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On Linear Bornologies and Linear Convergence Structures

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The purpose of this note is to state several fundamental relations between the two notions in the title. The main result is that the convergence in the Mackey sense defined by Hogbe-Nlend [2] is nothing more than that under a linear convergence structure determined by the given linear bornology (Theorem 3 and Theorem 4 below).

Linear spaces considered in this note are defined on the field K of real or complex numbers. The neighborhood filter of $0 \in K$ for the usual topology on K will be denoted by $\mathcal{W}$. For every positive real number ε , V_ε stands for the set of all $\xi \in K$ with $|\xi| \leq \varepsilon$.

Let E be a linear space. If Φ and Ψ are filters on E and $\lambda \in K$ then $\Phi + \Psi$, $\lambda\Phi$ and $\mathcal{W}\Phi$ denote the filters on E generated by the filter bases $\{A + B \mid A \in \Phi, B \in \Psi\}$, $\{\lambda A \mid A \in \Phi\}$ and $\{VA \mid V \in \mathcal{W}, A \in \Phi\}$ respectively. Further if A is a subset of E then $\mathcal{W}A$ denotes the filter on E generated by the filter base $\{VA \mid V \in \mathcal{W}\}$; the filter $\mathcal{W}\{x\}$ is also written $\mathcal{W}x$, where $x \in E$. A *linear convergence structure* on E is a convergence structure on E under which the mappings $+: E \times E \rightarrow E$ and $\cdot: K \times E \rightarrow E$ are continuous. A *linear bornology* $\mathcal{B}$ on E is a cover of E satisfying the following conditions:

- (1) If $A \subset B \in \mathcal{B}$ then $A \in \mathcal{B}$.

(2) If $A, B \in \mathcal{B}$ then $A + B \in \mathcal{B}$.

(3) If $B \in \mathcal{B}$ and $\lambda \in K$ then $\lambda B \in \mathcal{B}$.

(4) If $B \in \mathcal{B}$ then $V_1 B \in \mathcal{B}$.

Let τ be a linear convergence structure on E . A subset B of E is said to be τ -bounded if $\mathbb{W}B$ belongs to τ_0 . If τ is a linear topology on E , then a subset of E is τ -bounded if and only if it is bounded in the topological linear space (E, τ) . Let $\mathcal{B}(\tau)$ denote the set of all τ -bounded subsets of E . We have then the following

Theorem 1. *If τ is a linear convergence structure on a linear space E , then $\mathcal{B}(\tau)$ is a linear bornology on E .*

Proof. Since $\mathbb{W}x \in \tau_0$ for every $x \in E$, $\mathcal{B}(\tau)$ is a cover of E . If $A \subset B \in \mathcal{B}(\tau)$ then since $\mathbb{W}B \subset \mathbb{W}A$ and $\mathbb{W}B \in \tau_0$, we have $A \in \mathcal{B}(\tau)$. If $A, B \in \mathcal{B}(\tau)$, then $\mathbb{W}A + \mathbb{W}B \in \tau_0$, and so the relation $\mathbb{W}A + \mathbb{W}B \subset \mathbb{W}(A + B)$ yields $A + B \in \mathcal{B}(\tau)$. Assume $B \in \mathcal{B}(\tau)$ and $\lambda \in K$. If $\lambda = 0$ then $\lambda B = \{0\} \in \mathcal{B}(\tau)$; if $\lambda \neq 0$ then $\mathbb{W}(\lambda B) = \mathbb{W}B \in \tau_0$ and hence $\lambda B \in \mathcal{B}(\tau)$. Now if $B \in \mathcal{B}(\tau)$ then $V_1 B \in \mathcal{B}(\tau)$ since $\mathbb{W}(V_1 B) = \mathbb{W}B$. This completes the proof.

Following Fischer [1], we say that a convergence structure τ on a set X is *finer than* another convergence structure τ' on X if τx is contained in $\tau' x$ for all $x \in X$. Obviously we obtain:

Theorem 2. *Let τ, τ' be two linear convergence structures on a linear space. If τ is finer than τ' then $\mathcal{B}(\tau) \subset \mathcal{B}(\tau')$.*

However the converse is not true, since each compatible topology with a given dual system determines the same family of bounded sets (see for example Horváth [3]).

