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PDF issue: 2026-11-16

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(Citation)
Mathematics Seminar Notes,2:11-13

(Issue Date)
1974

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/0100491896

(URL)
https://hdl.handle.net/20.500.14094/0100491896

KOBE UNIVERSITY

MATHEMATICS SEMINAR NOTES
Vol. 2 (1974) January 1974

# Fixed Point Theorem in Banach Spaces

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There are many generalizations of the classical contraction mapping theorem of S. Banach (see the bibliography of I. A. Rus [2]).

On the other hand, recently some fixed point theorems on non-contraction mappings on a Banach space have been obtained by several mathematicians. One of them is due to K. Goebel and E. Zlotkiewicz [1].

In this paper, we shall generalize a fixed point theorem by K. Goebel and E. Zlotkiewicz [1].

**Theorem.** *Let $F$ be a mapping of a Banach space $X$ into itself. If $F$ satisfies conditions*

1) $F^2 = I$, *where $I$ is the identity mapping,*

2) $\|F(x) - F(y)\| \leq \alpha\|x - y\| + \beta(\|x - F(x)\| + \|y - F(y)\|)$

*for every $x, y \in X$, where $0 \leq \alpha$, $\beta$ and $0 < \alpha + 4\beta < 2$, then $F$ has at least one fixed point.*

*Proof.* Let $x$ be a point of $X$, and put

$$y = \frac{1}{2}(F + I)(x). \quad z = F(y), \quad u = 2y - z.$$

Then, by conditions 1), 2), we have

$$\|z - x\| = \|F(y) - x\|$$

$$= \|F(y) - F^2(x)\| \leq \alpha\|y - F(x)\| + \beta(\|y - F(y)\|$$

$$+ \|F(x) - F^2(x)\|) < \alpha\|\tfrac{1}{2}(F + I)(x) - F(x)\|$$
$$+ \beta(\|x - F(x)\| + \|y - F(y)\|)$$
$$= (\tfrac{\alpha}{2} + \beta)\|x - F(x)\| + \beta\|y - F(y)\|,$$

and

$$\|u - x\| = \|2y - F(y) - x\|$$
$$= \|(F + I)(x) - F(y) - x\| \le \|F(x) - F(y)\| \le \alpha\|x - y\|$$
$$+ \beta(\|x - F(x)\| + \|y - F(y)\|$$
$$= \alpha\|x - \tfrac{1}{2}(F + I)(x)\| + \beta(\|x - F(x)\| + \|y - F(y)\|)$$
$$= (\tfrac{\alpha}{2} + \beta)\|x - F(x)\| + \beta\|y - F(y)\|.$$

Hence

$$\|z - u\| \le (\alpha + 2\beta)\|x - F(x)\| + 2\beta\|y - F(y)\|.$$

On the other hand, we have

$$\|z - u\| = 2\|y - F(y)\|.$$

Therefore

$$2\|y - F(y)\| \le (\alpha + 2\beta)\|x - F(x)\| + 2\beta\|y - F(y)\|,$$

which implies

$$2(1 - \beta)\|y - F(y)\| \le (\alpha + 2\beta)\|x - F(x)\|.$$

Hence we have the following inequality.

(1) $\|y - F(y)\| \le \dfrac{\alpha + 2\beta}{2(1 - \beta)}\|x - F(x)\|$.

Let $G = \frac{1}{2}(F + I)$, then, for any $x \in X$, $\|G^2(x) - G(x)\| = \|G(y) - y\|$

$$= \|\tfrac{1}{2}(F + I)(y) - y\|$$
$$= \tfrac{1}{2}\|y - F(y)\| \le \tfrac{1}{2}\,\frac{\alpha + 2\beta}{2(1 - \beta)}\|x - F(x)\|$$
$$= \tfrac{1}{2}\,\frac{\alpha + 2\beta}{2(1 - \beta)}\|x - (2G(x) - x)\| = \frac{\alpha + 2\beta}{2(1 - \beta)}\|G(x) - x\|.$$

By the hypothese, we have $0 \le \dfrac{\alpha + 2\beta}{2(1 - \beta)} < 1$. Therefore $\{G^n(x)\}$ is a Cauchy sequence in $X$. By the completeness, $\{G^n(x)\}$ converges to some element $x_0$ in $X$, i.e. $\lim_{n\to\infty} G^n(x) = x_0$. This implies $G(x_0) = x_0$.

Hence $F(x_0) = x_0$, i. e. $x_0$ is a fixed point of $F$, which completes the proof.

Remark 1. If $\beta = 0$, then we have Theorem 1 of K. Goebel and E. Zlotkiewicz [1].

## References

[1] K. Goebel and E. Zlotkiewicz: Some fixed point theorems in Banach spaces, Colloquium Math., 23 (1971), 103-106.

[2] I. A. Rus: Theoria punctului fix, II, Universitatea Babes-Bolyai din Cluj, 1973.