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Remarks on Some Fixed Point Theorems

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In this note, we shall show that certain fixed point theorems for mappings of Ćirić-type (see Ćirić [3]) obtained by Ćirić [3] and Iséki [5] can be easily derived from slight modifications of results obtained in [7] and [6] respectively, and that the fixed point theorem in partially ordered sets obtained in [8] is equivalent to the axiom of choice. Moreover a generalization of a fixed point theorem of DeMarr [4] is obtained together with a stronger version of Zorn's lemma.

1. On mappings of Ćirić-type

In this section, we make free use of the notations and terminology in [6]. Let $(X, \rightarrow)$ be an L-space. A mapping f of X into itself is said to be *orbitally continuous* if $x \in X$ and $f^{n(i)}(x) \rightarrow a \in X$ as $i \rightarrow \infty$ imply $f(f^{n(i_k)}(x)) \rightarrow f(a)$ as $k \rightarrow \infty$ for some subsequence $\{n(i_k)\}_{k \in \omega}$ of $\{n(i)\}_{i \in \omega}$. The proof of Theorem 1 of [6] shows that Theorem 1 of [6] remains valid if f is orbitally continuous (and hence so does Theorem 2 of [6]; note that Theorem 3 of [6] also holds for an orbitrally continuous mapping f):

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Theorem A. *Let* $(X, \to)$ *be a separated* L-*space which is* d-*complete for a nonnegative extended real valued function* d *on* $X \times X$, *and* f *be an orbitally continuous mapping of* X *into itself satisfying the following conditions for some* α, β *with* $0 \le \alpha < 1$ *and* $0 < \beta \le \infty$:

(1) $d(f^2(x), f(x)) \le \alpha d(f(x), x)$ *for every* $x \in X$ *with* $d(f(x), x) < \beta$.

(2) $d(f(a), a) < \beta$ *for some* $a \in X$.

Then f *has a fixed point, and the sequence* $\{f^n(a)\}_{n \in \omega}$ *converges to a fixed point of* f.

As an immediate consequence of this theorem, we have at once the following

Corollary. *Let* $(X, \to)$ *be a separated* L-*space which is* d-*complete for a nonnegative extended real valued function* d *on* $X \times X$ *such that* $d(x, y) = 0$ *if and only if* $x = y$. *Suppose that* f *is an orbitally continuous mapping of* X *into itself satisfying the following conditions for some* α, β *with* $0 \le \alpha < 1$ *and* $0 < \beta \le \infty$:

(1') $$\min\{d(f(x), f(y)), d(f(x), x), d(f(y), y)\} - \min\{d(f(y), x), d(f(x), y)\} \le \alpha d(x, y)$$

for all $x, y \in X$ *with* $d(x, y) < \beta$.

(2') $d(f(a), a) < \beta$ *for some* $a \in X$.

Then f *has a fixed point, and the sequence* $\{f^n(a)\}_{n \in \omega}$ *converges to a fixed point of* f.

Proof. It suffices to show that f satisfies the condition (1) of Theorem A. Suppose $d(f(x), x) < \beta$. If $d(f(x), x) = 0$ then we have

$$d(f^2(x), f(x)) = d(f(x), x) = 0 = \alpha d(f(x), x),$$

since $f(x) = x$. So we can assume $d(f(x), x) \neq 0$. Now (1') yields

$$\begin{aligned}&\min\{d(f^2(x), f(x)), d(f(x), x)\}\\ &= \min\{d(f^2(x), f(x)), d(f(x), x)\} - \min\{d(f(x), f(x)), d(f^2(x), x)\}\\ &\leq \alpha d(f(x), x).\end{aligned}$$

Since $d(f(x), x) \neq 0$, this shows that $\min\{d(f^2(x), f(x)), d(f(x), x)\} = d(f^2(x), f(x))$. Thus we have $d(f^2(x), f(x)) \leq \alpha d(f(x), x)$.

Theorem 1 of Ćirić [3] is a special case of this corollary, and its generalization due to Iséki [5; Theorem 1]*) follows from this corollary by considering the function d^* defined by $d^*(x, y) = d(y, x)$ for all $x, y \in X$.

