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Some Theorems on Fixed Points in Banach Spaces

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Recently, many authors have proved fixed point theorems for operators mapping a Banach space X into itself (see for example [1], [2], [3]). In each of these theorems, it is assumed that the space X is either uniformly convex or reflexive.

In this paper we shall concern with the mapping T of a Banach space X into itself satisfying the following inequality

$$(1) \quad \|Tx - Ty\| \leq \alpha \cdot \max\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \frac{1}{2}\|x - Ty\|, \frac{1}{2}\|y - Tx\|\}$$

for $x, y \in X$, where $0 \leq \alpha < 1$. Some fixed point theorems for such a mapping X will be stated in the first section. It may be noted that the space here is not necessarily uniformly convex or reflexive. In the second section, we shall discuss Reiner's iteration scheme (cf. [4]) in Banach and Hilbert spaces converging to the fixed point of the mapping satisfying the inequality (1).

1. Fixed point theorems in Banach spaces

Theorem 1. *Let M be a closed convex subset of a Banach space X and T be a selfmapping on M such that it satisfies (1) for all $x, y \in M$ and*

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for some $0 \leq \alpha < 1$. Let $x_0 \in M$, $t \in (0, 1)$ and $x_{n+1} = (1 - t)x_n + tTx_n$ for each integer $n \geq 0$. If the sequence $\{x_n\}_{n=0}^{\infty}$ converges to a point u in M , then u is a unique fixed point of T .

Proof. By using (1), we have

$$\begin{aligned} (2) \quad \|x_{n+1} - Tu\| &\leq (1 - t)\|x_n - Tu\| + t\|Tx_n - Tu\| \\ &\leq (1 - t)\|x_n - Tu\| + t\alpha \cdot \max\{\|x_n - u\|, \|x_n - Tx_n\|, \\ &\quad \|u - Tu\|, \frac{1}{2}\|x_n - Tu\|, \frac{1}{2}\|u - Tx_n\|\} \\ &\leq (1 - t)\|x_n - Tu\| + t\alpha[\|x_n - u\| + \|x_n - Tx_n\| + \|u - Tu\|]. \end{aligned}$$

Since $\{x_n\}$ converges to u and $x_{n+1} - x_n = t(Tx_n - x_n)$, the sequence $\{\|Tx_n - x_n\|\}_{n=1}^{\infty}$ tends to zero. Therefore letting $n \rightarrow \infty$ in (2), we have

$$\begin{aligned} \|u - Tu\| &\leq (1 - t)\|u - Tu\| + \alpha t\|u - Tu\| \\ &\leq (1 - (1 - \alpha)t)\|u - Tu\| \\ &= \beta\|u - Tu\|, \end{aligned}$$

where $\beta = 1 - (1 - \alpha)t \in (0, 1)$. Hence $u = Tu$. We now show the uniqueness of the fixed point. Suppose $v (\neq u)$ be another fixed point of T . Then from (1) we obtain $\|u - v\| = \|Tu - Tv\| \leq \alpha\|u - v\|$, which is impossible since $\alpha < 1$. Hence $u = v$, and this completes the proof.

Theorem 2. Let X be a Banach space and let T be a mapping of X into itself. If there exists a mapping K of X into itself which has a right inverse K^{-1} (i.e. $KK^{-1} = I$, the identity mapping) such that $K^{-1}TK$ satisfies the condition

$$(3) \quad \|K^{-1}TKx - K^{-1}TKy\| \leq \alpha \cdot \max\{\|x - y\|, \|x - K^{-1}TKx\|, \|y - K^{-1}TKy\|, \\ \frac{1}{2}\|x - K^{-1}TKy\|, \frac{1}{2}\|y - K^{-1}TKx\|\}$$

for all $x, y \in X$, where $\alpha \in (0, 1)$. Then T has a unique fixed point.

Proof. Since $K^{-1}TK$ satisfies the condition (3) and X is a Banach

space, $K^{-1}TK$ has a unique fixed point $u \in X$. Hence $TKu = KK^{-1}TKu = Ku$. If $v \in X$ is a fixed point of T , then since $K^{-1}TKK^{-1}v = K^{-1}v$, we have $K^{-1}v = u$. Therefore Ku is a unique fixed point of T .

Theorem 3. Let T be a mapping of a Banach space X into itself satisfying the condition (1) for all $x, y \in X$, where $\alpha \in (0, 1)$. For each positive integer n , let $a_n \in X$ be a solution of the equation $u - Tu = A_n$, where $A_n \in X$. If $A_n \rightarrow 0$ as $n \rightarrow \infty$, then the sequence $\{a_n\}_{n=1}^{\infty}$ converges to the solution of the equation $u = Tu$.

