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A Fixed Point Theorem for Mappings
With an Iterate at a Point

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In his paper [5], Sehgal proved the following result:

Theorem A. Let (M, d) be a complete metric space. If T is a continuous mapping of M into itself with a contractive iterate at each point of M , then T has a unique fixed point $a \in M$ and $T^n x \rightarrow a$ as $n \rightarrow \infty$ for each $x \in M$.

This result is generalized by Guseman [1] to mappings which are not necessarily continuous and which have a contractive iterate at each point in a subset of the space. Guseman proved the following result:

Theorem B. Let T be a mapping of a metric space (M, d) into itself, and let $a, b \in M$ be such that $T^n b \rightarrow a$ as $n \rightarrow \infty$. If T has a contractive iterate at a , then a is a unique fixed point of T and $T^n x \rightarrow a$ as $n \rightarrow \infty$ for each $x \in M$.

Recently Hardy and Rogers [2] have proved the following theorem:

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Theorem C. Let T be a mapping of a complete metric space (M, d) into itself. If there exist nonnegative real numbers $\alpha_1, \alpha_2, \dots, \alpha_5$ such that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$ and

$$d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) + \alpha_4 d(x, Ty) + \alpha_5 d(y, Tx)$$

for all $x, y \in M$, then T has a unique fixed point.

The purpose of this note is to prove the following theorem which is a generalization of Theorem B.

Theorem. Let T be a mapping of a metric space (M, d) into itself, and let $a, b \in M$ be such that $T^n b \rightarrow a$ as $n \rightarrow \infty$. If there exist a positive integer m and nonnegative real numbers $\alpha_1, \alpha_2, \dots, \alpha_5$ such that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$, $\alpha_1 + 2\alpha_2 + \alpha_4 + \alpha_5 < 1$, and

$$(*) \quad d(T^m x, T^m a) \leq \alpha_1 d(x, a) + \alpha_2 d(x, T^m x) + \alpha_3 d(a, T^m a) \\ + \alpha_4 d(x, T^m a) + \alpha_5 d(a, T^m x)$$

for all $x \in M$, then a is a unique fixed point of T and $T^n x \rightarrow a$ as $n \rightarrow \infty$ for each $x \in M$.

Proof. For each positive integer n , we have

$$\begin{aligned} d(a, T^m a) &\leq d(a, T^n b) + d(T^n b, T^{n+m} b) + d(T^{n+m} b, T^m a) \\ &\leq d(a, T^n b) + d(T^n b, T^{n+m} b) \\ &\quad + \alpha_1 d(T^n b, a) + \alpha_2 d(T^n b, T^{n+m} b) + \alpha_3 d(a, T^m a) + \\ &\quad + \alpha_4 d(T^n b, T^m a) + \alpha_5 d(a, T^{n+m} b) \\ &\leq \frac{1 + \alpha_1 + \alpha_4}{1 - \alpha_3 - \alpha_4} d(T^n b, a) + \frac{1 + \alpha_2}{1 - \alpha_3 - \alpha_4} d(T^n b, T^{n+m} b) \\ &\quad + \frac{\alpha_5}{1 - \alpha_3 - \alpha_4} d(a, T^{n+m} b), \end{aligned}$$

where the sequence $\{T^{n+m} b\}$ converges to a as $n \rightarrow \infty$, since it is a

subsequence of $\{T^n b\}$. Hence we have $a = T^m a$. Using (*) we can easily show that a is the unique fixed point of T^m . So $Ta = T^m T^m a = T^m Ta$ implies $Ta = a$. Clearly a is the unique fixed point of T .

Now we shall show that $T^n x \rightarrow a$ as $n \rightarrow \infty$ for each $x \in M$. Put $p(x) = \max\{d(a, T^i x) \mid i = 1, 2, \dots, m\}$, and let n be a positive integer. Then there are nonnegative integers q, r such that $n = qm + r$ and $r < m$. Putting $k = (q - 1)m$, we have

$$\begin{aligned} d(T^n x, a) &= d(T^m T^{k+r} x, T^m a) \\ &\leq \alpha_1 d(T^{k+r} x, a) + \alpha_2 d(T^{k+r} x, T^n x) + \alpha_3 d(a, T^m a) \\ &\quad + \alpha_4 d(T^{k+r} x, T^m a) + \alpha_5 d(a, T^n x) \\ &\leq \beta d(T^{k+r} x, a), \end{aligned}$$

where

$$\beta = \frac{\alpha_1 + \alpha_2 + \alpha_4}{1 - \alpha_2 - \alpha_5} < 1.$$

Therefore we obtain

$$d(T^{qm+r} x, a) \leq \beta d(T^{(q-1)m+r} x, a),$$

which yields

$$d(T^n x, a) = d(T^{qm+r} x, a) \leq \beta^q d(T^r x, a) \leq \beta^q p(x).$$

Since $n \rightarrow \infty$ implies $q \rightarrow \infty$, this shows that $T^n x \rightarrow a$ as $n \rightarrow \infty$. This completes the proof.

Note that Theorem A is generalized to other direction by Guseman [1], Khazanchi [4], and Iséki [3].

References

- [1] L. F. Guseman Jr.: Fixed point theorems for mappings with a contractive iterate at a point, Proc. Amer. Math. Soc., 26 (1970),

615-618.

- [2] G. E. Hardy and T. D. Rogers: A generalization of a fixed point theorem of Reich, *Canad. Math. Bull.*, 16 (1973), 201-206.
- [3] K. Iséki: A generalization of Sehgal-Khazanchi's fixed point theorems, these *NOTES*, 2 (1974), XV, Kobe University.
- [4] L. Khazanchi: Results on fixed points in complete metric space, *Math. Japonicae*, 19 (1974), 283-289.
- [5] V. M. Sehgal: A fixed point theorem for mappings with a contractive iterate, *Proc. Amer. Math. Soc.*, 23 (1969), 631-634.