\pi}{L_i} \right)^2 (n_i + \eta_i)^2 + M^2(\varphi) \right] \quad (2.4)$$

in order to obtain the effective potential on $S_\tau^1 \times R^{D-(p+1)} \times \prod_{i=1}^p S_i^1$ in one-loop approximation. Here, the $M(\varphi)$ is the field-dependent mass of the scalar (fermion) field,

$$M^2(\varphi) = m_s^2 + \frac{\lambda}{2} \varphi^2 \quad (M(\varphi) = m_F + g\varphi). \quad (2.5)$$

Hereafter, we denote $M(\varphi)$ by M for simplicity. The p_E denotes the $D - (p + 1)$ -dimensional Euclidean momentum. The f is the fermion number that is 0(1) for the boson (fermion) and the \mathcal{N} is the on-shell degrees of freedom. The n_0 denotes the Matsubara mode arising from the S_τ^1 , and the Kaluza-Klein mode $n_i (i = 1, \dots, p)$ comes from each $S_i^1 (i = 1, \dots, p)$. The parameter $\eta_i (i = 0, 1, \dots, p)$ is given in Eqs. (2.2) and (2.3).

Let us make use of the zeta-function regularization in order to evaluate Eq. (2.4). By defining

$$I(s) \equiv \left(\prod_{i=0}^p \frac{1}{L_i} \sum_{n_i=-\infty}^{\infty} \right) \int \frac{d^{D-(p+1)} p_E}{(2\pi)^{D-(p+1)}} \left[p_E^2 + \left(\frac{2\pi}{L_0} \right)^2 (n_0 + \eta_0)^2 + \sum_{i=1}^p \left(\frac{2\pi}{L_i} \right)^2 (n_i + \eta_i)^2 + M^2 \right]^{-s}, \quad (2.6)$$

then, the V_{eff} is written as

$$V_{\text{eff}} = (-1)^f \mathcal{N} \frac{1}{2} \left(-\frac{d}{ds} I(s) \right) \Big|_{s \rightarrow 0}. \quad (2.7)$$

Performing the p_E integration with the formula

$$A^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{-At}, \quad (2.8)$$

and employing the Poisson summation

$$\sum_{n_j=-\infty}^{\infty} e^{-\left(\frac{2\pi}{L_j}\right)^2 (n_j + \eta_j)^2 t} = \sum_{m_j=-\infty}^{\infty} \frac{L_j}{2\pi} \left(\frac{\pi}{t} \right)^{\frac{1}{2}} e^{-\frac{(m_j L_j)^2}{4t} + 2\pi i m_j \eta_j}, \quad (2.9)$$

we obtain

$$V_{\text{eff}} = (-1)^{f+1} \mathcal{N} \frac{\pi^{\frac{D}{2}}}{2(2\pi)^D} \sum_{m_0=-\infty}^{\infty} \cdots \sum_{m_p=-\infty}^{\infty} \int_0^\infty dt t^{-\frac{D}{2}-1} e^{-\frac{1}{4t}[(m_0 L_0)^2 + \cdots + (m_p L_p)^2] - M^2 t + 2\pi i(m_0 \eta_0 + \cdots + m_p \eta_p)}. \quad (2.10)$$

Hereafter, we call $m_j (j = 0, 1, \dots, p)$ in Eq. (2.9) or Eq. (2.10) the winding modes, while n_0 and $n_i (i = 1, 2, \dots, p)$ in Eq. (2.9) or Eq. (2.4) the Matsubara and the Kaluza-Klein modes, respectively.

It is useful to separate each summation with respect to m_j in Eq. (2.10) into the zero mode ($m_j = 0$) and the nonzero ones ($m_j \neq 0$) and to express Eq. (2.10) into the form

$$V_{\text{eff}} = \sum_{n=0}^{p+1} F^{(n)D}(M) = \sum_{n=0}^{p+1} \sum_{0 \leq i_1 < i_2 < \cdots < i_n \leq p} F_{L_{i_1}, L_{i_2}, \dots, L_{i_n}}^{(n)D}(M), \quad (2.11)$$

where

$$F_{L_{i_1}, L_{i_2}, \dots, L_{i_n}}^{(n)D}(M) = (-1)^{f+1} \mathcal{N} \frac{\pi^{\frac{D}{2}}}{2(2\pi)^D} \sum'_{m_{i_1}=-\infty}^{\infty} \cdots \sum'_{m_{i_n}=-\infty}^{\infty} \int_0^\infty dt t^{-\frac{D}{2}-1} e^{-\frac{1}{4t}[(m_{i_1} L_{i_1})^2 + \cdots + (m_{i_n} L_{i_n})^2] - M^2 t + 2\pi i(m_{i_1} \eta_{i_1} + \cdots + m_{i_n} \eta_{i_n})}. \quad (2.12)$$

The prime of the summation in $\sum'_{m_j=-\infty}^{\infty}$ means that the zero mode ($m_j = 0$) is removed.

The $F^{(0)}$ in Eq. (2.11) corresponds to the contribution from all the zero modes $m_0 = m_1 = \dots = m_p = 0$ in Eq. (2.10) and is found to be

$$\begin{aligned} F^{(0)D}(M) &= (-1)^{f+1} \mathcal{N} \frac{\pi^{\frac{D}{2}}}{2(2\pi)^D} \int_0^\infty dt t^{-\frac{D}{2}-1} e^{-M^2 t} \\ &= (-1)^{f+1} \mathcal{N} \frac{\pi^{\frac{D}{2}}}{2(2\pi)^D} \Gamma\left(-\frac{D}{2}\right) (M^2)^{\frac{D}{2}}. \end{aligned} \quad (2.13)$$

It must be understood that $F^{(0)D}(M)$ is regularized by the dimensional regularization for $D = \text{even}$. On the other hand, for $D = \text{odd}$, it yields [13]

$$\begin{aligned} F_{L_{i_1}, L_{i_2}, \dots, L_{i_n}}^{(n)D}(M) &= (-1)^{f+1} \mathcal{N} \frac{2^n}{(2\pi)^{\frac{D}{2}}} \sum_{m_{i_1}=1}^\infty \dots \sum_{m_{i_n}=1}^\infty \left(\frac{M^2}{(m_{i_1} L_{i_1})^2 + \dots + (m_{i_n} L_{i_n})^2} \right)^{\frac{D}{4}} \\ &\times K_{\frac{D}{2}} \left(\sqrt{M^2 \{ (m_{i_1} L_{i_1})^2 + \dots + (m_{i_n} L_{i_n})^2 \}} \right) \cos(2\pi m_{i_1} \eta_{i_1}) \dots \cos(2\pi m_{i_n} \eta_{i_n}). \end{aligned} \quad (2.16)$$

The effective potential is given by the modified Bessel function of the second kind accompanied with the multiple mode summation with respect to the winding mode m_{i_j} ($j = 1, \dots, n$) associated with the $S_{i_j}^1$. The winding mode is derived from the Matsubara/Kaluza-Klein mode n_{i_j} through the Poisson summation (2.9). We will use the inverse process in the next section, which is employed to prove a mode recombination formula. The results in this section have already been obtained in the previous paper (see Eq. (2.10) of Ref. [13]) and they are the starting point for the discussion in the present paper.

III. MODE RECOMBINATION FORMULA

In this section, we present and prove the formula called a mode recombination formula. The formula plays a crucial role to obtain a new form of the effective potential, which is different from Eq. (2.11) and is convenient for discussing the nonanalytic terms. These terms can be obtained from the effective potential by use of the mode recombination formula in easier and more transparent way, as we will see in the next section.

