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Fixed Point Theorem in 2-Metric Spaces

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S. Gähler [2] introduced a notion called 2-metric (normed) space and these spaces have been developed by S. Gähler, A. G. White [4] and C. Diminnie.

S. Gähler defines a *2-metric space* to be a set satisfying the following conditions:

(1a) For each pair x, y with $x \neq y$, there exists a point $z \in X$ such that

$$\rho(x, y, z) \neq 0,$$

(1b) $\rho(x, y, z) = 0$ only when at least two of three points are equal,

(2) $\rho(x, y, z) = \rho(x, z, y) = \rho(y, z, x)$,

(3) $\rho(x, y, z) \leq \rho(x, y, u) + \rho(x, y, z) + \rho(u, y, z)$.

Let $y = z$ in (3), then we have

$$0 \leq \rho(x, y, u) + \rho(x, u, y) = 2\rho(x, y, u).$$

Hence ρ *is non-negative.*

Definition 1. If $\rho(x, y, z)$ is bounded, the 2-metric space is called to be *bounded.* By $\delta(X)$, we mean its diameter, i.e. $\sup_{x, y, z \in X} \rho(x, y, z)$.

Definition 2. If $\rho(x_n, x, a)$ converges to 0 for all $a \in X$, we say that the sequence $\{x_n\}$ converges to x, and x is a limit of $\{x_n\}$.

Definition 3. If, in a 2-metric space X, $\rho(x_m, x_n, a) \to 0$ $(m, n \to \infty)$ for all $a \in X$, the sequence $\{x_n\}$ is called a *Cauchy sequence*. If every Cauchy sequence is convergent, X is called *complete*.

The present author proved a fixed point theorem about contraction type mapping on a 2-metric space in a joint paper with P. L. Sharma and S. K. Sharma [3].

In this Note, we shall prove a fixed point theorem which is proved for a 1-metric space by L. B. Ćirić [1].

To state theorem, we shall introduce a notion of orbitally continuity.

Definition 4. Let T be a mapping of a 2-metric space X into itself. If for all $a \in X$

$$\rho(T^{n_i}x, u, a) \to 0 \qquad (i \to \infty)$$

implies

$$\rho(TT^{n_i}x, Tu, a) \to 0 \qquad (i \to \infty)$$

then T is called *orbitally continuous*.

Theorem. *Let T be an orbitally continuous mapping of a bounded, complete 2-metric space T into itself. If T satisfies the condition:*

$$(4)\quad \operatorname{Min}\{\rho(Tx, Ty, a), \rho(x, Tx, a), \rho(y, Ty, a), - \operatorname{Min} \rho(x, Ty, a), \rho(y, Tx, a)\} \leq q \cdot \rho(x, y, a)$$

for all $x, y, a \in X$ and for some q with $0 < q < 1$ then for each $x \in X$, the sequence $\{T^n x\}$ $(n = 1, 2, \cdots)$ converges to a fixed point of T.

Proof. Let x be an arbitrary point of X. We define a sequence:

$$x_0, x_1 = Tx_0, x_2 = Tx_1, \cdots, x_n = Tx_{n-1}, \cdots.$$

If for some x, $x_n = x_{n+1}$, then $\{x_n\}$ is a Cauchy sequence, and the limit of $\{x_n\}$ is a fixed point of T.

Suppose that $x_n \neq x_{n+1}$ for every $n = 0, 1, 2, \cdots$. By (4), we have

$$\begin{aligned}&\mathrm{Min}\{\rho(Tx_{n-1}, Tx_n, a), \rho(x_{n-1}, Tx_{n-1}, a), \rho(x_n, Tx_n, a)\}\\ &- \mathrm{Min}\{\rho(x_{n-1}, Tx_n, a), \rho(x_n, Tx_{n-1}, a)\}\\ &= \mathrm{Min}\{\rho(x_n, x_{n+1}, a), \rho(x_{n-1}, x_n, a)\} \leq qd(x_{n-1}, x_n, a).\end{aligned}$$

Since X is a 2-metric space, $\rho(x_{n-1}, x_n, a) \neq 0$ for some $a \in X$.

Hence $q\rho(x_{n-1}, x_n, a) < \rho(x_{n-1}, x_n, a)$, which implies

$$\rho(x_n, x_{n+1}, a) \leq q\rho(x_{n-1}, x_n, a).$$

Therefore we have

$$\rho(x_n, x_{n+1}, a) \leq q^n\rho(x_0, x_1, a)$$

for $n = 1, 2, \cdots$. For any m, we have

$$\begin{aligned}\rho(x_n, x_{n+m}, a) &\leq \rho(x_n, x_{n+m}, x_{n+1}) + \rho(x_n, x_{n+1}, a) + \rho(x_{n+1}, x_{n+m}, a)\\ &\leq \rho(x_n, x_{n+1}, x_{n+m}) + \rho(x_n, x_{n+1}, a) + \rho(x_{n+1}, x_{n+m}, a)\\ &\leq \rho(x_n, x_{n+1}, x_{n+m}) + \rho(x_n, x_{n+1}, a)\\ &+ \rho(x_{n+1}, x_{n+m}, x_{n+2}) + \rho(x_{n+1}, x_{n+2}, a) + \rho(x_{n+2}, x_{n+m}, a)\\ &\leq \cdots\\ &\leq \rho(x_n, x_{n+1}, x_{n+m}) + \rho(x_n, x_{n+1}, a) + \rho(x_{n+1}, x_{n+2}, x_{n+m})\\ &+ \rho(x_{n+1}, x_{n+2}, a) + \cdots\\ &+ \rho(x_{n+m-2}, x_{n+m-1}, x_{n+m}) + \rho(x_{n+m-1}, x_{n+m}, a)\\ &\leq \Sigma_{k=n}^{n+m} q^k\rho(x_0, x_1, x_{n+m}) + \Sigma_{k=n}^{n+m} q^k\rho(x_0, x_1, a) \leq 2k\Sigma_{k=n}^{n+m} q^k\\ &\leq \frac{2kq^n}{1-q}.\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence. X is complete, so $\{x_n\}$ converges to a point u.

Hence we have for all $a \in X$,

$$\lim_{n\to\infty} \rho(T^n x_0, u, a) = 0.$$

Since T is orbitally continuous, we have

$$\lim_{n\to\infty} (T^{n+1}x_0, Tu, a) = 0$$

for all $a \in X$.

Therefore

$$\rho(u, Tu, a) \le \rho(u, Tu, T^{n+1}x_0) + \rho(u, T^{n+1}x_0, a)$$
$$+ \rho(T^{n+1}x_0, Tu, a) \to 0 \ (n \to \infty).$$

Hence $\rho(u, Tu, a) = 0$. If $u \neq Tu$, then $\rho(u, Tu, a) \neq 0$ for some a. This implies $u = Tu$.

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