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4-Manifolds of Covering Number 2

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1. Introduction

K. Kobayashi and Y. Tsukui introduced the concept of the ball coverings of manifolds in [1]. For a manifold W, the minimum number of the balls of the ball coverings of W is called the *covering number* of W and is denoted by $b(W)$.

In [1], they obtained the following result.

Theorem. *For a closed n-manifold W, $2 \leqq b(W) \leqq n + 1$ holds.*

Clearly, W is an n-sphere if and only if W is a closed n-manifold with $b(W) = 2$. It is difficult, however, to determine the n-manifold W (with $b(W) = 2$) when $n \geqq 4$ and W has non-empty boundary. The answer to this problem implies the classification of closed n-manifolds W with $b(W) = 3$.

Recently, Y. Tsukui obtained a complete answer to the problem under the conditions $n = 4$ and $H_2(W) = 0$ in [2].

In the present paper, we are interested in the case $n = 4$ and $H_2(W) = Z$ (the additive group of integers).

2. Definitions

First of all, we explain some notations.

For a manifold W, we denote the boundary of W by $\dot{W}$ and the interior of W by $\mathring{W}$. For topological spaces X and Y, $X + Y$ means the disjoint union of X and Y, and $X \vee Y$ a one-point union of X and Y.

For groups G and H, we denote the direct sum of G and H by $G + H$. For a group G and a positive integer p, pG means the direct sum $G + \cdots + G$ (p times).

Definition. For two non-negative integers p and q, we define $C(p, q)$ to be the set of 4-manifolds W satisfying the following three conditions.

(1) $\dot{W} \neq \emptyset$,

(2) $b(W) = 2$,

(3) $H_1(W) = p\mathrm{Z}$ and $H_2(W) = q\mathrm{Z}$.

The subset $\overline{C}(p, q)$ of $C(p, q)$ is defined by the condition

(4) $\dot{W} = S^3$ (the 3-sphere).

Let W be an element of $C(p, q)$. From the definition, there exist two 4-balls A and B such that $W = A \cup B$ and $A \cap B = \dot{A} \cap \dot{B}$ is a 3-manifold. Put $A \cap B = F$. The triple $(A, B; F)$ is called a *realization* of W and is written just W_F.

Suppose W is an element of $\overline{C}(p, q)$. By $\hat{W}$, we denote the closed 4-manifold obtained from W by attaching a 4-ball C to $\dot{W}$. The 4-manifold $\hat{W}$ is called the *completion* of W.

Let $W_F = (A, B; F)$ be a realization of an element W of $C(p, q)$. Putting $F' = \dot{A} - \mathring{F}$, we call the pair (F, F') a *splitting* of $\dot{A}$. Suppose W belongs to $\overline{C}(p, q)$. Let (F, F') and (F, F'') be splittings of $\dot{A}$ and $\dot{B}$, respectively. The completion $\hat{W}$ of W can be regarded as the union of 4-balls A, B and C. Immediately, we obtain $A \cap C = F'$ and $B \cap C = F''$. Thus, we obtain two realizations $W_{F'}$ and $W_{F''}$ from W_F via $\hat{W}$. $W_{F'}$ and $W_{F''}$ are called the *induced realizations* of W_F.

3. Good realizations

By caluculating the homology groups, we obtain the following lemma easily.

Lemma 1. *Let W be an element of $C(p, q)$ and W_F a realization of W. Then, $\tilde{H}_{k-1}(F) = H_k(W)$ and $H_1(\dot{F}) = 2qZ$, where k is an integer.*

This implies that the number of connected components of F should be $p + 1$. Thus, we write $F = F_0 + \cdots + F_p$. If W_F is a realization of W of $C(p, 1)$, exactly one connected component of $\dot{F}$ is a torus $S^1 \times S^1$ and the others are all 2-spheres. From now on, without loss of generality, we assume that F_0 contains the boundary component $S^1 \times S^1$.

Let r_i $(i = 0, \cdots, p)$ denote number of the boundary components of F_i each of which is homeomorphic to the 2-sphere. It is not hard to see that each F_i $(i = 1, \cdots, p)$ is obtained from a 3-ball by removing $r_i - 1$ disjoint 3-balls in its interior.

Let $W_F = (A, B; F)$ be a realization of an element W of $C(p, 1)$ and (F, F') the splitting of $\dot{A}$. Let us consider F_0 and $K = \dot{F}_0 - S^1 \times S^1$. K consists of r_0 2-spheres each of which bounds a 3-ball in $\dot{A} - \mathring{F}_0$. Observe that these r_0 3-balls are disjoint. Then, we obtain a 3-manifold $\hat{F}_0$ in $\dot{A}$ by taking the union of F_0 and the r_0 3-balls above. We call the pair $(\hat{F}_0, \hat{F}'_0)$ a *capping* of W_F.

The following lemmas are proved.

Lemma 2. *Making use of the notations above, we obtain* $\hat{F}_0 \cup \hat{F}'_0 = \dot{A}$ *and* $\hat{F}_0 \cap \hat{F}'_0 = F_0 \cap F'_0 = S^1 \times S^1$.

Lemma 3. *At least one of* $\hat{F}_0$ *and* $\hat{F}'_0$ *is a solid torus.*

We define a good realization of an element of $C(p, 1)$ as follows.

