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(Citation)

Mathematics Seminar Notes, 3:1-12

(Issue Date)

1975

(Resource Type)

journal article

(Version)

Version of Record

(JaLCD0I)

<https://doi.org/10.24546/0100491929>

(URL)

<https://hdl.handle.net/20.500.14094/0100491929>



On Ideals in BCK-Algebras

By Kiyoshi ISEKI

I. Ideals and quotient algebras

The notion of BCK-algebra was introduced by the present author [3], and the algebras were studied by Y. Imai and S. Tanaka. B. Bosbach discussed a connection between the BCK-algebras and the complemented semigroups [1]. In my Note [4], we considered some special classes of BCK-algebra.

By a *BCK-algebra*, we mean a general algebra $M = \langle X; *, 0 \rangle$ of type $\langle 2, 0 \rangle$ satisfying the following axioms:

- (1) $(x * y) * (x * z) \leq z * y,$
- (2) $x * (x * y) \leq y,$
- (3) $x \leq x,$
- (4) $x \leq y, y \leq x$ imply $x = y,$
- (5) $0 \leq x,$

where $x \leq y$ is defined as $x * y = 0.$

Some elementary properties of a BCK-algebra is given in my Note [4].

In this Note, we shall use these properties without reference.

We first introduce the notion of an ideal in a BCK-algebra.

Definition 1. Let A be a non-empty subset of BCK-algebra M . A is

called to be an *ideal*, if

- 1) $0 \in A$,
- 2) $x \in A$ and $y * x \in A$ imply $y \in A$.

By Definition 1, the simplest examples of ideals are $\{0\}$ and X . Let A be an ideal of M . If $x \in A$ and $y * x = 0$, then $y \in A$. Hence if x is an element of an ideal A and $y \leq x$, then $y \in A$.

But, in a BCK-algebra M , the set $A(a)$ of all elements x satisfying $x \leq a$ for a fixed element a of X does not be an ideal in general.

Theorem 1. Any set $A(a)$ is an ideal if and only if $x * y \leq z$ and $y \leq z$ imply $x \leq z$ for x, y, z of X .

Proof. Let any set $A(a)$ be an ideal of M , and let $x * y \leq z$ and $y \leq z$. The set $A(z)$ is an ideal. Hence $x * y \in A(z)$ and $y \in A(z)$ imply $x \in A(z)$, and we have $x \leq z$. Conversely, consider $A(z)$ for any element z of M , and let $x * y \in A(z)$ and $y \in A(z)$. Hence $x * y \leq z$ and $y \leq z$. By the assumption, we have $x \leq z$. This means $z \in A(z)$.

Next we shall consider the quotient algebra M/A of a BCK-algebra M by an ideal A .

For any x, y of M , $x \sim y$ is defined as

$$x * y \in A, \text{ and } y * x \in A.$$

Then $\sim$ is an equivalent relation on X . Since $0 \in A$, it is obvious that $\sim$ is reflexive, i.e. $x \sim x$. By the definition, if $x \sim y$, then $y \sim x$, which shows that $\sim$ is symmetric. To prove that $\sim$ is transitive, let $x \sim y$ and $y \sim z$. Then $x * y, y * z \in A$. Since A is an ideal, and $y * z \in A$, we have $(x * z) * (x * y) \in A$ by $(x * z) * (x * y) \leq y * z$. Hence, by the definition of an ideal, $x * z \in A$. Similarly, from $(z * x) *$

$(z * y) \leq y * x$, we have $z * x \in A$. Therefore $x \sim y$ and $y \sim z$ imply $x \sim z$.

Therefore, X is decomposed by the equivalent relation $\sim$. The class containing x will be denoted by C_x . Then we have

$$(6) \quad A = C_0.$$

Let $x \in A$, then $x * 0 = x \in A$, and $0 * x = 0 \in A$. Hence $x \in C_0$. Next, let $x \in C_0$, then $x * 0 \in A$. Since A is an ideal, we have $x \in A$. Hence $C_0 \subset A$, which shows (6).

