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Closeness Spaces and Convergence Spaces

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The purpose of this note is to show that every convergence structure ("Limitierung" of Fischer [2]) can be described by a family, called a *closeness*, of closure-like operations.

After stating several elementary properties of operations on the power set of a set, we shall introduce new notions "closeness" and "closeness space". Then some fundamental relations between closenesses and convergence structures will be established.

In what follows, the power set of a set X will be denoted by $\mathfrak{P}(X)$, and the value of a mapping $\alpha:\mathfrak{P}(X) \to \mathfrak{P}(X)$ at $A \in \mathfrak{P}(X)$ by A^{α}. The complement of $A \in \mathfrak{P}(X)$ in X will be written A^{c}. For each $x \in X$, $\dot{x}$ denotes the filter on X consisting of all $A \in \mathfrak{P}(X)$ with $x \in A$.

1. Throughout this section X denotes an arbitrary set. Let α be a mapping of $\mathfrak{P}(X)$ into itself. For each $x \in X$, we denote by $\Phi_{\alpha}(x)$ the set of all $A \in \mathfrak{P}(X)$ such that $x \notin A^{c\alpha}$. Evidently Φ_{α} is a mapping of X into $\mathfrak{P}\mathfrak{P}(X) = \mathfrak{P}(\mathfrak{P}(X))$.

The following four lemmas may be easily verified, and we omit the proofs.

Lemma 1. *Let α be a mapping of $\mathfrak{P}(X)$ into itself and let $x \in X$. Then the following statements hold:*

(1) $\Phi_\alpha(x) \neq \emptyset$ *if and only if* x *does not belong to* $\bigcap \{A^\alpha \mid A \in \mathfrak{P}(X)\}$.

(2) $\emptyset \notin \Phi_\alpha(x)$ *if and only if* $x \in X^\alpha$.

Lemma 2. *Let* α *be a monotone mapping of* $\mathfrak{P}(X)$ *into itself. Then* $x \in \{x\}^\alpha$ *for every* $x \in X$ *if and only if* $A \subset A^\alpha$ *for every* $A \in \mathfrak{P}(X)$.

Lemma 3. *Let* α *be a monotone mapping of* $\mathfrak{P}(X)$ *into itself, and let* $A \in$ (X). *Then* $x \in A^\alpha$ *if and only if* $S \cap A \neq \emptyset$ *for every* $S \in \Phi_\alpha(x)$.

Lemma 4. *Let* α, β *be two monotone mappings of* $\mathfrak{P}(X)$ *into itself. Then* $\Phi_\alpha(x) \subset \Phi_\beta(x)$ *for every* $x \in X$ *if and only if* $A^\beta \subset A^\alpha$ *for every* $A \in \mathfrak{P}(X)$.

Let Ψ be a mapping of X into $\mathfrak{P}\mathfrak{P}(X)$. For each $A \in \mathfrak{P}(X)$, we denote by $A^{\kappa(\Psi)}$ the set of all $x \in X$ for which we have $S \cap A \neq \emptyset$ for every $S \in \Psi(x)$. Obviously $\kappa(\Psi)$ is a monotone mapping of $\mathfrak{P}(X)$ into itself. Conversely, as an immediate consequence of Lemma 3, we have the following

Lemma 5. *If* α *is a monotone mapping of* $\mathfrak{P}(X)$ *into itself, then* $\alpha = \kappa(\Phi_\alpha)$.

Now for each subset $\mathscr{A}$ of $\mathfrak{P}(X)$, we denote by $[\mathscr{A}]$ the set of all $S \in \mathfrak{P}(X)$ containing at least one member of $\mathscr{A}$.

Lemma 6. *Let* Ψ *be a mapping of* X *into* $\mathfrak{P}\mathfrak{P}(X)$. *Then*

$$\Phi_{\kappa(\Psi)}(x) = [\Psi(x)] \qquad \text{for every } x \in X.$$

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*) A mapping α of $\mathfrak{P}(X)$ into itself is called *monotone* if $A \subset B$ implies $A^\alpha \subset B^\alpha$ for every $A, B \in \mathfrak{P}(X)$.

Proof. Clearly $A \in \Phi_{K(\Psi)}(x)$ is equivalent to the fact that $S \cap A^{c} = \emptyset$ for some $S \in \Psi(x)$, and $S \cap A^{c} = \emptyset$ if and only if $S \subset A$.