Definition. Let W_F be a realization of an element W of $C(p, 1)$. If W_F has a capping $(\hat{F}_0, \hat{F}'_0)$ such that both $\hat{F}_0$ and $\hat{F}'_0$ are solid tori, W_F is said to be a *good realization* of W.

In the definition, saying in the words of knot theory, we require that the center line of the solid torus obtained in Lemma 3 is a trivial knot in $\dot{A}$.

Now, we obtain a very useful theorem.

Theorem 1. *Let* W_F *be a realization of* W *of* $\overline{C}(p, 1)$ *and* $W_{F'}$ *and* $W_{F''}$ *the induced ones. Then, two of* $W_F, W_{F'}$ *and* $W_{F''}$ *are good realizations of* W.

Corollary to Theorem 1. *Let* W *be an element of* $\overline{C}(0, 1)$. *Then,* W *collapses to a 2-sphere which is locally unknotted in* W *except at most one point.*

4. Finding a 1-handle of W of $\overline{C}(p, 1)$

The purpose of this section is to find a good realization W_F of W of $\overline{C}(p, 1)$ such that F contains a 3-ball as a connected component.

We say that a good realization W_F of W of $\overline{C}(p, 1)$ a *better* one of W if $r_0 \geqq 1$.

Lemma 4. *If* $p \geqq 1$, *there exists a better realization for any element of* $\overline{C}(p, 1)$.

We obtain the following from the homology groups of W and F.

Lemma 5. *Let* W_F *be a better realization of an element* W *of* $\overline{C}(p, 1)$. *Then,* $r_0 + \cdots + r_p = 2p$.

This together with the statement about F_i in section 3 lead us to the following theorem.

Theorem 2. *Let* W_F *be a better realization of an element* W *of* $\overline{C}(p, 1)$. *Then, at least one connected component of* F *is a 3-ball.*

5. Main results

We construct a correspondence between $\overline{C}(p, 1)$ and $\overline{C}(p-1, 1)$. Suppose $p \geqq 1$. Let W be an element of $\overline{C}(p, 1)$ and $W_F = (A, B; F)$ a better realization of W. Without loss of generality, we assume that F_p, a connected component of F, is a 3-ball. Then, the second derived neighborhood H of F_p in W can be regarded as a 1-handle of W. We define the 4-manifold V to be the closure of $W - H$. Put $X = A \frown V$, $Y = B \frown V$ and $G = F_0 + \cdots + F_{p-1}$.

Then, we obtain the following lemmas.

Lemma 6. V *belongs to* $C(p-1, 1)$ *and* $V_G = (X, Y; G)$ *is a good realization of* V.

Lemma 7. $\dot{V}$ *consists of exactly two connected components each of which is a 3-sphere.*

Let S denote a boundary component of V. By W^*, we denote the 4-manifold obtained from V by attaching a 4-ball to S.

Immediately, we obtain the following.

Lemma 8. *$\dot{W}^*$ is a 3-sphere and $H_k(W^*) = H_k(V)$ for $k = 1, 2$.*

From the construction, we have $b(W^*) \leqq 3$. By making use of a method of [1], we can prove the following.

Lemma 9. $b(W^*) = 2$.

Lemma 8 and Lemma 9 assert that W^* actually belongs to the set $\overline{C}(p-1, 1)$.

Remark. From the proof of Lemma 9, it is seen that V_G induces a good realization $W^*_{F^*}$ of W^*. Especially, $W^*_{F^*}$ can be chosen to be better if $p \geqq 2$. In this sense, W^* inherits the structure of W.

Now, we obtain the following theorem.

Theorem 3. *For any element W of $\overline{C}(p, 1)$ with $p \geqq 1$, there exists an element W^* in $\overline{C}(p-1, 1)$ such that W can be constructed from W^* as follows.*

(1) *Let $W^*_{F^*} = (A^*, B^*; F^*)$ be a good realization of W^*. Further, we assume $W^*_{F^*}$ to be better for $p \geqq 1$. Let x be a point of $\mathring{F}^*$ and let N denote the 2-nd derived neighborhood of x in W^*. Define $A' = A^* - (A^* \cap \mathring{N})$, $B' = B^* - (B^* \cap \mathring{N})$, $G = F^* - (F^* \cap \mathring{N})$ and $V = W^* - \mathring{N}$.*

(2) *Let $H = D^1 \times D^3$, the cartesian product of a 1-ball D^1 and a 3-ball D^3. Let F_p and L denote the connected components of $\dot{D}^1 \times D^3$. Define $A = A'$ and a 4-ball $B = B' \cup H$, where $B' \cap H = L$ is contained in*

$\dot{B}' \cap (\dot{B}^* - \dot{A}^*)$. *And define* $F = G + F_p$.

(3) *Then,* W *is constructed from* $V \cup H = A \cup B$ *by an identification of* F_p *with a 3-ball in* $(\dot{A} \cap \dot{N})^\circ$ *and* $(A, B; F)$ *is a better realization of* W.

Remark. The identification in the final statement of Theorem 3 depends only on the orientability of W.

Finally, we obtain the following theorem.

Theorem 4. *Let* W *be an element of* $\overline{C}(p, 1)$. *Then,* W *has a spine homeomorphic to the one-point union of* p *1-spheres, a 2-sphere and* p *3-spheres, that is,*

$$W \searrow (S_1^1 \vee \cdots \vee S_p^1) \vee S^2 \vee (S_1^3 \vee \cdots \vee S_p^3).$$

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