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Jaiswal, Anuradha
Singh, Bijendra

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The Hahn-Banach Extension Theorem in Normed Linear Spaces over Topological Semifield

By Anuradha JAISWAL and Bijendra SINGH

The object of this Note is to generalize the well known Hahn-Banach extension theorem. We consider a normed linear space over a topological semifield. For related concepts, see [1] and [2].

Let X be a normed linear space over a topological semifield K , and M a subspace of X . We define

$$\|f\|_M = \sup_{x \in M, \|x\| \ll 1} |f(x)|.$$

Consider the set $M + \alpha x_0$ of all elements $x + \alpha x_0, x_0 \notin M$, where $x \in M$ and α is an arbitrary element in K . It is easy to see that $M + \alpha x_0$ is uniquely represented in the form of $x + \alpha x_0$. If $x', x'' \in M$, then, we get

$$\begin{aligned} f(x') - f(x'') &= f(x' - x'') \ll \|f\| \|x' - x''\| \\ &\ll \|f\| \{ \|x' + x_0\| + \|x'' + x_0\| \}^{1)} \end{aligned}$$

it follows that

$$f(x') - \|f\| \|x' + x_0\| \ll f(x'') + \|f\| \|x'' + x_0\|.$$

Elements x', x'' are arbitrary in K , then, by a result in [1, p. 10], we have

¹⁾ The symbol $\ll$ means the order in K (see [1, p. 7]).

$$n = \inf_{x \in M} \{f(x) - \|f\| \|x + x_0\|\} \ll \inf_{x \in M} \{f(x) + \|f\| \|x + x_0\|\} = N,$$

where n, N are contained in K . Therefore, we can take an element η in K such that

$$(1) \quad n \ll \eta \ll N.$$

Let $y \in \{M + \alpha x_0\}$, then we define a function $p(y)$ as

$$p(y) = f(x) - \alpha\eta.$$

Obviously the functional $p(y)$ is additive and homogeneous in M . Now, we shall show that $p(y)$ is bounded and possesses the same norm as $f(x)$.

For, consider $\alpha \gg 0$, $\frac{x}{\alpha} \in M$, then from (1) we have

$$p(y) = \alpha \{f(\frac{x}{\alpha}) - \eta\} \ll \alpha \{ \|f\| \|\frac{x}{\alpha} + x_0\| \} = \|f\| \|x + \alpha x_0\| = \|f\| \cdot \|y\|,$$

so that, $p(y) \ll \|f\| \cdot \|y\|$. Again, if $\alpha \ll 0$, then from (1), we obtain

$$f(\frac{x}{\alpha}) - \eta \gg -\|f\| \cdot \|\frac{x}{\alpha} + x_0\| = -\frac{1}{|\alpha|} \|f\| \cdot \|y\| = \frac{1}{\alpha} \|f\| \cdot \|y\|.$$

This implies that

$$p(y) = \alpha \{f(\frac{x}{\alpha}) - \eta\} \ll \alpha \frac{1}{\alpha} \|f\| \cdot \|y\| = \|f\| \cdot \|y\|.$$

which holds for every y in $\{M + \alpha x_0\}$. We replace y by $-y$, we have

$$-p(y) \ll \|f\| \cdot \|y\|.$$

Hence

$$\|p\| \ll \|f\|.$$

Since p is an extension of f from M to $\{M + \alpha x_0\}$, so we have

$$\|p\| \gg \|f\|.$$

Therefore

$$\|p\| = \|f\|.$$

and we have the following

Theorem. *Let X be a normed linear space over a topological semifield K , M a subspace of X and $p(x)$ an additive and homogeneous functional*

defined on M . Then there exists a functional f defined on X such that i) $f(x)$ is an extension of $p(x)$ and ii) $\|f\|_M = \|p\|_M$.

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References

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