By virtue of Lemma 5 and Lemma 6, we have at once the following

Corollary. *If* α *is a monotone mapping of* $\mathfrak{P}(X)$ *into itself, then* $[\Phi_{\alpha}(x)] = \Phi_{\alpha}(x)$ *for every* $x \in X$.

Lemma 7. *Let* α *be a mapping of* $\mathfrak{P}(X)$ *into itself. If* $(A \cup B)^{\alpha} = A^{\alpha} \cup B^{\alpha}$ *for every* $A, B \in \mathfrak{P}(X)$, *then* $\Phi_{\alpha}(x)$ *is a filter on* X *for each* $x \in X^{\alpha} \setminus \emptyset^{\alpha}$.

Proof. Let $x \in X^{\alpha} \setminus \emptyset^{\alpha}$. Then by Lemma 1, the set $\Phi_{\alpha}(x)$ is nonempty and $\emptyset \notin \Phi_{\alpha}(x)$. On the other hand, the mapping α is monotone as can readily be seen. Hence according to the above Corollary we have $[\Phi_{\alpha}(x)] = \Phi_{\alpha}(x)$. Now if $A, B \in \Phi_{\alpha}(x)$, then since $x \notin A^{c\alpha}$ and $x \notin B^{c\alpha}$, we have

$$x \notin A^{c\alpha} \cup B^{c\alpha} = (A^{c} \cup B^{c})^{\alpha} = (A \cap B)^{c\alpha},$$

which shows that $A \cap B \in \Phi_{\alpha}(x)$. This completes the proof.

Lemma 8. *For each mapping* Ψ *of* X *into* $\mathfrak{P}\mathfrak{P}(X)$, *the following statements hold*:

(1) $\Psi(x) \neq \emptyset$ *for every* $x \in X$ *if and only if* $\emptyset^{K(\Psi)} = \emptyset$.

(2) *If* $x \in X$, *then* $\emptyset \notin \Psi(x)$ *if and only if* $x \in X^{K(\Psi)}$.

Proof. To prove (1), suppose first $\emptyset^{K(\Psi)} \neq \emptyset$. Then there is an $x \in \emptyset^{K(\Psi)}$. Hence if $\Psi(x)$ has a member S, then we have a contradiction $S \cap \emptyset \neq \emptyset$. Conversely if $\Psi(x) = \emptyset$ for some $x \in X$, then since $[\Psi(x)] = \emptyset$, we have, in view of Lemma 6 and (1) of Lemma 1,

$$x \in \bigcap \{ A^{K(\Psi)} \mid A \in \mathfrak{P}(X) \} \subset \emptyset^{K(\Psi)},$$

and so $\emptyset^{\kappa(\Psi)} \neq \emptyset$. On the other hand, since $\emptyset \notin \Psi(x)$ if and only if $\emptyset \notin [\Psi(x)]$, the statement (2) follows immediately from Lemma 6 and (2) of Lemma 1.

A mapping α of $\mathfrak{P}(X)$ into itself is called a *semiclosure* on X if it satisfies the following conditions:

(1) $\emptyset^{\alpha} = \emptyset$ and $X^{\alpha} = X$.

(2) $(A \cup B)^{\alpha} = A^{\alpha} \cup B^{\alpha}$ for every $A, B \in \mathfrak{P}(X)$.

Lemma 7 yields obviously the following

Theorem 1. *If α is a semiclosure on X, then $\Phi_{\alpha}(x)$ is a filter on X for each $x \in X$.*

We have moreover the following

Theorem 2. *Let Ψ be a mapping of X into $\mathfrak{P}\mathfrak{P}(X)$. Then $\kappa(\Psi)$ is a semiclosure on X if and only if $\Psi(x)$ is a filter base on X for each $x \in X$.*

Proof. If $\kappa(\Psi)$ is a semiclosure on X, then by Lemma 7, $\Phi_{\kappa(\Psi)}(x)$ is a filter on X for each $x \in X$. But then since $\Phi_{\kappa(\Psi)}(x) = [\Psi(x)]$ by Lemma 6, $\Psi(x)$ is a filter base on X.

