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Common Fixed Point Theorems for Mappings

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In my previous note [2], we proved a fixed point theorem which is a generalization of M. G. Maia's theorem [4]. The theorem is written as follows.

Lemma [2]. *Let X be a metric space with two metrics ρ and δ. If X satisfies*

(i) $\rho(x, y) \leq \delta(x, y)$ for all x, y in X,

(ii) X is complete with respect to ρ,

(iii) two mappings U, V: $X \to X$ are continuous with respect to ρ,

and

(1) $$\delta(U(x), V(x)) \leq \alpha[\delta(x, U(x)) + \delta(y, U(y))] + \beta[\delta(x, V(y)) + \delta(y, V(x))] + \gamma\delta(x, y)$$

for all x, y in X, where α, β and γ are non-negative, and $2\alpha + 2\beta + \gamma < 1$, then U, V have a unique common fixed point.

The above Lemma is useful to prove some fixed point theorem (see [2]). In this note, we shall prove the following

Theorem 1. *Let X be a metric space with two metric ρ and δ, $\{T_i\}$ a family of continuous mappings with respect to ρ of X into itself. If X and $\{T_i\}$ satisfy the following conditions*

(i) $\rho(x, y) \leq \delta(x, y)$ *for all* x, y *in* X,

(ii) (X, ρ) *is complete,*

(iii) *there are non-negative integers* $\{m_i\}$ *and non-negative numbers* α, β *and* γ *such that for all* x, y *in* X *and every pair* i, j *with* $i \neq j$,

$$\begin{aligned}(2)\quad \delta(T_i^{m_i}(x), T_j^{m_j}(x)) &\leq \alpha[\delta(x, T_i^{m_i}(x)) + \delta(y, T_j^{m_j}(y))] \\ &\quad + \beta[\delta(x, T_j^{m_j}(y)) + \delta(y, T_i^{m_i}(x))] \\ &\quad + \gamma\delta(x, y),\end{aligned}$$

where $2\alpha + 2\beta + \gamma < 1$, *then the family of mappings* $\{T_i\}$ *has a unique common fixed point.*

Proof. Let $U_i = T_i^{m_i}$, then for $i \neq j$, we have

$$\begin{aligned}(3)\quad \delta(U_i(x), U_j(y)) &\leq \alpha[\delta(x, U_i(x)) + \delta(y, U_j(y))] \\ &\quad + \beta[\delta(x, U_j(y)) + \delta(y, U_i(x))] + \gamma\delta(x, y)\end{aligned}$$

and each U_i is continuous with respect to ρ. By Lemma, for any fixed i, j, U_i and U_j have a unique common fixed point x_0, i.e.

$$U_i(x_0) = U_j(x_0) = x_0.$$

Hence we have

$$\begin{aligned}T_i(x_0) &= T_i(U_i(x_0)) = T_i(T_i^{m_i}(x_0)) \\ &= T_i^{m_i}(T_i(x_0)) = U_i(T_i(x_0)),\end{aligned}$$

and

$$T_j(x_0) = T_j(U_j(x_0)) = U_j(T_j(x_0)).$$

This means that $T_i(x_0)$ and $T_j(x_0)$ are fixed points of U_i and U_j respectively. By (3), we have

$$\begin{aligned}\delta(T_i(x_0), T_j(x_0)) &= \delta(U_i(T_i(x_0)), U_j(T_j(x_0))) \\ &\leq \alpha[\delta(T_i(x_0), U_i(T_i(x_0))) + \delta(T_j(x_0), U_j(T_j(x_0)))] \\ &\quad + \beta[\delta(T_i(x_0), U_j(T_j(x_0))) + \delta(T_j(x_0), U_i(T_i(x_0)))]\end{aligned}$$

$$+ \gamma\delta(T_i(x_0), T_j(x_0))$$
$$\leq 2\beta\delta(T_i(x_0), T_j(x_0)) + \gamma\delta(T_i(x_0), T_j(x_0)).$$

Since $2\beta + \gamma \leq 2\alpha + 2\beta + \gamma < 1$, $\delta(T_i(x_0), T_j(x_0)) = 0$, which implies that $T_i(x_0)(= T_j(x_0))$ is a common fixed point of T_i and T_j.

