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On Common Fixed Point Theorem of Mappings

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In this recent book [1], V. I. Istrăţescu proved some common fixed point theorems about contraction mappings. In this Note, we shall generalize his results.

Let (X, ρ) be a complete metric space, and T_k $(k = 1, 2, \cdots,)$ a family of mappings of X into itself.

Theorem 1. *If* T_k $(k = 1, 2, \cdots, n)$ *satisfies*

1) $T_kT_\ell = T_\ell T_k$ $(k, \ell = 1, 2, \cdots, n)$,

2) *There is a system of positive integers* $m_1, m_2, \cdots, m_n$ *such that*

(1) $\rho(T_1^{m_1}T_2^{m_2} \cdots T_n^{m_n}x, T_1^{m_1}T_2^{m_2} \cdots T_n^{m_n}y)$

$\leq \alpha\rho(x, y) + \beta[\rho(x, T_1^{m_1}T_2^{m_2} \cdots T_n^{m_n}x) + \rho(y, T_1^{m_1}T_2^{m_2} \cdots T_n^{m_n}y)]$

$+ \gamma[\rho(x, T_1^{m_1}T_2^{m_2} \cdots T_n^{m_n}y) + \rho(y, T_1^{m_1}T_2^{m_2} \cdots T_n^{m_n}x)]$

for every x, y *of* X, *where* α, β, γ *are non-negative and* $\alpha + 2\beta + 2\gamma < 1$, *then* T_k $(k = 1, 2, \cdots, n)$ *have the unique common fixed point.*

Proof. To prove Theorem, we use I. Rus theorem [2]. Let $U = T_1^{m_1}T_2^{m_2} \cdots T_n^{m_n}$, then by (1), we have

$\rho(Ux, Uy) \leq \alpha(x, y) + \beta[\rho(x, Ux) + \rho(y, Uy)] + \gamma[\rho(x, Uy) + \rho(y, Ux)]$

for all x, y of X. Hence by I. Rus theorem, U has the unique fixed

point ξ in X. Therefore $U\xi = \xi$, then we have

(2) $T_i(U\xi) = T_i\xi$ $(i = 1, 2, \cdots, n)$

By the commutativity of $\{T_k\}$, (2) implies

$$U(T_i\xi) = T_i\xi.$$

Since U has the unique fixed point ξ, we obtain $T_i\xi = \xi$ $(i = 1, 2, \cdots, n)$. Hence ξ is a common fixed point of the family $\{T_k\}$.

Let ξ, η be common fixed points of $\{T_k\}$, then by (1), we have

$$\rho(\xi, \eta) = \rho(U\xi, U\eta) \leq \alpha\rho(\xi, \eta) + \beta[\rho(\xi, U\xi) + \rho(\eta, U\eta)] + \gamma[\rho(\xi, U\eta) + \rho(\eta, U\xi)],$$

which implies

$$\rho(\xi, \eta) \leq \alpha\rho(\xi, \eta) + 2\gamma\rho(\xi, \eta).$$

From $\alpha + 2\gamma < 1$, we have $\rho(\xi, \eta) = 0$, i.e. $\xi = \eta$.

We have the uniqueness, and we complete the proof.

Theorem 2. *If* $\{T_k\}$ *satisfies the conditions*:

1) $T_1T_2 \cdots T_n$ *commutes with every* T_i,

2) *for every* x, y *of* X,

(3) $\rho(T_1T_2 \cdots T_nx, T_nT_{n-1} \cdots T_1y) \leq \alpha\rho(x, y) + \beta[\rho(x, T_1T_2 \cdots T_nx) + \rho(y, T_nT_{n-1} \cdots T_1y)] + \gamma[\rho(x, T_nT_{n-1} \cdots T_1y) + (y, T_1T_2 \cdots T_nx)]$,

where α, β, γ *are non-negative, and* $\alpha + 2\beta + 2\gamma < 1$, *then* T_k $(k = 1, 2, \cdots, n)$ *have the unique common fixed point.*

Proof. Let $U = T_1T_2 \cdots T_n$, $V = T_nT_{n-1} \cdots T_1$, then by (3), we have

$$\rho(Ux, Vy) \leq \alpha\rho(x, y) + \beta[\rho(x, Ux) + \rho(y, Vy)] + \gamma[\rho(x, Vy) + \rho(y, Ux)]$$

for all x, y of X. By I. Rus theorem [2], U and V have the unique

common fixed point ξ. Then $U\xi = V\xi = \xi$.

For any T_i, $T_i(U\xi) = T_i\xi$. By the assumption, $U(T_i\xi) = T_i\xi$. Similary $V(T_i\xi) = T_i\xi$. Hence $T_i\xi = \xi$ $(i = 1, 2, \cdots, n)$, which means that ξ is a common fixed point of $\{T_k\}$. It is easily seen that ξ is the unique common fixed point of $\{T_k\}$. This completes the proof.

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References

[1] V. I. Istrăţescu: Introducere in teoria punctelor fixe, Bucarest 1973.

[2] I. A. Rus: On common fixed points, Studia Universitatis Babes-Bolyai fasc., 1 (1973), 31-33.