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Some Applications of Banach Type
Contraction Principles

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Recently I have seen a new publication [1] which has been presented by Professor Ioan A. Rus. In the publication, I have found an interesting result of Perov. I have not known this result until receiving the publication. I have been strongly influenced by Perov's theorem. I think that his theorem is a result of a useful application of Banach type contraction mapping theorems. In this note, we shall prove some fixed point theorems by the same ideas.

Throughout this note, we denote by (X, d) a nonempty complete metric space, and by f, g two mappings of the product space $X \times X$ into X .

In his manuscript [3], Chi-Lin Yen obtains the following fixed point theorem which is a special case of his fixed point theorem for multivalued mappings:

Lemma. If S, T are mappings of X into itself such that there exist $k, k' > 0$ with $k + 2k' < 1$ satisfying

$d(Sx, Ty) \leq k \cdot \max\{d(Sx, x), d(Ty, y), d(x, y)\} + k'[d(Sx, y) + d(Ty, x)]$
for all x, y in X , then S, T have a unique common fixed point.

Applying this lemma, we have the following

Theorem 1. If for some $\alpha, \beta > 0$ with $\alpha + 3\beta < 1$,

$$(1) \quad d(f(x, y), g(x', y')) \leq \alpha \cdot \max\{d(f(x, y), x), d(g(x', y'), x'), d(x, x')\} + \beta[d(f(x, y), x') + d(g(x', y'), x) + d(y, y')]$$

for all x, x', y, y' in X , then there exists exactly one point $a \in X$ such that

$$f(a, a) = a = g(a, a)$$

Proof. Letting $y = y'$ in (1), we obtain

$$d(f(x, y), d(x', y)) \leq \alpha \cdot \max\{d(f(x, y), x), d(g(x', y), x'), d(x, x')\} + \beta[d(f(x, y), x') + d(g(x', y), x)].$$

For any fixed $y \in X$, by Lemma ($\alpha + 3\beta < 1$ implies $\alpha + 2\beta < 1$), there exists a unique $x(y)$ in X such that

$$(2) \quad x(y) = f(x(y), y) = g(x(y), y).$$

For any y, y' in X , this relation and (1) imply

$$\begin{aligned} d(x(y), x(y')) &= d(f(x(y), y), g(x(y'), y')) \\ &\leq \alpha \cdot \max\{d(f(x(y), y), x(y)), d(g(x(y'), y'), x(y')), d(x(y), x(y'))\} \\ &\quad + \beta[d(f(x(y), y), x(y')) + d(g(x(y'), y'), x(y)) + d(y, y')] \\ &= (\alpha + 2\beta)d(x(y), x(y')) + \beta d(y, y'). \end{aligned}$$

Therefore we have

$$d(x(y), x(y')) \leq \frac{\beta}{1 - (\alpha + 2\beta)} d(y, y').$$

Since $\alpha + 3\beta < 1$, the Banach contraction theorem implies that the mapping $x(\cdot)$ of X into itself has exactly one fixed point $a \in X$, i.e., $x(a) = a$. Therefore by (2), we have

$$a = x(a) = f(a, a) = g(a, a).$$

If $b \in X$ satisfies

$$b = f(b, b) = g(b, b),$$

then by (1),

$$\begin{aligned} d(a, b) &= d(f(a, a), g(b, b)) \\ &\leq \alpha \cdot \max \{ d(a, a), d(b, b), d(a, b) \} \\ &\quad + \beta [d(a, b) + d(b, a) + d(a, b)]. \end{aligned}$$

Therefore we obtain

$$d(a, b) \leq (\alpha + 3\beta) d(a, b),$$

which implies $a = b$. This completes the proof.

Remark. Under the condition (1), if $x, y \in X$ satisfy the equations $x = f(x, y)$ and $y = g(x, y)$, then $x = y$. In fact, (1) implies

$$\begin{aligned} d(x, y) &= d(f(x, y), g(x, y)) \\ &\leq \alpha \cdot \max \{ d(f(x, y), x), d(g(x, y), y), d(x, x) \} \\ &\quad + \beta [d(f(x, y), y) + d(g(x, y), x) + d(y, y)] \\ &= 2\beta d(x, y). \end{aligned}$$

Theorem 2. If for some nonnegative $\alpha, \beta, \gamma, \delta$ with $2\alpha + 2\beta + \gamma < 1$, $2\beta + \gamma + \delta < 1$,

$$\begin{aligned} (3) \quad &d(f(x, y), g(x', y')) \\ &\leq \alpha [d(x, f(x, y)) + d(x', g(x', y'))] \\ &\quad + \beta [d(x, g(x', y')) + d(x', f(x, y))] + \gamma d(x, x') + \delta d(y, y'), \end{aligned}$$

then there exists only one a in X such that $a = f(a, a) = g(a, a)$.

Proof. Let $y = y'$. Then by (3), we have

$$\begin{aligned} d(f(x, y), g(x', y)) &\leq \alpha [d(x, f(x, y)) + d(x', g(x', y))] \\ &\quad + \beta [d(x, g(x', y)) + d(x', f(x, y))] + \gamma d(x, x'). \end{aligned}$$

For a fixed point $y \in X$, by Rus's Theorem [2], we have exactly one $x(y) \in X$ such that $x(y) = f(x(y), y) = g(x(y), y)$. Then for any $y, y' \in X$, by (3)

$$\begin{aligned}
d(x(y), x(y')) &= d(f(x(y), y), g(x(y'), y')) \\
&\leq \alpha \cdot 0 + 2\beta d(x(y), x(y')) + \gamma d(x(y), x(y')) + \delta d(y, y') \\
&= (2\beta + \gamma) d(x(y), x(y')) + \delta d(x, y').
\end{aligned}$$

Therefore we have

$$d(x(y), x(y')) \leq \frac{\delta}{1 - (2\beta + \gamma)} d(y, y').$$

By the assumption, there exists only one a in X such that $a = x(a)$.

Hence $a = f(a, a) = g(a, a)$. The uniqueness is obvious, which completes the proof.

References

- [1] G. Coman, G. Pavel, I. Rus and I. A. Rus: Introducere in teoria ecuatiilor operatoriale, Cluj-Napoca, 1976.
- [2] I. A. Rus: Teoria punctului fix, II, Cluj-Napoca, 1973.
- [3] C. L. Yen: some fixed point theorems, Preprint.