Now, for each linear bornology $\mathcal{B}$ on a linear space E , we define a mapping $\gamma(\mathcal{B})$ of E into the set $\mathcal{F}(E)$ of all filters on E as follows:

$$\gamma(\mathcal{B})(0) = \{\Phi \in \mathcal{F}(E) \mid \forall B \in \mathcal{B} \quad \Phi \subset B\}$$

and

$$\gamma(\mathcal{B})(x) = \gamma(\mathcal{B})(0) + x$$

for every $x \in E$. Then we have the following

Theorem 3. *If $\mathcal{B}$ is a linear bornology on a linear space E , then $\gamma(\mathcal{B})$ is a linear convergence structure on E .*

Proof. Since $\{0\} \in \mathcal{B}$ and $\mathcal{W}\{0\}$ coincides with the filter $\dot{0}$ on E generated by the filter base $\{\{0\}\}$, $\dot{0}$ belongs to $\gamma(\mathcal{B})(0)$. It is clear that $\Phi \in \gamma(\mathcal{B})(0)$ and $\Phi \subset \Psi \in \mathcal{F}(E)$ imply $\Psi \in \gamma(\mathcal{B})(0)$. If Φ, Ψ are in $\gamma(\mathcal{B})(0)$, then we can choose $A, B \in \mathcal{B}$ such that $\mathcal{W}A \subset \Phi$ and $\mathcal{W}B \subset \Psi$; since $V_1A + V_1B \in \mathcal{B}$ and

$$\mathcal{W}(V_1A + V_1B) \subset \mathcal{W}A + \mathcal{W}B \subset \Phi + \Psi,$$

$\Phi + \Psi$ belongs to $\gamma(\mathcal{B})(0)$. Let $\Phi \in \gamma(\mathcal{B})(0)$ and $\lambda \in K$. Then we have $\mathcal{W}B \subset \Phi$ for some $B \in \mathcal{B}$. If $\lambda = 0$ then $\lambda\Phi = \dot{0} \in \gamma(\mathcal{B})(0)$; if $\lambda \neq 0$ then $\mathcal{W}B \subset \lambda\mathcal{W}B \subset \lambda\Phi$. This places $\lambda\Phi$ in $\gamma(\mathcal{B})(0)$. Now $\mathcal{W}B \subset \Phi$ implies $\mathcal{W}B = \mathcal{W}\mathcal{W}B \subset \mathcal{W}\Phi$, and consequently if Φ belongs to $\gamma(\mathcal{B})(0)$ then so does $\mathcal{W}\Phi$. If x is in E , then since $\{x\} \in \mathcal{B}$, $\mathcal{W}x$ belongs to $\gamma(\mathcal{B})(0)$. This completes the proof.

The following result shows that a filter Φ on a linear space E with a linear bornology $\mathcal{B}$ converges in the Mackey sense to $x \in E$ if and only if $\Phi \in \gamma(\mathcal{B})(x)$.

Theorem 4. *If $\mathcal{B}$ is a linear bornology on a linear space E then*

$\gamma(\mathcal{B})(0)$ coincides with the set $\mathcal{O}$ of all $\Phi \in \mathcal{F}(E)$ with the property that there is a $B \in \mathcal{B}$ such that $\lambda B \in \Phi$ for every non-zero $\lambda \in K$.

Proof. If $B \in \mathcal{B}$ and $WB \subset \Phi$, then we have $V_1 B \in \mathcal{B}$ and $\lambda V_1 B = V_1 |\lambda| B \in WB \subset \Phi$ for every non-zero $\lambda \in K$, which shows that $\gamma(\mathcal{B})(0)$ is contained in $\mathcal{O}$. Conversely if $B \in \mathcal{B}$ and if λB belongs to a filter Φ on E for every non-zero $\lambda \in K$, then for each positive real number ε , we have $\varepsilon B \in \Phi$ and $\varepsilon B \subset \varepsilon V_1 B = V_\varepsilon B$, which imply $V_\varepsilon B \in \Phi$, and thus $\Phi \in \gamma(\mathcal{B})(0)$.

Theorem 5. Let $\mathcal{B}, \mathcal{B}'$ be two linear bornologies on a linear space. Then $\gamma(\mathcal{B})$ is finer than $\gamma(\mathcal{B}')$ if and only if $\mathcal{B} \subset \mathcal{B}'$.