Conversely assume that $\Psi(x)$ is a filter base on X for each $x \in X$. Then $\emptyset^{\alpha} = \emptyset$ and $X^{\alpha} = X$ by Lemma 8. Let $x \in (A \cup B)^{\kappa(\Psi)}$. If $S \cap A \neq \emptyset$ for every $S \in \Psi(x)$, then $x \in A^{\kappa(\Psi)} \subset A^{\kappa(\Psi)} \cup B^{\kappa(\Psi)}$. If $S_0 \cap A = \emptyset$ for some $S_0 \in \Psi(x)$, then for each $S \in \Psi(x)$ the set $S \cap S_0$ contains some $S_1 \in \Psi(x)$, and hence we have

$$S \cap B = \emptyset \cup (S \cap B) = (S_0 \cap A) \cup (S \cap B)$$

$$\supset (S_1 \cap A) \cup (S_1 \cap B) = S_1 \cap (A \cup B) \neq \emptyset,$$

which implies that $x \in B^{K(\Psi)} \subset A^{K(\Psi)} \cup B^{K(\Psi)}$. Thus $(A \cup B)^{K(\Psi)} \subset A^{K(\Psi)} \cup B^{K(\Psi)}$. Now let x be in $A^{K(\Psi)} \cup B^{K(\Psi)}$; one can assume $x \in A^{K(\Psi)}$. We have then

$$S \cap (A \cup B) = (S \cap A) \cup (S \cap B) \supset S \cap A \neq \emptyset$$

for every $S \in \Psi(x)$. It follows that $x \in (A \cup B)^{K(\Psi)}$. Therefore $(A \cup B)^{K(\Psi)} \supset A^{K(\Psi)} \cup B^{K(\Psi)}$. This completes the proof.

2. Let Γ be a set of semiclosures on a set X. The ordered pair (X, Γ) is called a *closeness space*, and Γ is called a *Closeness* on X if the following conditions are satisfied:

(C1) For every $x \in X$, there exists an $\alpha \in \Gamma$ such that $x \in \{x\}^{\alpha}$.

(C2) For every $\alpha, \beta \in \Gamma$, there exists a $\gamma \in \Gamma$ such that

$$A^{\alpha} \cup A^{\beta} \subset A^{\gamma} \quad \text{for all} \quad A \in \mathcal{P}(X).$$

Let Γ, Γ' be two closenesses on a set X. We say that Γ' is *finer then* Γ(or Γ is coarser than Γ') if for every $x \in X$ and for every $\alpha \in \Gamma$, there exists a $\beta \in \Gamma'$ such that $\Phi_{\beta}(x) \subset \Phi_{\alpha}(x)$. Γ and Γ' are said to be *equivalent* or $\Gamma \equiv \Gamma'$ if Γ is finer than Γ' and if Γ' is finer than Γ. It is easy to see that $\equiv$ is an equivalence relation on the set of all closenesses on X.

Theorem 3. *Let X be a set. For each closeness Γ on X, there exists a unique convergence structure τ on X such that, for every $x \in X$, $\Psi \in \tau(x)$ if and only if $\Phi_{\alpha}(x) \subset \Psi$ for some $\alpha \in \Gamma$.*

Proof. It clearly suffices to show that the mapping τ of X into the power set of the set $\mathcal{F}(X)$ of all filters on X defined by

$$\tau(x) = \{ \Psi \in \mathcal{F}(X) \mid \Phi_{\alpha}(x) \subset \Psi \text{ for some } \alpha \in \Gamma \} \text{ for every } x \in X,$$

is a convergence structure on X. Theorem 1 shows that the mapping τ is

well-defined. Let $x \in X$ and let $\Phi, \Psi \in \tau(x)$. Then we have $\Phi_\alpha(x) \subset \Phi$ and $\Phi_\beta(x) \subset \Psi$ for some $\alpha, \beta \in \Gamma$. Hence the condition (C2) ensures the existence of a $\gamma \in \Gamma$ such that $A^\alpha \cup A^\beta \subset A^\gamma$ for all $A \in \mathcal{P}(X)$. Now if $A \in \Phi_\gamma(x)$, then since $x \notin A^{c\gamma}$, we have $x \notin A^{c\alpha}$ and $x \notin A^{c\beta}$, which imply

$$A \in \Phi_\alpha(x) \cap \Phi_\beta(x) \subset \Phi \cap \Psi.$$

Consequently we have $\Phi_\gamma(x) \subset \Phi \cap \Psi$, and hence $\Phi \cap \Psi \in \tau(x)$. It remains to prove that $\dot{x} \in \tau(x)$ for each $x \in X$. Let x be in X. Then by (C1) one can find an $\alpha \in \Gamma$ such that $x \in \{x\}^\alpha$. If A is a member of $\Phi_\alpha(x)$, then since $x \notin A^{c\alpha}$, the set A^c cannot contain $\{x\}$, and so $x \in A$. Thus we have $\Phi_\alpha(x) \subset \dot{x}$ as desired.

