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Induced Bornological Representations of Bornological Algebras I

By Shunsuke FUNAKOSI

Let G be a finite group, and let H be a subgroup of G . The restriction functor $W \rightarrow W_H$ which assigns to each G -module (equivalently, representation of G) W , the corresponding H -module, W_H , has both a left adjoint and a right adjoint. That is ([2], p. 57), there are functors $V \rightarrow {}^G_V$ and $V \rightarrow V^G$ which assign, to each H -module V , corresponding induced G -modules G_V and V^G , such that

$$\mathrm{Hom}_G({}^G_V, W) \cong \mathrm{Hom}_H(V, W_H),$$

and

$$\mathrm{Hom}_G(W, V^G) \cong \mathrm{Hom}_H(W_H, V),$$

respectively (the Frobenius reciprocity theorem can be considered to be the statement of these equivalences).

In [3] (resp. [4]), it is shown that the situation described above is true for uniformly bounded strongly continuous Banach space representations (resp. for continuous locally convex space representations) of locally compact groups.

Our purpose is to investigate the extent to which the situation is also true for bounded bornological space representations of bornological groups.

In this paper, we shall treat the induced bornological representations which is hereafter referred to as bornological modules, of bornological algebras.

§ 1. Bornological structure [1], [5].

Let S be a set. By a *bornological structure* on S we shall mean a family $\mathcal{B}$ of subsets of S such that

- (1) $B_1 \cup B_2 \in \mathcal{B}$ if $B_1, B_2 \in \mathcal{B}$,
- (2) $B' \in \mathcal{B}$ if $B' \subset B \in \mathcal{B}$,
- (3) S is the union of all the elements of $\mathcal{B}$.

A set S together with a bornological structure $\mathcal{B}$ we shall call a *bornological space* $(S, \mathcal{B})$. We shall often omit $\mathcal{B}$ and speak simply of the bornological space S . We call the members of $\mathcal{B}$ the *bounded sets* of S .

Let $(S_1, \mathcal{B}_1), (S_2, \mathcal{B}_2)$ be two bornological spaces. A mapping $m: S_1 \rightarrow S_2$ is said to be a *morphism* if $m(B) \in \mathcal{B}_2$ whenever $B \in \mathcal{B}_1$.

The family of all (B_1, B_2) , where $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$, defines a bornological structure in $S_1 \times S_2$. The product set $S_1 \times S_2$ with this bornological structure is called a *bornological product space* of $(S_1, \mathcal{B}_1)$ and $(S_2, \mathcal{B}_2)$.

Let E be a vector space over the real or complex field K . Suppose K is equipped with the canonical bornological structure which is defined by the absolute value.

A bornological structure in E is called a *vector bornological struc-*

ture if the following two conditions are satisfied:

- (1) $(x, y) \mapsto x + y$ is a morphism of $E \times E$ into E .
- (2) $(k, x) \mapsto kx$ is a morphism of $K \times E$ into E .

Where $E \times E$ and $K \times E$ are bornological product spaces.

A vector space E together with a vector bornological structure is called a *vector bornological space*. Let E, F be two vector bornological spaces. A mapping $m: E \rightarrow F$ is a *morphism* if m is linear and morphism for the bornological structure.

Furthermore, let E be an algebra. The bornological structure on E is an *algebraic bornological structure* if it is a vector bornological structure such that the multiplication of two bounded sets is bounded. Then such an E is called a *bornological algebra*.

A bornological algebra is called a *bornological *-algebra*, if an operation is defined in E which assigns to each element x in E an element x^* in E in such a way that following conditions are satisfied, for each $x, y \in E$

- (1) $(x^*)^* = x$,
- (2) $(x \circ y)^* = y^* \circ x^*$ where $\circ$ is a ring multiplication,
- (3) $(x + y)^* = x^* + y^*$,
- (4) $(kx)^* = \bar{k}x^*$ for every scalar k ,
- (5) mapping: $x \mapsto x^*$ is morphism.