Let us recall Eq. (2.11) and first write it as

$$V_{\text{eff}} = F^{(0)D}(M) + \sum_{n=1}^{p+1} \sum_{0 \leq i_1 < i_2 < \dots < i_n \leq p} F_{L_{i_1}, L_{i_2}, \dots, L_{i_n}}^{(n)D}(M), \quad (3.1)$$

where we have separated the $n = 0$ term from the summation in Eq. (2.11) for later convenience. We focus on the scale L_0 in L_{i_j} ($i_j = 0, \dots, p$) and separate the term

$$F^{(0)D}(M) = (-1)^{f+1} \mathcal{N} \frac{(-1)^{\frac{D+1}{2}}}{2^{\frac{D+1}{2}} \pi^{\frac{D-1}{2}} D!!} M^D. \quad (2.14)$$

By using the formula

$$\int_0^\infty dt t^{-\nu-1} e^{-At - \frac{B}{t}} = 2 \left(\frac{A}{B} \right)^{\frac{\nu}{2}} K_\nu(2\sqrt{AB}), \quad (2.15)$$

where the $K_\nu(z)$ is the modified Bessel function of the second kind, Eq. (2.12) for $n \geq 1$ becomes

associated with L_0 from the terms without L_0 on the right-hand side of Eq. (3.1) as²

$$\begin{aligned} &\sum_{n=1}^{p+1} \sum_{0 \leq i_1 < i_2 < \dots < i_n \leq p} F_{L_{i_1}, L_{i_2}, \dots, L_{i_n}}^{(n)D}(M) \\ &= F_{L_0}^{(1)D}(M) + \sum_{n=1}^p \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq p} \left(F_{L_{i_1}, \dots, L_{i_n}}^{(n)D}(M) \right. \\ &\quad \left. + F_{L_0, L_{i_1}, \dots, L_{i_n}}^{(n+1)D}(M) \right). \end{aligned} \quad (3.2)$$

Let us show that the second and third terms on the right-hand side of Eq. (3.2) can be combined into a single expression

$$\begin{aligned} &F_{L_{i_1}, \dots, L_{i_n}}^{(n)D}(M) + F_{L_0, L_{i_1}, \dots, L_{i_n}}^{(n+1)D}(M) \\ &= \frac{1}{L_0} \sum_{n_0=-\infty}^\infty F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)}), \end{aligned} \quad (3.3)$$

²Let us comment that the effective potential obtained by Dolan-Jackiw, Eq. (3.13c) in their paper [1], is reduced to $F_{L_0}^{(1)D}(M)$ by applying the formula [14]

$$\begin{aligned} K_{\frac{1}{2}}(z) &= \sqrt{\frac{\pi}{2z}} e^{-z} \quad \text{and} \\ \int_0^\infty dx K_\nu(\alpha \sqrt{x^2 + z^2}) \frac{x^{2\mu+1}}{(x^2 + z^2)^{\frac{\nu}{2}}} &= \frac{2^\mu \Gamma(\mu+1)}{\alpha^{\mu+1} z^{\nu-\mu-1}} K_{\nu-\mu-1}(\alpha z) \end{aligned}$$

to the series expansion of the logarithm in Eq. (3.13c). The mode in the series expansion plays the same role with the winding mode m_0 in $F_{L_0}^{(1)D}(M)$.

where

$$M_{(0)}^2 \equiv M^2 + \left(\frac{2\pi}{L_0}\right)^2 (n_0 + \eta_0)^2. \quad (3.4)$$

Here, the n_0 is the original Matsubara mode appeared in Eq. (2.4).

Let us prove Eq. (3.3). By using Eq. (2.16), the two terms on the left-hand side of Eq. (3.3) are combined into a single expression, including the $m_0 = 0$ mode,³ as

$$\begin{aligned} & F_{L_{i_1}, \dots, L_{i_n}}^{(n)D}(M) + F_{L_0, L_{i_1}, \dots, L_{i_n}}^{(n+1)D}(M) \\ &= (-1)^{f+1} \mathcal{N} \frac{2^n}{(2\pi)^{\frac{D}{2}}} \sum_{m_0=-\infty}^{\infty} \sum_{m_{i_1}=1}^{\infty} \cdots \sum_{m_{i_n}=1}^{\infty} \left(\frac{M^2}{(m_0 L_0)^2 + (m_{i_1} L_{i_1})^2 + \cdots + (m_{i_n} L_{i_n})^2} \right)^{\frac{D}{4}} \\ & \quad \times K_{\frac{D}{2}} \left(\sqrt{M^2 \{ (m_0 L_0)^2 + (m_{i_1} L_{i_1})^2 + \cdots + (m_{i_n} L_{i_n})^2 \}} \right) e^{2\pi i m_0 \eta_0} \cos(2\pi m_{i_1} \eta_{i_1}) \cdots \cos(2\pi m_{i_n} \eta_{i_n}). \end{aligned} \quad (3.5)$$

With the help of the formula (2.15), the modified Bessel function of the second kind in Eq. (3.5) is written in the integral form, and we inversely use the Poisson summation (2.9) for the m_0 mode, which is then converted into the Matsubara mode n_0 . Thus, we obtain

$$\begin{aligned} & F_{L_{i_1}, \dots, L_{i_n}}^{(n)D}(M) + F_{L_0, L_{i_1}, \dots, L_{i_n}}^{(n+1)D}(M) = \frac{1}{L_0} (-1)^{f+1} \mathcal{N} \frac{2^n}{2(4\pi)^{\frac{D-1}{2}}} \sum_{n_0=-\infty}^{\infty} \sum_{m_{i_1}=1}^{\infty} \cdots \sum_{m_{i_n}=1}^{\infty} \int_0^{\infty} dt t^{-\frac{D-1}{2}-1} \\ & \quad \times e^{-[M^2 + (\frac{2\pi}{L_0})^2 (n_0 + \eta_0)^2] t - \frac{1}{4t} [(m_{i_1} L_{i_1})^2 + \cdots + (m_{i_n} L_{i_n})^2]} \cos(2\pi m_{i_1} \eta_{i_1}) \cdots \cos(2\pi m_{i_n} \eta_{i_n}) \\ &= \frac{1}{L_0} (-1)^{f+1} \mathcal{N} \frac{2^n}{(2\pi)^{\frac{D-1}{2}}} \sum_{n_0=-\infty}^{\infty} \sum_{m_{i_1}=1}^{\infty} \cdots \sum_{m_{i_n}=1}^{\infty} \left(\frac{M_{(0)}^2}{(m_{i_1} L_{i_1})^2 + \cdots + (m_{i_n} L_{i_n})^2} \right)^{\frac{D-1}{4}} \\ & \quad \times K_{\frac{D-1}{2}} \left(\sqrt{M_{(0)}^2 \{ (m_{i_1} L_{i_1})^2 + \cdots + (m_{i_n} L_{i_n})^2 \}} \right) \cos(2\pi m_{i_1} \eta_{i_1}) \cdots \cos(2\pi m_{i_n} \eta_{i_n}), \end{aligned} \quad (3.6)$$

where we have used Eq. (2.15) again to rewrite the integral from into the Bessel function in the last equality. Equation (3.6) is nothing but the right-hand side of Eq. (3.3) and we have proved Eq. (3.3).

We shall call Eq. (3.3) the mode recombination formula⁴ in the present paper. Let us note that the Matsubara mass squared, the second term in Eq. (3.4), is recovered by the inverse use of the Poisson summation (2.9). From Eqs. (3.2) and (3.3), we obtain an important relation

$$\sum_{n=1}^{p+1} \sum_{0 \leq i_1 < i_2 < \cdots < i_n \leq p} F_{L_{i_1}, L_{i_2}, \dots, L_{i_n}}^{(n)D}(M) = F_{L_0}^{(1)D}(M) + \sum_{n=1}^p \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq p} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)}). \quad (3.7)$$

Some comments are in order. It is important to note that the first term on the right-hand side of Eq. (3.7), which is the contribution of the single mode summation with respect to the winding mode m_0 associated with the S_t^1 having the focused scale L_0 , is separated from the contributions of the multiple mode summations in the process of deriving the mode recombination formula. The Matsubara mass squared, the second term in Eq. (3.4), arises after using

³Let us note that this is allowed because Eq. (2.16) is regularized with respect to $m_{i_j} = 0 (j = 1, \dots, n)$.

⁴The idea of the mode recombination has been developed in a different context by the authors [10].

inversely the Poisson summation. As is clear from the discussions given above, the formula (3.7) holds irrespective of whether the field is the scalar or fermion and also of the boundary condition for the spatial $S_i^1 (i = 1, \dots, p)$ direction. The spacetime dimensions in $F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)})$ is effectively reduced to $D - 1$. This is interpreted as that the particle with mass squared M^2 having the Matsubara mass squared $(\frac{2\pi}{L_0})^2 (n_0 + \eta_0)^2$ to be considered on the $D - 1$ dimensional spacetime, $R^{D-(p+1)} \times \prod_{i=1}^p S_i^1$. One can choose another scale, say $L_j (j \in \{1, \dots, p\})$ instead of L_0 , then, the $M_{(0)}^2$ is replaced by $M_{(j)}^2$ with the Kaluza-Klein mass squared $(\frac{2\pi}{L_j})^2 (n_j + \eta_j)^2$.

