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Knotting Problems of 2-Spheres in 4-Sphere

By Shin'ichi SUZUKI

Preface

The purpose of this note is to survey the knotting problems (or placement problems) of 2-spheres in the 4-sphere. The knotting problems of 1-spheres in the 3-sphere have been studied extensively as 'classical' knot theory. Recently, through the development of the general theory of differential and piecewise linear topology, the knotting problems of n -spheres in the $(n+2)$ -sphere, with $n \geq 3$, have been made great progress toward 'higher dimensional' or 'modern' knot theory. In the classical knot theory, 2-dimensional diagrams through the regular projection are useful and some infinite group theory is essential, while in the modern knot theory general position arguments are useful and homological algebraic technique is essential. Many results coming out of the both theories may be extended to those in the knot theory of 2-spheres in the 4-sphere (abbreviated as the 2-knot theory) without serious difficulties, however, many fundamental properties still remain unsolved.

The 2-knot theory is essentially the 4-dimensional topology, so we can not avoid treating the second homotopy groups, and the general position arguments are useless. On the other hand, we have the so-called 'motion picture method' to describe a 2-manifold in the 4-space, but it is yet

incomplete. We will attempt to formulate properties and problems in the 2-knot theory geometrically as far as we can.

We do not claim that this survey is complete and we apologize to anyone whose favorite theorem is omitted. For example, we do not discuss the Smith conjecture of the 4-sphere and isotopy types of 2-links, etc..

Throughout the article, we shall only be concerned with the piecewise linear category, consisting of simplicial complexes and piecewise linear maps. For the piecewise linear category, the main references are Glaser [G], Hudson [H], Rourke-Sanderson [R-S], Stallings [S] and Zeeman [Ze]. We assume the reader is familiar with the basic theorems of the classical knot theory (see, for example, Crowell-Fox [C-F], Murasugi [M], Neuwirth [N₁], Reidemeister [Re] and Rolfsen [Ro]). We also recommend the survey article Fox [3].

This work was developed through the geometric topology seminars at the University of Toronto in 1975 and the Kobe University in 1976. I wish to express my gratitude to Professor Kunio Murasugi for helpful suggestions. I would also like to thank the members of Kobe Topology Seminar for numerous discussions.

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1. Introduction

1.1. The following notation is fixed throughout the note :

$\mathbb{Z}$: set of all integers,

$\mathbb{J}$: the infinite cyclic multiplicative group generated by t ,

$\mathbb{Q}$: set of all rational numbers,

$\mathbb{R}$: set of all real numbers,

$I = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$: the closed unit interval,

$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}\}$: the euclidean n -space ($n > 0$),

$\mathbb{R}_+^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$,

$\mathbb{R}_-^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_n \leq 0\}$,

$D^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2} \leq 1\}$: the n -disk,

$S^{n-1} = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2} = 1\}$: the $(n-1)$ -sphere,

$S_+^{n-1} = \{(x_1, x_2, \dots, x_n) \in S^{n-1} \mid x_n \geq 0\}$,

$S_-^{n-1} = \{(x_1, x_2, \dots, x_n) \in S^{n-1} \mid x_n \leq 0\}$.

We always assume that $\mathbb{R}^n$ and the above standard subspaces have the right-handed orientation, unless otherwise stated. We identify $\mathbb{R}^n$ (and the standard subspaces X^n , respectively) with the subspace of $\mathbb{R}^{n+m}$ (and X^{n+m}) having all components after the n -th equal to 0.

The piecewise linear structures of $\mathbb{R}^n$, D^n and S^n are unique except for $\mathbb{R}^4$, D^4 and S^4 ; Stallings [1]. We always assume that $\mathbb{R}^4$, D^4 and

S^4 have the standard piecewise linear structures which are compatible with the affine structure.

∂X : the boundary of X ,

$^\circ X$: the interior of X ,

$\text{Cl}(X)$: the closure of X ,

$\cong$: homeomorphic to, and isomorphic to,

$\simeq$: homotopic to, and homotopy equivalence.

1.2. Definitions. An $(n, n+m)$ -link with μ components is an embedding $\ell : \bigcup_{i=1}^{\mu} S_i^n \rightarrow S^{n+m}$ of a disjoint union $\bigcup_{i=1}^{\mu} S_i^n$ of oriented n -spheres into the oriented $(n+m)$ -sphere S^{n+m} . In particular, an $(n, n+m)$ -link with one component will be called an $(n, n+m)$ -knot.

Two $(n, n+m)$ -links $\ell_j : \bigcup_{i=1}^{\mu} S_i^n \rightarrow S^{n+m}$ with μ components, $j=1, 2$, are said to be *equivalent* (or *of the same type*) iff there exists an orientation preserving homeomorphism $\psi : S^{n+m} \rightarrow S^{n+m}$ such that $\psi \ell_1 = \ell_2$.

Then it is trivial that this is an equivalence relation. We call the equivalence class of an $(n, n+m)$ -link ℓ the $(n, n+m)$ -link type (or *knot type*) of ℓ . Our problem is to describe and classify the link types and knot types.

Remember that everything is piecewise linear. For smooth (or differential) knots and links, good references are Haefliger [1] and Levine [2]. The general (or wild) knots and links are nicely covered in Rushing [Ru].

1.3. Definition. Those $(n, n+m)$ -knots equivalent to the natural inclusion $k_0 : S^n \rightarrow S^{n+m}$ are called *trivial* (or *unknotted*), and constitute

the *trivial type*.

By the homogeneity theorem of Newman-Gugenheim, Gugenheim[1], we have:

1.4. Proposition. An $(n, n+m)$ -knot $k : S^n \rightarrow S^{n+m}$ is trivial iff there exists an embedding $\tilde{k} : D^{n+1} \rightarrow S^{n+m}$ with $\tilde{k}|_{\partial D^{n+1}} = k$.

1.5. Definition. An $(n, n+m)$ -link $\ell : \sum_{i=1}^{\mu} S_i^n \rightarrow S^{n+m}$ with μ components is called *trivial* iff there exists an embedding $\tilde{\ell} : \sum_{i=1}^{\mu} D_i^{n+1} \rightarrow S^{n+m}$ with $\tilde{\ell}|_{\sum_{i=1}^{\mu} \partial D_i^{n+1}} = \ell$.

If $m > 1$, then any two trivial $(n, n+m)$ -links with μ components are equivalent ; the equivalence class will be called the *trivial type*.

From the definition, we have :

1.6. Theorem. Let $\ell : \sum_{i=1}^{\mu} S_i^n \rightarrow S^{n+m}$ be an $(n, n+m)$ -link. If ℓ is trivial and $m=2$, then the fundamental group $\pi_1(S^{n+m} - \ell(\sum_{i=1}^{\mu} S_i^n))$ is a free group of rank μ .

1.7. Theorem. (Jordan-Schoenflies-Alexander Theorem) For $m > 1$, any $(0, m)$ -link with μ components is trivial. Any $(1, 2)$ -knot and any $(2, 3)$ -knot are trivial.

1.8. PL Schoenflies Conjecture ($n \geq 3$) : Any $(n, n+1)$ -knot is trivial. That is, let K^n be a PL n -sphere in S^{n+1} . Then, the closures of the complementary domains of K^n are PL $(n+1)$ -disks.

1.9. Theorem. (Alexander-Newman Theorem) Let $k : D^n \rightarrow S^n$ be a PL embedding. Then, $Cl(S^n - k(D^n))$ is a PL n -disk.

1.10. Theorem. (Zeeman [1 ; 2]) For $m > 2$, any $(n, n+m)$ -knot is trivial.

For smooth knots, this Theorem need not be true, see Haefliger [1].

