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Iseki, Kiyoshi

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B C K - Algebras

By Kiyoshi ISEKI

In this lecture, I will talk about a new class of algebra called a BCK-algebra which was introduced in my seminar of mathematical logic and algebras. As you know, we have many kinds of algebras obtained from algebraic formulations of propositional calculi. For example, we know Boolean algebras, Lukasiewicz algebras, Post algebras, Heyting algebras etc. Some of these algebras have the representation spaces by prime ideals, maximal ideals etc. Great many interesting results have been obtained by many mathematicians (for details, see R. Bables and Ph. Dwinger [4]).

First we introduced three algebras called BCK, BCI and Griss algebras (see Y. Imai and K. Iséki [21], Y. Imai and K. Iséki [22], and K. Iséki [23]). These algebras are obtained from algebraic formulations of some propositional calculi.

We begin with the definitions of these algebras.

A BCK-algebra is a general algebra of type (2, 2, 0) which is defined as follows:

By a *BCK-algebra*, we mean an algebra which satisfies the following conditions: Let X be an abstract set with two binary operations $*$ and $\leqq$. We suppose that X has a constant denoted by 0.

(1) $\quad (x * y) * (x * z) \leqq z * y,$

(2) $x * (x * y) \leqslant y$,

(3) $x \leqslant x$,

(4) $x \leqslant y$, $y \leqslant x$ implies $x = y$,

(5) $0 \leqslant x$,

(6) $x \leqslant y$ iff $x * y = 0$.

On the other hand, a BCI-algebra is defined as follows: By a *BCI-algebras*, we mean an algebra with two binary operations $*$ and $\leqslant$, and a constant 0 satisfying the following conditions:

(1) $(x * y) * (x * z) \leqslant z * y$,

(2) $x * (x * y) \leqslant y$,

(3) $x \leqslant x$,

(4) $x \leqslant y$, $y \leqslant x$ implies $x = y$,

(5) $x \leqslant 0$ implies $x = 0$,

(6) $x \leqslant y$ iff $x * y = 0$.

These algebras are essentially different. In a BCI-algebra, we do not assume $0 * x = 0$, i.e. $0 \leqslant x$. Roughly speaking, 0 is the least element in a BCK-algebra, but a minimal element in a BCI-algebra. Until now, we have not yet studied a BCI-algebra in detail.

Next we shall give a definition of a Griss algebra. A Griss algebra is an algebra of type $(2, 2, 2, 0)$. By a *Griss algebra*, we mean an algebra with three binary operations $*$, $\leqslant$, $\vee$ and a constant 0 satisfying the following conditions:

(1) $x \vee x \leqslant x$,

(2) $x \vee y \leqslant y \vee x$,

(3) $(x \vee y) * (x \vee z) \leqslant y \vee z$,

(4) $x * y \leqslant (x * z) \vee (z * y)$,

(5) $x \leqslant y \vee x$,

(6) $0 \leqslant x$,

(7) $x \leqslant y$, $y \leqslant x$ imply $x = y$,

(8) $x \leqslant y$ if and only if $x * y = 0$.

This algebra has been studied in Y. Imai and K. Iseki [22] and A. I. Arruda [1].

In my lecture, I will be mainly concerned with a BCK-algebra. In a BCK-algebra, we have the following properties:

Theorem 1. *The following relations hold in a BCK-algebra.*

(1) $x \leqslant y$ *implies* $z * y \leqslant z * x$,

(2) $x \leqslant y$, $y \leqslant z$ *implies* $x \leqslant z$,

(3) $(x * y) * z = (x * z) * y$,

(4) $x * y \leqslant z$ *implies* $x * z \leqslant y$,

(5) $x \leqslant y$ *implies* $x * z \leqslant y * z$,

(6) $x * y \leqslant x$.

(7) $x * 0 = x$.

Theorem 1 is fundamental for the calculation to lead various results.

Definition 1. We define $x \wedge y$ as $y * (y * x)$.

Then $x \wedge y$ is a common lower bound of x and y, and we have $x \wedge x = x$, $x \wedge 0 = 0 \wedge x = 0$.

Definition 2. If there is an element 1 of a BCK-algebra satisfying $x \leqslant 1$ for all x. then 1 is called *unit*. A BCK-algebra with unit is called *bounded*.

In a bounded BCK-algebra, we denote $1 * x$ by Nx.