As is easily verified, each result in [7]**) still holds for an orbitally continuous mapping f. For example, we have the following result corresponding to Theorem 2 of [7].

Theorem B. *Let f and g be mappings of an* L-*space* $(X, \to)$ *into itself, and d a continuous nonnegative extended real valued function on the product space $X \times X$ with the property that $d(x, y) = 0$ implies $x = y$. Suppose that the following conditions are satisfied for some α, β with $0 \leq \alpha < 1$ and $0 < \beta \leq \infty$:*

(0) *f is orbitally continuous and g is continuous.*

(1) *$d(f^2(x), g(f(x))) < d(f(x), g(x))$ for every $x \in X$ with* $0 < d(f(x), g(x)) < \beta$.

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*) In Theorem 1 of [5], it is assumed that $d(x, x) = 0$ for each x in X, but it should read " $d(x, y) = 0$ if and only if $x = y$ ".

**) "$\{f^n(a)\}_{n\in\omega}$" on the third line of Proof of Theorem 2 in [7] should read "$\{f^n(b)\}_{n\in\omega}$", and "$d(f(x), g(f(x)))$" in (1*) of [7] "$d(f(x), g(f(y)))$".

(2) *There exist* $a, b \in X$ *such that the sequence* $\{f^n(b)\}_{n\in\omega}$ *has a subsequence converging to* a.

Then one of the following statements 1°-3° *holds*:

1° $f(a) = g(a)$.

2° $f(x) = g(x)$ *for some* $x = f^n(b)$, *where* $n \in \omega$.

3° $d(f(a), g(a)) \geq \beta$.

Moreover if $(X, \to)$ *is separated and* $f\circ g = g\circ f$ *on* $\{f^n(b) \mid n \in \omega\}$, *then either* 1° *or* 3° *holds.*

Now this theorem yields the following result which extends Theorem 3 of Ćirić [3]:

Corollary. *Let* f *be an orbitally continuous mapping of a separated* L-*space* $(X, \to)$ *into itself, and* d *a continuous nonnegative extended real valued function on the product space* $X \times X$ *with the property that* $d(x, y) = 0$ *if and only if* $x = y$. *Suppose that the following conditions are satisfied for some* α, β *with* $0 \leq \alpha < 1$ *and* $0 < \beta \leq \infty$:

(1') $$\min \{d(f(x), f(y)), d(f(x), x), d(f(y), y)\} - \min \{d(f(y), x), d(f(x), y)\} < d(x, y)$$

for all $x, y \in X$ *with* $0 < d(x, y) < \beta$.

(2') *There exist* $a, b \in X$ *such that the sequence* $\{f^n(b)\}_{n\in\omega}$ *has a subsequence converging to* a.

Then $d(f(a), a) \geq \beta$ *or* $f(a) = a$.

Proof. If $0 < d(f(x), x) < \beta$, then it follows from (1') that

$$\min \{d(f^2(x), f(x)), d(f(x), x)\}$$
$$= \min \{d(f^2(x), f(x)), d(f(x), x)\} - \min \{d(f(x), f(x)), d(f(x), x)\}$$
$$< d(f(x), x),$$

which implies $d(f^2(x), f(x)) < d(f(x), x)$. Therefore Theorem B yields the desired conclusion.

2. Partially ordered sets

In [8], it was shown that the fixed point theorem of Caristi-Kirk (see [1], [2] or [9]) and a theorem of Smithson [11] are consequences of the following fixed point theorem:

Theorem C. *Let $\mathscr{F}$ be a family of mappings f of a partially ordered set $(X, \leq)$ into itself such that $f(x) \leq x$ for all $x \in X$. If each chain in X which contains a fixed element e of X has a lower bound, then $\mathscr{F}$ has a common fixed point.*

We shall show here that this theorem is an equivalent form of the axiom of choice. To this end, we derive Zorn's lemma from this theorem.

Let $(X, \leq)$ be a nonempty partially ordered set such that each chain of which has a lower bound, and let $\mathscr{F}$ be the set of all mappings f with the property that $f(x) \leq x$ for all $x \in X$. Then Theorem C implies that $\mathscr{F}$ has a common fixed point $a \in X$. If $b \in X$ and $b \leq a$, then $a = b$, since the mapping f defined by

$$(*) \qquad f(x) = \begin{cases} b & \text{if } x = a, \\ x & \text{otherwise,} \end{cases}$$

belongs to $\mathscr{F}$. Therefore a is a minimal element of X.

