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Some Fixed Point and Coincidence Theorems in L-Spaces

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In a previous note [3], it was shown that several known fixed point theorems still hold in L-spaces. In this note, we shall show that some fixed point theorems due to Kannan [1] and Sehgal [4] and a coincidence theorem obtained in [2] also hold in L-spaces.

We use the notations and terminology in [3], however we need one more definition and notation. A mapping f of an L-space $(X, \rightarrow)$ into another $(X', \rightarrow')$ is said to be *continuous at* $a \in X$ if $x_n \rightarrow a$ implies $f(x_{n_i}) \rightarrow' f(a)$ for some subsequence $\{x_{n_i}\}_{i \in \omega}$ of $s = \{x_n\}_{n \in \omega}$; $f(s)$ stands for the sequence $\{f(x_n)\}_{n \in \omega}$.

Theorem 1. *Let f and g be mappings of an L-space $(X, \rightarrow)$ into itself, and d be a continuous nonnegative extended real valued function on the product space $X \times X$ with the property that $d(x, y) = 0$ ($x, y \in X$) implies $x = y$. Suppose that the following conditions are satisfied for some α, β with $0 \leq \alpha < 1$ and $0 < \beta \leq \infty$.*

- (1) $d(f^2(x), g(f(x))) \leq \alpha d(f(x), g(x))$ for every $x \in X$ with $d(f(x), g(x)) < \beta$.
- (2) f and g are continuous at some $a \in X$.
- (3) For some $b \in X$ with $d(f(b), g(b)) < \beta$, the sequence $\{f^n(b)\}_{n \in \omega}$ has a subsequence converging to a .

Then $f(a) = g(a)$.

Proof. By induction, we have $d(f^{n+1}(b), g(f^n(b))) \leq \alpha^n d(f(b), g(b))$ for every $n \in \omega$. It follows that $d(f^{n+1}(b), g(f^n(b))) \rightarrow 0$. Let s be a subsequence of $\{f^n(b)\}_{n \in \omega}$ converging to a . Then $f(s') \rightarrow f(a)$ for some subsequence s' of s . Since $s' \rightarrow a$, we can choose a subsequence s'' of s' with $g(s'') \rightarrow g(a)$. Hence the continuity of d implies that $d(f(s^*), g(s^*)) \rightarrow d(f(a), g(a))$ for some subsequence s^* of s'' . But since $d(f(s^*), g(s^*))$ is a subsequence of $\{d(f^{n+1}(b), g(f^n(b)))\}_{n \in \omega}$, we have $d(f(s^*), g(s^*)) \rightarrow 0$. Therefore we obtain $d(f(a), g(a)) = 0$, which implies the desired conclusion.

The above theorem remains valid if the condition (1) is replaced by the following condition:

$$(1') \quad d(f^2(x), g(f(x))) < \infty \quad \text{and}$$

$$d(f(x), g(f(y))) \leq \alpha \cdot \max\{d(x, g(y)), d(f(x), g(x)), d(f(y), g(y))\}$$

for every $x \in X$ with $d(f(x), g(x)) < \beta$.

In fact, if $d(f(x), g(x)) < \beta$ then

$$\begin{aligned} d(f^2(x), g(f(x))) &\leq \alpha \cdot \max\{d(f(x), g(x)), d(f^2(x), g(f(x)))\} \\ &= \alpha d(f(x), g(x)). \end{aligned}$$

Therefore we have the following

Corollary 1. Let f and g be mappings of an L-space $(X, \rightarrow)$ into itself, and d be a continuous nonnegative real valued function on the product space $X \times X$ with the property that $d(x, y) = 0$ implies $x = y$. Suppose that the following conditions are satisfied for some α, β with $0 \leq \alpha < 1$ and $0 < \beta \leq \infty$.

$$(1) \quad d(f(x), g(f(y))) \leq \alpha \cdot \max\{d(x, g(y)), d(f(x), g(x)), d(f(y), g(y))\}$$

for every $x \in X$ with $d(f(x), g(x)) < \beta$.