From Eq. (3.7), the effective potential (3.1) is written by

$$V_{\text{eff}} = F^{(0)D}(M) + F_{L_0}^{(1)D}(M) + \sum_{n=1}^p \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq p} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)}). \quad (3.8)$$

We repeat the same discussion for the third term in Eq. (3.8) by focusing next on the scale L_1 . One immediately sees that the term, aside from the scale L_0 , can be written as

$$\begin{aligned} & \sum_{n=1}^p \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq p} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)}) \\ &= \sum_{n_0=-\infty}^{\infty} F_{L_1}^{(1)D-1}(M_{(0)}) + \sum_{n=1}^{p-1} \sum_{2 \leq i_1 < i_2 < \dots < i_n \leq p} \sum_{n_0=-\infty}^{\infty} \left(F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)}) + F_{L_1, L_{i_1}, \dots, L_{i_n}}^{(n+1)D-1}(M_{(0)}) \right) \\ &= \sum_{n_0=-\infty}^{\infty} F_{L_1}^{(1)D-1}(M_{(0)}) + \sum_{n=1}^{p-1} \sum_{2 \leq i_1 < i_2 < \dots < i_n \leq p} \sum_{n_0=-\infty}^{\infty} \frac{1}{L_1} \sum_{n_1=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-2}(M_{(0,1)}), \end{aligned} \quad (3.9)$$

where we have defined

$$M_{(0,1)}^2 = M^2 + \sum_{i=0}^1 \left(\frac{2\pi}{L_i} \right)^2 (n_i + \eta_i)^2 \quad (3.10)$$

and have used Eq. (3.3) with $L_0, D-1, M$ and $M_{(0)}$ being replaced by $L_1, D-2, M_{(0)}$ and $M_{(0,1)}$, respectively in the last equality. We observe that the first term on the right-hand side of Eq. (3.9), the contribution of the single mode summation with respect to the winding mode m_1 associated

with the S_1^1 having the focused scale L_1 , is separated from the contributions of the multiple mode summations. The Kaluza-Klein mass squared $(\frac{2\pi}{L_1})^2 (n_1 + \eta_1)^2$ turns out to be added to $M_{(0)}^2$ through the formula (3.3) in Eq. (3.10).

We repeat the same discussion given above by focusing on the scales L_2, L_3, \dots, L_{p-1} sequentially for the multiple mode summation obtained at each step such as the second term on the right-hand side of Eq. (3.7) [or Eq. (3.9)]. We finally find that the effective potential is recast into

$$\begin{aligned} V_{\text{eff}} &= F^{(0)D}(M) + F_{L_0}^{(1)D}(M) + \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_1}^{(1)D-1}(M_{(0)}) + \frac{1}{L_0 L_1} \sum_{n_0=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} F_{L_2}^{(1)D-2}(M_{(0,1)}) \\ &\quad + \dots + \frac{1}{L_0 \dots L_{p-1}} \sum_{n_0=-\infty}^{\infty} \dots \sum_{n_{p-1}=-\infty}^{\infty} F_{L_p}^{(1)D-p}(M_{(0,1,\dots,p-1)}) \\ &= F^{(0)D}(M) + \sum_{k=0}^p \left(\frac{1}{\prod_{i=0}^{k-1} L_i} \right) \left(\prod_{j=0}^{k-1} \sum_{n_j=-\infty}^{\infty} \right) F_{L_k}^{(1)D-k}(M_{(0,1,\dots,k-1)}), \end{aligned} \quad (3.11)$$

where we have defined

$$M_{(0,1,\dots,k-1)}^2 \equiv M^2 + \sum_{i=0}^{k-1} \left(\frac{2\pi}{L_i} \right)^2 (n_i + \eta_i)^2. \quad (3.12)$$

The second term $F_{L_0}^{(1)D}(M)$ in the first line of Eq. (3.11) is incorporated into the summation with respect to k as the $k=0$ term in Eq. (3.11), where it is understood that we set

$$\prod_{i=0}^{k-1} L_i \Big|_{k=0} = \prod_{j=0}^{k-1} \sum_{n_j=-\infty}^{\infty} \Big|_{k=0} = 1, \quad M_{(0,1,\dots,k-1)}^2 \Big|_{k=0} = M^2. \quad (3.13)$$

Equation (3.11) is the effective potential rewritten by using successively the mode recombination formula (3.3) and gives the starting point to discuss the nonanalytic terms in the present paper. Let us note that Eq. (3.11) holds for any boundary condition for the spatial S_i^1 direction, reflecting the fact that the mode recombination formula holds irrespective of the boundary condition for any spatial direction.

It may be instructive to mention that the effective potential (3.1) is invariant under the exchange of the scales L_i and L_j ($i \neq j$). The invariance, however, does not become manifest in the new form of the effective potential (3.11). Instead of losing the manifest invariance, the effective potential has a remarkable feature that except for the first term in Eq. (3.11), all the contributions to the effective potential are given by the single mode summation with respect to the winding mode m_k ($k = 0, \dots, p$) associated with the S_k^1 , even though the M^2 has the Kaluza-Klein (Matsubara) mass squared, the second term in Eq. (3.12). One can further simplify Eq. (3.11) by taking account of the discussion on the zero modes in Eq. (3.12), as we will see in the next section.

We shall study the nonanalytic terms for the case of the fermion field as well as the scalar one in the present paper. It may be appropriate here to present the convenient form of the effective potential for the discussion in the case of the fermion. One can show that the first and the second terms in Eq. (3.8) are combined into a single expression

$$F^{(0)D}(M) + F_{L_0}^{(1)D}(M) = \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F^{(0)D-1}(M_{(0)}). \quad (3.14)$$

Then, the effective potential (3.8) is written as

$$V_{\text{eff}} = \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F^{(0)D-1}(M_{(0)}) + \sum_{n=1}^p \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq p} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)}) \quad (3.15)$$

or

$$V_{\text{eff}} = \sum_{n=0}^p \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq p} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)}), \quad (3.16)$$

where the first term in Eq. (3.15) is incorporated into the summation with respect to n as the $n = 0$ term in Eq. (3.16).

Let us prove Eq. (3.14). By setting $m_{i_1} = \dots = m_{i_n} = 0$, $n = 0$ and dropping the summation \sum_{m_j} ($j = 1, \dots, n$) in the first equality of Eq. (3.6), the left-hand side of Eq. (3.14) is given by

$$\begin{aligned} F^{(0)D}(M) + F_{L_0}^{(1)D}(M) &= \frac{1}{L_0} (-1)^{f+1} \mathcal{N} \frac{1}{2(4\pi)^{\frac{D-1}{2}}} \sum_{n_0=-\infty}^{\infty} \int_0^{\infty} dt t^{-\frac{D-1}{2}-1} e^{-[M^2 + (\frac{2\pi}{L_0})^2(n_0 + \eta_0)^2]t} \\ &= \frac{1}{L_0} (-1)^{f+1} \mathcal{N} \frac{\pi^{\frac{D-1}{2}}}{2(2\pi)^{D-1}} \Gamma\left(-\frac{D-1}{2}\right) \sum_{n_0=-\infty}^{\infty} (M_{(0)}^2)^{\frac{D-1}{2}}, \end{aligned} \quad (3.17)$$

where we have used Eq. (2.8) in the last equality. From Eq. (2.13), this is nothing but the right-hand side of Eq. (3.14) and we have proved Eq. (3.14).

We will use Eqs. (3.15) and (3.16) in Sec. V, where we discuss the nonanalytic terms in the effective potential for the case of the fermion. It can be said that Eq. (3.14) is also the mode recombination formula and is regarded as the $n = 0$ case in Eq. (3.3). Let us note that we can always write the effective potential as Eq. (3.16) for any boundary condition of the spatial S_i^1 direction.

For later convenience, we introduce the integral representation for the modified Bessel function of the second kind on the complex plane [14,15]

$$K_{\nu}(x) = \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma(t) \Gamma(t-\nu) \left(\frac{x}{2}\right)^{-2t+\nu}. \quad (3.18)$$

The constant c should be understood to be a point located on the real axis which is greater than all the poles of the gamma functions in the integrand. By using the formula (3.18), one obtains the integral form of the right-hand side of Eq. (3.3) with $n \geq 1$ as

$$\begin{aligned} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)}) &= \frac{1}{L_0} (-1)^{f+1} \mathcal{N} \frac{2^n}{(2\pi)^{\frac{D-1}{2}}} \sum_{n_0=-\infty}^{\infty} \left(\frac{M_{(0)}^2}{2}\right)^{\frac{D-1}{2}} \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma\left(t - \frac{D-1}{2}\right) \left(\frac{M_{(0)}}{2}\right)^{-2t} \\ &\quad \times S^{(n)}(t; L_{i_1}, \dots, L_{i_n}), \end{aligned} \quad (3.19)$$

where we have defined

$$S^{(n)}(t; L_{i_1}, \dots, L_{i_n}) \equiv \Gamma(t) \sum_{m_{i_1}=1}^{\infty} \cdots \sum_{m_{i_n}=1}^{\infty} \{(m_{i_1} L_{i_1})^2 + \cdots + (m_{i_n} L_{i_n})^2\}^{-t} \cos(2\pi m_{i_1} \eta_{i_1}) \cdots \cos(2\pi m_{i_n} \eta_{i_n}), \quad (3.20)$$

and $M_{(0)}$ is given by Eq. (3.4). Let us note that the $n = 0$ case in Eq. (3.3) is given by Eq. (3.17).

