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On ∧-Commutative Algebras

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In 1966, the notion of BCK-algebras was introduced by K. Iséki (see [1], [2]). Recently K. Iséki has obtained some results about BCK-algebras (see [3]). In this Note we shall show the relation between BCK-algebras and ∧-commutative algebras. Further we shall prove some properties of ∧-commutative algebras.

By a *BCK-algebra*, we mean a general algebra $M = \langle X; *, 0\rangle$ of type $\langle 2, 0\rangle$ satisfying the following axioms:

(1) $(x * y) * (x * z) \leq z * y$,

(2) $x * (x * y) \leq y$,

(3) $x \leq x$,

(4) $x \leq y$, $y \leq x$ imply $x = y$,

(5) $0 \leq x$,

where $x \leq y$ is defined as $x * y = 0$ (see [1]).

By a *∧-commutative algebra*, we mean a general algebra $N = \langle Y; *, 0\rangle$ of type $\langle 2, 0\rangle$ satisfying the following axioms:

(6) $x \leq x$,

(7) $x \leq y$, $y \leq x$ imply $x = y$,

(8) $x \leq y$, $y \leq z$ imply x z,

(9) $0 \leq x$,

(10) $(x * y) * z \leq (x * z) * y$,

(11) $x * (x * y) \leq y * (y * x)$,

where $x \leq y$ means $x * y = 0$ (see [4]).

In this note, we shall use some properties of both algebras without reference. These properties are contained in [3], [4].

Theorem 1. *In a ∧-commutative algebra, the following hold:*

(12) $(x * y) * (x * z) \leq z$,

(13) $(x * y) * (x * z) \leq z * y$,

(14) $(x * y) * (z * y) \leq x * z$,

(15) $x \leq y$ *implies* $z * y \leq z * x$,

(16) $x \leq y$ *implies* $x * z \leq y * z$.

Proof. (12) follows from

$$(x * y) * (x * z) = (z \wedge x) * y \leq z \wedge x \leq z.$$

To prove (13), note that the following statements are equivalent:

$$(x * y) * (x * z) \leq z * y,\ (z \wedge x) * y \leq z * y,$$

$$(x \wedge z) * (z * y) \leq y,\ \{z * (z * x)\} * (z * y) \leq y.$$

The last statement follows from (12). Therefore (13) holds.

(14), (15) follow from (13). (16) follows from (14).

Theorem 2. *A ∧-commutative algebra is a BCK-algebra.*

Proof. It is sufficient to show that (1) and (2) hold in a ∧-commutative algebra. (1) is obvious by the theorem 1. (2) follows from $x * y \leq x * y$.

Theorem 3. *In a ∧-commutative algebra, the following conditions are equivalent:*

(17) $x \leq x * (y * x)$,

(18) $x * y \leq (x * y) * y$,

(19) $(x * y) * (z * y) \leq (x * z) * y$.

Proof. (17) $\Rightarrow$ (18). Since $x \leq x * (y * x) \leq x$, then $x = x * (y * x)$. Hence we have $(x * y) * \{(x * y) * y\} = y \wedge (x * y) = (x * y) \wedge y = y * \{y * (x * y)\} = y * y = 0$.

(18) $\Rightarrow$ (17). Since $x * y \leq (x * y) * y \leq x * y$, then $x * y = (x * y) * y$. Hence we have $x * \{x * (y * x)\} = (y * x) \wedge x = x \wedge (y * x) = (y * x) * \{(y * x) * x\} = (y * x) * (y * x) = 0$.

(18) = (19). From (14) and (16) we have $\{(x * y) * (z * y)\} * y \leq (x * z) * y$ and so

1) $[\{(x * y) * y\} * (z * y)] * \{(x * z) * y\} = 0$.

On the other hand by (13) and (16) we have $\{(x * y) * (x * z)\} * w \leq (z * y) * w$ and so $\{(x * y) * w\} * (x * z) \leq (z * y) * w$. Substitute $x * y$ for x, $z * y$ for y, $(x * y) * y$ for z, $(x * z) * y$ for w, then we have $[\{(x * y) * (z * y)\} * \{(x * z) * y\}] * [(x * y) * \{(x * y) * y\}] \leq [\{(x * y) * y\} * (z * y)] * \{(x * z) * y\}$. The right side is equal to 0 by 1). And the second term in the left side is equal to 0 by (18). Hence by (7)' and (9) we have

$$(x * y) * (z * y) \leq (x * z) * y.$$

(19) $\Rightarrow$ (18). Substitute $z * y$ for x, $(z * x) * y$ for y, $x * y$ for z in (19). Then we have

$$[(z * y) * \{(z * x) * y\}] * [(x * y) * \{(z * x) * y\}]$$
$$\leq \{(z * y) * (x * y)\} * \{(z * x) * y\}.$$

