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A Theorem on Fixed Points

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It is the purpose of this note to prove the following theorem:

Theorem. *Let T be a self-mapping of a nonempty complete metric space (X, d) satisfying*

$$(*) \quad d(Tx, Ty) \leq \alpha[d(x, Tx) \cdot d(y, Ty)]^{1/2}$$

for all $x, y \in X$ and $0 < \alpha < 1$. Then T has a unique fixed point, and the iterate $T^n x$ converges to the fixed point for every $x \in X$.

Proof. Let x be an arbitrary point of X . Then by $(*)$, we get for each $n \geq 1$,

$$d(T^{n+1}x, T^n x) \leq \alpha[d(T^n x, T^{n+1}x) \cdot d(T^{n-1}x, T^n x)]^{1/2},$$

which implies

$$d(T^{n+1}x, T^n x) \leq \beta d(T^n x, T^{n-1}x)$$

where $\beta = \alpha^2$. It follows that

$$d(T^{n+1}x, T^n x) \leq \beta^n d(x, Tx)$$

for all $n \geq 1$. So if $m > n$, then

$$\begin{aligned} d(T^m x, T^n x) &\leq d(T^m x, T^{m-1}x) + d(T^{m-1}x, T^{m-2}x) + \cdots + d(T^{n+1}x, T^n x) \\ &\leq [\beta^{m-1} + \beta^{m-2} + \cdots + \beta^{n-1}]d(x, Tx) \\ &\leq \frac{\beta^{n-1}}{1 - \beta}d(x, Tx) \end{aligned}$$

Since $\beta < 1$, $\{T^n x\}$ is a Cauchy sequence in X and so has a limit z in

X , since X is complete. Now we have

$$\begin{aligned} d(z, Tz) &\leq d(z, T^n x) + d(T^n x, Tz) \\ &\leq d(z, T^n x) + \alpha \{ d(T^{n-1} x, T^n x) \cdot d(z, Tz) \}^{1/2}. \end{aligned}$$

Letting $n \rightarrow \infty$, this yields $z = Tz$. Hence z is a fixed point of T . To prove the uniqueness of the fixed point, let $z, z' \in X$ be fixed points of T . Then

$$\begin{aligned} d(z, z') &= d(Tz, Tz') \\ &\leq \alpha \{ d(z, Tz) \cdot d(z', Tz') \}^{1/2} \\ &= \alpha \{ d(z, z) \cdot d(z', z') \}^{1/2} \\ &= 0. \end{aligned}$$

That is to say $z = z'$. So the fixed point is unique. This completes the proof.

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