We have succeeded in representing the effective potential in the various forms like Eqs. (3.8), (3.11), (3.15) and (3.16). It may be helpful to comment on which forms are useful for the analysis of the scalar or fermion field. The expressions (3.8) and (3.11) of the effective potential will be used for the scalar with the periodic boundary condition. On the other hand, the expressions (3.15) and (3.16) turn out to be useful for the fermion and the scalar with the antiperiodic boundary condition.

IV. NONANALYTIC TERMS FOR SCALAR FIELD WITH PERIODIC BOUNDARY CONDITION

In this section, we derive the nonanalytic terms of the effective potential for a real scalar ($f = 0, \mathcal{N} = 1, \eta_0 = 0$) with the periodic boundary condition in all the spatial directions ($\eta_1 = \cdots = \eta_p = 0$), although the results have already been obtained in the previous paper [13]. The purpose of this section is to show that our formulation presented in this paper is easier and more transparent to obtain the nonanalytic terms than that in the previous paper. To this end, we examine the right-hand side of Eq. (3.3). From Eq. (3.19), we have

$$\begin{aligned} & \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)}) \\ &= -\frac{2^n}{(2\pi)^{\frac{D-1}{2}} L_0} \left(\frac{M^2}{2}\right)^{\frac{D-1}{2}} \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma\left(t - \frac{D-1}{2}\right) \left(\frac{M}{2}\right)^{-2t} \tilde{S}^{(n)}(t; L_{i_1}, \dots, L_{i_n}) \\ & \quad - \frac{2^{n+1} \pi^{\frac{D-1}{2}}}{L_0^D} \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma\left(t - \frac{D-1}{2}\right) \sum_{n_0=1}^{\infty} \left\{ n_0^2 + \left(\frac{ML_0}{2\pi}\right)^2 \right\}^{\frac{D-1}{2}-t} \left(\frac{\pi}{L_0}\right)^{-2t} \tilde{S}^{(n)}(t; L_{i_1}, \dots, L_{i_n}), \end{aligned} \quad (4.1)$$

where we have defined

$$\tilde{S}^{(n)}(t; L_{i_1}, \dots, L_{i_n}) \equiv \Gamma(t) \sum_{m_{i_1}=1}^{\infty} \cdots \sum_{m_{i_n}=1}^{\infty} \{(m_{i_1} L_{i_1})^2 + \cdots + (m_{i_n} L_{i_n})^2\}^{-t} \quad (4.2)$$

and have separated the zero mode ($n_0 = 0$) from the nonzero modes ($n_0 \neq 0$).

We show that the second term in Eq. (4.1) never has nonanalytic terms, in other words, all the terms are given by positive integer powers of the mass squared M^2 . The second term in Eq. (4.1) is written, after changing the variable $\bar{t} = t - \frac{D-1}{2}$, as

$$-\frac{2^{n+1}}{\pi^{\frac{D-1}{2}} L_0} \frac{1}{4\pi i} \int_{\bar{c}-i\infty}^{\bar{c}+i\infty} d\bar{t} \Gamma(\bar{t}) \sum_{n_0=1}^{\infty} \left\{ n_0^2 + \left(\frac{ML_0}{2\pi}\right)^2 \right\}^{-\bar{t}} \left(\frac{\pi}{L_0}\right)^{-2\bar{t}} \tilde{S}^{(n)}\left(\bar{t} + \frac{D-1}{2}; L_{i_1}, \dots, L_{i_n}\right), \quad (4.3)$$

where we have again denoted \bar{t} by t . Let us recall the following formula used in the previous paper [13,16]

$$\begin{aligned} & \Gamma(t) \sum_{n_0=1}^{\infty} \{(n_0 L_0)^2 + z^2\}^{-t} \\ &= -\frac{1}{2} \frac{\Gamma(t)}{z^{2t}} + \frac{\sqrt{\pi} \Gamma(t - \frac{1}{2})}{2L_0 z^{2(t-\frac{1}{2})}} + \frac{2\pi^t}{L_0^{t+\frac{1}{2}} z^{t-\frac{1}{2}}} \sum_{m=1}^{\infty} m^{t-\frac{1}{2}} K_{t-\frac{1}{2}}\left(\frac{2\pi m}{L_0} z\right) \\ &= -\frac{1}{2} \frac{\Gamma(t)}{z^{2t}} + \frac{\sqrt{\pi} \Gamma(t - \frac{1}{2})}{2L_0 z^{2(t-\frac{1}{2})}} + \frac{1}{\sqrt{\pi}} \left(\frac{\pi}{L_0}\right)^{2t} \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma\left(t_1 - t + \frac{1}{2}\right) \zeta(2t_1 - 2t + 1) \Gamma(t_1) \left(z \frac{\pi}{L_0}\right)^{-2t_1}, \end{aligned} \quad (4.4)$$

where we have used Eq. (3.18) in the last equality.

We apply the formula (4.4) with $L_0 = 1$ and $z = \frac{ML_0}{2\pi}$ to Eq. (4.3), which becomes

$$-\frac{2^{n+1}}{\pi^{\frac{D-1}{2}} L_0} \frac{1}{4\pi i} \int_{\bar{c}-i\infty}^{\bar{c}+i\infty} dt \left\{ -\frac{1}{2} \Gamma(t) \left(\frac{ML_0}{2\pi} \right)^{-2t} + \frac{\sqrt{\pi}}{2} \Gamma\left(t - \frac{1}{2}\right) \left(\frac{ML_0}{2\pi} \right)^{-2(t-\frac{1}{2})} \right. \\ \left. + \frac{\pi^{2t}}{\sqrt{\pi} 2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma\left(t_1 - t + \frac{1}{2}\right) \zeta(2t_1 - 2t + 1) \Gamma(t_1) \left(\frac{ML_0}{2} \right)^{-2t_1} \right\} \left(\frac{\pi}{L_0} \right)^{-2t} \tilde{S}^{(n)}\left(t + \frac{D-1}{2}; L_{i_1}, \dots, L_{i_n}\right). \quad (4.5)$$

We deform the integration path with respect to t_1 in Eq. (4.5) in such a way that it encloses all the poles in the integrand and perform the t_1 integration by the residue theorem. Among the poles $t_1 = t - \frac{1}{2} - \ell$ ($\ell = 1, 2, \dots$) of $\Gamma(t_1 - t + \frac{1}{2})$, only the pole $t_1 = t - \frac{1}{2}$ is relevant because of the property $\zeta(-2\ell) = 0$ ($\ell = 1, 2, \dots$), which always follows from the combination $\Gamma(t_1 - t + \frac{1}{2}) \zeta(2t_1 - 2t + 1)$ and is frequently used throughout our discussions. The residue integral from the pole $t_1 = t - \frac{1}{2}$ with $\zeta(0) = -1/2$ cancels the second term in Eq. (4.5) and the integral from the pole $t_1 = t$ of $\zeta(2t_1 - 2t + 1)$ with $\Gamma(1/2) = \sqrt{\pi}$ does the first terms in Eq. (4.5). We are left with the contribution from the pole $t_1 = -\bar{n}$ ($\bar{n} = 0, 1, \dots$) of $\Gamma(t_1)$, so that we find that the second term in Eq. (4.1) becomes

$$-\frac{2^{n+1}}{\pi^{\frac{D}{2}} L_0} \frac{1}{4\pi i} \int_{\bar{c}-i\infty}^{\bar{c}+i\infty} dt \sum_{\bar{n}=0}^{\infty} \frac{(-1)^{\bar{n}}}{\bar{n}!} \Gamma\left(-\bar{n} - t + \frac{1}{2}\right) \zeta(-2\bar{n} - 2t + 1) \left(\frac{ML_0}{2} \right)^{2\bar{n}} \left(\frac{1}{L_0} \right)^{-2t} \tilde{S}^{(n)}\left(t + \frac{D-1}{2}; L_{i_1}, \dots, L_{i_n}\right). \quad (4.6)$$

