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PDF issue: 2026-11-14

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(Citation)
Mathematics Seminar Notes,3:55-58

(Issue Date)
1975

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/0100491935

(URL)
https://hdl.handle.net/20.500.14094/0100491935

KOBE UNIVERSITY

MATHEMATICS SEMINAR NOTES
Vol. 3 (1975) March 1975

# On a Common Fixed Point Theorem

By Kiyoshi ISÉKI

In my Note [1], we generalized two fixed point theorems of R. Kannan [2]. In this Note, we shall prove a common fixed point theorem which is a generalization of our results ([1], [2]).

**Theorem.** *Let* $(X, \rho)$ *be a metric space. Let* $T_1$, $T_2$ *be mappings of* $X$ *into itself satisfying*

1) $\rho(T_1(x), T_2(y)) \leq \alpha\rho(x, y) + \beta(\rho(x, T_1(x)) + \rho(y, T_2(y)) + \gamma(\rho(x, T_2(y)) + \rho(y, T_1(x)))$

*for all* $x, y$ *of* $X$, *where* $0 \leq \alpha, \beta, \gamma$ *and* $\alpha + 2\beta + 2\gamma < 1$,

2) $T_1$, $T_2$ *are continuous at a point* $x_0$ *of* $X$,

3) *there is a point* $\xi$ *such that the sequence* $x_1 = T_1(\xi)$, $x_2 = T_2(x_1)$, $\cdots$, $x_{2n-1} = T_1(x_{2n-2})$, $x_{2n} = T_2(x_{2n-1})$, $\cdots$ *has a subsequence converging to* $x_0$.

*Then* $x_0$ *is a unique common fixed point of* $T_1$, $T_2$.

*Proof.* Let $\{x_{n_i}\}$ be a convergent subsequence of $\{x_n\}$ which converges to $x_0$. Then the set of indices $\{n_i\}$ contains infinitely many even or odd. Suppose $n_i$ contains infinitely many even. These indices are written in the form of $\{2m_i\}$. Therefore the sequence $\{x_{2m_i}\}$ converges to $x_0$. Suppose that $x_0$ is not a fixed point of $T_1$, then $\rho(x_0, T_1(x_0))$

$> 0$. Hence there is a positive number $\varepsilon$ such that

$$\varepsilon < \frac{1}{3}\rho(x_0, T_1(x_0)).$$

For $\varepsilon$, there exists a positive integer $N$ such that

$$\rho(x_{2m_i}, x_0) < \varepsilon,$$

and

$$\rho(T_1(x_{2m_i}), T_1(x_0)) < \varepsilon$$

for all $i \geq N$, by the assumption 2). Hence for $i \geq N$, we have

(1) $\rho(x_{2m_i}, T_1(x_{2m_i}))$

$$\geq \rho(x_0, T_1(x_0)) - \rho(x_0, x_{2m_i}) \quad \rho(T_1(x_0), T_1(x_{2m_i})) > 3\varepsilon - 2\varepsilon = \varepsilon.$$

On the other hand,

$$\begin{aligned}
\rho(x_{2m_i+1}, x_{2m_i+2}) &= \rho(T_1(x_{2m_i}), T_2(x_{2m_i+1})) \\
&\leq \alpha\rho(x_{2m_i}, x_{2m_i+1}) \\
&\quad + \beta[\rho(x_{2m_i}, x_{2m_i+1}) + \rho(x_{2m_i+1}, x_{2m_i+2})] \\
&\quad + \gamma[\rho(x_{2m_i}, x_{2m_i+2}) + \rho(x_{2m_i+1}, x_{2m_i+1})] \\
&\leq \alpha\rho(x_{2m_i}, x_{2m_i+1}) \\
&\quad + \beta[\rho(x_{2m_i}, x_{2m_i+1}) + \rho(x_{2m_i+1}, x_{2m_i+2})] \\
&\quad + \gamma[\rho(x_{2m_i}, x_{2m_i+1}) + \rho(x_{2m_i+1}, x_{2m_i+2})]
\end{aligned}$$

Hence we have

$$\rho(x_{2m_i+1}, x_{2m_i+2}) \leq \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\rho(x_{2m_i}, x_{2m_i+1}).$$

Therefore, for $i > k > N$, we have

$$\begin{aligned}
\rho(x_{2m_i+1}, x_{2m_i+2}) &\leq \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\rho(x_{2m_i}, x_{2m_i+1}) \\
&\leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right)^2 \rho(x_{2m_i-1}, x_{2m_i}) \leq \cdots \\
&\leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right)^{n_i - n_k} \rho(x_{2m_k+1}, x_{2m_k+2}).
\end{aligned}$$

Hence the left side of the last expression tends to 0, as $i \to \infty$, since $\alpha + 2\beta + \gamma < 1$. This contradicts to (1). Hence $T_1(x_0) = x_0$.

By the assumption 1),

$$\begin{aligned}\rho(x_0, T_2(x_0)) &= \rho(T_1(x_0), T_2(x_0)) \\ &\le \alpha\rho(x_0, x_0) + \beta[\rho(x_0, T_1(x_0)) + \rho(x_0, T_2(x_0))] \\ &\quad + \gamma[\rho(x_0, T_2(x_0)) + \rho(x_0, T_1(x_0))] \\ &= \beta\rho(x_0, T_2(x_0)) + \gamma\rho(x_0, T_2(x_0)).\end{aligned}$$

Since $\alpha + 2\beta + 2\gamma < 1$, we have $\rho(x_0, T_2(x_0)) = 0$, which means that $x_0$ is a fixed point of $T_2$.

Let $y_0$ be a common fixed point of $T_1$, $T_2$.

$$\begin{aligned}\rho(x_0, y_0) &= \rho(T_1(x_0), T_2(y_0)) \\ &\le \alpha\rho(x_0, y_0) + \beta[\rho(x_0, T_1(x_0)) + \rho(y_0, T_2(y_0))] \\ &\quad + \gamma[\rho(x_0, T_2(y_0)) + \rho(y_0, T_1(x_0))]\end{aligned}$$

Hence we have

$$\rho(x_0, y_0) \le \alpha\rho(x_0, y_0) + 2\gamma\rho(x_0, y_0),$$

which implies $x_0 = y_0$.

Therefore we complete the proof.

As a special case, we only state the following

**Corollary.** *Let* $(X, \rho)$ *be a metric space, and* $T_1$, $T_2$ *continuous mappings of* $X$ *into itself, satisfying*

$$\rho(T_1(x), T_2(y)) \le \alpha\rho(x, y)$$

*for all* $x$, $y$ *of* $X$, *where* $0 < \alpha < 1$. *If for some* $\xi$ *of* $X$, *the sequence* $x_1 = T_1(\xi)$, $x_2 = T_2(x_1)$, $x_3 = T_1(x_2)$, $\cdots$ *contains a convergent subsequences which converges to* $x_0$, *then* $x_0$ *is a unique common fixed point of* $T_1$, $T_2$.

## References

[1] K. Iséki: Generalization of Kannan fixed point theorems, Math. Seminar Notes, Kobe University, 2 (1974), 19-22.

[2] R. Kannan: Some results on fixed points - II, Amer. Math. Monthly, 76 (1969), 405-408.