Let $(E_1, \mathcal{B}_1), (E_2, \mathcal{B}_2)$ be vector bornological spaces. $M(E_1, E_2)$ is the vector space of all morphisms of E_1 into E_2 .

Now we define a bornological structure in $M(E_1, E_2)$ by the following way: a subset $B \subset M(E_1, E_2)$ is bounded if $B(S) = \bigcup_{m \in B} m(S)$ belongs to $\mathcal{B}_2$

for all $S \in \mathfrak{S}_1$. We write $M(E)$ in place of $M(E, E)$.

§ 2. Bornological modules.

In this section, we study vector bornological spaces which are module over a bornological algebra.

Definition 2.1. Let A be a bornological algebra. A *bornological left A -module* is a vector bornological space which is a left A -module in the algebraic sense such that the mapping: $(a, x) \mapsto ax$ of $A \times E \rightarrow E$ is a morphism and for each $a \in A$ the mapping: $x \mapsto ax$ is in the vector bornological space $M(E)$.

A *bornological right A -module* is defined in the analogous fashion.

Let A be a bornological algebra, E a bornological right A -module, F a bornological left A -module, and L a vector bornological space. We shall call a bilinear mapping f of $E \times F$ into L is *A -balanced* if $f(xa, y) = f(x, ay)$ for all $a \in A, x \in E, y \in F$.

We shall denote by $B(E, F; L)$ the set of all A -balanced bilinear mappings of $E \times F$ into L . We shall write $B(E, F)$ in place of $B(E, F; C)$, where C is the complex numbers field.

For each pair $(x, y) \in E \times F$, the mapping: $f \mapsto f(x, y)$ is a linear form on $B(E, F)$ and hence is an element $x \otimes y$ of the algebraic dual $B(E, F)^*$. The mapping $\chi: (x, y) \mapsto x \otimes y$ of $E \times F$ into $B(E, F)^*$ is bilinear and A -balanced. The linear span of $\chi(E, F)$ in $B(E, F)^*$ is called the *tensor product* of E and F , and is written $E \otimes_A F$.

We intend to study bornological structure in $E \otimes_A F$. Let $(E, \mathfrak{S}_E)$, $(F, \mathfrak{S}_F)$ and $(L, \mathfrak{S}_L)$ be vector bornological spaces. A bilinear mapping

f of $E \times F$ into L is said to be *morphism* if, for each $S \in \mathfrak{S}_E$ and each $T \in \mathfrak{S}_F$, $f(S \times T) \in \mathfrak{S}_L$.

Let $M_A(E, F; L)$ be the set of all A -balanced morphisms of bilinear mapping of $E \times F$ into L . Let H be a subset of $M_A(E, F; L)$. If for each $S \in \mathfrak{S}_E$ and each $T \in \mathfrak{S}_F$, there is a bounded set $U \in \mathfrak{S}_L$ such that $m(S \times T) \subset U$ for all m in H , then H is said to be an *equimorphism*. Let m be a linear mapping of $E \otimes_A F$ into L , then $m \circ \chi$ is a mapping of $E \times F$ into L which is A -balanced. Moreover the mapping $f: m \mapsto m \circ \chi$ is an algebraic isomorphism of the space of all linear mappings of $E \otimes_A F$ into L onto the space of all A -balanced bilinear mappings of $E \times F$ into L .

Lemma 2.1. [1] *Let $(E, \mathfrak{S}_E)$, $(F, \mathfrak{S}_F)$ be vector bornological spaces. There exists in $E \otimes_A F$ a unique bornological structure $\mathfrak{T}$ such that for each vector bornological space L the algebraic isomorphism f maps $M((E \otimes_A F, \mathfrak{T}), L)$ onto $M_A(E, F; L)$.*

Moreover, a subset H of $M((E \otimes_A F, \mathfrak{T}), L)$ is an equimorphism if and only if $f(H)$ is an equimorphism subset of $M_A(E, F; L)$.