Contrary to Theorems 1.7 and 1.10, we have :

1.11. Theorem. (Artin [1], etc.) For any $n \geq 1$, non-trivial $(n, n+2)$ -knots exist. For any $n \geq m - 1$ and $\mu \geq 2$, non-trivial $(n, n+m)$ -links with μ components exist.

1.12. Example (*Suspension*): (Artin [1], cf. Andrews-Curtis [1]). Let $k^1 : S^1 \rightarrow S^3$ be a $(1, 3)$ -knot. Given points $a \in {}^\circ S_+^4$ and $b \in {}^\circ S_-^4$, let K^2 be the join of $k^1(S^1)$ with $a \cup b$. We give an orientation on K^2 so that the orientation of the 2-disk $k^1(S^1) * a$ is coherent to that of $k^1(S^1)$. Now we have a $(2, 4)$ -knot $k^2 : S^2 \rightarrow S^4$ with $k^2(0, 0, 0, 1) = a$, $k^2(0, 0, 0, -1) = b$, $k^2(S^2) = K^2$; we call the k^2 the *suspension* of k^1 .

It holds that :

$$\pi_1(S^4 - K^2) \cong \pi_1(S^3 - k^1(S^1)).$$

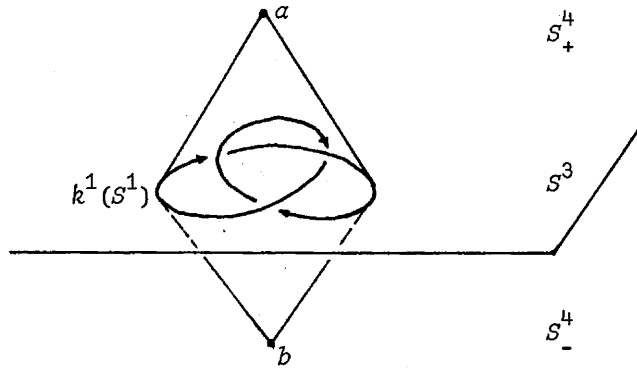


Figure 1.1

1.13. Example (*Spin*): (Artin [1]) Let R_+^3 be the half-space defined by $x_4 = 0$, $x_3 \geq 0$ in R^4 . We spin R_+^3 according to the equations :

$$\begin{aligned} \bar{x}_1 &= x_1, & \bar{x}_2 &= x_2, \\ \bar{x}_3 &= x_3 \cos \theta - x_4 \sin \theta, & \bar{x}_4 &= x_3 \sin \theta + x_4 \cos \theta. \end{aligned}$$

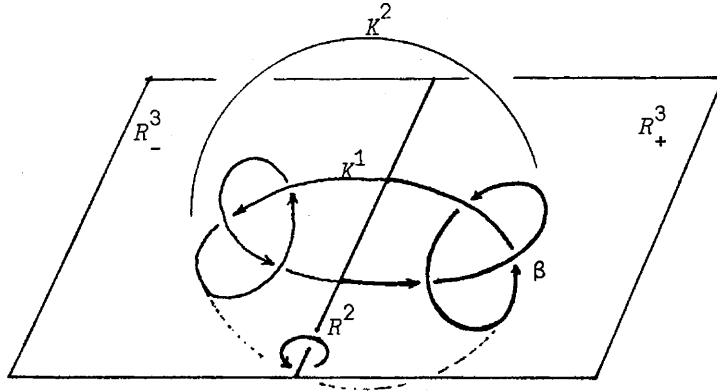


Figure 1.2

Let R^2 be the plane defined by $x_3 = x_4 = 0$, so that R^3_+ is spun about R^2 . Let $k^1 : S^1 \rightarrow R^3_+$ be an embedding such that $k^1(S^1) \cap R^2$ consists of a single arc α . Then spinning of $\beta = \text{Cl}(k^1(S^1) - \alpha)$ sweeps out a 2-sphere K^2 . If R^4 is compactified with a point ∞ (at infinity) to give S^4 , then R^3 becomes S^3_0 and R^2 becomes S^2_0 . $K^2 \cap S^3_0$ consists of a simple loop K^1 . We give an orientation on K^1 which is coherent to that of β . Then K^1 is a representative of the $(1,3)$ -knot $k^1 \# (-k^1)$. Now K^2 is oriented so that the orientation of the 2-disk $K^2 \cap S^4_+$ is coherent to that of K^1 . We have a $(2,4)$ -knot $k^2 : S^2 \rightarrow S^4$ with $k^2(S^2) = K^2$. We refer to the $(2,4)$ -knot k^2 as a *spun* of the k^1 . Since every loop in $S^4 - K^2$ can be deformed continuously in $R^3_+ - \beta$, we have :

$$\pi_1(S^4 - K^2) \cong \pi_1(R^3_+ - \beta) \cong \pi_1(R^3 - k^1(S^1)).$$

In the same way as that of the spun $(2,4)$ -knot k^2 , we can construct *spun* $(2,4)$ -links with μ components from some non-trivial $(1,3)$ -links with μ components. We can also construct *spun* $(n, n+2)$ -links from some $(n-1, n+1)$ -links. Refer to Epstein [1], Zeeman [3], Cappell [1], etc..

1.14. *Singularities or Local Knots* : In Example 1.12, there exist two singular points a and b in $k^2(S^2)$. We will measure such a point as the following way : Let $f : M^n \rightarrow W^{n+m}$ be an embedding of an oriented n -manifold M^n into an oriented $(n+m)$ -manifold W^{n+m} . Suppose $\partial M^n = \emptyset$ and $\partial W^{n+m} = \emptyset$. For a point $x \in M^n$, let $St(x; M^n)$ and $St(f(x); W^{n+m})$ denote the closed star neighborhoods of x in M^n and $f(x)$ in W^{n+m} , respectively. The boundary $\partial St(x; M^n)$ is an $(n-1)$ -sphere with an orientation inherited from that of $St(x; M^n)$, and the boundary $\partial St(f(x); W^{n+m})$ is an $(n+m-1)$ -sphere with an orientation inherited from that of $St(f(x); W^{n+m})$. The knot type, denoted by $\kappa(x)$, of the embedding

$$f|_{\partial St(x; M^n)} : \partial St(x; M^n) \rightarrow \partial St(f(x); W^{n+m})$$

is called the *singularity* or *local knot type* of the embedding f at x . When $\kappa(x)$ is of trivial type, we may say that f is *locally flat* at x . We say f is *locally flat* iff it is locally flat at each point of M^n . When $\kappa(x)$ is of non-trivial type, we may say that f is *locally knotted* at x . It will be noticed that the locally knotted points occur only at the vertices of M^n and that if $m > 2$ then any embedding f is locally flat by Theorem 1.10.

In the suspension $(2,4)$ -knot k^2 in Example 1.12, if the $(1,3)$ -knot k^1 is trivial, k^2 is locally flat (and k^2 is trivial), and if k^1 is non-trivial, then k^2 is locally knotted at two points whose local knot types are the types of k^1 and $-k^1$.

The spun $(2,4)$ -knot k^2 in Example 1.13 is locally flat.

For the singularities, we refer the reader to Fox-Milnor [2], Noguchi [1], [No].

1.15. Definition. In usual manner, we will also call an embedding

$\ell : \bigcup_{i=1}^{\mu} S_i^n \rightarrow R^{n+2}$ an $(n, n+2)$ -link with μ components. In particular, an $(n, n+2)$ -link with 1 component will be called an $(n, n+2)$ -knot. Under the definition, we also define the link types in the same way as that of 1.2. That is, two $(n, n+2)$ -links $\ell_j : \bigcup_{i=1}^{\mu} S_i^n \rightarrow R^{n+2}$ with μ components, $j=1, 2$, are said to be *equivalent* (or of the same type) iff there exists an orientation preserving homeomorphism $\psi : R^{n+2} \rightarrow R^{n+2}$ with $\psi \ell_1 = \ell_2$.