Next we shall define a binary operation $*$ in the quotient M/A . $C_x * C_y$ is defined as the class containing $x * y$. To show that the operation $*$ is well defined, let $u \sim x, v \sim y$. For any $w \in C_x * C_y, w * (x * y) \in A$. On the other hand,

$$(7) \quad (x * y) * (x * v) \leq v * y \in A,$$

hence $(x * y) * (x * v) \in A$. Similarly $(x * v) * (x * y) \in A$, and we have $x * y \sim x * v$. And $x * v \sim u * v$ implies $(x * y) * (u * v) \in A$. Then by

$$(w * (u * v)) * (w * (x * y)) \leq (x * y) * (u * v),$$

we have $(w * (u * v)) * (w * (x * y)) \in A$. Since A is an ideal, $w * (u * v) \in A$.

To prove $(u * v) * w \in A$, consider

$$(8) \quad ((u * v) * w) * ((u * v) * (x * y)) \leq (x * y) * w.$$

Then $(x * y) * w \in A$ implies the left element of (8) is contained in A .

Further by the same argument (7) $(u * v) * (x * y) \in A$. Hence we have

$(u * v) * w \in A$, which shows $u * v \sim w$. Therefore $w \in C_u * C_v$. From this, if $x \sim u$ and $y \sim v$, then $C_x * C_y = C_u * C_v$, we have the following lemmas.

Lemma 1. If $x \sim u, y \sim v$, then $x * y \sim u * v$.

Lemma 2. If $x \leq y$, $x \sim u$, $y \sim v$, then $u \leq v$.

Next, we shall prove that M/A is a BCK-algebra.

To do so, we shall verify the axioms (1) ~ (5) of a BCK-algebra.

If $C_x * C_y = C_0$, then we define $C_x \leq C_y$. Since $C_0 * C_x = C_{0*x} = C$, we have $C_0 \leq C_x$ for every x of M .

Let $C_x * C_y = C_y * C_x = 0$, then $x * y \sim 0$, $y * x \sim 0$. If $u \in C_x$, then $u \sim x$. By Lemma 1, we have $u * y \sim 0$, $y * u \sim 0$, which imply $u \in C_y$.

Hence $C_x \subset C_y$. Similarly we have $C_y \subset C_x$. Therefore $C_x \leq C_y$ and $C_y \leq C_x$ imply $C_x = C_y$.

$C_x * C_x = C_{x+x} = C_0$ implies $C_x \leq C_x$.

Therefore the axioms (3), (4) and (5) holds for M/A .

To verify the axioms (1), (2), it is sufficient to see that $x \leq y$ implies $C_x \leq C_y$.

From $x \leq y$, we have $x * y = 0$ and

$$C_x * C_y = C_{x*y} = C_0.$$

Hence if $x \leq y$, then $C_x \leq C_y$.

Therefore we have

$$\begin{aligned} & (C_x * C_y) * (C_x * C_z) \\ &= C_{x*y} * C_{x*z} = C_{(x*y)*(x*z)} \\ &\leq C_{z*y} = C_z * C_y. \end{aligned}$$

Similarly the axiom (2) holds for M/A . We have the following

Theorem 2. If A is an ideal of a BCK-algebra M , then the quotient M/A is also a BCK-algebra.

The theory of positive implicative algebras and the ideal theory has been extensively studied by A. Diego [2]. He calls this algebra a Hilbert

algebra (see H. Rasiowa [7]). Recently E. L. Marsden ([5], [6]) has obtained some new results about the positive implicative algebras.

The following result is an analog of a result in the positive implicative algebra, and important in the ideal theory of BCK-algebras.

Theorem 3. *Let $M = \langle X; *, 0 \rangle$ be a BCK-algebra, and let A be a nonempty subset of X . Then the set B composed of all $x \in X$ such that there are $a_0, a_1, \dots, a_n \in A$ satisfying*

$$(9) \quad (\dots((x * a_0) * a_1) * \dots) * a_n = 0$$

is a minimal ideal containing A .