Indeed, the family $\left\{ \overline{S \otimes T} \mid S \in \mathfrak{S}_E, T \in \mathfrak{S}_F \right\}$, where $S \otimes T = \left\{ s \otimes t \mid s \in S, t \in T \right\}$ and $\overline{S \otimes T}$ is the balanced hull of $S \otimes T$, is a base of the vector bornological space $E \otimes_A F$.

Definition 2.2. The space $E \otimes_A F$ with the vector bornological structure of Lemma 2.1 is called the *bornological tensor product*.

Let A, B be bornological algebras. A *bornological (A, B) -bimodule* E is a bornological left A -module which is also a bornological right

B -module and satisfies $a(xb) = (ax)b$ for all $a \in A$, $b \in B$ and $x \in E$.

Proposition 2.1. *Let E be a bornological (B, A) -bimodule and F a bornological left A -module. Then $E \otimes_A F$ is a bornological left B -module.*

Proof. As can be easily seen, for each $b \in B$, the mapping $(x, y) \mapsto bx \otimes y$ is an A -balanced bilinear morphism. By Lemma 2.1, there is a morphism f_b of $E \otimes_A F$ into itself such that $f_b(x \otimes y) = bx \otimes y$. We define

$$b(\sum x_i \otimes y_i) = f_b(\sum x_i \otimes y_i).$$

Then the mapping: $(b, u) \mapsto bu$ of $B \times (E \otimes_A F) \rightarrow E \otimes_A F$ is a morphism and for each $b \in B$ the mapping: $u \mapsto bu$ is in $M(E \otimes_A F)$, which follows immediately from the fact that the family $\left\{ \overline{S \otimes T} \mid S \in \mathcal{S}_E, T \in \mathcal{S}_F \right\}$ forms a base of the bornological tensor product $E \otimes_A F$ where $\mathcal{S}_E$ (resp. $\mathcal{S}_F$) is a bornological structure on E (resp. F).

Let $(E, \mathcal{S}_E)$, $(F, \mathcal{S}_F)$ be bornological left (resp. right) A -modules. A morphism m of E into F is called A -module if $m(ax) = a(m(x))$ (resp. $m(xa) = (m(x))a$) for all $a \in A$, $x \in E$.

Let $M_A(E, F)$ be the set of all A -module morphisms of E into F . We define a vector bornological structure on $M_A(E, F)$ by the following way: a subset $U \subset M_A(E, F)$ is called bounded if $U(S) = \bigcup_{u \in U} u(S)$ belongs to $\mathcal{S}_F$ for all $S \in \mathcal{S}_E$.

Proposition 2.2. *Let E_1 and F_1 be bornological right A -modules and E_2 and F_2 be bornological left A -modules. Moreover let $m_1: E_1 \rightarrow F_1$ and $m_2: E_2 \rightarrow F_2$ be A -module morphisms. Then there is a unique morphism $m_1 \otimes m_2$ of $E_1 \otimes_A E_2$ into $F_1 \otimes_A F_2$ such that*

$$(m_1 \otimes m_2)(x_1 \otimes x_2) = m_1(x_1) \otimes m_2(x_2).$$

If, in addition, E_1 and F_1 are bornological (B, A) -bimodule and m_1 is also a B -module morphism, then $m_1 \otimes m_2$ is a B -module morphism.

Proof. The mapping: $(x_1, x_2) \mapsto m_1(x_1) \otimes m_2(x_2)$ is A -balanced and morphism, so the existence and uniqueness of the morphism $m_1 \otimes m_2$ follow immediately from Lemma 2.1.

