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On Common Fixed Points of Mappings

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Recently B. Ray [1] and C. S. Wong [3] proved some interesting theorems about common fixed points of mappings of a complete metric space in itself. In this note, we shall prove some theorems about common fixed points which are generalizations of results in B. Ray [1].

Theorem 1. *Let* X *be a complete metric space,* T_n $(n = 1, 2, \cdots)$ *a sequence of mappings of* X *into itself. Suppose that there are non-negative numbers* α, β, γ *such that for* $x, y \in X$

$$\rho(T_i(x), T_j(y)) \leq \alpha(\rho(x, T_i(x)) + \rho(y, T_j(y))) + \beta(\rho(x, T_j(y)) + \rho(y, T_i(x))) + \gamma\rho(x, y),$$

where $2\alpha + 2\beta + \gamma < 1$. *Then the sequence of mappings* $\{T_n\}$ *has a unique common fixed point.*

Proof. Let $x_0 \in X$, put

$$x_n = T_n(x_{n-1}) \qquad (n = 1, 2, \cdots),$$

then we have

$$\begin{aligned}\rho(x_1, x_2) &= \rho(T_1(x_0), T_2(x_1)) \\ &\leq \alpha(\rho(x_0, x_1) + \rho(x_1, x_2)) + \beta(\rho(x_0, x_2) + \rho(x_1, x_1)) + \gamma\rho(x_0, x_1) \\ &= (\alpha + \gamma)\rho(x_0, x_1) + \alpha\rho(x_1, x_2) + \beta\rho(x_0, x_2)\end{aligned}$$

$$\leq (\alpha + \gamma)\rho(x_0, x_1) + \alpha\rho(x_1, x_2) + \beta(\rho(x_0, x_1) + \rho(x_1, x_2)).$$

Hence

$$\rho(x_1, x_2) \leq \frac{\alpha + \beta + \gamma}{1 - \alpha - \beta}\rho(x_0, x_1).$$

Similarly we have

$$\rho(x_2, x_3) = \rho(T(x_1), T(x_2))$$

$$\leq (\alpha + \gamma)\rho(x_1, x_2) + \alpha\rho(x_2, x_3) + \beta(\rho(x_1, x_2) + \rho(x_2, x_3)).$$

Therefore, we have

$$\rho(x_2, x_3) \leq \frac{\alpha + \beta + \gamma}{1 - \alpha - \beta}\rho(x_1, x_2).$$

In general, we have

$$\rho(x_n, x_{n+1}) \leq (\frac{\alpha + \beta + \gamma}{1 - \alpha - \beta})^n \rho(x_0, x_1).$$

This means that the sequence $\{x_n\}$ is a Cauchy sequence. Hence, by the completeness of X, $\{x_n\}$ converges to some point x in X. For the point x,

$$\begin{aligned}
\rho(x, T_n(x)) &\leq \rho(x, x_{m+1}) + \rho(x_{m+1}, T_n(x)) \\
&= \rho(x, x_{m+1}) + \rho(T_{m+1}(x_m), T_n(x)) \\
&\leq \rho(x, x_{m+1}) + \alpha(\rho(x_m, T_{m+1}(x_m)) + \rho(x, T_n(x))) \\
&\quad + \beta(\rho(x_m, T_n(x)) + \rho(x, T_{m+1}(x_m))) + \gamma\rho(x_m, x) \\
&= \rho(x, x_{m+1}) + \alpha(\rho(x_m, x_{m+1}) + \rho(x, T_n(x))) \\
&\quad + \beta(\rho(x_m, T_n(x)) + \rho(x, x_{m+1})) + \gamma\rho(x_m, x).
\end{aligned}$$

Letting $m \to \infty$, then we have

$$\rho(x, T_n(x)) \leq (\alpha + \beta)\rho(x, T_n(x)).$$

Therefore $\rho(x, T_n(x)) = 0$, i. e. the point x is a common fixed point of all T_n.

To show that x is a unique common fixed point of all T_n, we consider a point y in X such that $T_n(y) = y$ for every n. Then we have

$$\rho(x, y) = \rho(T_n(x), T_n(y))$$

$$\leq \alpha(\rho(x, T_n(x)) + \rho(y, T_n(y))) + \beta(\rho(x, T_n(y)) + \rho(y, T_n(x)))$$
$$+ \gamma\rho(x, y)$$
$$= (2\beta + \gamma)\rho(x, y).$$

Hence $\rho(x, y) = 0$, i. e. $x = y$. This completes the proof of Theorem 1.

Theorem 2. *Let* $\{T_n\}$ *be a sequence of mappings of a complete metric space* X *into itself. Let* x_n *be a fixed point of* T_n $(n = 1, 2, \cdots)$, *and suppose that* T_n *converges uniformly to* T_0. *If* T_0 *satisfies the condition*

(1) $\rho(T_0(x), T_0(y))$
$\leq \alpha(\rho(x, T_0(x)) + \rho(y, T_0(y)))$
$+ \beta(\rho(x, T_0(y)) + \rho(y, T_0(x))) + \gamma\rho(x, y)$,

where α, β, γ *are non-negative and* $2\alpha + 2\beta + \gamma < 1$, *then* $\{x_n\}$ *converges to the fixed point* x_0 *of* T_0.

Under the condition (1), T_0 has a unique fixed point by a result of L. B. Ćirić (see I. A. Rus [2], p. 21.).