The convergence structure whose existence is ensured by Theorem 3 is called the *convergence structure associated with* Γ and is denoted by τ_Γ, that is

$$\tau_\Gamma(x) = \{ \Psi \in \mathcal{F}(X) \mid \Phi_\alpha(x) \subset \Psi \text{ for some } \alpha \in \Gamma \}$$

for every $x \in X$, where $\mathcal{F}(X)$ denotes the set of all filters on X.

It is easy to verify the following theorem, and the proof is therefore omitted.

Theorem 4. *Let Γ, Γ' be two closenesses on a set X. Then Γ is finer than Γ' if and only if τ_Γ is finer than $\tau_{\Gamma'}$.*

Thus we have the following

Corollary. *Two closenesses Γ, Γ' on a set X are equivalent if and only if $\tau_\Gamma = \tau_{\Gamma'}$.*

We shall now prove the following

Theorem 5. *For each convergence structure* τ *on* X, *there exists a closeness* Γ *on* X *such that* $\tau = \tau_\Gamma$. *The closeness* Γ *can be chosen to satisfy moreover the condition*

(C1') There exists an $\alpha \in \Gamma$ such that $A \subset A^\alpha$ for every $A \in \mathfrak{P}(X)$.

Proof. Let Γ denotes the set of all $\kappa(\Psi)$ where Ψ runs through the set $\Pi\{\tau(x) \mid x \in X\}$. By Theorem 2, each element of Γ is a semiclosure on X. We shall show that Γ satisfies the condition (C1') which implies (C1). Since $\dot{x} \in \tau(x)$ for each $x \in X$, there is a $\Psi \in \Pi\{\tau(x) \mid x \in X\}$ such that $\Psi(x) = \dot{x}$ for every $x \in X$; by Lemma 2, it is sufficient to prove that $x \in \{x\}^{\kappa(\Psi)}$ for every $x \in X$. Let $x \in X$; then for each $S \in \Psi(x)$, we have $S \cap \{x\} \neq \emptyset$, and consequently $x \in \{x\}^{\kappa(\Psi)}$. In order to verify (C2), let $\alpha, \beta \in \Gamma$. Then $\alpha = \kappa(\Psi_1)$ and $\beta = \kappa(\Psi_2)$ for some $\Psi_1, \Psi_2 \in \Pi\{\tau(x) \mid x \in X\}$, and hence we can find a $\Psi_0 \in \Pi\{\tau(x) \mid x \in X\}$ such that $\Psi_0(x) = \Psi_1(x) \cap \Psi_2(x)$ for all $x \in X$. Let us denote by γ the semiclosure $\kappa(\Psi_0) \in \Gamma$, and let $A \in \mathfrak{P}(X)$. If $x \in A^\alpha$, then since $\Psi_0(x) \subset \Psi_1(x)$, we have $S \cap A \neq \emptyset$ for every $S \in \Psi_0(x)$, which shows that $x \in A^\gamma$. It follows that $A^\alpha \subset A^\gamma$. Thus we have $A^\alpha \cup A^\beta \subset A^\gamma$. It remains only to prove that $\tau = \tau_\Gamma$. Let x be in X. For each $\mathscr{F} \in \tau(x)$, one can find a $\Psi \in \Pi\{\tau(x) \mid x \in X\}$ for which we have $\Psi(x) = \mathscr{F}$; then since $\Phi_{\kappa(\Psi)}(x) = [\Psi(x)] = [\mathscr{F}] = \mathscr{F}$ by Lemma 6, we have $\mathscr{F} \in \tau_\Gamma(x)$. Consequently $\tau(x) \subset \tau_\Gamma(x)$. Conversely for each $\mathscr{F} \in \tau_\Gamma(x)$, there is a $\Psi \in \Pi\{\tau(x) \mid x \in X\}$ such that $\Phi_{\kappa(\Psi)}(x) \subset \mathscr{F}$; and hence by Lemma 6 again, we have $\Phi_{\kappa(\Psi)}(x) = [\Psi(x)] = \Psi(x) \in \tau(x)$ which implies $\mathscr{F} \in \tau(x)$. Therefore we have $\tau_\Gamma(x) \subset \tau(x)$. Thus $\tau(x) = \tau_\Gamma(x)$ for every $x \in X$.