Even though there would appear the poles in the multiple mode summations $\tilde{S}^{(n)}(t + \frac{D-1}{2}; L_{i_1}, \dots, L_{i_n})$ that contribute to the t integration in Eq. (4.6) by the residue theorem, it does not affect the power of the mass squared $(M^2)^{\bar{n}}$ in Eq. (4.6). Thus, Eq. (4.6) has only positive integer powers of the mass squared M^2 , i.e. $(M^2)^{\bar{n}}$. This observation implies that only the first term in Eq. (4.1) potentially can have the nonanalytic terms, so that from Eq. (3.3), we obtain the important relation

$$\left(F_{L_{i_1}, \dots, L_{i_n}}^{(n)D}(M) + F_{L_0, L_{i_1}, \dots, L_{i_n}}^{(n+1)D}(M) \right) \Big|_{\text{n.a.}} \\ = \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)}) \Big|_{\text{n.a.}} \\ = \frac{1}{L_0} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M) \Big|_{\text{n.a.}} \quad (4.7)$$

The abbreviation denoted by “n.a.” in Eq. (4.7) means *nonanalytic terms*, which is used throughout the paper. It should be stressed that only the zero mode ($n_0 = 0$) in $M_{(0)}$ is relevant for the existence of the nonanalytic terms in the mode recombination formula (3.3). The relation (4.7) is crucial for the continuing discussion given below.

The relevant part for the nonanalytic terms in the effective potential (3.8) is given by replacing $M_{(0)}$ by M without the summation because of Eq. (4.7). Considering the discussion led to Eq. (4.7), we understand that the relevant part for the nonanalytic terms in Eq. (3.9) is given by replacing the $M_{(0)}, M_{(0,1)}$ by $M, M_{(1)}$, respectively and

dropping the n_0 summation. By repeating the same discussion as above and applying it to the effective potential (3.11), the relevant part of the nonanalytic terms in the effective potential is given by picking up only the zero modes in $M_{(0,1,\dots,k-1)}$ ($k = 1, 2, \dots, p$), that is,

$$V_{\text{eff}}|_{\text{n.a.}} = F^{(0)D}(M)|_{\text{n.a.}} + \sum_{k=0}^p \left(\frac{1}{\prod_{i=0}^{k-1} L_i} \right) F_{L_k}^{(1)D-k}(M)|_{\text{n.a.}} \quad (4.8)$$

It turns out that the relevant part of the nonanalytic terms in the effective potential drastically reduces to the simple expression (4.8) and is given by the contribution of the *single* mode summation of the winding mode m_k ($k = 0, \dots, p$) associated with the S_k^1 . This is quite different from that given in the previous paper where we have analyzed the multiple mode summations in order to obtain the nonanalytic terms. It should be emphasized that Eq. (4.8) is obtained as the consequence of taking account of the zero mode ($n_0 = \dots = n_p = 0$) alone in the Matsubara and Kaluza-Klein modes.

Let us confirm the results obtained in the previous paper by using Eq. (4.8). We will soon recognize that the calculations based on Eq. (4.8) are easier and more transparent compared with that in the previous paper. We first study the nonanalytic terms of $F_{L_k}^{(1)D-k}(M)$ in Eq. (4.8). From Eqs. (2.16) and (3.18), one has (remember $\eta_0 = 0$, $\mathcal{N} = 1, \eta_k = 0$ in this section)

$$\begin{aligned}
F_{L_k}^{(1)D-k}(M) &= \frac{-2}{(2\pi)^{\frac{D-k}{2}}} \sum_{m_k=1}^{\infty} \left(\frac{M^2}{(m_k L_k)^2} \right)^{\frac{D-k}{4}} K_{\frac{D-k}{2}}(m_k M L_k) \\
&= \frac{-2}{(2\pi)^{\frac{D-k}{2}}} \left(\frac{M^2}{2} \right)^{\frac{D-k}{2}} \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma\left(t - \frac{D-k}{2}\right) \Gamma(t) \zeta(2t) \left(\frac{M L_k}{2} \right)^{-2t}, \quad (4.9)
\end{aligned}$$

where we have used the definition of the zeta function $\zeta(z) = \sum_{m=1}^{\infty} m^{-z}$. Since the scale dependence on M in Eq. (4.9) is $(M^2)^{\frac{D-k}{2}-t}$, the poles $t = \frac{D-k}{2} - \ell$ ($\ell = 0, 1, \dots$) of $\Gamma(t - \frac{D-k}{2})$ yields analytic terms $(M^2)^\ell$, which are outside of our interest. Only the pole $t = 0$ among the poles $t = -m$ ($m = 0, 1, \dots$) of $\Gamma(t)$ contributes to the residue integration because of the property $\zeta(-2m) = 0$ ($m = 1, 2, \dots$). Hence, there are two poles, $t = 0$ of $\Gamma(t)$ and $t = \frac{1}{2}$ of $\zeta(2t)$ that contribute to the residue integration in Eq. (4.9) for obtaining the nonanalytic terms.⁵

Since the nonanalytic terms are given by positive odd integer powers of M , whether the pole of $t = 0$ or $t = 1/2$ can produce a nonanalytic term depends on whether D and k are even or odd. For $(D, k) = (\text{even}, \text{even})$ or (odd, odd) , the nonanalytic terms in Eq. (4.9) is given by the pole $t = \frac{1}{2}$ as

$$F_{L_k}^{(1)D-k}(M)|_{\text{n.a.}} = \frac{(-1)(-1)^{\frac{D-k}{2}}}{2^{\frac{D-k}{2}} \pi^{\frac{D-k-2}{2}} (D-k-1)!!} \frac{M^{D-k-1}}{L_k}, \quad (4.10)$$

while for $(D, k) = (\text{even}, \text{odd})$ or $(\text{odd}, \text{even})$, the pole $t = 0$ yields the nonanalytic term

$$F_{L_k}^{(1)D-k}(M)|_{\text{n.a.}} = \frac{(-1)^{\frac{D-k+1}{2}}}{2^{\frac{D-k+1}{2}} \pi^{\frac{D-k-1}{2}} (D-k)!!} M^{D-k}. \quad (4.11)$$

Here, we have used

$$\Gamma\left(\frac{1-D}{2}\right) = \frac{(-1)^{\frac{D}{2}} 2^{\frac{D}{2}}}{(D-1)!!} \sqrt{\pi} \quad \text{for } D = \text{even}. \quad (4.12)$$

We are ready to calculate the nonanalytic terms in the effective potential based on Eq. (4.8) by using Eqs. (4.10) and (4.11). It is convenient to study the terms for each case of even/odd D and $p+1$. Let us remind that $p+1$ is the total number of S^1 .

⁵The log M is another type of the nonanalytic term in the effective potential, which is obtained by the residue integration for the double pole of the integrand in Eq. (4.9). We do not study such the term in this paper.

A. $(D, p+1) = (\text{even}, \text{odd})$

Let us first introduce

$$\begin{aligned}
A_{2n+1} &\equiv \frac{1}{\prod_{i=0}^{2n} L_i} F_{L_{2n+1}}^{(1)D-(2n+1)}(M)|_{\text{n.a.}} \\
&= \frac{(-1)^{\frac{D-2n}{2}}}{2^{\frac{D-2n}{2}} \pi^{\frac{D-2n-2}{2}} (D-(2n+1))!!} \frac{M^{D-(2n+1)}}{L_0 \cdots L_{2n}}, \quad (4.13)
\end{aligned}$$

$$\begin{aligned}
B_{2n} &\equiv \frac{1}{\prod_{i=0}^{2n-1} L_i} F_{L_{2n}}^{(1)D-2n}(M)|_{\text{n.a.}} \\
&= \frac{(-1)(-1)^{\frac{D-2n}{2}}}{2^{\frac{D-2n}{2}} \pi^{\frac{D-2n-2}{2}} (D-2n-1)!!} \frac{M^{D-2n-1}}{L_0 \cdots L_{2n}}, \quad (4.14)
\end{aligned}$$

where we have used Eq. (4.11) [Eq. (4.10)] in the equality of Eq. (4.13) [Eq. (4.14)]. One immediately observes that

$$A_{2n+1} = -B_{2n} \quad \text{for } n = 0, 1, \dots \quad (4.15)$$

Let us note that the definition for B_0 in Eq. (4.14) is consistently equal to $F_{L_0}^{(1)D}(M)$ by setting $\prod_{i=0}^{2n-1} L_i|_{n=0} = 1$, which originally corresponds to the $k = 0$ term in Eq. (3.11).