The second statement is obtained by an easy calculation. Indeed, for each $b \in B$,

$$\begin{aligned} (m_1 \otimes m_2)(b(x_1 \otimes x_2)) &= (m_1 \otimes m_2)(bx_1 \otimes x_2) \\ &= m_1(bx_1) \otimes m_2(x_2) \\ &= (bm_1(x_1)) \otimes m_2(x_2) \\ &= b(m_1(x_1) \otimes m_2(x_2)) \\ &= b((m_1 \otimes m_2)(x_1 \otimes x_2)). \end{aligned}$$

Proposition 2.3. Let $(E, \mathcal{E}_E)$ be a bornological (A, B) -bimodule and $(F, \mathcal{E}_F)$ be a bornological left A -module, where A (resp. B) is a bornological algebra with a bornological structure $\mathcal{B}$ (resp. $\mathcal{C}$). Then $M_A(E, F)$ is a bornological left B -module.

Proof. For $m \in M_A(E, F)$ and $b \in B$, define bm by

$$bm(x) = m(xb).$$

Then bm is an A -module morphism, i.e.

$$bm(ax) = m(axb) = am(xb) = abm(x).$$

Since the mapping: $x \mapsto xb$ is a morphism, bm is a morphism.

To show that the mapping: $(b, m) \mapsto bm$ of $B \times M_A(E, F)$ into $M_A(E, F)$ is a morphism, let V be a bounded set in $M_A(E, F)$. If $W \in \mathcal{C}$ then

$$WV(S) = \bigcup_{m \in V} Wm(S) = \bigcup_{m \in V} m(SW) = V(SW)$$

belongs to $\mathcal{E}_F$ for each $S \in \mathcal{E}_E$ and hence so WV is a bounded set in

$M_A(E, F)$.

Proposition 2.4. *Let E be a bornological (B, A) -bimodule, F and L be bornological left B -modules. For each $n \in M_B(F, L)$, define a mapping $n_*: M_B(E, F) \rightarrow M_B(F, L)$ by*

$$n_*(m) = n \circ m.$$

Then n_ is an A -module morphism.*

Proof. For each $a \in A$, $m \in M_B(E, F)$ and $n \in M_B(F, L)$, we define

$$am(x) = m(ax).$$

Then

$$an_*(x) = an(m(x)) = n(am(x))$$

for each $x \in E$.

We use [2] as standard references for categorical results.

Let A (resp. B) be the category of bornological left A -modules (resp. left B -modules) and A -module (resp. B -module) morphisms, where A (resp. B) is a bornological algebra. Let E be a bornological (B, A) -bimodule.

Propositions 2.1 - 2.4 tell us that we can define functors $E \otimes_A -: A \rightarrow B$ and $\text{mor}_B(E, -): B \rightarrow A$ as follows. For objects F, L in A and $m \in M_A(F, L)$ define

$$(E \otimes_A -)(F) = (E \otimes_A F)$$

and

$$(E \otimes_A -)(m) = 1_E \otimes m,$$

where 1_E notes the identity mapping: $E \rightarrow E$.

For objects H, K in B and $n \in M_B(H, K)$ define

$$\text{mor}_B(E, -)(H) = M_B(E, H)$$

and

$$\text{mor}_B(E, -)(n) = n_*.$$

Theorem 2.1. $E \otimes_A -$ is the adjoint of $\text{mor}_B(E, -)$. That is, for each bornological left A -module $(F, \mathfrak{S}_F)$ and for each bornological left B -module $(L, \mathfrak{S}_L)$, there is a natural isomorphism

$$f_{FL}: M_B(E \otimes_A F, L) \rightarrow M_A(F, \text{mor}_B(E, L)).$$

Proof. For $m \in M_B(E \otimes_A F, L)$, define f_{FL}^m by

$$f_{FL}^m(y)(x) = m(x \otimes y)$$

for each $x \in E, y \in F$. Let χ be the canonical mapping: $E \times F \rightarrow E \otimes_A F$.

Then

$$f_{FL}^m(y)(x) = m \circ \chi(x, y)$$

and $m \circ \chi$ is a morphism of $E \times F \rightarrow L$ by Lemma 2.1. It follows easily that $f_{FL}^m(y) \in \text{mor}_B(E, L) (= M_B(E, L))$.