Theorem 2. *In a bounded BCK-algebra, we have*

(1) $N1 = 0$, $N0 = 1$,

(2) $NNx \leqslant x$,

(3) $Nx * Ny \leqslant y * x$,

(4) $y \leqslant x$ implies $Nx \leqslant Ny$,

(5) $Nx * y = Ny * x$,

(6) $1 \wedge x = x$,

(7) $x \wedge 1 = NNx$,

(8) $NNNx = Nx$.

Definition 3. If an element x satisfies $NNx = x$, then x is called an *involution* (This terminology is due to Gr. C. Moisil).

Theorem 3. *The set S of involutions of a bounded BCK-algebra is also a bounded BCK-algebra (and nonempty).*

Definition 4. If, for each pair of x, y of a BCK-algebra,

$$x \wedge y = y \wedge x,$$

then the algebra is called *commutative*.

The following theorem is very interesting:

Theorem 4. *Any commutative BCK-algebra is a semilattice with respect to $\wedge$.*

Therefore we have

Theorem 5. *Any bounded commutative BCK-algebra is a lattice.*

In this case, we define $x \vee y$ by $N(Nx \wedge Ny)$. S. Tanaka [44] proved the following theorem:

Theorem 6. *Any commutative BCK-algebra is an algebra with two binary operations and a constant 0 satisfying the following conditions:*

(1) $x \leqslant x$,

(2) $x \leqslant y$, $y \leqslant x$ imply $x = y$.

(3) $x \leqslant y$, $y \leqslant z$ imply $x \leqslant z$,

(4) $0 \leqslant x$,

(5) $(x * y) * z = (x * z) * y$,

(6) $x * (x * y) = y * (y * x)$,

(7) $x \leqslant y$ if and only if $x * y = 0$.

We introduce the notion of an ideal in a BCK-algebra.

Definition 5. Let A be a nonempty subset of a BCK-algebra. A is called an *ideal*, if

(1) $0 \in A$,

(2) $x,\ y * x \in A \Longrightarrow y \in A$.

A is called an *implicative ideal*,

(3) $0 \in A$

(4) $(y * x) * z \in A$ and $x * z \in A$ imply $y * z \in A$.

Theorem 7. *If* x *is in an ideal* A, *and* $y \leqslant x$, *then* $y \in A$. *Any set* $A(a) = \{ x \mid x \leqslant a \}$ *is an ideal if and only if* $x * y \leqslant z$ *and* $y \leqslant z$ *imply* $x \leqslant z$ *for any* $x,\ y,\ z$.

Theorem 8. *The zero-ideal* $\{0\}$ *is implicative, if and only if every* $A(a)$ *is an ideal.*

Next we shall define the direct product of BCK-algebras.

Let M_α be a family of BCK-algebras. For $f,\ g \in \P M_\alpha$, we define

$$(f * g)(\alpha) = f(\alpha) * g(\alpha)$$

for every α and 0 by

$$0(\alpha) = 0_\alpha.$$

Then we have the following interesting

Theorem 9. *If* I *is an ideal of* $M_1 \times M_2$, *then* $I = I_1 \times I_2$, *where* $I_1,\ I_2$ *are the projections of* I *to the factor BCK-algebras.*

A similar result is obtained for the case of implicative ideals. A relation between a positive implicative algebra and a BCK-algebra is given by the following theorem.

Theorem 10. *If for every* x *of a BCK-algebra,* $A(x)$ *is an impli-*

cative ideal, then it is a positive implicative algebra.

The notion of a positive implicative algebra was introduced by an algebraic formulation of the positive implicative propositional calculus. If, in a BCK-algebra, the equality

$$(^\circ) \qquad x * y = (x * y) * y$$

holds, then we have

$$(x * z) * (y * z) \leqslant (x * y) * z.$$

Therefore, we have the following result (see K. Iseki [27]). *If* $(^\circ)$ *holds in a BCK-algebra, then it is positive implicative.* Theorem 9 follows from this proposition.

A homomorphism of a BCK-algebra into a BCK-algebra is given by a mapping f satisfying

$$f(y * x) = f(y) * f(x).$$

Then we have a category BCK *with BCK-algebras as objects and homomorphisms as morphisms.*

I think that it is an important problem to study relationships between categories of lattices, distributive lattices, Boolean algebras, Post algebras etc and BCK. Further, we must determine projective and injective objects and other objects of BCK. These are research problems about the category BCK.

The reference contains useful papers to develop the category theory of BCK-algebras.

This note is a résumé of my lecture given at Department of Mathematics, Jeonbug National University, Korea on June 30, 1976.

In conclusion, I express my cordial thank to Professor Kean Lee who invited me to give a lecture.

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