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Examples of BCK-Algebras

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The notion of BCK-algebras was introduced by K. Iséki (see [2]). The present author introduced the notion of $\wedge$ -commutative algebras and showed that any $\wedge$ -commutative algebra is a BCK-algebra (see [3], [4]). In this Note we shall give examples for a BCK-algebra and examples for a $\wedge$ -commutative algebra.

By a *BCK-algebra*, we mean a general algebra $M = \langle X; *, 0 \rangle$ of type $\langle 2, 0 \rangle$ satisfying the following axioms:

- (1) $(x * y) * (x * z) \leq z * y,$
- (2) $x * (x * y) \leq y,$
- (3) $x \leq x,$
- (4) $x \leq y, y \leq x$ imply $x = y,$
- (5) $0 \leq x,$

where $x \leq y$ is defined as $x * y = 0.$

A *$\wedge$ -commutative algebra* is characterized by (1), (3), (4), (5) and

- (6) $(x * y) * z \leq (x * z) * y,$
- (7) $x * (x * y) \leq y * (y * x).$

I. Example of BCK-algebra

Consider an ordered set X with the least element 0 such that every pair of non-zero distinct elements is incomparable.

Then $M = \langle X; *, 0 \rangle$ is a BCK-algebra if and only if for any element x, y , we have the following condition:

$$x * y = x \text{ if } x \not\leq y, \text{ and } x * y = 0 \text{ if } x \leq y.$$

where $\leq$ is the order on X .

To prove sufficiency, we shall verify axioms of a BCK-algebra. We first verify the condition (1). Let $x = 0$. Then we have

$$(x * y) * (x * z) = 0 * 0 = 0 \leq z * y.$$

Let $x \neq 0$. Then we have

$$(x * y) * (x * z) = x * x = 0 \leq z * y.$$

Next we verify the condition (2). Let $x = 0$. We have

$$x * (x * y) = 0 * 0 = 0 \leq x.$$

Let $x \neq 0$. Then $x * (x * y) = x * x = 0 \leq x$. It is obvious that the conditions (3), (4), (5) hold in M .

We shall prove the necessity. Note that $(x * y) * z = (x * z) * y$ holds in a BCK-algebra (see [2]). Suppose that $x * y = u \neq x$ for some non-zero x and y with $x \not\leq y$. Then $u \neq 0$, since $x \neq 0, x \not\leq y$.

Then we have

$$(x * x) * y = 0 * y = 0,$$

$$(x * y) * x = u * x.$$

Hence by $(x * y) * z = (x * z) * y$, we have $u * x = 0$. This implies $u = 0$ or $u = x$, which contradict. Hence $x * y = x$ if $x \not\leq y$.

Moreover M is a $\wedge$ -commutative algebra.

To verify this, we show that (7) holds. For $x = 0$, $x * (x * y) = 0 = y * y = y * (y * 0) = y * (y * x)$. Similarly it is obvious that (7) holds for $y = 0$. Let $x \neq 0$. If $x = y$, then we have

$$x * (x * y) = x * 0 = x = y = y * 0 = y * (y * x).$$

If $x \neq y$ ($y \neq 0$), then we have

$$x * (x * y) = x * x = 0 = y * y = y * (y * x).$$

Furthermore M is an I -algebra (see [1]). To prove this we shall show that $x * y \leq x$, $x \leq x * (y * x)$ hold in M . The former is obvious. To verify the latter we put $x = 0$, $x = 0 = 0 * (y * x) = x * (y * x)$. For $x \neq 0$ and $y = x$, we have

$$x = x * 0 = x * (y * x).$$

For $x \neq 0$ and $y = x$, we have

$$x = x * y = x * (y * x).$$

II. Example of $\wedge$ -commutative algebra

Consider the set $X = \{0, 1, 2, \dots\}$, i.e., the set of all natural numbers.