To show that f_{FL}^m is a morphism, let $T \in \mathfrak{S}_F$. Since $m \circ \chi$ is a morphism, $f_{FL}^m(T)$ is a bounded set in $\text{mor}_B(E, L)$.

Let $m' \in M_A(F, \text{mor}_B(E, L))$, and define a bilinear mapping $m^*: E \times F \rightarrow$ by $m^*(x, y) = m'(y)(x)$. The m^* is A -balanced, because $m^*(ax, y) = m'(y)(ax) = m'(\alpha y)(x) = m^*(x, \alpha y)$. Since $m'(V)(U)$ ($V \in \mathfrak{S}_F$ and $U \in \mathfrak{S}_E$, belongs to $\mathfrak{S}_L$), m^* is a morphism of $E \times F \rightarrow L$.

By Lemma 2.1 there is an $m \in M_B(E \otimes_A F, L)$ such that $m \circ \chi = m^*$. It follows easily that $f_{FL}^m = m'$, and that $m \in M_B(E \otimes_A F, L)$ since $m(b(x \otimes y)) = m(bx \otimes y) = m^*(bx, y) = m'(y)(bx) = bm'(y)(x) = bm(x \otimes y)$.

It is obvious that the one-to-one of the mapping f_{FL} .

Let A be a bornological algebra with a unit element and consider A as a bornological (B, A) -bimodule, where B is a unitary subalgebra of A .

We have the following proposition by Lemma 2.1.

Proposition 2.5. *Let E be a bornological left A -module. Then the mapping: $a \otimes x \mapsto ax$ is a bornological isomorphism of $A \otimes_A E \rightarrow E$.*

In the next proposition we consider A as a bornological (A, B) -bimodule.

Proposition 2.6. *Let $(E, \mathfrak{S})$ be a bornological left A -module. Then there is a B -module bornological isomorphism $f_E: E \rightarrow M_A(A, E)$.*

Proof. For each $x \in E$, let f_E be defined by

$$f_E(x)(a) = ax.$$

It is easily seen that $f_E(x) \in M_A(A, E)$ and that f_E is a B -module linear mapping. Moreover, for each $S \in \mathfrak{S}$, $U \in \mathfrak{T}$, where $\mathfrak{T}$ is a bornological structure in A , $f_E(S)(U) = SU$ is a bounded set of E , i.e. a elements of $\mathfrak{S}$. Thus $f_E(x) \in M_A(A, E)$.

Now define $g_E: M_A(A, E) \rightarrow E$ by

$$g_E(m) = m(e)$$

where e is the unit element of A . It is obvious that g_E is a morphism.

It is straightforward to conclude that g_E is a B -module morphism and $f_E \circ g_E, g_E \circ f_E$ are identity mapping.

We shall now prove the associativity of tensor products.

Proposition 2.7. *Let E be a bornological (A, B) -bimodule, F a bornological right A -module and L a bornological left B -module. Then there is a unique natural bornological isomorphism*

$$m_{FL}: (F \otimes_A E) \otimes_B L \rightarrow F \otimes_A (E \otimes_B L)$$

such that $m_{FL}((x \otimes y) \otimes z) = x \otimes (y \otimes z)$.

Proof. For each $z \in L$, $n_z: y \mapsto y \otimes z$ is an A -module morphism of $E \rightarrow$

$E \otimes_A L$. Put $m_z = \ell_z \otimes n_z$, then m_z is a morphism of $F \otimes_A E$ into $F \otimes_A (E \otimes_B L)$ by Proposition 2.2. The mapping: $(x, z) \mapsto m_z(x)$ of $(F \otimes_A E) \times L$ into $F \otimes_A (E \otimes_B L)$ is an A -balanced bilinear morphism, thus, by Lemma 2.1, there is a unique morphism

$$m_{FL}: (F \otimes_A E) \otimes_B L \rightarrow F \otimes_A (E \otimes_B L)$$

such that $m_{FL}(x \otimes z) = m_z(x)$, which immediately gives $m_{FL}((x \otimes y) \otimes z) = x \otimes (y \otimes z)$.