The terminology is justified by the following :

1.16. Proposition. Let $\ell_j : \bigcup_{i=1}^{\mu} S_i^n \rightarrow R^{n+2}$, $j=1, 2$, be embeddings (i.e. $(n, n+2)$ -links) and let $\tilde{\ell}_j : \bigcup_{i=1}^{\mu} S_i^n \rightarrow S^{n+2}$, $j=1, 2$, be the natural extensions, where S^{n+2} is the one-point compactification of R^{n+2} . Then, ℓ_1 and ℓ_2 are equivalent if and only if $\tilde{\ell}_1$ and $\tilde{\ell}_2$ are equivalent.

Proof. The necessity is obvious. Since $S^{n+2} - \ell_j(\bigcup_{i=1}^{\mu} S_i^n)$ is connected, the sufficiency follows from the following Fisher's theorem.

1.17. Proposition. (Fisher [1]) An orientation preserving homeomorphism of S^n onto itself is isotopic to the identity.

1.18. Remark : Knotting problems of n -spheres in R^{n+2} do not always agree with the problems in S^{n+2} . But little distinction will be drawn between them. In the following, sometimes R^{n+2} is convenient, sometimes S^{n+2} , because R^{n+2} has the standard coordinate-system and S^{n+2} is compact.

1.19. Convention : The topics discussed in this note are $(2, 4)$ -knots as stated in the Preface. From now on, we shall discuss only locally flat

knots and links. Usually, we shall say '*n-knots*' and '*n-links*' for short instead of *locally flat* $(n, n+2)$ -knots and *locally flat* $(n, n+2)$ -links, respectively. As an n -link $\ell : \bigcup_{i=1}^{\mu} S_i^n \rightarrow S^{n+2}$ (or $\ell : \bigcup_{i=1}^{\mu} S_i^n \rightarrow R^{n+2}$), we refer to the image $(\ell(\bigcup_{i=1}^{\mu} S_i^n) \subset S^{n+2}) = (K_1^n \cup \dots \cup K_{\mu}^n \subset S^{n+2})$ (or $(\ell(\bigcup_{i=1}^{\mu} S_i^n) \subset R^{n+2}) = (K_1^n \cup \dots \cup K_{\mu}^n \subset R^{n+2})$), where $K_i^n = \ell(S_i^n)$ for $i = 1, \dots, \mu$. We also refer to $(L_{\mu}^n \subset S^{n+2})$, $(L_{\mu}^n \subset R^{n+2})$, $K_1^n \cup \dots \cup K_{\mu}^n$ and L_{μ}^n as the n -link $(K_1^n \cup \dots \cup K_{\mu}^n \subset S^{n+2})$ or $(K_1^n \cup \dots \cup K_{\mu}^n \subset R^{n+2})$, unless confusion.

If two n -links $(K_1^n \cup \dots \cup K_{\mu}^n \subset S^{n+2})$ and $(K_1^n \cup \dots \cup K_{\mu}^n \subset R^{n+2})$ are of the same type, we denote $(K_1^n \cup \dots \cup K_{\mu}^n \subset S^{n+2}) \cong (K_1^n \cup \dots \cup K_{\mu}^n \subset R^{n+2})$, and the same notation will be used for n -links in R^{n+2} .

We refer to $[K_1^n \cup \dots \cup K_{\mu}^n \subset S^{n+2}]$ and $[K_1^n \cup \dots \cup K_{\mu}^n \subset R^{n+2}]$, respectively, as the link types of $(K_1^n \cup \dots \cup K_{\mu}^n \subset S^{n+2})$ and $(K_1^n \cup \dots \cup K_{\mu}^n \subset R^{n+2})$.

The notation is justified by the following, which follows from 1.17.

1.20. Proposition. Let $\ell_j : \bigcup_{i=1}^{\mu} S_i^n \rightarrow S^{n+2}$, $j=1,2$, be n -links with μ components. If there exists an orientation preserving homeomorphism $\phi : S^{n+2} \rightarrow S^{n+2}$ such that $\phi \ell_1(\bigcup_{i=1}^{\mu} S_i^n) = \ell_2(\bigcup_{i=1}^{\mu} S_i^n)$ and $\phi|_{\ell_1(\bigcup_{i=1}^{\mu} S_i^n)}$ is orientation preserving. Then, there exists an orientation preserving homeomorphism $\psi : S^{n+2} \rightarrow S^{n+2}$ with $\psi \ell_1 = \ell_2$ (i.e. ℓ_1 and ℓ_2 are equivalent). This remains valid if R^{n+2} is substituted for S^{n+2} .

1.21. Remark. By virtue of the PL normal bundle theory for locally flat PL embedding with codimension 2, (cf. Kato [1], Rourke-Sanderson [1], Wall [2]), and by the Cairns-Hirsch smoothing theory, (cf. Hirsch [1] or Lashof-Rothenberg [1]), '*n-knot*' and '*n-link*' in all the statements in the

note can be replaced by 'smooth n -knot' and 'smooth n -link', unless otherwise stated.

1.22. Remarks. In reading the literature, you should be aware that the definitions of n -links and n -link types are not universally accepted. We refer the reader to, for example, Crowell-Fox [C-F ; Chapter I], Zeeman[4].

(a) *Unoriented n -links and equivalence* : All spaces S^n , S^{n+2} and R^{n+2} do not endowed with any orientation, and so all of which ψ is not required to 'preserve'.

(b) *Isotopy types* : Two n -links $\ell_j : \sum_{i=1}^H S_i^n \rightarrow S^{n+2}$, $j=1,2$, are said to be *isotopic* (or of the same isotopy type) iff ℓ_1 and ℓ_2 are isotopic.

(c) *Ambient isotopy types* : Two n -links $\ell_j : \sum_{i=1}^H S_i^n \rightarrow S^{n+2}$, $j=1,2$, are said to be *ambient isotopic* (or of the same ambient isotopy type) iff ℓ_1 and ℓ_2 are ambient isotopic.

1.23. Definitions. Let X and Y be topological spaces.

(1) Two homeomorphisms (or embeddings) $f, g : X \rightarrow Y$ are *isotopic* iff there exists a homeomorphism (or embedding) $H : X \times I \rightarrow Y \times I$ such that

(i) level preserving (that is, $H(x, t) = (\eta_t(x), t)$, where $\eta_t : X \rightarrow Y$ is a homeomorphism (or embedding)), and

(ii) $\eta_0 = f$ and $\eta_1 = g$.

(2) An *isotopic deformation* of X is a homeomorphism $H : X \times I \rightarrow X \times I$ such that

(i) level preserving (that is, $H(x, t) = (\eta_t(x), t)$, where $\eta_t : X \rightarrow X$ is a homeomorphism), and

(ii) starts with the identity, $\eta_0 = 1_X$.

(3) Two embeddings $f, g : X \rightarrow Y$ are *ambient isotopic* iff there exists an isotopic deformation $H : Y \times I \rightarrow Y \times I$ of Y with $\eta_1 f = g$.

(4) Two subspaces X_1 and X_2 of Y are *ambient isotopic* iff there exists an isotopic deformation $H : Y \times I \rightarrow Y \times I$ of Y with $\eta_1(X_1) = X_2$.

For isotopy types of n -links, we refer the reader to Rolfsen [1; 2].