Proof. Since

$$\begin{aligned} \|a_n - a_m\| &\leq \|a_n - Ta_n\| + \|a_m - Ta_m\| + \|Ta_n - Ta_m\| \\ &\leq \|A_n\| + \|A_m\| + \alpha \cdot \max\{\|a_n - Ta_n\|, \|a_n - a_m\|, \\ &\quad \|a_m - Ta_m\|, \frac{1}{2}\|a_n - Ta_n\|, \frac{1}{2}\|a_m - Ta_m\|\} \\ &\leq \frac{1+\alpha}{1-\alpha} (\|A_n\| + \|A_m\|), \end{aligned}$$

the sequence $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Hence it converges to some $a \in X$. But then we have

$$\begin{aligned} \|a - Ta\| &\leq \|a - a_n\| + \|a_n - Ta_n\| + \|Ta_n - Ta\| \\ &\leq \|a - a_n\| + \|a_n - Ta_n\| \\ &\quad + \alpha \cdot \max\{\|a - a_n\|, \|a_n - Ta_n\|, \|a - Ta\|, \frac{1}{2}\|a_n - Ta_n\|, \frac{1}{2}\|a - Ta\|\} \\ &\leq \|a - a_n\| + \|a_n - Ta_n\| + \alpha(\|a - a_n\| + \|a_n - Ta_n\| + \|a - Ta\|) \\ &\leq \frac{1+\alpha}{1-\alpha} (\|a - a_n\| + \|A_n\|) \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Hence it follows that $a = Ta$.

2. Fixed point iterations using infinite matrices

Let X be a normed linear space, M a non-empty closed bounded convex subset of X , T a mapping of M into itself possessing at least one fixed

point in M . In his paper [4], Reinerman defined a summability matrix A $= (a_{nk})$ by

$$(4) \quad a_{nk} = \begin{cases} c_k \prod_{j=k+1}^n (1 - c_j) & \text{if } k < n, \\ c_n, & \text{if } k = n, \\ 0, & \text{if } k > n, \end{cases}$$

where $\{c_n\}_{n=0}^{\infty}$ is a real sequence satisfying (i) $c_0 = 1$, (ii) $0 < c_n < 1$ for $n \geq 1$, and (iii) $\sum_k c_k \rightarrow \infty$. He then defined $x_{n+1} = \sum_{k=0}^n a_{nk} T x_k$, $x_0 \in M$, which can be written in the form

$$(5) \quad x_{n+1} = (1 - c_n)x_n + c_n T x_n.$$

The same iteration scheme was defined independently by Outlaw and Groetsch [5] and Dotson [6].

Theorem 4. *Let T be a mapping of a Banach space X into itself satisfying (1). Suppose that $A = (a_{nk})$ is an infinite matrix defined by (4) with $\{c_n\}_{n=0}^{\infty}$ satisfying (i), (ii) and bounded away from zero. If the sequence defined by (5) converges to u , then u is the unique fixed point of T .*

Proof. Since $x_n \rightarrow u$ and $\{c_n\}_{n=0}^{\infty}$ is bounded away from zero, it follows from $x_{n+1} - x_n = c_n(Tx_n - x_n)$ that $\|Tx_n - x_n\| \rightarrow 0$. On the other hand, because of (1), we have

$$\begin{aligned} \|Tx_n - Tu\| &\leq \alpha \cdot \max\{\|x_n - u\|, \|x_n - Tx_n\|, \|u - Tu\|, \\ &\quad \frac{1}{2}\|x_n - Tu\|, \frac{1}{2}\|u - Tx_n\|\} \\ &\leq \alpha[\|x_n - u\| + \|x_n - Tx_n\| + \|Tx_n - Tu\|], \end{aligned}$$

which implies

$$\|Tx_n - Tu\| \leq \frac{\alpha}{1 - \alpha} [\|x_n - u\| + \|x_n - Tx_n\|].$$

Therefore $Tx_n \rightarrow Tu$ as $n \rightarrow \infty$. Consequently

$$\|Tu - u\| \leq \|Tu - Tx_n\| + \|Tx_n - x_n\| + \|x_n - u\| \rightarrow 0$$

as $n \rightarrow \infty$, whence we get $Tu = u$.

Theorem 5. Let E be a closed convex subset of a Hilbert space H , and T a mapping of E into itself. Let $A = (a_{nk})$ be an infinite matrix defined by (4) with $\{c_n\}_{n=0}^{\infty}$ satisfying (i) - (iii) and $\overline{\lim}_n c_n < 1 - \alpha^2$. Then the sequence defined by (5) converges to the fixed point of T .