To prove that x_0 is a unique common fixed point of T_i and T_j, let y_0 be a common fixed point of T_i and T_j, then, by (2) we have

$$\delta(x_0, y_0) \leq \beta[\delta(x_0, y_0) + \delta(y_0, x_0)]$$
$$+ \gamma\delta(x_0, y_0).$$

Since $2\beta + \gamma < 1$, we have $x_0 = y_0$. Hence $x_0 = T_i(x_0) = T_j(x_0)$, and x_0 is a unique common fixed point of T_i and T_j. We conclude that every pair of mappings T_i, T_j has a unique common fixed point x_{ij}.

To show that these common fixed points x_{ij} coincide, let x_{ij}, $x_{k\ell}(i \neq k)$ be fixed points of T_i, T_j and T_k, T_ℓ respectively. Since T_i, T_k satisfy the condition (2), we have

$$\delta(x_{ij}, x_{k\ell}) \leq \beta[\delta(x_{ij}, x_{k\ell}) + \delta(x_{k\ell}, x_{ij})]$$
$$+ \gamma\delta(x_{ij}, x_{k\ell}),$$

which implies $x_{ij} = x_{k\ell}$ $(i \neq k)$. It is easily seen that the family of mappings $\{T_i\}$ has a unique common fixed point. This completes the proof of Theorem.

Since the Banach contraction theorem was obtained, many theorems about fixed points have been proved. We shall prove a unified theorem which contains these many results. For example, the theorem contains Theorem 1 in [1] as special cases.

Theorem 2. *Let $\{T_i\}$ be a family of mappings of a complete metric space X into itself. Suppose that there is a non-negative integer m_i for each T_i, and non-negative numbers α, β, γ such that for all x, y*

in X *and every pair* i, j *with* $i \neq j$,

$$(4)\quad \rho(T_i^{m_i}(x), T_j^{m_j}(y)) \leq \alpha[\rho(x, T_i^{m_i}(x)) + \rho(y, T_j^{m_j}(y))] + \beta[\rho(x, T_j^{m_j}(y)) + \rho(y, T_i^{m_i}(x))] + \gamma\rho(x, y),$$

where $2\alpha + 2\beta + \gamma < 1$. *Then the family of mappings* $\{T_i\}$ *has a unique common fixed point.*

Proof. Let $U_i = T_i^{m_i}$, then (4) is written in the form of

$$(5)\quad \rho(U_i(x), U_j(y)) \leq \alpha[\rho(x, U_i(x)) + \rho(y, U_j(y))] + \beta[\rho(x, U_j(y)) + \rho(y, U_i(x))] + \gamma\rho(x, y).$$

For any fixed i, j we may apply I. A. Rus theorem [5] to get a common fixed point of U_i, and U_j. Therefore U_i, U_j have a unique common fixed point x_0 in X, i.e. $U_i(x_0) = U_j(x_0) = x_0$. Hence we have

$$T_i(x_0) = T_i(U_i(x_0)) = U_i(T_i(x_0)),$$

and

$$T_j(x_0) = T_j(U_j(x_0)) = U_j(T_j(x_0)).$$

This shows that $T_i(x_0)$ and $T_j(x_0)$ are fixed points of U_i and U_j respectively. By (5), we have

$$\begin{aligned}\rho(T_i(x_0), T_j(x_0)) &= \rho(U_i(T_i(x_0)), U_j(T_j(x_0)))\\ &\leq \alpha[\rho(T_i(x_0), U_i(T_i(x_0))) + \rho(T_j(x_0), U_j(T_j(x_0)))]\\ &\quad + \beta[\rho(T_i(x_0), U_j(T_j(x_0))) + \rho(T_j(x_0), U_i(T_i(x_0)))]\\ &\quad + \gamma\rho(T_i(x_0), T_j(x_0)),\end{aligned}$$

which implies

$$\rho(T_i(x_0), T_j(x_0)) \leq \beta[\rho(T_i(x_0), T_j(x_0)) + \rho(T_j(x_0), T_i(x_0))] + \gamma\rho(T_i(x_0), T_j(x_0)).$$

Hence, since $2\beta + \gamma \leq 2\alpha + 2\beta + \gamma < 1$, we have $\rho(T_i(x_0), T_j(x_0)) = 0$, i.e. $T_i(x_0) = T_j(x_0)$. Therefore $T_i(x_0) = T_j(x_0)$ is a common fixed point of U_i and U_j. By the uniqueness, we have $x_0 = T_i(x_0) = T_j(x_0)$, which means that x_0 is a common fixed point of T_i and T_j.