Isotopy types of n -knots are not interesting, as the following shows.

1.24. Proposition. *Any n -knot is isotopic to an unknotted n -knot.*

Proof. For a vertex v of K^n , let $St(v; K^n)$ and $St(v; S^{n+2})$ be the star neighborhoods of v in K^n and S^{n+2} , respectively. Since K^n is locally flat, the $(n, n+2)$ -disk pair $(St(v; K^n) \subset St(v; S^{n+2}))$ is equivalent to the standard $(n, n+2)$ -disk pair $(D^n \subset D^{n+2})$, (that is, there exists an orientation preserving homeomorphism $\psi : St(v; S^{n+2}) \rightarrow D^{n+2}$ such that $\psi(St(v; K^n)) = D^n$). Let $B^{n+2} = S^{n+2} - \circ St(v; S^{n+2})$; B^{n+2} is an $(n+2)$ -disk by Theorem 1.9. Then, by the Alexander's trick in B^{n+2} , we can conclude the proposition; see Rolfsen [1]. $\square$

1.25. Proposition. *Two n -links $\ell_j : \bigcup_{i=1}^n S_i^n \rightarrow S^{n+2}$, $j=1,2$, are ambient isotopic if and only if they are equivalent.*

Proof. The sufficiency follows from Proposition 1.17; and the necessity follows from the definition of 'ambient isotopic'. $\square$

2. Standard Representatives of 2-Knot Types

We will consider a method of describing 2-knots on the paper. A well-known method of describing a 2-manifold in R^4 is to exhibit its cross

sections by a family of parallel hyperplanes $R_u^3 = R^3 \times \{u\}$ in $R^4 = R^3 \times R^1$. We refer the reader to Fox [3, §6], Gluck [3, §III], Hosokawa [1], Kinoshita [1], etc.. After establishing the method, we will consider standard representatives of 2-knot types. In this section, we discuss exclusively 2-knots and 2-links in the 4-space R^4 , but the reader may replace R^4 with S^4 without difficulty (see Note in 2.2 below).

For the sake of convenience, we first introduce the following :

2.1. *Fusions and Fissions of n -links* : Let $(K_1^n \cup \dots \cup K_\mu^n \subset R^{n+2})$ be an n -link with μ components ($\mu > 1$). Let $B^+ : I \times D^n \rightarrow R^{n+2}$ be an embedding of an $(n+1)$ -disk such that $B^+(I \times D^n) \cap (K_1^n \cup \dots \cup K_\mu^n) = B^+(\partial I \times D^n)$ that is oriented coherently with that of $K_1^n \cup \dots \cup K_\mu^n$, and $B^+(\{0\} \times D^n)$ and $B^+(\{1\} \times D^n)$ are contained in different components of $K_1^n, \dots, K_\mu^n$. Then, $(K_1^n \cup \dots \cup K_\mu^n) + B^+(\partial(I \times D^n)) \equiv \tilde{K}_1^n \cup \dots \cup \tilde{K}_{\mu-1}^n$ is an n -link with $\mu-1$ components, where $+$ means homological addition. We shall say that $(\tilde{K}_1^n \cup \dots \cup \tilde{K}_{\mu-1}^n \subset R^{n+2})$ is obtained by a *fusion* of $(K_1^n \cup \dots \cup K_\mu^n \subset R^{n+2})$ along B^+ .

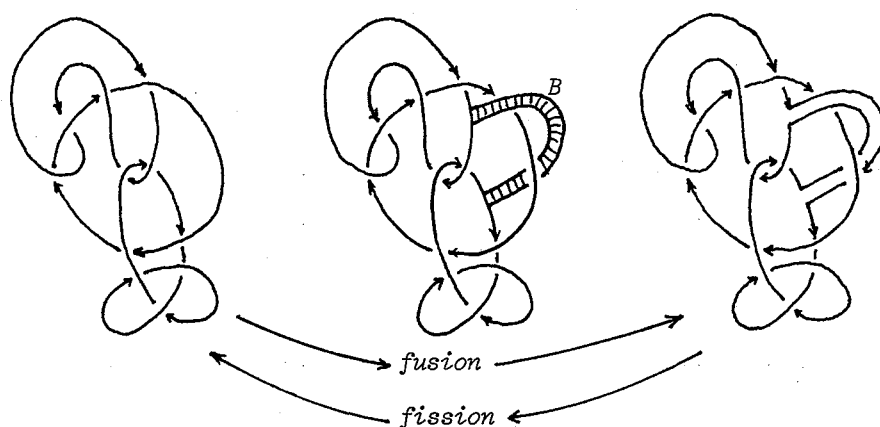


Figure 2.1

Let $B^- : D^n \times I \rightarrow R^{n+2}$ be an embedding of an $(n+1)$ -disk such that $B^-(D^n \times I) \cap (K_1^n \cup \dots \cup K_\mu^n) = B^-(\partial D^n \times I)$ that is oriented coherently with that of $K_1^n \cup \dots \cup K_\mu^n$ and if $n = 1$, $B^-(\partial D^1 \times I)$ is contained in one of $K_1^n, \dots, K_\mu^n$. Then, $(K_1^n \cup \dots \cup K_\mu^n) + B^-(\partial(D^n \times I)) = \tilde{K}_1^n \cup \dots \cup \tilde{K}_{\mu+1}^n$ is an n -link with $\mu + 1$ components. We say that $(\tilde{K}_1^n \cup \dots \cup \tilde{K}_{\mu+1}^n \subset R^{n+2})$ is obtained by a *fission* of $(K_1^n \cup \dots \cup K_\mu^n \subset R^{n+2})$ along B^- .

It will be noticed that if $\tilde{K}_1^n \cup \dots \cup \tilde{K}_{\mu-1}^n$ is obtained by a fusion of $K_1^n \cup \dots \cup K_\mu^n$ along B^+ , then $K_1^n \cup \dots \cup K_\mu^n$ is obtained by a fission of the $\tilde{K}_1^n \cup \dots \cup \tilde{K}_{\mu-1}^n$ along some B^- with $B^-(D^n \times I) = B^+(I \times D^n)$.

In general, let $B_1^+, \dots, B_r^+$ be embeddings of $I \times D^n$ into R^{n+2} such that $B_j^+(I \times D^n) \cap B_k^+(I \times D^n) = \emptyset$ for $j \neq k$, and $B_j^+(I \times D^n) \cap (K_1^n \cup \dots \cup K_\mu^n) = B_j^+(\partial I \times D^n)$ that is oriented coherently with that of $K_1^n \cup \dots \cup K_\mu^n$. Suppose that $(K_1^n \cup \dots \cup K_\mu^n) + (\bigcup_{j=1}^r B_j^+(\partial(I \times D^n)))$ is an n -link with $\mu - r$ components, say $\tilde{K}_1^n \cup \dots \cup \tilde{K}_{\mu-r}^n$. We say that the n -link $\tilde{K}_1^n \cup \dots \cup \tilde{K}_{\mu-r}^n$ is obtained by a *fusion* of $K_1^n \cup \dots \cup K_\mu^n$ along $B_1^+ \cup \dots \cup B_r^+$.

In the same way, we define a *fission* of an n -link $K_1^n \cup \dots \cup K_\mu^n$ along $B_1^- \cup \dots \cup B_r^-$, provided that embeddings B_j^- of $D^n \times I$, $j = 1, \dots, r$, are nicely defined.

Remark : The above definitions are slight different from those of Hosokawa [1; 2]. In fact, the fusion (or fission) illustrated in Figure 2.1 is not a fusion (or fission) in the sense of Hosokawa.