Proof. Since T satisfies (1), it has a unique fixed point $u \in E$. Let $p, q \in H$ and λ, μ be two nonnegative real numbers with $\lambda + \mu = 1$. Then expanding the inner products $(\lambda p + \mu q, \lambda p + \mu q)$ and $(\lambda(p - q), \mu(p - q))$ we obtain

$$(6) \quad \|\lambda p + \mu q\|^2 = \lambda\|p\|^2 + \mu\|q\|^2 - \lambda\mu\|p - q\|^2.$$

Since $x_{n+1} - u = (1 - c_n)(x_n - u) + c_n(Tx_n - u)$ by (5), applying (6) we have

$$(7) \quad \|x_{n+1} - u\|^2 = (1 - c_n)\|x_n - u\|^2 + c_n\|Tx_n - u\|^2 - c_n(1 - c_n)\|Tx_n - x_n\|^2.$$

Since T satisfies (1), we get

$$\begin{aligned} \|Tx_n - Tu\| &\leq \alpha \cdot \max\{\|x_n - u\|, \|x_n - Tx_n\|, \|u - Tu\|, \\ &\quad \frac{1}{2}\|x_n - Tu\|, \frac{1}{2}\|u - Tx_n\|\} \\ &\leq \alpha \cdot \max\{\|x_n - u\|, \|x_n - Tx_n\|\}. \end{aligned}$$

If the maximum in this inequality is $\|x_n - u\|$, then from (7), we have

$\|x_{n+1} - u\|^2 \leq (1 - c_n)\|x_n - u\|^2 + c_n\alpha^2\|x_n - u\|^2 - c_n(1 - c_n)\|Tx_n - x_n\|^2$, and if the maximum is $\|x_n - Tx_n\|$, then $\|x_{n+1} - u\|^2 \leq (1 - c_n)\|x_n - u\|^2 + c_n\alpha^2\|x_n - Tx_n\|^2 - c_n(1 - c_n)\|Tx_n - x_n\|^2$. In either case we get

$$(8) \quad \|x_{n+1} - u\|^2 \leq \|x_n - u\|^2 - c_n(1 - c_n - \alpha^2)\|Tx_n - x_n\|^2.$$

From (8) we observe that $\|x_{n+1} - u\|$ decreases with n for all sufficiently large n . On the other hand, since $\{c_n\}_{n=0}^{\infty}$ satisfies (iii) and $\overline{\lim}_n c_n < 1 - \alpha^2$, there exists a subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$ such that $\|Tx_{n_k} - x_{n_k}\| \rightarrow 0$ as $k \rightarrow \infty$. Since T satisfies (1) we have

$$\begin{aligned} \|Tx_{n_k} - Tx_{n_\ell}\| &\leq \alpha \cdot \max \{ \|x_{n_k} - x_{n_\ell}\|, \|x_{n_k} - Tx_{n_k}\|, \|x_{n_\ell} - Tx_{n_\ell}\|, \\ &\quad \frac{1}{2}\|x_{n_k} - Tx_{n_\ell}\|, \frac{1}{2}\|x_{n_\ell} - Tx_{n_k}\| \} \end{aligned}$$

implying

$$\begin{aligned} \|Tx_{n_k} - Tx_{n_\ell}\| &\leq \frac{\alpha}{1 - \alpha} (\|x_{n_k} - Tx_{n_k}\| + \|x_{n_\ell} - Tx_{n_\ell}\|) \\ &\rightarrow 0 \end{aligned}$$

as $k, \ell \rightarrow \infty$. Thus $\{Tx_{n_k}\}_{k=0}^{\infty}$ is a Cauchy sequence and hence convergent.

Let its limit be v , so that

$$\lim_k Tx_{n_k} = \lim_k x_{n_k} = v.$$

Now from (1), we have

$$\begin{aligned} \|Tv - v\| &\leq \|Tv - Tx_{n_k}\| + \|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - v\| \\ &\leq \alpha \cdot \max \{ \|v - x_{n_k}\|, \|v - Tv\|, \|Tx_{n_k} - x_{n_k}\|, \\ &\quad \frac{1}{2}\|v - Tx_{n_k}\|, \frac{1}{2}\|x_{n_k} - Tv\| \} \\ &\quad + \|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - v\| \\ &\leq \alpha (\|x_{n_k} - v\| + \|Tx_{n_k} - x_{n_k}\| + \|Tv - v\|) \\ &\quad + \|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - v\|, \end{aligned}$$

which implies

$$\|Tv - v\| \leq \frac{1 + \alpha}{1 - \alpha} (\|v - x_{n_k}\| + \|Tx_{n_k} - x_{n_k}\|).$$

Consequently, we can conclude that $Tv = v$. So we obtain $u = v$, since T has a unique fixed point. Because of the fact that $x_{n_k} \rightarrow u$ as $k \rightarrow \infty$ and that $\|x_n - u\|$ decreases with n for all sufficiently large n , we

get $x_n \rightarrow u$ as $n \rightarrow \infty$. This completes the proof of the theorem.

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