$M = \langle X; *, 0 \rangle$ is a $\wedge$ -commutative algebra if and only if for any element x, y we have

$$x * y = \begin{cases} 0, & \text{if } x \leq y, \\ x - y, & \text{if } y < x. \end{cases}$$

To prove the sufficiency we shall verify axioms of a $\wedge$ -commutative algebra. It is known that M is a BCK-algebra (see [3]). We show that the condition (7) holds. If $x < y$, then we have

$$x * (x * y) = x * 0 = x,$$

$$y * (y * x) = y * (y - x) = y - (y - x) = x.$$

If $x = y$, then we easily see that (7) holds. If $x > y$, then we have

$$x * (x * y) = x * (x - y) = x - (x - y) = y,$$

$$y * (y * x) = y * 0 = y.$$

Next we shall prove the necessity. We first remark that $A(a) = \{x | x \leq a\}$ is a subalgebra of M . Let $x \in A(a)$. $y \in A(a)$, then we have $x \leq a$

and so $x * y \leq x \leq a$. Hence $x * y \in A(a)$. Therefore $A(a)$ is closed under the binary operation $*$ for any a .

(8) For any $i \geq 1$, $i * 0 = i$.

Proof. Let $i * 0 = p$, $x = i$. $y = 0$ in (6), then we have

$$x * (x * y) = i * (i * 0) = i * p,$$

$$y * (y * x) = 0 * (0 * i) = 0 * 0 = 0.$$

Hence $i * p = 0$ and so $i \leq p$. Since $A(i)$ is closed, $p \leq i$

Therefore $i = p$.

(9) For $j < i$, $i * j \neq 0$.

Proof. $i * j = 0 \nRightarrow i \leq j$.

(10) For $j < i$, $i * j = k$ implies $i * k = j$.

Proof. By (7), $i * k = i * (i * j) = j * (j * i) = j * 0 = j$.

(11) For any i , $i * j$ is a decreasing sequence with respect to j .

We show that $i * j = p$, $i * (j + 1) = q$ imply $p > q$.

Case 1. Suppose $p < q$. $(i * (j + 1)) * p = q * p$. Since $A(q)$ is closed, $0 < q * p \leq q$. On the other hand, from (6) and (10),

$$(i * (j + 1)) * p = (i * p) * (j * 1) = j * (j + 1) = 0,$$

which contradicts.

Case 2. Suppose $p = q$. By (10), we have $i * p = j$. Similarly from $i * (j + 1) = q = p$, we have $i * p = j + 1$, which contradicts. Therefore the assertion is proved.

Remark. From the above facts we see that $i * j$ ($j = 0, 1, 2, \dots$) is equal to

$$i, i - 1, i - 2, \dots, 1, 0, 0, \dots.$$

M is not an I-algebra. To prove this, put $x = 1, y = 2$, then we have

$$\begin{aligned} x * (x * (y * x)) &= 1 * (1 * (2 * 1)) = 1 * (1 * 1) \\ &= 1 * 0 = 1 > 0. \end{aligned}$$

Let $X = \{0, 1, \dots, m\}$. $M = \langle X; *, 0 \rangle$ is a $\wedge$ -commutative algebra, but not an I-algebra.

Let $X = \{0, 1, \dots, m\}$. $M = \langle X; *, 0 \rangle$ is an I-algebra if and only if $m \leq 1$.

III. Example of $\wedge$ -commutative algebra

Consider m chains A_i ($i = 1, \dots, m$), where A_i consists of

$$0 = a_{i0} < a_{i1} < \dots < a_{ij} < \dots$$

Let $X = \bigcup_{i=1}^m A_i$. For $x, y \in X$, we define

$$(12) \quad x * y = \begin{cases} 0, & \text{if } x \leq y, \\ a_{i, k-l}, & \text{if } y < x \text{ and } x = a_{ik}, y = a_{il}, \\ x, & \text{otherwise.} \end{cases}$$

of course $\leq$ and $<$ are the order on X . Then $M = \langle X; *, 0 \rangle$ is a $\wedge$ -commutative algebra.

We first verify the condition (1). If x, y, z are in three different chains A_i, A_j, A_k , respectively, then (1) holds by the example I. If x, y, z are in the same chain A_i , then (1) holds by the example II. For the remainder, we consider three cases.

Case 1: $x, y \in A_i, z \in A_j$.

If $x \leq y$, then $(x * y) * (x * z) = 0 * z = 0 \leq z * y$. If there is no confusion, we write a_k for a_{ik} . If $y < x$, then $(x * y) * (x * z)$
 $= (a_k * a_l) * (a_k * z) = a_{k-l} * a_k = 0 \leq z * y$.

