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On Fixed Points in Partially Ordered Sets
and Kirk-Caristi Theorem

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In a recent paper [2], Kirk and Caristi stated an interesting fixed point theorem:

Let f be a mapping of a complete nonempty metric space (X, d) into itself. If there exists a lower semicontinuous function Φ mapping X into the nonnegative real numbers such that for each $x \in X$,

$$d(x, f(x)) \leq \Phi(x) - \Phi(f(x)),$$

then f has a fixed point.

In this note, we shall show that this theorem can be derived from a simple fixed point theorem for families of mappings of a partially ordered set (abbreviated *ordered set* in what follows) into itself, which includes also a fixed point theorem due to Smithson [3] as a special case.

Theorem 1. *Let $\mathcal{F}$ be a family of mappings of an ordered set $(X, \leq)$ into itself such that $f(x) \leq x$ for all $f \in \mathcal{F}$ and for all $x \in X$. If each chain in X which contains a fixed element $e \in X$ has a lower bound, then $\mathcal{F}$ has a common fixed point.*

Proof. By Zorn's lemma, there exists a maximal chain C in X contain-

ing e . Let a be a lower bound of C . The maximality of C implies $a \in C$. Let $f \in \mathcal{F}$. Then since $f(a) \leq a$, the maximality of C places $f(a)$ in C , and hence $a \leq f(a)$. Thus $f(a) = a$.

The following result generalizes Theorem 2.1 of Smithson [3].

Corollary. *Let $\mathcal{F}$ be a family of increasing mappings of an ordered set $(X, \leq)$ into itself such that $e \leq f(e)$ for all $f \in \mathcal{F}$, where e is a fixed element of X . If each chain in X which contains e has a least upper bound in X , then $\mathcal{F}$ has a common fixed point.*

Proof. We shall prove the dual of the corollary. The set Y of all $x \in X$ with $f(x) \leq x$ for all $f \in \mathcal{F}$ becomes an ordered set under the induced ordering of $\leq$ into Y . It suffices only to prove that each chain C in Y which contains e has a lower bound. Since C is a chain in X , it has a greatest lower bound $a \in X$. Let $f \in \mathcal{F}$. Then for each $x \in C$, we have $f(a) \leq f(x) \leq x$, since $a \leq x$. So $f(a)$ is a lower bound of C , and consequently $f(a) \leq a$. This shows that a is in Y .

We shall now concern with L-spaces (see [1]). An extended real valued function φ on an L-space $(X, \rightarrow)$ is said to be *lower semicontinuous* if $x_n \rightarrow a$ and $\varphi(x_n) \leq \alpha$ for all n imply $\varphi(a) \leq \alpha$.

Theorem 2. *Let $(X, \rightarrow)$ be a nonempty L-space which is d -complete for a premetric d on X such that the function $x \mapsto d(x, y)$ is lower semicontinuous for every $y \in X$ and $d(x, y) = 0$ implies $x = y$, and let $\mathcal{F}$ be a family of mappings of X into itself. If there exists a lower semicontinuous nonnegative real valued function Φ on X such that for each $f \in \mathcal{F}$ and for each $x \in X$,*

$$d(f(x), x) \leq \Phi(x) - \Phi(f(x)),$$

then $\mathcal{F}$ has a common fixed point.

Proof. Define $\leq$ by $x \leq y$ if and only if $d(x, y) \leq \Phi(y) - \Phi(x)$. It is easy to see that $(X, \leq)$ is an ordered set and $f(x) \leq x$ for all $f \in \mathcal{F}$ and for all $x \in X$. Let e be an arbitrary but fixed element of X , and C a chain in $(X, \leq)$ which contains e . Denote by α the greatest lower bound of the set $\{\Phi(x) \mid x \in C\}$. If $\Phi(a) = \alpha$ for some $a \in C$, then a is a lower bound of C . For, if $x \leq a$ for some x in $C \setminus \{a\}$, then we have $d(x, a) \leq \Phi(a) - \Phi(x) \leq 0$, which yields $d(x, a) = 0$, a contradiction. Therefore one can assume $\Phi(x) \neq \alpha$ for all $x \in C$. Then the set $S(n, x)$ of all $y \in C$ with $y \leq x$ and

$$(*) \quad \alpha < \Phi(y) < \alpha + \frac{1}{n+1}$$

is nonempty for each $(n, x) \in \omega \times C$, where ω denotes the set of all non-negative integers. In fact, there is a $y \in C$ satisfying (*), and so y belongs to $S(n, x)$ if $y \leq x$; if $x \leq y$ then since $0 \leq d(x, y) \leq \Phi(y) - \Phi(x)$, we have $\alpha < \Phi(x) \leq \Phi(y) < \alpha + (n+1)^{-1}$, which shows that x belongs to $S(n, x)$. Let $\mathcal{P}$ be a choice function for the family of all nonempty subsets of C . Then by the recursion theorem, there is a sequence $\{x_n\}_{n \in \omega}$ in C such that $x_0 = e$ and $x_{n+1} = \mathcal{P}(S(n, x_n))$ for all $n \in \omega$. Since $x_{n+1} \leq x_n$ for all $n \in \omega$, we have $\sum_{n=0}^m d(x_{n+1}, x_n) < \Phi(e)$ for all $m \in \omega$, and so $\sum_{n=0}^{\infty} d(x_{n+1}, x_n) < \infty$. Hence the d -completeness implies that $x_n \rightarrow a$ for some $a \in X$. Now let x be in C . Then we can find an $m \in \omega$ such that $\Phi(x_m) < \alpha + (1/m) < \Phi(x)$. Since x and x_m are in the chain C , this implies $x_m \leq x$. Therefore we have

$$(**) \quad \psi(x_n) = d(x_n, x) + \Phi(x_n) \leq \Phi(x)$$

for every $n \geq m$. Since ψ is lower semicontinuous, it follows from (**) that $d(a, x) + \Phi(a) \leq \Phi(x)$, or equivalently $a \leq x$. This shows that a is a lower bound of C , and the proof is completed.

The following result which generalizes the theorem of Kirk and Caristi is now obvious:

Corollary. Let $\mathcal{F}$ be a family of mappings of a complete nonempty metric space (X, d) into itself. If there exists a lower semicontinuous nonnegative real valued function Φ on X such that for each $f \in \mathcal{F}$ and for each $x \in X$,

$$d(x, f(x)) \leq \Phi(x) - \Phi(f(x)),$$

then $\mathcal{F}$ has a common fixed point.

References

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