(2) f and g are continuous at some $a \in X$.

(3) For some $b \in X$ with $d(f(b), g(b)) < \beta$, the sequence $\{f^n(b)\}_{n \in \omega}$ has a subsequence converging to a .

Then $f(a) = g(a)$.

As can easily be seen, Theorem 1 still holds if the condition (1) is replaced by the following condition (1'') provided $d(x, x) = 0$ for all $x \in X$.

(1'') $d(f^2(x), g(f(x))) < \infty$ and

$$d(f(x), g(f(y))) \leq \alpha \cdot \max\{d(x, f(y)), d(x, g(y)), d(f(x), g(x)), d(f(y), g(y))\}$$

for every $x \in X$ with $d(f(x), g(x)) < \beta$.

Now these coincidence theorems yield obviously fixed point theorems.

For example, we have the following

Corollary 2. Let f be a mapping of an L-space $(X, +)$ into itself, and d be a continuous nonnegative real valued function on the product space $X \times X$ with the property that $d(x, y) = 0$ if and only if $x = y$. Suppose that the following conditions are satisfied for some α, β with $0 \leq \alpha < 1$ and $0 < \beta \leq \infty$.

(1) $d(f(x), f(y)) \leq \alpha \cdot \max\{d(x, y), d(f(x), x), d(f(y), y)\}$

for every $x \in X$ with $d(f(x), x) < \beta$.

(2) f is continuous at some $a \in X$.

(3) For some $b \in X$ with $d(f(b), b) < \beta$, the sequence $\{f^n(b)\}_{n \in \omega}$ has a subsequence converging to a .

Then a is a fixed point of f . If in addition

(L) $d(x, y) < \beta$ for all fixed points $x, y \in X$ of f ,
then a is a unique fixed point of f .

A fixed point theorem of Kannan [1] is a special case of this corollary.
The following result is a generalization of a coincidence theorem in [2].

Theorem 2. Let f and g be continuous mappings of an L-space $(X, \rightarrow)$ into itself, and d be a continuous nonnegative extended real valued function on the product space $X \times X$ with the property that $d(x, y) = 0$ implies $x = y$. Suppose that the following conditions are satisfied for some β with $0 < \beta \leq \infty$.

- (1) $d(f^2(x), g(f(x))) < d(f(x), g(x))$ for every $x \in X$ with $0 < d(f(x), g(x)) < \beta$.
- (2) There exist $a, b \in X$ such that the sequence $\{f^n(b)\}_{n \in \omega}$ has a subsequence converging to a

Then one of the following statements 1°-3° holds:

- 1° $f(a) = g(a)$.
- 2° $f(x) = g(x)$ for some $x = f^n(b)$ where $n \in \omega$.
- 3° $d(f(a), g(a)) \geq \beta$.

Moreover if $(X, \rightarrow)$ is separated and $f \circ g = g \circ f$ on $\{f^n(b) \mid n \in \omega\}$, then either 1° or 3° holds.

Proof. Assume $d(f(a), g(a)) < \beta$. Because of the condition (2) and the continuity of f, g and d , we can choose, by the same argument used in the proof of Theorem 1, a subsequence s of $\{f^n(b)\}_{n \in \omega}$ such that $d(f(s), g(s)) \rightarrow d(f(a), g(a))$. Hence we have $d(f(c), g(c)) < \beta$ for some c in $\{f^n(b) \mid n \in \omega\}$. Suppose that $f(x) \neq g(x)$ for every $x \in A = \{f^n(b) \mid n \in \omega\} \cup \{a\}$. Then by induction, we have

$$(*) \quad 0 < d(f(f^n(c)), g(f^n(c))) < \beta$$

for all $n \in \omega$. Consider the nonnegative real valued function h on A defined by