The right side is equal to 0 by (19). Hence we have $(z * y) * \{(z * x) * y\} \leq (x * y) * \{(z * x) * y\}$. We put $z = x$, $x = y$, then we have

$$(x * y) * \{(x * y) * y\} \leq (y * y) * \{(x * y) * y\} = 0.$$

Hence we have

$$x * y \leq (x * y) * y.$$

Theorem 4. *In a bounded $\wedge$-commutative algebra, if there exist the compliments of each element, then they are unique.*

Proof. Suppose y and y' are compliments of $x \in Y$. $x \wedge y = 0$, i.e., $y * (y * x) = 0$ and so $y \leq y * x \leq y$, from which $y = y * x$. Similarly $y' = y' * x$. By these and $y' \vee x = 1$, we have

$$y * y' = (y * x) * (y' * x) = y * (y' \vee x) = y * 1 = 0,$$

which shows $y \leq y'$. Similarly we have $y' \leq y$. Therefore $y = y'$, which completes the proof of the theorem.

Theorem 5. *In a bounded $\wedge$-commutative algebra, the following conditions are equivalent*:

(17) $x \leq x * (y * x)$,

(18) $x * y \leq (x * y) * y$,

(19) $(x * y) * (z * y) \leq (x * z) * y$,

(20) $x \wedge Nx = 0$,

(21) $x \vee Nx = 1$,

(22) $x \leq x * Nx$,

(23) $Nx \leq Nx * x$,

(24) $x * (x * Ny) \leq x * y$,

(25) $x * (x * y) \leq x * Ny$,

(26) $\{x * (y * z)\} * (x * y) \leq x * Nz$.

Proof. We have already proved that (17), (18) and (19) are equivalent

each other.

(17) $\Rightarrow$ (20). $x \wedge Nx = Nx \wedge x = x * (x * Nx) = x * \{x * (1 * x)\} = 0$.

(20) $\Leftrightarrow$ (21). $x \wedge Nx = 0 \Leftrightarrow N(x \wedge Nx) = N0 \Leftrightarrow Nx \vee NNx = 1 \Leftrightarrow x \vee Nx = Nx \vee x = 1$.

(20) $\Leftrightarrow$ (22). $x * \{x * Nx\} = Nx \wedge x = 0 \Leftrightarrow x \leq x * Nx$.

(22) $\Leftrightarrow$ (23). Let $x = Nx$ in (22), then $Nx \leq Nx * NNx$.

(24) $\Leftrightarrow$ (25). This follows from $x * (x * Ny) \leq x * y \Leftrightarrow x * (x * y) \leq x * Ny$.

(22) $\Rightarrow$ (25). By (13) we have

$$\{x * (y * z)\} * \{x * (y * w)\} \leq (y * w) * (y * w) \leq z * w.$$

Hence by (8), we have $\{x * (y * z)\} * (z * w) \leq x * (y * w)$. In this let $y = x$, $z = y$, $w = Nx$, then by (22), we have $\{x * (x * y)\} * (y * Nx) \leq x * (x * Nx) = 0$. Therefore

$$x * (x * y) \leq y * Nx = x * Ny.$$

(25) $\Leftrightarrow$ (26). First we shall prove $y \wedge x = x * Ny = y * Nx$. $y \wedge x = x * (x * y) \leq x * Ny$. On the other hand by (13) $(x * Ny) * \{x * (x * y)\} \leq (x * y) * Ny = (x * y) * (1 * y) \leq x * 1 = 0$ and so $x * Ny \leq x * (x * y) = y \wedge x$. Hence $x \wedge y = y \wedge x = x * Ny = y * Nx$. From this and (13)

$$\begin{aligned} \{x * (y * z)\} * (x * y) &= \{x * (x * y)\} * (y * z) \\ &= (y \wedge x) * (y * z) \\ &= (y * Nx) * (y * z) \leq z * Nx = x * Nz. \end{aligned}$$

(26) $\Leftrightarrow$ (19). Since $x \leq 1$, then by (16) we have $x * y \leq 1 * y$ and so $(x * y) * Ny = 0$. In (26) substitute $x * y$ for x, we have by above $\{(x * y) * (z * y)\} * \{(x * y) * z\} \leq (x * y) * Ny = 0$. Hence by (10) $(x * y) * (z * y) \leq (x * y) * z \leq (x * z) * y$.

Remarks. An axiom system of a $\wedge$-commutative algebra is characterized by (6), (7), (9), (10), (11) and (13). (20) or (21) shows that a bounded $\wedge$-commutative algebra with (17) is a complimented lattice with respect to $\wedge$ and $\vee$. The condition (25) shows that in a bounded $\wedge$-commutative algebra with (17),

$$x \wedge y = y \wedge x = x * Ny = y * Nx$$

$$x \vee y = y \vee x = N(Nx * y) = N(Ny * x).$$

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