In an analogous manner one can define a morphism

$$n_{FL}: F \otimes_A (E \otimes_B L) \rightarrow (F \otimes_A E) \otimes_B L$$

such that $n_{FL}(x \otimes (y \otimes z)) = (x \otimes y) \otimes z$.

It follows that $m_{FL} \circ n_{FL}$ and $n_{FL} \circ m_{FL}$ are identity mappings. The remainder of the proof is straightforward.

We close this section with a result which will be used later. Let A be a bornological $*$ -algebra. Let E be a bornological left A -module. Let E^b be the bornological dual of E , i.e. $E^b = M(E, K)$, where K is a real or complex field with the canonical bornological structure which is defined by the absolute value.

For $f \in E^b$ and $a \in A$ we define af by

$$af(x) = f(ax).$$

Then we have

Proposition 2.8. E^b is a bornological left A -module.

§ 3. Induced representations of bornological left A -module.

In this section, we assume that A is a bornological algebra with a unit element e , and B is a unitary subalgebra of A .

Let E be a bornological left A -module, then by restricting the mapping: $(a, x) \mapsto ax$ to $B \times E$, E is a bornological left B -module which we shall write E_B . In this way we obtain a functor from the category of bornological left A -modules which we shall call the *restriction functor*. We apply the results of section 2 in order to show that the restriction functor has both an adjoint and a coadjoint.

Theorem 3.1. *The functor $A \otimes_B -$ is the adjoint of the restriction functor, and the functor $\text{mor}_B(A, -)$ is the coadjoint.*

Proof. By Theorem 2.1, we have the bornological isomorphism

$$M_A(A \otimes_B E, F) \cong M_B(E, \text{mor}_B(A, F)).$$

By Proposition 2.6, $\text{mor}_A(A, F)$ is bornologically isomorphic to F_B and it is easy to see that this isomorphism is natural. This shows that $A \otimes_B -$ is the adjoint of the restriction functor. Again by Theorem 2.1, we have the bornological isomorphism

$$M_B(A \otimes_A E, F) = M_A(E, \text{mor}_B(A, F)).$$

By Proposition 2.5, $A \otimes_A E$ is bornologically isomorphic to E_B and it is seen that this isomorphism is natural. Thus $\text{mor}_B(A, -)$ is the coadjoint of the restriction functor.

From now on, we shall write ${}^A E$ in place of $A \otimes_B E$ and E^A in place of $\text{mor}_B(A, E)$. With this notation the results of Theorem 3.1 can be written as follows

$$M_A({}^A E, F) \cong M_B(E, F_B)$$

and

$$M_A(F, E^A) \cong M_B(F_B, E).$$

Corollary. Let A be a bornological algebra and B, C unitary subalgebras of A . Let E be a bornological left B -module and F a bornological left C -module. Then

$$M_C(({}^A E)_C, F) \cong M_B(E, (F^A)_B).$$

The next theorem concerns induction in stages.

Theorem 3.2. Let C be a unitary subalgebra of B . Then

$${}^A E \cong {}^A ({}^B E) \quad \text{and} \quad E^A \cong (E^B)^A$$

for any bornological left C -module E , and the isomorphism is natural.

Proof. Considering B as a bornological (B, C) -bimodule, we have the bornological isomorphism

$$\begin{aligned} {}^A ({}^B E) &\cong A \otimes_B (B \otimes_C E) \\ &\cong (A \otimes_B B) \otimes_C E && \text{(by Proposition 2.7)} \\ &\cong A \otimes_C E && \text{(by Proposition 2.5)} \end{aligned}$$

which proves the first assertion.