Proof. It will suffice to prove the "only if" part. To this end, let $B \in \mathcal{B}$ be arbitrary. Then since $WB \in \gamma(\mathcal{B})(0) \subset \gamma(\mathcal{B}')(0)$, we have $WB' \subset WB$ for some $B' \in \mathcal{B}'$. Hence we can find an $\varepsilon > 0$ such that $V_\varepsilon B \subset V_1 B'$, and so we obtain $\varepsilon B \subset \varepsilon V_1 B = V_\varepsilon B \subset V_1 B'$. Since $V_1 B' \in \mathcal{B}'$, this shows that $\varepsilon B \in \mathcal{B}'$, and so $B \in \mathcal{B}'$.

Theorem 6. For each linear space E the following statements hold.

- (1) If τ is a linear convergence structure on E , then $\gamma(\mathcal{B}(\tau))$ is finer than τ .
- (2) If $\mathcal{B}$ is a linear bornology on E , then $\mathcal{B}(\gamma(\mathcal{B})) = \mathcal{B}$.

Proof. If Φ is in $\gamma(\mathcal{B}(\tau))$, then there is a $B \in \mathcal{B}(\tau)$ for which $WB \subset \Phi$; since WB belongs to τ_0 , so does Φ . This establishes (1). Now since $B \in \mathcal{B}$ implies $WB \in \gamma(\mathcal{B})(0)$, to prove (2) it suffices to verify that $\mathcal{B}(\gamma(\mathcal{B})) \subset \mathcal{B}$. Let $B \in \mathcal{B}(\gamma(\mathcal{B}))$ be arbitrary. Then since $WB \in \gamma(\mathcal{B})(0)$, we have $WA \subset WB$ for some $A \in \mathcal{B}$. Hence we can find an $\varepsilon > 0$

such that $\varepsilon V_1 B \subset V_1 A$, and so $\varepsilon V_1 B \in \mathcal{B}$. Therefore we have

$$B \subset V_1 B = \frac{1}{\varepsilon}(\varepsilon V_1 B) \in \mathcal{B}.$$

which implies $B \in \mathcal{B}$.

Remark 1. As is shown in [2], if τ is a linear topology on a linear space E , then $\gamma(\mathcal{B}(\tau)) = \tau$ if and only if there is a τ -bounded τ -neighborhood of $0 \in E$. Hence in general, $\gamma(\mathcal{B}(\tau))$ does not coincide with τ . In [2], a topological linear space (E, τ) is said to satisfy the *condition of Mackey convergence for filters* in case $\gamma(\mathcal{B}(\tau)) = \tau$.

Remark 2. Let $\mathcal{B}$ be a linear bornology on a linear space E . From Theorem 6 it follows that $\gamma(\mathcal{B})$ is the finest linear convergence structure τ on E for which all members of $\mathcal{B}$ are τ -bounded.

Theorem 7. *If τ is a separated linear convergence structure on a linear space E , then the linear bornology $\mathcal{B}(\tau)$ is separated.*

Proof. Suppose that $\mathcal{B}(\tau)$ is not separated. Then there is an $x \in E \setminus \{0\}$ such that Kx is τ -bounded. Hence we have $WKx \in \tau 0$, and so $WKx = WKx + x \in \tau x$, which is absurd.

It is easy to see that the converse of Theorem 7 holds for linear topology τ on E . However the author does not know whether it is generally true or not.

Theorem 8. *A linear bornology $\mathcal{B}$ on a linear space E is separated if and only if the linear convergence structure $\gamma(\mathcal{B})$ is separated.*

Proof. If $\gamma(\mathcal{B})$ is separated, then $\mathcal{B} = \mathcal{B}(\gamma(\mathcal{B}))$ is separated by Theorem 7. To prove the "only if" part, suppose that there are an $x \in$

$E \setminus \{0\}$ and a $\Phi \in \gamma(\mathcal{B})(0) \cap \gamma(\mathcal{B})(x)$. Then by Theorem 4, we can find B , $B' \in \mathcal{B}$ such that $\lambda B \in \Phi$ and $\lambda B' \in \Phi - x$ for every non-zero $\lambda \in K$. Put $B'' = B \cup B'$, and let λ be an arbitrary non-zero element of K . Since $\lambda^{-1}B''$ and $\lambda^{-1}B'' + x$ are members of the filter Φ , they have a common element y , and so both $y - x$ and y belong to $\lambda^{-1}B''$; consequently we have $x = y - (y - x) \in \lambda^{-1}B'' - \lambda^{-1}B''$ or $\lambda x \in B'' - B''$. This implies that $Kx \subset B'' - B'' \in \mathcal{B}$, proving $Kx \in \mathcal{B}$.