In terms of Eqs. (4.13) and (4.14), the nonanalytic term in Eq. (4.8) is given by

$$V_{\text{eff}}|_{\text{n.a.}} = F^{(0)D}(M)|_{\text{n.a.}} + \sum_{n=0}^{\frac{p}{2}-1} (A_{2n+1} + B_{2n}) + B_p = B_p, \quad (4.16)$$

where we have used Eq. (4.15) and the fact that from Eq. (2.13), the $F^{(0)D}(M)$ does not possess any nonanalytic term for $D = \text{even}$. The B_p is given by Eq. (4.14) with $n = \frac{p}{2}$ to yield, by noting that $(-1)(-1)^{\frac{D-p}{2}} = (-1)^{\frac{D+p+2}{2}}$ for $p = \text{even}$,

$$V_{\text{eff}}|_{\text{n.a.}} = B_p = \frac{(-1)^{\frac{D+p+2}{2}}}{2^{\frac{D-p}{2}} \pi^{\frac{D-p-2}{2}} (D-(p+1))!!} \frac{M^{D-(p+1)}}{L_0 L_1 \cdots L_p}. \quad (4.17)$$

B. $(D, p+1) = (\text{even}, \text{even})$

In this case, the nonanalytic term in Eq. (4.8) is written by using Eqs. (4.13) and (4.14) as

$$V_{\text{eff}}|_{\text{n.a.}} = F^{(0)D}(M)|_{\text{n.a.}} + \sum_{n=0}^{\frac{p-1}{2}} (A_{2n+1} + B_{2n}) = 0, \quad (4.18)$$

where we have used Eq. (4.15) in the last equality. The effective potential does not have any nonanalytic term in this case.

C. $(D, p+1) = (\text{odd}, \text{odd})$

For the case of $D = \text{odd}$, from Eqs. (4.10) and (4.11), it is convenient to introduce

$$\begin{aligned} C_{2n-1} &\equiv \frac{1}{\prod_{i=0}^{2n-2} L_i} F_{L_{2n-1}}^{(1)D-(2n-1)}(M)|_{\text{n.a.}} \\ &= \frac{(-1)(-1)^{\frac{D-(2n-1)}{2}}}{2^{\frac{D-(2n-1)}{2}} \pi^{\frac{D-2n-1}{2}} (D-2n)!!} \frac{M^{D-2n}}{L_0 \cdots L_{2n-1}}, \end{aligned} \quad (4.19)$$

$$\begin{aligned} D_{2n} &\equiv \frac{1}{\prod_{i=0}^{2n-1} L_i} F_{L_{2n}}^{(1)D-2n}(M)|_{\text{n.a.}} \\ &= \frac{(-1)^{\frac{D-2n+1}{2}}}{2^{\frac{D-2n+1}{2}} \pi^{\frac{D-2n-1}{2}} (D-2n)!!} \frac{M^{D-2n}}{L_0 \cdots L_{2n-1}}, \end{aligned} \quad (4.20)$$

respectively. We see that the relation

$$C_{2n-1} = -D_{2n} \quad \text{for } n = 0, 1, \dots \quad (4.21)$$

holds. It should be understood that in Eq. (4.21) we have defined $C_{2n-1}|_{n=0} \equiv F^{(0)D}(M)$ and $D_{2n}|_{n=0} \equiv F_{L_0}^{(1)D}(M)$. One immediately confirms that the $C_{-1} = -D_0$ is satisfied by using the explicit expressions for $F^{(0)D}(M)$ in Eq. (2.14) and $F_{L_0}^{(1)D}(M)$ [13] in Eq. (4.11) for $D = \text{odd}$.

In terms of Eqs. (4.19) and (4.20), the nonanalytic part of the effective potential in Eq. (4.8) is calculated as

$$V_{\text{eff}}|_{\text{n.a.}} = \sum_{n=0}^{\frac{p}{2}} (C_{2n-1} + D_{2n}) = 0, \quad (4.22)$$

thanks to Eq. (4.21).

D. $(D, p+1) = (\text{odd}, \text{even})$

In this case, the nonanalytic term in Eq. (4.8) is obtained as

$$V_{\text{eff}}|_{\text{n.a.}} = \sum_{n=0}^{\frac{p-1}{2}} (C_{2n-1} + D_{2n}) + C_p = C_p, \quad (4.23)$$

where we have used Eq. (4.21). The C_p is given by Eq. (4.19) with $n = \frac{p+1}{2}$ to yield

$$V_{\text{eff}}|_{\text{n.a.}} = C_p = \frac{(-1)^{\frac{D+p}{2}}}{2^{\frac{D-p}{2}} \pi^{\frac{D-p-2}{2}} (D-(p+1))!!} \frac{M^{D-(p+1)}}{L_0 L_1 \cdots L_p}, \quad (4.24)$$

where we have used the fact $(-1)(-1)^{\frac{D-p}{2}} = (-1)^{\frac{D+p}{2}}$ for $p = \text{odd}$.

The results (4.17), (4.18), (4.22) and (4.24) are found to exactly agree with those given in the previous paper, as they should be. Even though they have already been obtained in the previous paper, our formalism developed in this paper is easier and more transparent to derive the nonanalytic terms of the effective potential, and furthermore makes it possible to analyze the case of the fermion field, as we will see in the next section. In addition to it, a new insight on the nonanalytic term is obtained. The nonanalytic term Eq. (4.17) [or Eq. (4.24)] depends on each scale, L_0, \dots, L_p of all the S^1 's. This may reflect the fact that only the zero modes in the Matsubara and Kaluza-Klein modes are relevant for the existence of the nonanalytic terms, as shown in Eq. (4.8).

V. NONANALYTIC TERMS FOR FERMION FIELD

Let us study the nonanalytic terms in the effective potential for the case of the fermion field ($f=1, \eta_0=\frac{1}{2}$) satisfying arbitrary boundary condition for the spatial $S^1_i (i=1, 2, \dots, p)$ direction. The effective potential (3.15) rather than Eq. (3.11) is appropriate to discuss the terms in this case.

We first consider the $n \geq 1$ term, the second one in Eq. (3.15), whose integral form on the complex plane is given from Eq. (3.19) by

$$\begin{aligned} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)}) &= \frac{1}{L_0} \mathcal{N} \frac{2^n}{\pi^{\frac{D-1}{2}}} \frac{1}{4\pi i} \int_{\bar{c}-i\infty}^{\bar{c}+i\infty} dt \Gamma(t) \sum_{n_0=-\infty}^{\infty} \left\{ \left(n_0 + \frac{1}{2} \right)^2 + \left(\frac{ML_0}{2\pi} \right)^2 \right\}^{-t} \\ &\quad \times \left(\frac{\pi}{L_0} \right)^{-2t} S^{(n)} \left(t + \frac{D-1}{2}; L_{i_1}, \dots, L_{i_n} \right), \end{aligned} \quad (5.1)$$

where we have changed the variable $\bar{t} = t - \frac{D-1}{2}$ and have denoted \bar{t} by t again.

One needs to develop the analytical extension for the mode summation with respect to n_0 in Eq. (5.1). To this end, we employ the formula (2.8) and the Poisson summation (2.9). Then, we find

$$\Gamma(t) \sum_{n_0=-\infty}^{\infty} \left\{ \left(n_0 + \frac{1}{2} \right)^2 + z^2 \right\}^{-t} = \sqrt{\pi} \sum_{m_0=-\infty}^{\infty} \int_0^{\infty} d\bar{t} \bar{t}^{(t-\frac{1}{2})-1} e^{-\frac{(2\pi m_0)^2}{4\bar{t}} + \pi i m_0 - z^2 \bar{t}}. \quad (5.2)$$