To prove that x_0 is a unique common fixed point of T_i, T_j let y_0 be a common fixed point of T_i and T_j. Then by (4), we have

$$\begin{aligned}\rho(x_0, y_0) &= \rho(T_i^{m_i}(x_0), T_j^{m_j}(y_0)) \\ &\leq \beta[\rho(x_0, T_j^{m_j}(y_0)) + \rho(y_0, T_i^{m_i}(x_0))] \\ &\quad + \gamma\rho(x_0, y_0),\end{aligned}$$

which implies $x_0 = y_0$. Therefore T_i and T_j have a unique common fixed point in X.

Let x_{ij}, $x_{k\ell}$ be common fixed points of T_i, T_j and T_k, T_ℓ respectively. We may suppose $i \neq k$. Hence, by (4), we have

$$\rho(x_{ij}, x_{k\ell}) \leq \beta[\rho(x_{ij}, x_{k\ell}) + \rho(x_{k\ell}, x_{ij})] + \gamma\rho(x_{ij}, x_{k\ell}).$$

This implies $x_{ij} = x_{k\ell}$, and then this shows that the fixed points of every pair T_i, T_j coincide. Therefore the proof completes.

Theorem 3. *Let T_1, T_2 be mappings of a complete metric space X into itself. Suppose that there are non-negative α, β, γ with $2\alpha + 2\beta + \gamma < 1$ such that*

$$\begin{aligned}(6)\quad \rho(T_1T_2(x), T_2T_1(x)) &\leq \alpha[\rho(x, T_1T_2(x)) + \rho(y, T_2T_1(y))] \\ &\quad + \alpha[\rho(x, T_2T_1(y)) + \rho(y, T_1T_2(x))] \\ &\quad + \gamma\rho(x, y),\end{aligned}$$

for every x, y of X. Then T_1, T_2 have a unique common fixed point in X.

Proof. Let $U = T_1T_2$, $V = T_2T_1$, then by (6) we have

$$(7)\quad \rho(U(x), V(x)) \leq \alpha[\rho(x, U(x)) + \rho(y, V(y))] + \beta[\rho(x, V(y)) + \rho(y, U(x))] + \gamma\rho(x, y)$$

for every x, y of X. Hence by I. A. Rus theorem [5], there is a unique common fixed point x_0 of U and V, i.e. $U(x_0) = V(x_0) = x_0$.

Then we have

$$T_1(x_0) = T_1V(x_0) = (T_1T_2)T_1(x_0) = UT_1(x_0),$$
$$T_2(x_0) = T_2U(x_0) = (T_2T_1)T_2(x_0) = VT_2(x_0).$$

Therefore $T_1(x_0)$, $T_2(x_0)$ are fixed points of U, V respectively. In the above inequality (7), put $x = T_1(x_0)$, $y = T_2(x_0)$, then we have

$$\rho(T_1(x_0), T_2(x_0)) \leq \alpha[\rho(T_1(x_0), T_1(x_0)) + \rho(T_2(x_0), T_2(x_0))] + \beta[\rho(T_1(x_0), T_2(x_0)) + \rho(T_2(x_0), T_1(x_0))] + \gamma\rho(T_1(x_0), T_2(x_0)),$$

which implies

$$\rho(T_1(x_0), T_2(x_0)) \leq (2\beta + \gamma)\rho(T_1(x_0), T_2(x_0)).$$

Hence $T_1(x_0) = T_2(x_0)$. This means that $T_1(x_0)$ $(= T_2(x_0))$ is a common fixed point of U, V and by the uniqueness of the common fixed point of U, V we have $x_0 = T_1(x_0) = T_2(x_0)$. This completes the proof of Theorem 3. It is trivial that the common fixed point of T_1, T_2 is unique.

We remark that the commutativity of T_1 and T_2 is not needed in Theorem 3.

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