2.2. Notation. By $H_{(a,b)}$, $H[a,b]$, $H[a,b)$ and $H(a,b]$, $-\infty \leq a, b \leq \infty$, we denote subspaces of the 4-space R^4 defined as follows :

$$H_{(a,b)} = \{ (x_1, x_2, x_3, x_4) \in R^4 \mid a < x_4 < b \} ,$$

$$H[a,b] = \{ (x_1, x_2, x_3, x_4) \in R^4 \mid a \leq x_4 \leq b \} ,$$

$$H_{[a,b)} = \{(x_1, x_2, x_3, x_4) \in R^4 \mid a \leq x_4 < b\},$$

$$H_{[a,b]} = \{(x_1, x_2, x_3, x_4) \in R^4 \mid a \leq x_4 \leq b\}.$$

Note : If $R^4 = R^3 \times R^1$ is compactified with a point ∞ to give S^4 , then a hyperplane $R_u^3 = R^3 \times \{u\}$ becomes a 3-sphere, say S_u^3 , and the above subspaces $H_{(a,b)}$, $H_{(a,b]}$, $H_{[a,b)}$ and $H_{[a,b]}$ become unions of such the 3-spheres, say $S_{(a,b)}$, $S_{(a,b]}$, $S_{[a,b)}$ and $S_{[a,b]}$. Using the notation, all statements in this section hold for 2-knots in S^4 .

2.3. Now, we are going to give the method of describing a 2-manifold in R^4 , but for brevity we will only consider (locally flat) 2-knots.

Let $(K^2 \subset R^4)$ be a 2-knot. By well-known general position argument, we may assume that vertices of K^2 are in general position with respect to (x_1, x_2, x_3) and also x_4 ; that is, for any two vertices $v = (x_1, x_2, x_3, x_4)$ and $v' = (x'_1, x'_2, x'_3, x'_4)$ it holds that $(x_1, x_2, x_3) \neq (x'_1, x'_2, x'_3)$ and $x_4 \neq x'_4$.

We assume now that $K^2 \subset H_{(-2,2)}$.

An *ordinary* cross section $K^2 \cap R_{u_0}^3$ is empty or consists of one or more simple oriented loops, that is, a 1-link in the 3-space $R_{u_0}^3$. Since K^2 is compact, the number of *exceptional* cross sections is finite, and for a critical value u_0 , $K^2 \cap R_{u_0}^3$ contains a single point which may be a vertex of K^2 . Let $x_4 = u_0$ be a critical value, and let $v \in K^2 \cap R_{u_0}^3$ be a critical point. Then, a 4-dimensional neighborhood of v is one of the following as illustrated in Figure 2.2; which will be called a *maximum*, a *minimum* and a *saddle-point* (for the x_4 coordinate), respectively.

Especially, a transformation

$$(K^2 \cap R_{u_0+\epsilon}^3 \subset R_{u_0+\epsilon}^3) \implies (K^2 \cap R_{u_0}^3 \subset R_{u_0}^3) \implies (K^2 \cap R_{u_0-\epsilon}^3 \subset R_{u_0-\epsilon}^3)$$

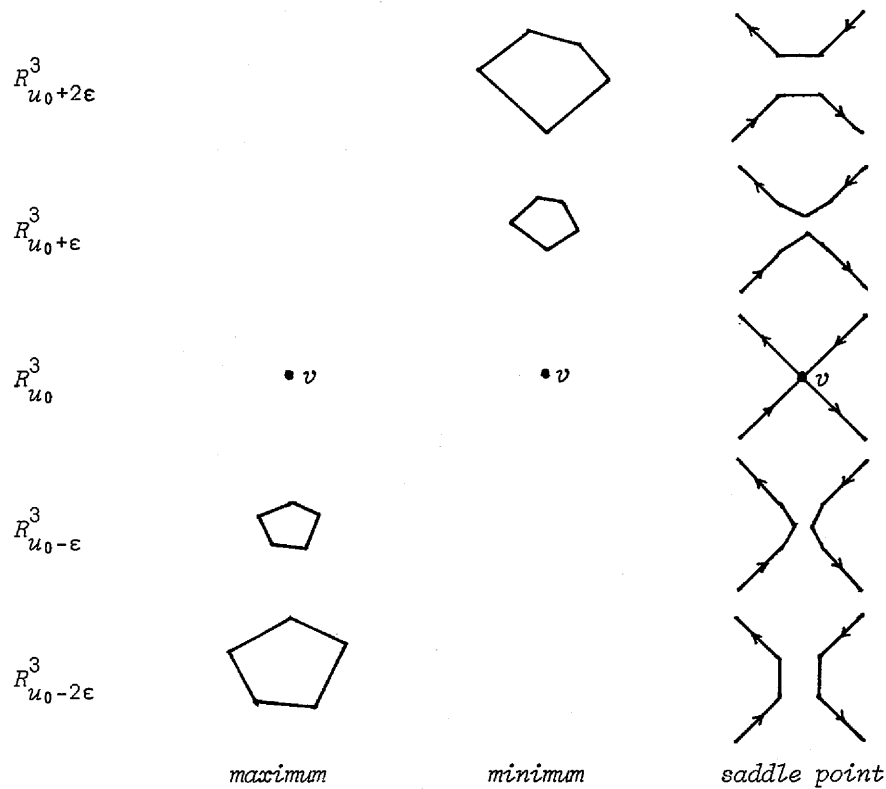


Figure 2.2

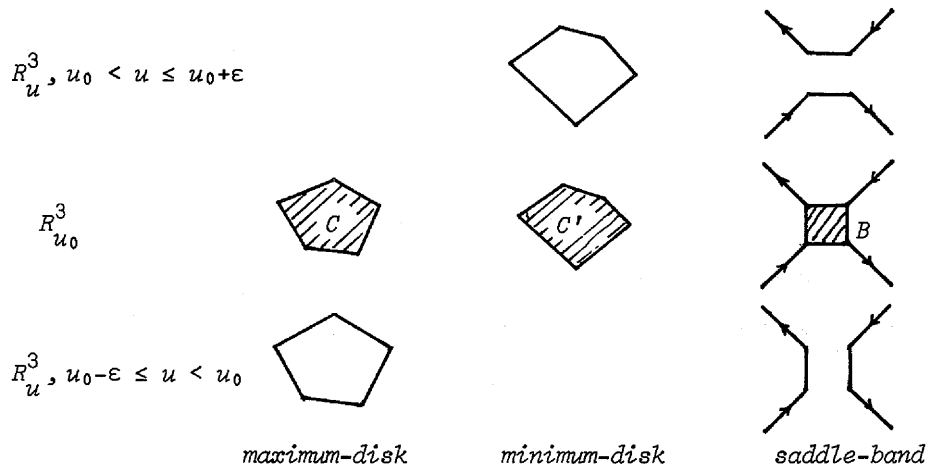


Figure 2.3

will be called an *elliptic transformation* if v is a maximum or a minimum and a *hyperbolic transformation* if v is a saddle-point.

We can make the neighborhood of a critical point v look like Figure 2.3 by an isotopic deformation of R^4 that is the identity outside the neighborhood. (The deformed K^2 is of course no longer in general position with respect to x_u .) Then, we will say the cross section $K^2 \cap R_{u_0}^3$ has a *maximum-disk* C , a *minimum-disk* C' or a *saddle-band* B , respectively, instead of a maximum v , a minimum v or a saddle-point v . If in Figure 2.3(3), $K_i \neq K_j$ in $R_{u_0+\varepsilon}^3$, then the hyperbolic transformation $(K^2 \cap R_{u_0+\varepsilon}^3 \subset R_{u_0+\varepsilon}^3) \implies (K^2 \cap R_{u_0-\varepsilon}^3 \subset R_{u_0-\varepsilon}^3)$ corresponds to a fusion along B , and if $K_i = K_j$ it corresponds to a fission along B . (Here, we do not distinguish embeddings $B^+ : I \times D^1 \rightarrow R_{u_0}^3$ and $B^- : D^1 \times I \rightarrow R_{u_0}^3$ from the images $B^+(I \times D^1) = B$ and $B^-(D^1 \times I) = B$.)