As an immediate consequence of Theorem 5 and Corollary of Theorem 4, we

have the following

Corollary. *For each closeness* Γ *on a set* X, *there exists a closeness* Γ' *on* X *satisfying the conditions* (C1') *and* $\Gamma \equiv \Gamma'$.

Let X be a set and let α be a mapping of $\mathfrak{P}(X)$ into itself. Then by Lemma 2, if $\{\alpha\}$ is a closeness on X then $A \subset A^{\alpha}$ for every $A \in \mathfrak{P}(X)$. Consequently $\{\alpha\}$ is a closeness on X if and only if the following conditions are satisfied:

(P1) $\emptyset^{\alpha} = \emptyset$.

(P2) $A \subset A^{\alpha}$ for every $A \in \mathfrak{P}(X)$.

(P3) $(A \cup B)^{\alpha} = A^{\alpha} \cup B^{\alpha}$ for every $A, B \in \mathfrak{P}(X)$.

In other words, $\{\alpha\}$ is a closeness on X if and only if α is a structure of "pré-anhérence" of Choquet [1]. An operator α satisfying the conditions (P1)-(P3) is called a *closure topology* by Koutník [3]. On the other hand, Rehermann [4] has introduced the notions of "liaison" and "liaison space": a subset λ of $X \times (\mathfrak{P}(X) \setminus \{\emptyset\})$ is called a *liaison* and the pair (X, λ) a *liaison space* if

(L1) $x\lambda\{x\}$ for every $x \in X$, and

(L2) $x\lambda(A \cup B)$ if and only if $x\lambda A$ or $x\lambda B$, for every $x \in X$ and for every $A, B \in \mathfrak{P}(X)$,

where $x\lambda A$ means $(x, A) \in \lambda$. In a liaison space (X, λ), Rehermann defines the *capusle* $A^{\alpha(\lambda)}$ of each $A \in \mathfrak{P}(X)$ by

$$A^{\alpha(\lambda)} = \{ x \in X \mid x\lambda A \}.$$

As is shown in [4], the mapping $\alpha(\lambda)$ of $\mathfrak{P}(X)$ into itself satisfies the conditions (P1)-(P3), and hence $\{\alpha(\lambda)\}$ is a closeness on X. Conversely if $\{\alpha\}$ is a closeness on X, then as can be easily seen, we

have $\alpha = \alpha(\lambda)$ for the liaison $\lambda = \{(x, A) \in X \times (\mathcal{P}(X) \setminus \{\emptyset\}) \mid x \in A^{\alpha}\}$ on X. Thus a liaison and a closeness consisting of a single element define the same kind of structures. Moreover the structures of "préadhérence" of Choquet coincide with the principal convergence structures ("Hauptideal-Limitierung" of Fischer. See [2]). This leads us to the following

Theorem 6. *A closeness Γ on a set X is equivalent to a closeness on X which is a singleton if and only if τ_{Γ} is a principal convergence structure on X.*

Proof. It will be enough to prove the "if part". Assume that τ_{Γ} is a principal convergence structure on X. Then for each $x \in X$, there exists a unique filter $\Psi(x)$ on X such that $\tau_{\Gamma}(x)$ is the set of all filters on X finer than $\Psi(x)$. In order to prove that $\Gamma' = \{\kappa(\Psi)\}$ is a closeness on X, it clearly suffice to show that $x \in \{x\}^{\kappa(\Psi)}$ for every $x \in X$. To this end, let $x \in X$. Then since $\Psi(x) \subset \dot{x}$, we have $S \cap \{x\} \neq \emptyset$ for every $S \in \Psi(x)$, and hence we have $x \in \{x\}^{\kappa(\Psi)}$. Therefore Γ' is a closeness on X. Now by Lemma 5, we have

$$\tau_{\Gamma'}(x) = \{\mathcal{F} \in \mathbb{F}(X) \mid \Phi_{\kappa(\Psi)}(x) \subset \mathcal{F}\} = \{\mathcal{F} \in \mathbb{F}(X) \mid [\Psi(x)] \subset \mathcal{F}\}$$
$$= \{\mathcal{F} \in \mathbb{F}(X) \mid \Psi(x) \subset \mathcal{F}\} = \tau_{\Gamma}(x)$$

for each $x \in X$, where $\mathbb{F}(X)$ denotes the set of all filters on X. Hence it follows from Corollary of Theorem 5 that Γ' and Γ are equivalent.

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