$$h(x) = \frac{d(f^2(x), g(f(x)))}{d(f(x), g(x))}$$

for each $x \in A$. From (1), we have $h(a) < 1$, and so we can find an α with $h(a) < \alpha < 1$. It is clear from (2) that $\{f^n(c)\}_{n \in \omega}$ has a subsequence s converging to a . Since f, g and d are continuous, by choosing subsequences repeatedly, we can conclude that

$$(**) \quad d(f(f^{N(i)}(c)), g(f^{N(i)}(c))) \rightarrow d(f(a), g(a))$$

and $h(f^{N(i)}(c)) \rightarrow h(a)$ for some subsequence $\{f^{N(i)}(c)\}_{i \in \omega}$ of s . So we can choose an $m \in \omega$ such that $h(f^{N(i)}(c)) < \alpha$ or equivalently

$$d(f^{N(i)+2}(c), g(f^{N(i)+1}(c))) < \alpha d(f^{N(i)+1}(c), g(f^{N(i)}(c)))$$

for all $i \geq m$. Therefore, keeping (*) and (1) in mind, we have

$$\begin{aligned} d(f^{N(i)+1}(c), g(f^{N(i)}(c))) &< d(f^{N(i)}(c), g(f^{N(i)-1}(c))) \dots \\ &< d(f^{N(i-1)+2}(c), g(f^{N(i-1)+1}(c))) \\ &< \alpha d(f^{N(i-1)+1}(c), g(f^{N(i-1)}(c))) \end{aligned}$$

for every $i > m$. Consequently we obtain

$$d(f^{N(i)+1}(c), g(f^{N(i)}(c))) < \alpha^{i-m} d(f^{N(m)+1}(c), g(f^{N(m)}(c)))$$

for every $i > m$. Thus the sequence $\{d(f(f^{N(i)}(c)), g(f^{N(i)}(c)))\}_{i \in \omega}$ converges to 0. Hence we have, from (**), $d(f(a), g(a)) = 0$, a contradiction. It follows that $f(x) = g(x)$ for some $x \in A$. Now assume that $f \circ g = g \circ f$ on $\{f^n(b) \mid n \in \omega\}$, and $f(f^m(b)) = g(f^m(b))$ for some $m \in \omega$. Then we have

$$f(f^{m+n}(b)) = g(f^{m+n}(b))$$

for every $n \in \omega$. Since f and g are continuous, it follows from (2) that $f(s) \rightarrow f(a)$ and $g(s) \rightarrow g(a)$ for some subsequence $s =$

$\{f^{n(i)}(b)\}_{i \in \omega}$ of $\{f^n(b)\}_{n \in \omega}$. But then, there is a $k \in \omega$ such that $n(i) \geq m$ for all $i \geq k$. Hence we have

$$f(f^{n(i)}(b)) = g(f^{n(i)}(b))$$

for all $i \geq k$, which implies $f(a) = g(a)$ if $(X, \rightarrow)$ is separated.

This completes the proof.

Theorem 2 still holds if (1) is replaced by the following condition:

$$(1^*) \quad d(f(x), g(f(y))) < \max\{d(x, g(y)), d(f(x), g(x)), d(f(y), g(y))\}$$

for every $x, y \in X$ with $0 < d(x, g(y)) < \beta$.

In fact, if $0 < d(f(x), g(x)) < \beta$, then

$$\begin{aligned} d(f^2(x), g(f(x))) &< \max\{d(f(x), g(x)), d(f^2(x), g(f(x)))\} \\ &= d(f(x), g(x)). \end{aligned}$$

Similarly, if the function d satisfies the condition $d(x, x) = 0$ for all $x \in X$, then (1) can be replaced by the following condition:

$$(1^{**}) \quad d(f(x), g(f(y))) < \max\{d(x, f(y)), d(x, g(y)),$$

$$d(f(x), g(x)), d(f(y), g(y))\}$$

for every $x, y \in X$ with $0 < d(x, g(y)) < \beta$.

These results imply also fixed point theorems in L-spaces, and a fixed point theorem due to Sehgal [4] is a special case of such a theorem.

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