We will prove the following :

2.4. Theorem. For any 2-knot $(K^2 \subset R^4)$, there exists a 2-knot $(\tilde{K}^2 \subset R^4)$ satisfying the following :

- (0) $(\tilde{K}^2 \subset R^4) \cong (K^2 \subset R^4)$,
- (1) all maximum-disks of $(\tilde{K}^2 \subset R^4)$ are in the hyperplane R_3^3 ,
- (2) all minimum-disks of $(\tilde{K}^2 \subset R^4)$ are in the hyperplane R_{-3}^3 ,
- (3) all saddle-bands of $(\tilde{K}^2 \subset R^4)$ are in the hyperplanes $R_1^3 \cup R_{-1}^3$,
- (4) $(\tilde{K}^2 \cap R_0^3 \subset R_0^3)$ is a 1-knot (i.e. $\tilde{K}^2 \cap R_0^3$ is connected) that is oriented coherently with that of the 2-disk $\tilde{K}^2 \cap R_+^4$.

We will call such a representative $(\tilde{K}^2 \subset R^4)$ a *standard representative* of the 2-knot type $[K^2 \subset R^4]$; it will be noticed that standard representatives of a 2-knot type are not uniquely determined. It should be noted

that the 1-knot $(\tilde{K}^2 \cap R_0^3 \subset R_0^3)$ is obtained by a fusion of the trivial 1-link $(\tilde{K}^2 \cap R_2^3 \subset R_2^3)$ along the saddle-bands in R_1^3 , and that the trivial 1-link $(\tilde{K}^2 \cap R_{-2}^3 \subset R_{-2}^3)$ is obtained by a fission of the 1-knot $(\tilde{K}^2 \cap R_0^3 \subset R_0^3)$ along the saddle-bands in R_{-1}^3 .

Proof of Theorem 2.4. We may assume that $K^2 \subset H_{(-2,2)}$. For each maximum $v = (v_1, v_2, v_3, v_4)$ of K^2 , we take a line which is parallel to the x_4 -axis and passes through v , and let ℓ_v denote the part with $v_4 \leq x_4 \leq 3$. Since $\dim(K^2) = 2$ and $\dim(\ell_v) = 1$, we may assume that $K^2 \cap \ell_v = v$. By our assumption, for any two maxima v and v' of K^2 , $\ell_v \cap \ell_{v'} = \emptyset$. By an isotopic deformation of R^4 that is the identity outside of a neighborhood of ℓ_v , we pull up v as far as R_3^3 , and we also deform its to a maximum-disk. By the quite similar way, we pull down minima of K^2 as far as R_{-3}^3 , and we deform them to minimum-disks. We have a new 2-sphere K' in R^4 with $(K' \subset R^4) \cong (K^2 \subset R^4)$, so that all maximum-disks are in R_3^3 , and all minimum-disks are in R_{-3}^3 , and all saddle-bands are in open slice $H_{(-2,2)}$. Let $u_1, u_2, \dots, u_r$ be critical values of K' with $-2 < u_1 < u_2 < \dots < u_r < 2$, and let $B_1, B_2, \dots, B_r$ be the corresponding saddle-bands.

To complete the proof of 2.4, it suffices to show that :

2.5. Lemma. We can make even the levels of any two saddle-bands B_i and B_{i+1} , and also we can change the levels of B_i and B_{i+1} , by an isotopic deformation of R^4 .

Proof. In the hyperplane $R_{u_{i+1}}$, there is a 1-link together with the saddle-band B_{i+1} . With decreasing u , $u_{i+1} \geq u \geq u_i$, we do not perform a fusion or a fission along B_{i+1} , and we take B_{i+1} along with $K' \cap R_u^3$.

In $R_{u_i}^3$ we have two bands B_i and B_{i+1} . If $\partial B_i \cap \partial B_{i+1} \neq \emptyset$, that is, if B_i attaches to a subarc of ∂B_{i+1} , we slide B_i along the side of B_{i+1} so that $\partial B_i \cap \partial B_{i+1} = \emptyset$, see Figure 2.4.

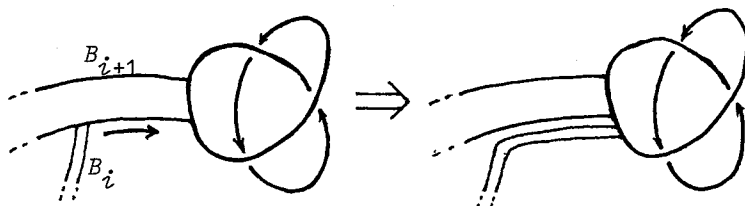


Figure 2.4

If $B_i \cap^\circ B_{i+1} \neq \emptyset$, that is, if B_i passes through B_{i+1} , we deform B_i in $R_{u_i}^3 - (K' \cap R_{u_i}^3 - B_i)$ so that $B_i \cap^\circ B_{i+1} = \emptyset$, see Figure 2.5.

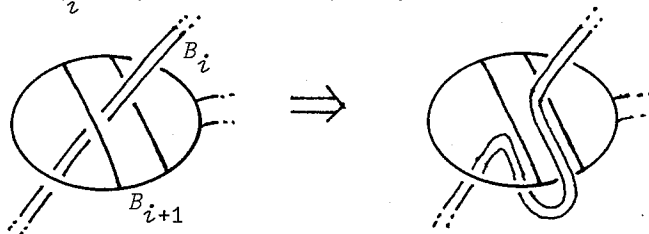


Figure 2.5

It will be noticed that these deformations of B_i can be realized by an isotopic deformation of R^4 . Now, the lemma is obvious, and completing the proof. $\square$

2.6. Theorem. For any 2-link $(K_1^2 \cup \dots \cup K_\mu^2 \subset R^4)$ with μ components, there exists a 2-link $(\tilde{K}_1^2 \cup \dots \cup \tilde{K}_\mu^2 \subset R^4)$ satisfying the following :

- (0) $(\tilde{K}_1^2 \cup \dots \cup \tilde{K}_\mu^2 \subset R^4) \cong (K_1^2 \cup \dots \cup K_\mu^2 \subset R^4)$,
- (1) each component $\tilde{K}_i^2$ satisfies the conditions (1), (2), (3) and
- (4) in Theorem 2.4, $1 \leq i \leq \mu$.

We call such a representative $(\tilde{K}_1^2 \cup \dots \cup \tilde{K}_\mu^2 \subset R^4)$ a standard representative of $[K_1^2 \cup \dots \cup K_\mu^2 \subset R^4]$.

2.7. Exercise. Prove that : for any locally flat embedding $k : M^2 \rightarrow R^4$ of an orientable, closed and connected 2-manifold M^2 of genus g into R^4 , there exists a locally flat embedding $h : M^2 \rightarrow R^4$ satisfying the following :

- (0) h and k are ambient isotopic,
- (1) all maximum-disks of $h(M^2)$ are in the hyperplane R_3^3 ,
- (2) all minimum-disks of $h(M^2)$ are in the hyperplane R_{-3}^3 ,
- (3) all saddle-bands of $h(M^2)$ are in hyperplanes $R_2^3 \cup R_1^3 \cup R_{-1}^3 \cup R_{-2}^3$,
- (4) both $h(M^2) \cap R_{3/2}^3$ and $h(M^2) \cap R_{-3/2}^3$ are connected,
- (5) $h(M^2) \cap R_0^3$ is a 1-link with $g + 1$ components,
- (6) both $h(M^2) \cap R_+^4$ and $h(M^2) \cap R_-^4$ are connected 2-manifolds of genus 0 with $g + 1$ boundary-components $h(M^2) \cap R_0^3$.