Proof. Let be give $\varepsilon > 0$, then there is a natural number N such that

(2) $\rho(T_n(x), T_0(x)) < \varepsilon$

for all $x \in X$ and $N \leq n$. Hence

$$\rho(x_n, x_0) = \rho(T_n(x_n), T_0(x_0))$$
$$\leq \rho(T_n(x_n), T_0(x_n)) + \rho(T_0(x_n), T_0(x_0))$$
$$\leq \rho(T_n(x_n), T_0(x_n)) + \alpha(\rho(x_n, T_0(x_n)) + \rho(x_0, T_0(x_0)))$$
$$+ \beta(\rho(x_n, T_0(x_0)) + \rho(x_0, T_0(x_n))) + \gamma\rho(x_n, x_0)$$
$$\leq \rho(T_n(x_n), T_0(x_n)) + (\alpha + \beta)\rho(T_n(x_n), T_0(x_n))$$
$$+ (\alpha + \beta)(\rho(x_0, x_n) + \rho(T_n(x_n), T_0(x_n))) + \gamma\rho(x_n, x_0)$$

$$= (1 + 2(\alpha + \beta))\rho(T_n(x_n), T_0(x_n)) + (\alpha + \beta + \gamma)\rho(x_n, x_0).$$

Hence

$$(1 - (\alpha + \beta + \gamma))\rho(x_n, x_0) \leq (1 + 2(\alpha + \beta))\rho(T_n(x_n), T_0(x_n)).$$

From the hypothese, $2(\alpha + \beta) + \gamma < 1$. Hence, for $n \geq N$, we have

$$\rho(x_n, x_0) \leq \frac{1 + 2(\alpha + \beta)}{1 - (\alpha + \beta + \gamma)}\varepsilon.$$

which shows that $\{x_n\}$ converges to x_0. We complete the proof.

Theorem 3. *Let* T_n $(n = 1, 2, \cdots)$ *be a sequence of mappings with fixed point* x_n *of a metric space* X *into itself. Suppose that*

$$(3)\quad \rho(T_n(x), T_n(y)) \leq \alpha(\rho(x, T_n(x)) + \rho(y, T_n(y))) + \beta(\rho(x, T_n(y) + \rho(y, T_n(x))) + \gamma\rho(x, y),$$

where α, β, γ *are non-negative and* $2\alpha + 2\beta + \gamma < 1$. *If* $\{T_n\}$ *converges to a mapping* T_0, *and* x_0 *is an accumulation point of* $\{x_n\}$, *then* x_0 *is fixed point of* T_0.

Proof. Since x_0 is an accumulation point of the set $\{x_n\}$, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges to x_0.

$$\rho(x_0, T_0(x_0)) \leq \rho(x_0, T_{n_i}(x_{n_i})) + \rho(T_{n_i}(x_{n_i}), T_{n_i}(x_0)) + \rho(T_{n_i}(x_0), T_0(x_0)).$$

Let $\varepsilon > 0$, then there is a natural number N such that

$$\rho(x_0, x_{n_i}) < \varepsilon$$

$$\rho(T_{n_i}(x_0), T_0(x_0)) < \varepsilon$$

for $N \leq n_i$. Hence for $N \leq n_i$, we have

$$(4)\quad \rho(x_0, T_0(x_0)) < 2\varepsilon + \rho(T_{n_i}(x_{n_i}), T_{n_i}(x_0)).$$

To estimate $\rho(T_{n_i}(x_{n_i}), T_{n_i}(x_0))$, we use the condition (3). Then

$$\rho(T_{n_i}(x_{n_i}), T_{n_i}(x_0)) \leq \alpha(\rho(x_{n_i}, T_{n_i}(x_{n_i})) + \rho(x_0, T_{n_i}(x_0))) + \beta(\rho(x_0, T_{n_i}(x_{n_i})) + \rho(x_{n_i}, T_{n_i}(x_0))) + \gamma\rho(x_{n_i}, x_0).$$

For $N \leq n_i$, we have

$$\rho(T_{n_i}(x_{n_i}), T_{n_i}(x_0)) \leq \alpha\rho(x_0, T_{n_i}(x_0)) + (\beta + \gamma)\varepsilon + \beta\rho(x_{n_i}, T_{n_i}(x_0)).$$

Hence

$$(5)\quad (1 - \beta)\rho(x_{n_i}, T_{n_i}(x_0)) \leq \alpha\rho(x_0, T_{n_i}(x_0)) + (\beta + \gamma)\varepsilon.$$

Next consider $\rho(x_0, T_{n_i}(x_0))$, then

$$\rho(x_0, T_{n_i}(x_0)) \leq \rho(x_0, x_{n_i}) + \rho(x_{n_i}, T_{n_i}(x_0)).$$

For $N \leq n_i$, we have

$$(6)\qquad \rho(x_0, T_{n_i}(x_0)) \leq \varepsilon + \rho(x_{n_i}, T_{n_i}(x_0)).$$

(5) and (6) imply

$$(1 - \beta)\rho(x_0, T_{n_i}(x_0)) \leq (1 - \beta)\varepsilon + \alpha\rho(x_0, T_{n_i}(x_0)) + (\beta + \gamma)\varepsilon.$$

Hence

$$(7)\qquad \rho(x_0, T_{n_i}(x_0)) \leq \frac{1 + \gamma}{1 - \alpha - \beta}\varepsilon.$$

From (4), (5) and (7), we have

$$\rho(x_0, T_0(x_0)) \leq 2\varepsilon + \frac{1}{1 - \beta}(\alpha\rho(x_0, T_{n_i}(x_0)) + (\beta + \gamma)\varepsilon)$$

$$\leq \left[2 + \frac{1}{1 - \beta}\left(\frac{1 + \gamma}{1 - \alpha - \beta} + (\beta + \gamma)\right)\right]\varepsilon.$$

This shows that x_0 is a fixed point of T_0. We complete the proof.

References

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