We shall now proceed to state a stronger version of Zorn's lemma together with a slight generalization of Theorem C by using the terminology of Rubin and Rubin [10].

Let R be a binary relation on a set X. We say that a mapping f of

X into itself is *R-downward* (resp. *R-upward*) if $f(x)Rx$ (resp. $xRf(x)$) for all $x \in X$. We need the following simple lemma:

Lemma. *Let R be a binary relation on a set X. An element a of X is R-minimal (resp. R-maximal) if and only if $aRf(a)$ (resp. $f(a)Ra$) for every R-downward (resp. R-upward) mapping f of X into itself.*

Proof. Only the proof of the "if" part is needed. Suppose $b \in X$ and bRa. Then since the mapping f defined by (*) is R-downward, we have aRb.

We are now in a position to prove the following proposition:

Proposition. *Each of the following statements is equivalent to the axiom of choice*:

(1) *Let R be a transitive relation on a set X with an element e. If each subset C of X containing e which is linearly ordered by R has an R-lower bound, then there is an R-minimal element of X.*

(2) *Let R be an anti-symmetric transitive relation on a set X, $\mathscr{F}$ a family of R-downward mappings of X into itself, and $e \in X$. If each subset C of X containing e which is linearly ordered by R has an R-lower bound, then $\mathscr{F}$ has an R-minimal common fixed point.*

Proof. First we shall show that the statement (1) follows from the dual of M 1 in Rubin and Rubin [10]: *If R is a transitive relation on a nonempty set X and if every subset of X which is linearly ordered by R has an R-lower bound, then there is an R-minimal element of X.*

Let X' be the set of all $a \in X$ with xRe or eRx or $x = e$, and let $R' = R \cap (X' \times X')$. If C' is a subset of X' which is linearly ordered by R', then since $C = C' \cup \{e\}$ is linearly ordered by R, the subset C has an R-lower bound $x \in X$, and hence $x \in X'$. So each subset of X' which is linearly ordered by R' has an R'-lower bound. Hence there is an R'-minimal element a of X' by the dual of M 1. Since eRa implies $\bar{a}Re$, we have aRe or $a = e$. Therefore if $x \in X$ and xRa, then $x \in X'$, and thus we have aRx.

Now using the above lemma, it is easy to see that (1) implies (2) and (2) implies the usual version of Zorn's lemma, which establishes the proposition.

We shall derive now, from (2) of the above proposition, the following result which slightly generalizes Theorem 1 of DeMarr [4] and Corollary of Theorem 1 in [8].

Theorem D. *Let $\mathscr{F}$ be a family of increasing mappings of a partially ordered set $(X, \leq)$ into itself. If there exists a chain-compact element**) $e \in X$ *such that* $f(e) \leq e$ *for all* $f \in \mathscr{F}$, *then* $\mathscr{F}$ *has a minimal common fixed point.*

Proof. Define $M = \{x \in X \mid f(x) \leq x \leq e$ for all $f \in \mathscr{F}\}$. Clearly $\mathscr{F}$ may be regarded as a family of $\leq$-downward mappings of M into M with the induced partial ordering $\leq$. Let C be a chain in M containing e. Then since e is chain-compact, $\inf C$ exists in X. Now for each $f \in \mathscr{F}$, $x \in C$ implies $f(\inf C) \leq x$, since $\inf C \leq x$ and $f(x) \leq x$. This shows that $f(\inf C) \leq \inf C$ or that $\inf C \in M$. Thus it follows from (2) of the above proposition that $\mathscr{F}$ has a minimal common fixed point $a \in M$.

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*) An element $e \in X$ is *chain-compact* if, for every nonempty chain C in X, where $x \leq e$ for all $x \in C$, $\inf C$ exists (see DeMarr [4]).

Let $b \in X$ be a common fixed point for $\mathscr{F}$ with $b \leq a$. Then we have $f(b) = b \leq e$ for all $f \in \mathscr{F}$, which shows $b \in M$. Thus we have $b = a$.

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