2.8. Example. We describe a standard representative of the 2-knot type of the spun 2-knot $(k^2(S^2) \subset R^4) = (K^2 \subset R^4)$ of the trefoil $(k^1(S^1) \subset R^3)$ given in Example 1.13.

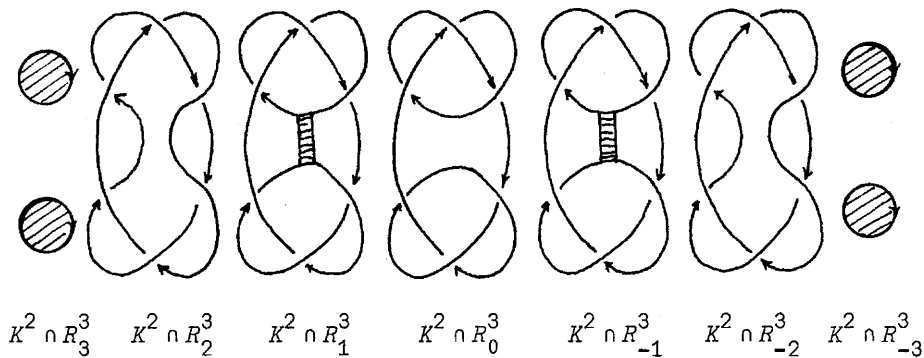


Figure 2.6

2.9. Definition. R.H.Fox [4, Problem 25] defined a *ribbon 1-knot*, and some characterizations are given ; refer to Fox [6], Hosokawa-Yanagawa [1],

Terasaka [1], etc.. That is, a 1-link (or 1-knot) $(K_1^1 \cup \dots \cup K_\mu^1 \subset R^3)$ is called a *ribbon* 1-link (or 1-knot) iff there is an integer $\lambda \geq \mu$ so that $K_1^1 \cup \dots \cup K_\mu^1$ is obtained by a fusion of a trivial 1-link $O_1^1 \cup \dots \cup O_\lambda^1$ with λ components.

Then, the equatorial cross section $(\tilde{K}^2 \cap R_0^3 \subset R_0^3)$ in Theorem 2.4(4) is a ribbon 1-knot, and the $((\tilde{K}_1^2 \cup \dots \cup \tilde{K}_\mu^2) \cap R_0^3 \subset R_0^3)$ in Theorem 2.6 is a ribbon 1-link with μ components.

Fox also defined in [4] a *slice* 1-link (or 1-knot), a *slice* 1-link in the strong sense and a *slice* 1-link in the weak sense. The cross section $h(M^2) \cap R_0^3$ in Exercise 2.7(5) is a slice 1-link in the weak sense.

2.10. Problem. (Fox [4, Problem 39]). (a) Which slice 1-knots are cross sections of trivial 2-knots? (b) Which slice 1-links with μ components are cross sections of 2-links $(K_1^2 \cup \dots \cup K_\lambda^2 \subset R^4)$, $\lambda \leq \mu$, such that K_j^2 is a trivial 2-knot (for $j=1, \dots, \lambda$)? (c) Which slice 1-links in the strong sense (with μ components) are cross sections of trivial 2-links with μ components?

We have now the following problem corresponding to 2.10(a) and (c) :

2.11. Problem. (a) Which ribbon 1-knots are equatorial cross sections $K^2 \cap R_0^3$ of standard representatives $(K^2 \subset R^4)$ of the trivial 2-knot type? (c) Which ribbon 1-links with μ components are equatorial cross sections $(K_1^2 \cup \dots \cup K_\mu^2) \cap R_0^3$ of standard representatives $(K_1^2 \cup \dots \cup K_\mu^2 \subset R^4)$ of the trivial 2-link type with μ components?

Some 1-knots are known which satisfy the condition of 2.10(a) and also 2.11(a), see Terasaka-Hosokawa [1], Hosokawa [1], Summers [7] and Fox[3].

Hosokawa [1] has given a sufficient geometric condition for 2.11(a), which will be discussed in Section 10; and Sumners [7] has given necessary algebraic conditions for 2.10(a), which will be noted in Section 11.

2.12. Definition. A 2-link $(K_1^2 \cup \dots \cup K_\mu^2 \subset R^4)$ with μ components is called a *ribbon* 2-link iff there exists an integer $\lambda \geq \mu$ such that $K_1^2 \cup \dots \cup K_\mu^2$ is obtained by a fusion of a trivial 2-link $O_1^2 \cup \dots \cup O_\mu^2$ with λ components along appropriate embeddings $B_1^+, \dots, B_{\lambda-\mu}^+ : I \times D^2 \rightarrow R^4$. When $\mu = 1$, we will call a *ribbon* 2-knot. A 2-link type will be called a *ribbon* type iff it has a ribbon 2-link as its representative.

A standard representative $(K_1^2 \cup \dots \cup K_\mu^2 \subset R^4)$ will be called *simple symmetric* iff $(K_1^2 \cup \dots \cup K_\mu^2) \cap R_+^4$ and $(K_1^2 \cup \dots \cup K_\mu^2) \cap R_-^4$ are symmetric with respect to the hyperplane $R^3 = R_0^3$.

2.13. Theorem. (Yanagawa [2]) A 2-link $(K_1^2 \cup \dots \cup K_\mu^2 \subset R^4)$ is a ribbon 2-link if and only if $[K_1^2 \cup \dots \cup K_\mu^2 \subset R^4]$ has a simple symmetric representative $(\tilde{K}_1^2 \cup \dots \cup \tilde{K}_\mu^2 \subset R^4)$.

Sketch Proof. Suppose that $K_1^2 \cup \dots \cup K_\mu^2$ is obtained by a fusion of a trivial 2-link $O_1^2 \cup \dots \cup O_\lambda^2$ along embeddings $B_1^+, \dots, B_{\lambda-\mu}^+ : I \times D^2 \rightarrow R^4$. We may assume that $(O_1^2 \cup \dots \cup O_\lambda^2) \cap R_+^4$ and $(O_1^2 \cup \dots \cup O_\lambda^2) \cap R_-^4$ are symmetric with respect to R_0^3 . Since $I \times D^2$ is a 3-disk, there exists an isotopic deformation $\{\eta_t\}$ of R^4 with $\eta_t(O_1^2 \cup \dots \cup O_\mu^2) = O_1^2 \cup \dots \cup O_\mu^2$ for each $0 \leq t \leq 1$, so that $\eta_1 B^+(I \times D^2) \cap R_+^4$ and $\eta_1 B^+(I \times D^2) \cap R_-^4$ are symmetric with respect to R_0^3 , $i=1, \dots, \lambda-\mu$. Conversely, let $B_1, \dots, B_r$ and $B'_1, \dots, B'_r$ be saddle-bands of $\tilde{K}_1^2 \cup \dots \cup \tilde{K}_\mu^2$ in R_1^3 and R_{-1}^3 , respectively. Since there are no critical points in $H_{(-1,1)}$, there exists

a homeomorphism $\phi : H_{(-1,1)} \rightarrow R_0^3 \times (-1,1)$ such that

$$\phi((\tilde{K}_1^2 \cup \dots \cup \tilde{K}_\mu^2) \cap H_{(-1,1)}) = ((\tilde{K}_1^2 \cup \dots \cup \tilde{K}_\mu^2) \cap R_0^3) \times (-1,1).$$