It is shown in [2] that every finite dimensional linear space has a unique separated linear bornology. Conversely we have the following

Theorem 9. *If a linear space E has a unique separated linear bornology, then E is finite dimensional.*

Proof. Consider the set $\mathcal{B}$ of all subsets of E which are contained in some finite sum $\sum_{i=1}^n V_1 x_i$ where $x_1, \dots, x_n \in E$. As can be readily seen, $\mathcal{B}$ is a linear bornology on E . Let $x_1, \dots, x_n \in E$ and let B a Hamel basis of the linear subspace F spanned by $\{x_1, \dots, x_n\}$. Then since the gauge of the absolute convex hull $\Delta(B)$ of B is a norm on F , no linear subspaces of one dimension are contained in $\Delta(B)$, and so in $\sum_{i=1}^n V_1 x_i$. Therefore $\mathcal{B}$ is separated. Now let A be a Hamel basis of E , and τ denote the norm topology on E with unit ball $\Delta(A)$, the absolute convex hull of A . Then since $\mathcal{B}(\tau)$ is a separated linear bornology on E by Theorem 7, we have $\mathcal{B}(\tau) = \mathcal{B}$, which implies $A \in \mathcal{B}$. Thus we have the desired conclusion.

Corollary. *If a linear space E has a unique separated linear convergence structure, then E is finite dimensional.*

Proof. If $\mathcal{B}$ and $\mathcal{B}'$ are separated linear bornologies on E , then by Theorem 8 both $\gamma(\mathcal{B})$ and $\gamma(\mathcal{B}')$ are separated linear convergence structures on E , and hence we have

$$\mathcal{B} = \mathcal{B}(\gamma(\mathcal{B})) = \mathcal{B}(\gamma(\mathcal{B}')) = \mathcal{B}'$$

by Theorem 6. Therefore E has a unique separated linear bornology by Theorem 7.

Hogbe-Nlend [2] defined M-open subsets of a linear space E with a linear bornology, and considered the topology τ_E determined by all M-open subsets of E . On the other hand, Fischer [1] introduced the notion of τ -open set in a convergence space (X, τ) , and discussed the topology $\bar{\omega}\tau$ defined by the τ -open sets. Concerning these notions we have the following

Theorem 10. *Let $\mathcal{B}$ be a linear bornology on a linear space E . Then a subset G of E is M-open if and only if it is $\gamma(\mathcal{B})$ -open, and hence $\tau_E = \bar{\omega}\gamma(\mathcal{B})$.*

Proof. Assume that G is M-open. By Theorem 4, for each $\Phi \in \gamma(\mathcal{B})(0)$, there is a $B \in \mathcal{B}$ such that $\lambda B \in \Phi$ for all non-zero $\lambda \in K$; if $x \in G$ then, $G - x$ being bornivore, we have $\lambda B \subset G - x$ for some non-zero $\lambda \in K$, which implies $G - x \in \Phi$ or $G \in \Phi + x$. This establishes the "only if" part. Conversely, assume that G is $\gamma(\mathcal{B})$ -open. Let $x \in G$ and $B \in \mathcal{B}$ be arbitrary. Then since VB belongs to $\gamma(\mathcal{B})(0)$, we have $G - x \in VB$ or equivalently $V_\varepsilon B \subset G - x$ for some $\varepsilon > 0$. Consequently $G - x$ is bornivore for each $x \in G$.

Remark 3. Let τ be a linear convergence structure on a linear space E . Fischer [1] called a subset B of E to be τ -bounded if it is $\Psi^0\tau$ -

bounded in our sense. Since τ is finer than $\psi^0\tau$ (see [1]), each τ -bounded subset of E is τ -bounded in the sense of Fischer. However the converse is not true as will be shown in a forthcoming note [4].

Problem: Let τ be a linear convergence structure on a linear space such that the linear bornology $\mathcal{B}(\tau)$ is separated. Is τ necessarily separated ?

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