By separating $m_0 = 0$ and $m_0 \neq 0$ modes in Eq. (5.2) and by using Eqs. (2.8) and (2.15), the right-hand side of Eq. (5.2) becomes

$$\begin{aligned} & \sqrt{\pi} \frac{\Gamma(t-\frac{1}{2})}{z^{2(t-\frac{1}{2})}} + 4\sqrt{\pi} \left(\frac{\pi}{z} \right)^{t-\frac{1}{2}} \sum_{m_0=1}^{\infty} (-1)^{m_0} m_0^{t-\frac{1}{2}} K_{t-\frac{1}{2}}(2\pi m_0 z) \\ &= \sqrt{\pi} \frac{\Gamma(t-\frac{1}{2})}{z^{2(t-\frac{1}{2})}} + 4\sqrt{\pi} \left(\frac{\pi}{z} \right)^{t-\frac{1}{2}} \left\{ - \sum_{m_0=1}^{\infty} m_0^{t-\frac{1}{2}} K_{t-\frac{1}{2}}(2\pi m_0 z) + 2 \sum_{m_0=1}^{\infty} (2m_0)^{t-\frac{1}{2}} K_{t-\frac{1}{2}}(4\pi m_0 z) \right\}, \end{aligned} \quad (5.3)$$

where we have used the property $K_{-\nu}(z) = K_{\nu}(z)$ and the formula

$$\sum_{m_0=1}^{\infty} (-1)^{m_0} f(m_0) = - \sum_{m_0=1}^{\infty} f(m_0) + 2 \sum_{m_0=1}^{\infty} f(2m_0) \quad (5.4)$$

for later convenience. We further recast Eq. (5.3) into the integral form on the complex plane by Eq. (3.18). Then, we obtain

$$\begin{aligned} \Gamma(t) \sum_{n_0=-\infty}^{\infty} \left\{ \left(n_0 + \frac{1}{2} \right)^2 + z^2 \right\}^{-t} &= \sqrt{\pi} \frac{\Gamma(t-\frac{1}{2})}{z^{2(t-\frac{1}{2})}} - \frac{4\pi^{2t}}{\sqrt{\pi}} \frac{1}{4\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma\left(t_1 - t + \frac{1}{2}\right) \zeta(2t_1 - 2t + 1) \Gamma(t_1) (\pi z)^{-2t_1} \\ &\quad + \frac{4(2\pi)^{2t}}{\sqrt{\pi}} \frac{1}{4\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma\left(t_1 - t + \frac{1}{2}\right) \zeta(2t_1 - 2t + 1) \Gamma(t_1) (2\pi z)^{-2t_1}. \end{aligned} \quad (5.5)$$

This is an analytical extension for the mode summation with respect to n_0 in Eq. (5.1). Eq. (5.5) does not have the term corresponding to the first term in Eq. (4.4) because of the lack of the zero mode due to the antiperiodic boundary condition for the fermion field in the Euclidean time direction.

By setting $z = \frac{ML_0}{2\pi}$ and inserting Eq. (5.5) into Eq. (5.1), we have

$$\begin{aligned} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)}) &= \frac{1}{L_0} \mathcal{N} \frac{2^n}{\pi^{\frac{D-1}{2}}} \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} dt \left\{ \sqrt{\pi} \Gamma\left(t - \frac{1}{2}\right) \left(\frac{ML_0}{2\pi} \right)^{-2(t-\frac{1}{2})} \right. \\ &\quad - \frac{4\pi^{2t}}{\sqrt{\pi}} \frac{1}{4\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma\left(t_1 - t + \frac{1}{2}\right) \zeta(2t_1 - 2t + 1) \Gamma(t_1) \left(\frac{ML_0}{2} \right)^{-2t_1} \\ &\quad + \frac{4(2\pi)^{2t}}{\sqrt{\pi}} \frac{1}{4\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma\left(t_1 - t + \frac{1}{2}\right) \zeta(2t_1 - 2t + 1) \Gamma(t_1) (ML_0)^{-2t_1} \Big\} \\ &\quad \times \left(\frac{\pi}{L_0} \right)^{-2t} S^{(n)}\left(t + \frac{D-1}{2}; L_{i_1}, \dots, L_{i_n}\right). \end{aligned} \quad (5.6)$$

We perform the residue integration with respect to t_1 by deforming the integration path in such a way that it encloses all the poles in the integrand. Among the poles $t_1 = t - \frac{1}{2} - \ell$ ($\ell = 0, 1, \dots$) of $\Gamma(t_1 - t + \frac{1}{2})$, only the pole $t_1 = t - \frac{1}{2}$ is relevant to the residue integration thanks to the property $\zeta(-2\ell) = 0$ ($\ell = 1, 2, \dots$). In addition to it, the poles $t_1 = t$ of $\zeta(2t_1 - 2t + 1)$ and $t_1 = -\bar{n}$ ($\bar{n} = 0, 1, \dots$) of $\Gamma(t_1)$ contribute to the t_1 integration by the residue theorem. It is easy to see that the contributions from the residue integration of the pole $t_1 = t - \frac{1}{2}$ in the second and third terms of Eq. (5.6) cancel the first term in Eq. (5.6) and that the contributions from the residue integration of the pole $t_1 = t$ of $\zeta(2t_1 - 2t + 1)$ in the second and the third terms of Eq. (5.6) are canceled each other. Thus, what is left is the contribution of the residue integration of the pole $t_1 = -\bar{n}$ ($\bar{n} = 0, 1, \dots$) of $\Gamma(t_1)$ alone, which is given by

$$\begin{aligned} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)}) &= \frac{1}{L_0} \mathcal{N} \frac{2^n}{\pi^{\frac{D-1}{2}}} \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} dt \sum_{\bar{n}=0}^{\infty} \left\{ -\frac{2}{\sqrt{\pi}} \frac{(-1)^{\bar{n}}}{\bar{n}!} \Gamma\left(-\bar{n}-t+\frac{1}{2}\right) \zeta(-2\bar{n}-2t+1) \left(\frac{ML_0}{2}\right)^{2\bar{n}} L_0^{2t} \right. \\ &\quad \left. + \frac{2}{\sqrt{\pi}} \frac{(-1)^{\bar{n}}}{\bar{n}!} \Gamma\left(-\bar{n}-t+\frac{1}{2}\right) \zeta(-2\bar{n}-2t+1) (ML_0)^{2\bar{n}} (2L_0)^{2t} \right\} S^{(n)}\left(t+\frac{D-1}{2}; L_{i_1}, \dots, L_{i_n}\right). \end{aligned} \quad (5.7)$$

One observes that the dependence of the power on M in Eq. (5.7) is given by the positive integer power of the mass squared $(M^2)^{\bar{n}}$, so that there exists no nonanalytic term in the second term in Eq. (3.15)

$$\sum_{n=1}^p \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq p} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)})|_{\text{n.a.}} = 0. \quad (5.8)$$

We note that the multiple mode summations in $S^{(n)}(t + \frac{D-1}{2}; L_{i_1}, \dots, L_{i_n})$ could produce poles that contribute to the

t integration in Eq. (5.7), but they do not change the power of $(M^2)^{\bar{n}}$ because of the nonexistence of M in the summations. It is important to note that Eq. (5.7) holds irrespective of the boundary condition for the spatial $S_i^1(i = 1, 2, \dots, p)$ direction and thus, so does Eq. (5.8).

Let us next study the $n = 0$ term, the first one in Eq. (3.15), which is given by Eq. (3.17) with $f = 0$ and $\eta_0 = \frac{1}{2}$. We make use of the analytical extension (5.4) in order to calculate the mode summation with respect n_0 in Eq. (3.17). By putting $t = -\frac{D-1}{2}$, $z = \frac{ML_0}{2\pi}$ in Eq. (5.5), the relevant part of Eq. (3.17) becomes

$$\begin{aligned} &\Gamma\left(-\frac{D-1}{2}\right) \sum_{n_0=-\infty}^{\infty} \left\{ \left(n_0 + \frac{1}{2}\right)^2 + \left(\frac{ML_0}{2\pi}\right)^2 \right\}^{\frac{D-1}{2}} \\ &= \sqrt{\pi} \Gamma\left(-\frac{D}{2}\right) \left(\frac{ML_0}{2\pi}\right)^D - \frac{4\pi^{-(D-1)}}{\sqrt{\pi}} \frac{1}{4\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma\left(t_1 + \frac{D}{2}\right) \zeta(2t_1 + D) \Gamma(t_1) \left(\frac{ML_0}{2}\right)^{-2t_1} \\ &\quad + \frac{4(2\pi)^{-(D-1)}}{\sqrt{\pi}} \frac{1}{4\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma\left(t_1 + \frac{D}{2}\right) \zeta(2t_1 + D) \Gamma(t_1) (ML_0)^{-2t_1}. \end{aligned} \quad (5.9)$$

We again evaluate the t_1 integration by the residue theorem by deforming the integration path in such a way that it encloses all the poles in the integrand. The nonanalytic terms⁶ in the second and the third terms in Eq. (5.9) are given by the residue integration of the poles that have the minus half odd integer values of t_1 , as seen from the scale dependence on M in Eq. (5.9).