Case 2: $x, z \in A_i, y \in A_j$.

If $x \leq z$, then $(x * y) * (x * z) = (a_k * y) * (a_k * a_l) = a_k * 0 = a_k < a_l = z * y$. If $z < x$, then $(x * y) * (x * z) = a_k * a_{k-l} = a_l \leq z * y$.

Case 3: $y, z \in A_i, x \in A_j$.

Then we have $(x * y) * (x * z) = x * x = 0 \leq z * y$.

Next we verify the condition (6). The proof is analogous to the proof of (1). We need to prove the following two cases.

Case 1: $x, y \in A_i, z \in A_j$.

If $x \leq y$, then $(x * y) * z = 0 * z = 0 = x * y = (x * z) * y$. If $y < x$, then $(x * y) * z = (a_k * a_l) * z = a_k * a_l = (a_k * z) * a_l = (x * z) * y$.

Case 2: $y, z \in A_j, x \in A_i$.

Then $(x * y) * z = x * z = x = x * y = (x * z) * y$.

Finally we verify the condition (7). If x, y are in the different chains A_i, A_j , then (7) holds by the example I. If x, y are in the same chain A_i , then (7) holds by the example II.

It is obvious the condition (3), (4) and (5) hold in M . Hence M is a $\wedge$ -commutative algebra. But M is not an I -algebra.

To prove this, put $x = a_1, y = a_2$. Then we have

$$\begin{aligned} x * \{x * (y * x)\} &= a_1 * \{a_1 * (a_2 * a_1)\} \\ &= a_1 * (a_1 * a_1) = a_1 * 0 = a_1 \neq 0. \end{aligned}$$

Further the example I is a special case of the example III. If in the example III, $A_i = \{a_{i0}, a_{i1}\}$ ($a_{i0} = 0$ and $a_{i0} < a_{i1}$), then we have the example I.

IV. Example of BCK-algebra

Consider the ordered set X which was constructed in the example III.

For $x, y \in X$, we define

$$x * y = \begin{cases} 0, & \text{if } x \leq y, \\ a_{i1} & \text{if } y < x \text{ and } y \neq 0 \text{ i.e., } x = a_{ik}, y = a_{il}, \\ x & \text{otherwise,} \end{cases}$$

Then $M = \langle X; *, 0 \rangle$ is a BCK-algebra.

K. Iséki has shown that if x, y, z are in the same chain A_i , then the axioms (1)-(5) hold (see [3]). For the remainder, we consider three cases.

Case 1: $x, y \in A_i, z \in A_j$.

If $x \leq y$, then $(x * y) * (x * z) = 0 * x = 0 \leq z * y$. If $y < x$ and $y \neq 0$, then since $a_1 < x$, $(x * y) * (x * z) = a_1 * x = 0 \leq z * y$. If $y < x$ and $y = 0$, then $(x * y) * (x * z) = x * x = 0 \leq z * y$.

Case 2: $x, z \in A_i, y \in A_j$.

If $x \leq z$, then $(x * y) * (x * z) = x * 0 = x \leq z = z * y$. If $z < x$ and $z \neq 0$, then since $a_1 < x$ and $a_1 < z$, we have $(x * y) * (x * z) = x * a_1 = a_1 \leq z = z * y$.

Case 3: $y, z \in A_i, x \in A_j$.

$(x * y) * (x * z) = x * x = 0 \leq z * y$. Therefore (1) holds in M .

Next we verify the condition (2). Suppose $x \in A_i$ and $y \in A_j$. Then $x * (x * y) = x * x = 0 \leq y$. Hence (2) holds in M .

The other conditions are easily verified. Therefore M is a BCK-algebra. But M is not a $\wedge$ -commutative algebra.

To prove this, put $x = a_2, y = a_3$. Then we have

$$x * (x * y) = a_2 \neq y * (y * x) = a_1.$$

Moreover M is not an I-algebra. To prove this, put $x = a_1, y = a_3$. Then we have

$$x * \{x * (y * x)\} = a_1 \neq 0.$$

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