Proof. It is obvious from (9) that 0 is contained in B . Let $y * x \in B$ and $x \in B$, then for some $a_1, \dots, a_m, b_1, \dots, b_n \in A$, we have

$$(10) \quad (\dots((y * x) * a_1) * \dots * a_m) = 0,$$

and

$$(11) \quad (\dots(x * b_1) * \dots * b_n) = 0.$$

By repeated applications of Corollary 1 in my Note [4], we have

$$(\dots((y * a_1) * a_2) * \dots * a_m) * x = 0,$$

hence

$$(\dots((y * a_1) * a_2) * \dots) * a_m \leq x.$$

By the application of (12) in my Note [4], we have

$$(\dots(y * a_1) * \dots * a_m) * b_1 \leq x * b_1.$$

By the same method, we have

$$\begin{aligned} & (\dots(((\dots(y * a_1) * \dots * a_m) * b_1) * b_2) * \dots * b_n) \\ & \leq (\dots((x * b_1) * b_2) * \dots * b_n) = 0. \end{aligned}$$

Therefore y is contained in B , and B is an ideal containing A .

Let C be an ideal containing A . To show $B \subset C$, let x be an element

of B . Then there are $a_1, a_2, \dots, a_m$ in A such that

$$(\dots((x * a_1) * a_2) * \dots) * a_m = 0.$$

The left side is contained in C . Since C is an ideal and $a_m \in A \subset C$, we have

$$(\dots(x * a_1) * \dots) * a_{n-1} \in C.$$

$a_{n-1} \in C$ implies $(\dots(x * a_1) * \dots) * a_{n-1} \in C$. Therefore we have $x * a_1 \in C$. By $a_1 \in A \subset C$, $x \in C$. Hence $B \subset C$. This completes the proof of

Theorem 3.

Definition 2. The ideal B in Theorem 3 is called the *ideal generated by A* .

II. Adjunction of unit

In this section, we prove that every BCK-algebra without unit is extended to a BCK-algebra with unit.

Let $M = \langle X; *, 0 \rangle$ be a BCK-algebra without unit, and let $1 \notin X$. We define, for all $x \in X$,

$$x * 1 = 0$$

and

$$1 * 1 = 0.$$

Next we define

$$1 * x = 1$$

for every $x \in X$.

Then we prove that the algebra M with 1 is a BCK-algebra. To show it, we only verify the axioms

$$(1) \quad (x * y) * (x * z) \leq z * y,$$

$$(2) \quad x * (x * y) \leq y$$

of a BCK-algebra.

To verify (1), let $y = 1$, then $x * y = 0$, and (1) holds. Next, let $x = 1$, then by the definition, we have

$$\begin{aligned}(x * y) * (x * z) &= (1 * y) * (1 * z) \\ &\leq 1 * 1 \text{ or } 0 * 1 \text{ or } 1 * 0 \\ &\text{or } 0 * 0 \leq z * y,\end{aligned}$$

which shows that (1) holds for the algebra with 1.

To verify (2), let $y = 1$, then the left side of (2) is equal to 0 or x . Since $x \leq 1$, we have (2). Next, let $x = 1$, if $y \neq 1$, then $x * (x * y) = 1 * (1 * y) = 1 * 1 = 0 \leq y$. If $y = 1$, then $x * (x * y) = 1 \leq y$, which shows that (2) holds for the algebra with 1.

Therefore *the algebra is a BCK-algebra*, and we can say that the algebra is obtained by adding an element to the algebra M . But this *extended algebra is not commutative*, because of, for $0 < x < 1$,

$$x * (x * 1) = x * 0 = x,$$

and

$$1 * (1 * x) = 1 * 1 = 0.$$

We have the following

Theorem 4. *Any BCK-algebra without unit is extended to a BCK-algebra with unit. The algebra is obtained by adding an element.*

In the above algebra M' obtained by adding 1, X is a maximal ideal of M' . Let $y * x \in X$, and $x \in X$. If $y = 1$, then $1 = y * x \in X$, which contradicts. Hence $y \in X$, which means that X is an ideal of M' . It is obvious that X is maximal.