To prove the second assertion define a mapping

$$f_E: E^A = \text{mor}_C(A, E) \rightarrow \text{mor}_B(A, \text{mor}_C(B, E)) = (E^B)^A$$

by $f_E^m(a)(b) = m(ba)$ where $m \in E^A$, $a \in A$, $b \in B$. Since ring multiplication is bounded, we have $f_E^m(a) \in \text{mor}_C(B, E)$, and for each bounded set U of A and for each bounded set W of B , $f_E^m(U)(W) = m(WU)$ is a bounded set in E and so f_E^m is a morphism.

We now show that f_E is a morphism. For any bounded set V of $\text{mor}_C(A, E)$, $f_E^V(U)(W) = \bigcup_{m \in V} m(WU)$ is a bounded set in E where U (resp. W) is a bounded set of A (resp. B). This shows that f_E is a morphism.

Now define a mapping $g_E: (E^B)^A \rightarrow E^A$ by $g_E^m(a) = m(a)e$, $m \in (E^B)^A$, $a \in A$, and e is the unit element of A . It is clear that the mapping is well

defined. By methods analogous to the proof that f_E is a morphism, it follows that the mapping g_E is a morphism.

Observe that $f_E \circ g_E$ and $g_E \circ f_E$ are identity mapping and the remainder of the proof is due to a simple algebraic argument.

Let A be a bornological $*$ -algebra and E a bornological left A -module. By Proposition 2.8, E^b is a bornological left A -module.

Proposition 3.1. *Let $(A, \mathfrak{C})$ be a bornological $*$ -algebra with a unit element, and B a unitary $*$ -subalgebra of A . Considering A as a bornological left B -module via the mapping: $(b, a) \rightarrow ab^*$, and as a bornological right B -module via that mapping: $(a, b) \rightarrow ab$, for any bornological left B -module $(E, \mathfrak{C})$,*

$$(E^b)^A \cong ({}^A E)^b$$

is a bornological isomorphism.

Proof. For each $a \in A$, $x \in E$ and $m' \in (A \otimes_B E)^b$, put

$$f_E^{m'}(a)(x) = m'(a \otimes x).$$

It follows by methods analogous to the proof of Theorem 3.1 that f_E is a natural algebraic isomorphism of $(A \otimes_B E)^b$ onto $\text{mor}_B(A, E^b)$.

To show that f_E is a morphism, let V be a bounded set of $(A \otimes_B E)^b$. If $U \in \mathfrak{C}$ and $S \in \mathfrak{C}$, then

$$f_E^V(U)(S) = V(U \otimes S) = \bigcup_{m \in V} m(U \otimes S)$$

is a bounded set of scalar field K with the canonical bornological structure, since $U \otimes S$ is a bounded set of $A \otimes_B E$.

By an analogous manner defines a morphism

$$g_E: \text{mor}_B(A, E^b) \rightarrow (A \otimes_B E)^b$$

such that $g_E^m(a \otimes x) = m(a)(x)$. It follows that $f_E \circ g_E$ and $g_E \circ f_E$

are identity mappings. Then the remainder of the proof is straightforward.

Definition 3.1. Let A be a bornological $*$ -algebra and let E, F be bornological left A -modules. An *intertwining form* for E and F is a bilinear form f on $E \times F$ such that

$$f(ax, y) = f(x, a^*y).$$

Let $J(E, F)$ be the set of all bounded intertwining forms on $E \times F$. For $f \in J(E, F)$ let $m: F \rightarrow E^b$ be defined by

$$\langle x, my \rangle = f(x, y).$$

It follows easily that m is a A -module morphism. Conversely, each $m \in \text{mor}_A(F, E^b)$ can be written the above form. Then we have

$$J(E, F) \cong \text{mor}_A(F, E^b).$$

Theorem 3.3. Let A be a bornological $*$ -algebra with unit element, and B a unitary $*$ -subalgebra of A . If E is a bornological left B -module and F a bornological left A -module, then

$$J({}^A E, F) \cong J(E, F_B).$$

Proof. This follows from Theorem 3.1 and Proposition 3.1.

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