For $D = \text{even}$, the relevant pole is $t_1 = -\frac{D-1}{2}$ of $\zeta(2t_1 + D)$, from which the second and the third terms in Eq. (5.9) yield

$$\begin{aligned} &-\frac{4\sqrt{\pi}}{\sqrt{\pi}2^2} \Gamma\left(-\frac{D-1}{2}\right) \left(\frac{ML_0}{2\pi}\right)^{D-1} \\ &+ \frac{4\sqrt{\pi}}{\sqrt{\pi}2^2} \Gamma\left(-\frac{D-1}{2}\right) \left(\frac{ML_0}{2\pi}\right)^{D-1} = 0. \end{aligned} \quad (5.10)$$

The first term in Eq. (5.9) is analytic for $D = \text{even}$, so that there is no nonanalytic term in Eq. (5.9) for $D = \text{even}$.

On the other hand, for $D = \text{odd}$, the relevant pole for the nonanalytic terms is given only by the pole $t_1 = -\frac{D}{2}$ among the poles $t_1 = -\frac{D}{2} - \ell$ ($\ell = 0, 1, \dots$) of $\Gamma(t_1 + \frac{D}{2})$ because of the property $\zeta(-2\ell) = 0$ ($\ell = 1, 2, \dots$). Then, the second and the third terms in Eq. (5.9) lead to

$$\begin{aligned} &\sqrt{\pi} \Gamma\left(-\frac{D}{2}\right) \left(\frac{ML_0}{2\pi}\right)^D - 2\sqrt{\pi} \Gamma\left(-\frac{D}{2}\right) \left(\frac{ML_0}{2\pi}\right)^D \\ &= -\sqrt{\pi} \Gamma\left(-\frac{D}{2}\right) \left(\frac{ML_0}{2\pi}\right)^D, \end{aligned} \quad (5.11)$$

which cancels the first term in Eq. (5.9). Thus, Eq. (5.9) has no nonanalytic term for $D = \text{odd}$.

We have shown that the first term in Eq. (3.15) does not possess the nonanalytic term

$$\frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{(0)}^{D-1}(M_{(0)})|_{\text{n.a.}} = 0. \quad (5.12)$$

⁶We do not consider the nonanalytic terms of the type, $\log M$ in this paper, which comes from the double pole in the integrand, as mentioned in the footnote of the Sec. IV.

We arrive at the important conclusion from Eqs. (5.8) and (5.12) that the effective potential (3.16) has no nonanalytic term for the case of the fermion

$$V_{\text{eff}}|_{\text{n.a.}} = \sum_{n=0}^p \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq p} \frac{1}{L_0} \sum_{n_0=-\infty}^{\infty} F_{L_{i_1}, \dots, L_{i_n}}^{(n)D-1}(M_{(0)})|_{\text{n.a.}} = 0. \quad (5.13)$$

It should be emphasized that Eq. (5.13) holds irrespective of the boundary condition for the spatial S_i^1 ($i = 1, 2, \dots, p$) direction.

VI. NONANALYTIC TERMS FOR SCALAR FIELD WITH ANTIPERIODIC BOUNDARY CONDITION

Taking account of the result obtained in the previous section, we can also study the case of the scalar field with some of the boundary condition for the spatial S_i^1 ($i = 1, \dots, p$) direction being antiperiodic. In this case, by regarding the direction that has the antiperiodic boundary condition as the Euclidean time direction, the effective potential essentially has the same with that of the case for the fermion field. Hence, we conclude that there is no nonanalytic term in the effective potential if the scalar field satisfies at least one antiperiodic boundary condition for the spatial S_i^1 direction.

VII. CONCLUSIONS AND DISCUSSIONS

We have studied the nonanalytic terms, which cannot be written in the form of any positive integer power of field-dependent mass squared, in the effective potential at finite temperature in one-loop approximation for the fermion and scalar fields on the D -dimensional spacetime, $S_\tau^1 \times R^{D-(p+1)} \times \prod_{i=1}^p S_i^1$. In doing it, we have developed the new formula called the mode recombination formula (3.3), which holds irrespective of whether the field is a fermion or a scalar and of the boundary condition for the spatial S_i^1 direction. The effective potential has been recast into the new forms (3.11) and (3.16) by using the formula, which is convenient to study the nonanalytic terms for both cases of the fermion and the scalar.

We have clarified the importance of the zero mode in the Kaluza-Klein mode for the existence of the nonanalytic terms through the mode recombination formula. This has drastically simplified the relevant part of the effective potential for calculating the nonanalytic terms for the case of the scalar field satisfying the periodic boundary condition for the spatial S_i^1 direction. The effective potential (4.8) is given in terms of the single mode summation of the winding mode for each S^1 and the integral form for the potential on the complex plane is easy to perform the residue integration. We have correctly reproduced the nonanalytic term (4.17) [and (4.24)] in easier and more transparent way.

The mode recombination formula has also provided the convenient form for studying the nonanalytic terms in the effective potential for the case of the fermion field satisfying arbitrary boundary conditions for the spatial S_i^1 ($i = 1, \dots, p$) directions. The antiperiodicity for the Euclidean time direction for the fermion has resulted the quite different pole structure of the analytical extension for the mode summation with respect to the Matsubara mode compared with that of the scalar. We have found that there is no nonanalytic term in the effective potential for the case of the fermion. The result for the case of the fermion has immediately led to the conclusion that there also exists no nonanalytic term for the case of the scalar field satisfying the antiperiodic boundary condition for at least one spatial S_i^1 direction.

We have obtained some insight on the nonanalytic terms in the effective potential. We have found that the nonanalytic term can appear only when there exists the zero mode associated with each of all the S^1 's in the Matsubara and Kaluza-Klein modes. This observation may explain that the nonanalytic term depends on all the scales L_0, L_1, \dots, L_p like in Eq. (4.17) [or Eq. (4.24)] and further that the effective potential has no nonanalytic term for the fermion due to the lack of the zero mode for the Euclidean time direction irrespective of the absence or presence of the zero mode for the spatial S_i^1 ($i = 1, 2, \dots, p$) direction.

Equipped with the result obtained for the cases of the scalar, we can also mention about the nonanalytic terms for the case of a higher dimensional gauge field on $S_\tau^1 \times R^{D-(p+1)} \times \prod_{i=1}^p S_i^1$. Let us consider the D -dimensional gauge field $A_M(\tau, x^k, y^i)$, whose component gauge fields are written as

$$A_M(\tau, x^k, y^i) = (A_\tau(\tau, x^k, y^i), A_k(\tau, x^k, y^i), A_i(\tau, x^k, y^i)). \quad (7.1)$$

We need to specify the boundary condition for the Euclidean time direction and the spatial S_i^1 direction. The boundary condition for the Euclidean time direction must be periodic, i.e.

$$A_M(\tau + L_0, x^k, y^i) = +A_M(\tau, x^k, y^i) \quad (7.2)$$

because of the quantum statistics. One can choose the boundary conditions of the $A_k(\tau, x^k, y^i)$ and the $A_i(\tau, x^k, y^i)$ for the spatial S_i^1 ($i = 1, \dots, p$) direction to be 0 or twisted under the assumption that the Lagrangian density must be single valued. The gauge field $A_M(\tau, x^k, y^i)$ can be massive through the Higgs mechanism and it has the field-dependent mass such as $M(\varphi)$. If we restrict the twisted boundary condition for the spatial S_i^1 direction to the antiperiodic boundary condition, the nonanalytic term in the effective potential for each gauge field in Eq. (7.1) is reduced to the cases of the scalar field studied in the

Secs. IV and VI. Thus, the result has already been obtained.⁷

In addition to the nonanalytic terms we have studied, there is another type of the nonanalytic term, $\log M$ in the effective potential, as mentioned in Sec. IV. Such the term arises from the double pole of the integrand, for example, in the residue integration of Eq. (4.9). In order to understand the whole nonanalytic structure of the effective potential with respect to M , one has to study such a term extensively

as well. Our analyses have been carried out at the level of one-loop approximation for the effective potential at finite temperature. One may wonder what type of the nonanalytic terms besides the one obtained in the paper can emerge beyond the one-loop calculation. In connection with higher loop calculations, we are also interested in the behavior of the $\log M$ term, which actually stands for genuine quantum effects. These are under investigation and will be reported elsewhere.

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⁷If the component gauge field A_i has the zero mode, it can acquire the vacuum expectation value (VEV) through the dynamics of the Wilson line phase [17]. The VEV is removed by the field redefinition to twist the boundary condition for the spatial direction of the matter field.

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