In my Note [3], we gave two examples of BCK-algebras without unit.

One of them was given by

$$x * y = \begin{cases} 0, & \text{if } x \leq y, \\ 1, & \text{if } y < x \text{ and } y \neq 0, \\ x, & \text{if } y < x \text{ and } y = 0 \end{cases}$$

on the set X of natural numbers. If, for $\omega \notin X$, we define

$$x * y = \begin{cases} 0, & \text{if } x \leq y \leq \omega, \\ 1, & \text{if } y < x < \omega \text{ and } y \neq 0, \\ \omega, & \text{if } y < x = \omega \text{ and } y \neq 0, \\ x, & \text{if } y < x \leq \omega \text{ and } y = 0, \end{cases}$$

then $X \cup \{\omega\}$ is a BCK-algebra with unit ω .

Another example was given by

$$x * y = \begin{cases} 0, & \text{if } x \leq y, \\ x - y, & \text{if } y < x \end{cases}$$

on the set X of natural numbers.

If, for $\omega \notin X$, we define

$$x * y = \begin{cases} 0, & \text{if } x \leq y \leq \omega, \\ x - y, & \text{if } y < x < \omega, \\ \omega, & \text{if } y < x = \omega, \end{cases}$$

then $X \cup \{\omega\}$ is a BCK-algebra with unit ω .

III. Implicative ideals

Definition 3. A subset A is called a *implicative ideal*, if for every x, y, z , $(y * x) * z \in A$ and $x * z \in A$ imply $y * z \in A$.

Theorem 5. The zero-ideal $\{0\}$ is implicative, if and only if, every section $A(a) = \{x: x \leq a\}$ is an ideal.

Proof. Assume that $\{0\}$ is a implicative ideal. Let $y * x \in A(a)$, and

$x \in A(a)$. Then $(y * x) * a = x * a = 0$. Since the ideal $\{0\}$ is implicative, then $y * a = 0$. Hence $y \in A(a)$, and $A(a)$ is an ideal.

Conversely, assume that every $A(a)$ is an ideal. Let $(y * x) * z \in \{0\}$, $x * z \in \{0\}$, then we have $y * x \leq z$ and $x \leq z$. By Theorem 1, $y \leq z$, and $y * z \in \{0\}$. Hence $\{0\}$ is a implicative ideal.

Definition 4. If every ideal of a BCK-algebra M is implicative, M is called an *implicative BCK-algebra*.

In an implicative BCK-algebra, for a fixed element a ,

$$B = \{x: x * a \in A\}$$

is an ideal.

To show that B is an ideal, let $y * x \in B$ and $x \in B$, then $(y * x) * a \in A$, and $x * a \in A$. Since M is an implicative, $y * a \in A$, and then $y \in B$. On the other hand, $0 * a = 0 \in A$ implies $0 \in B$. Hence B is an ideal.

Further, in an implicative BCK-algebra, for any ideal A and a fixed element a , $B = \{x: x * a \in A\}$ is the least ideal containing A and a .

Let C be an ideal containing A and a . For any element x of B , $x * a \in A$. Hence $x * a \in C$, and then $a \in C$ implies $x \in C$. Therefore $B \subseteq C$, and B is the least ideal containing A and a .

Let A be a maximal ideal in an implicative BCK-algebra. Then, for any elements x and y , $x * y \in A$ or $y * x \in A$.

Let $x, y \in A$, then $x * y \in A$, since A is an ideal. If $x \in A$ and $y \notin A$, then $x * y \leq x$ implies $x * y \in A$. Let $x \notin A$ and $y \notin A$. Suppose that $y * x \notin A$. Then

$$B = \{z: z * (y * x) \in A\}$$

is an ideal properly containing A . Hence $B = M$. Therefore

$$(x * y) * (y * x) \in A,$$

and then

$$(x * (y * x)) * y \in A.$$

On the other hand $(y * x) * y = 0 \in A$. Hence $x * y \in A$, which completes the proof.

IV. Homomorphisms

Let $M = \langle X; *, 0 \rangle$, $M' = \langle X'; *, 0 \rangle$ be BCK-algebras. A mapping $f: X \rightarrow X'$ is called a homomorphism, if for any $x, y \in X$, $f(x * y) = f(x) * f(y)$.

For a homomorphism f , $f(0) = f(0 * 0) = f(0) * f(0) = 0$. Hence $f(0) = 0$. For any x, y with $x \leq y$, we have $x * y = 0$, and then $0 = f(0) = f(x * y) = f(x) * f(y)$. Hence $x \leq y$ implies $f(x) \leq f(y)$, i.e. f is isotone. Therefore we have the following

Theorem 6. *If f is a homomorphism, then $f(0) = 0$ and f is isotone.*

Theorem 7. *Let $f: X \rightarrow X'$ be a homomorphism. Then the kernel $\ker(f)$ of f is an ideal of M .*

Proof. By Theorem 4, $f(0) = 0$. Hence $0 \in \ker(f)$. Let $y * x \in \ker(f)$, $x \in \ker(f)$, then

$$0 = f(y * x) = f(y) * f(x) = f(y),$$

since $f(x) = 0$. Therefore y is contained in $\ker(f)$. Hence $\ker(f)$ is an ideal.

If $\{0\}$ is an implicative ideal of M' , the kernel $\ker(f)$ of f is an implicative ideal.

Let $(y * x) * z \in \ker(f)$ and $x * z \in \ker(f)$, then $(f(y) * f(x)) * f(z)$

$= 0$ and $f(x) * f(z) = 0$. Hence, by the assumption, $f(y) * f(z) = 0$.

Therefore $y * z \in \ker(f)$, and $\ker(f)$ is implicative.

Definition 5. If a homomorphism $f: X \rightarrow X'$ is onto, f is called an *epimorphism*.

By Lemma 1, an epimorphism is isotone.

For a set Z with two elements $0, 1$, we define

$$0 * 0 = 0 * 1 = 1 * 1 = 0, 1 * 0 = 1.$$

Then $\langle Z, *, 0 \rangle$ is an implicative BCK-algebra.

This algebra is called a *BCK-algebra with two elements*.

Let f be an epimorphism from a BCK-algebra M to the BCK-algebra with two elements. Then the kernel of f is an implicative maximal ideal of M .

An ideal A is *maximal*, if it is proper and it is not a proper subset of any proper ideal of M .

By Theorem 4, then kernel A of f is an ideal. To show that A is maximal, let $x \in A$, and B an ideal containing A and x . For any $y \in X$, $f(y * x) = f(y) * f(x) = 0$, by $f(x) = 1$. Hence $y * x \in A$, and then $y * x \in B$. Since B is an ideal and $x \in B$, then $y \in B$. Therefore $X \subset B$, and A is a maximal ideal. It is obvious from the above remark that A is implicative.

Let f be an epimorphism of a BCK-algebra $M = \langle X; *, 0 \rangle$ onto a BCK-algebra $N = \langle Y; *, 0 \rangle$. Then the quotient algebra $M/\ker(f)$ is isomorphic to N .

By Theorem 7, $\ker(f)$ is an ideal of M . Moreover, if $f(x) = f(y)$, then

$$0 = f(x) * f(y) = f(x * y), \text{ and } f(y * x) = 0. \text{ Hence } f(x) = f(y)$$

implies $x * y \in \ker(f)$, $y * x \in \ker(f)$. Conversely $x \sim y \ (\ker(f))$ implies $f(x) = f(y)$.

Then $M/\ker(f) \ni C_x \rightarrow f(x) \in Y$ is a one-to-one correspondence between $M/\ker(f)$ and N . Further $C_x * C_y = C_{x*y}$ implies $f(x) * f(y) = f(x * y)$. Hence the correspondence $M/\ker(f) \ni C_x \rightarrow f(x) \in Y$ gives the required isomorphism.

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