Now we can deform $B_1 \cup \dots \cup B_r$ by an isotopic deformation $\{\eta_t\}$ of R^4 , so that $\eta_1(B_1 \cup \dots \cup B_r) = B'_1 \cup \dots \cup B'_r$ and the trace $\eta_t(B_1 \cup \dots \cup B_r)$, $t \in I$, forms r mutually disjoint 3-disks. We obtain embeddings $B_1^-, \dots, B_r^- : D^2 \times I \rightarrow R^4$ so that a fission of $\tilde{K}_1^2 \cup \dots \cup \tilde{K}_\mu^2$ along $B_1^- \cup \dots \cup B_r^-$ is a trivial 2-link $\tilde{O}_1^2 \cup \dots \cup \tilde{O}_{r+1}^2$, where $r+1$ is the number of the maximum-disks of $\tilde{K}_1^2 \cup \dots \cup \tilde{K}_\mu^2$. $\square$

2.14. Remark. Let $(K^1 \subset R_0^3)$ be a ribbon 1-knot obtained by a fusion of a trivial 1-link $(O_1^1 \cup \dots \cup O_\mu^1 \subset R^3)$ along embeddings $B_1^+, \dots, B_r^+ : I \times D^1 \rightarrow R_0^3$, $r = \mu - 1$. As in the sketch proof of 2.13, we have a simple symmetric representative $(K^2(K^1) \subset R^4)$ of a ribbon 2-knot as follows :

Let $(O_1^2 \cup \dots \cup O_\mu^2 \subset R^4)$ be a simple symmetric representative of the trivial 2-link type with μ components such that $(O_1^2 \cup \dots \cup O_\mu^2) \cap R_0^3 = O_1^1 \cup \dots \cup O_\mu^1$ and $(O_1^2 \cup \dots \cup O_\mu^2) \cap H_{(-3,3)} = (O_1^1 \cup \dots \cup O_\mu^1) \times (-3,3)$. Then, we have embeddings $\tilde{B}_1^+, \dots, \tilde{B}_r^+ : I \times D^2 \rightarrow R^4$ such that $\tilde{B}_i^+(I \times D^2) \cap R_0^3 = B_i^+(I \times D^1)$ and $\tilde{B}_i^+(I \times D^2) = B_i^+(I \times D^1) \times [-1,1]$ in $H_{[-1,1]} = R_0^3 \times [-1,1]$, for $i=1, \dots, r$. Now, $(K^2(K^1) \subset R^4)$ is obtained by a fusion of $O_1^2 \cup \dots \cup O_\mu^2$ along $\tilde{B}_1^+ \cup \dots \cup \tilde{B}_r^+$, which will be called an *associated ribbon 2-knot*. To give a standard (simple symmetric) representative of a ribbon 2-knot, it suffices to show a ribbon 1-knot as its equatorial cross section.

2.15. Proposition. Let $(K_1^1 \subset R_0^3)$ and $(K_2^1 \subset R_0^3)$ be ribbon 1-knots obtained by fusions of a trivial 1-link $O_1^1 \cup \dots \cup O_\mu^1$ along $B_{11}^+ \cup \dots \cup B_{1r}^+$ and $B_{21}^+ \cup \dots \cup B_{2r}^+$, respectively, $r = \mu - 1$. Let $q : R_0^3 \rightarrow R_0^2$ be the regular projection defined by $q(x_1, x_2, x_3) = (x_1, x_2, 0)$. If $B_{21}^+(I \times D^1) \cup$

$\dots \cup B_{2r}^+(I \times D^1)$ is obtained from $B_{11}^+(I \times D^1) \cup \dots \cup B_{1r}^+(I \times D^1)$ by changing some of the crossing of $q(B_{11}^+(I \times D^1) \cup \dots \cup B_{1r}^+(I \times D^1))$, then the associated ribbon 2-knots $(K^2(K_1^1) \subset R^4)$ and $(K^2(K_2^1) \subset R^4)$ are equivalent.

Proof. We choose a small unknotted simple loop $c_0 \subset R_0^3$ as shown in Figure 2.7(a). We pull up c_0 as far as $R_{3/2}^3$. Since $K^2(K^1) \cap R_{3/2}^3$ consists of a trivial 1-link $(O_1^1 \cup \dots \cup O_\mu^1) \times (3/2)$ alone, and $c_0 \times (3/2)$ does not link with this 1-link in $R_{3/2}^3$, c_0 bounds a locally flat 2-disk C^2 in $H_{[0, 3/2]}$. Let $\tilde{B}_{1i}^+ : I \times D^2 \rightarrow R^4$ be the embedding corresponding with B_{1i}^+ as in Remark 2.14. Along the 2-disk C^2 , we deform $\tilde{B}_{1i}^+(I \times D^2)$ by an isotopic deformation of R^4 so that $(\eta_1 \tilde{B}_{1i}^+(I \times D^2) \cap R_0^3) \cup B_{1j}^+(I \times D^1)$ is as illustrated in Figure 2.7(b) and $\eta_1 \tilde{B}_{1i}^+(I \times D^2) = \eta_1 B_{1i}^+(I \times D^1) \times [-1, 1]$ in $H_{[-1, 1]} = R_0^3 \times [-1, 1]$. $\square$

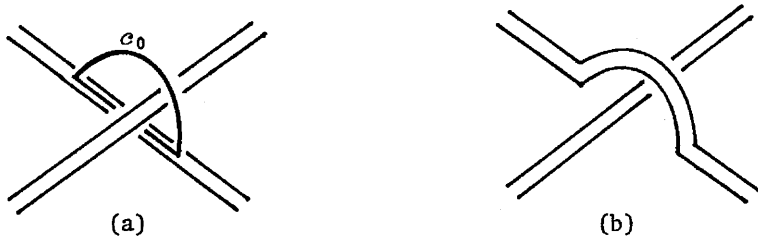


Figure 2.7

- 2.16. Theorem. (1) Every spun 2-knot is a ribbon 2-knot.
 (2) There exists a 2-knot type that is not obtained by spinning.
 (3) There exists a 2-knot type that is not a ribbon type.

We can check (1) easily. The proofs of (2) and (3) will be given in the next Section. On ribbon 2-knots, many properties are known by Yajima [2; 3], Kinoshita [1], Yanagawa [2; 3; 4], etc., and some of them will appear in the forthcoming Sections.

2.17. Definition. Let $k : S^n \rightarrow R^{n+2}$ be a locally flat embedding of the oriented n -sphere S^n into the R^{n+2} , that is, an n -knot. Let

$$\sigma : S^n \rightarrow S^n \quad \text{and} \quad \rho : R^{n+2} \rightarrow R^{n+2}$$

be orientation reversing homeomorphisms. Then, we have three new locally flat embeddings

$$k\sigma : S^n \rightarrow R^{n+2}, \quad \rho k : S^n \rightarrow R^{n+2}, \quad \rho k\sigma : S^n \rightarrow R^{n+2}.$$

When we represent $(k(S^n) \subset R^{n+2})$ by $(K^n \subset R^{n+2})$, then we denote $(k\sigma(S^n) \subset R^{n+2})$, $(\rho k(S^n) \subset R^{n+2})$ and $(\rho k\sigma(S^n) \subset R^{n+2})$ by $(-K^n \subset R^{n+2})$, $-(-K^n \subset R^{n+2})$ and $-(K^n \subset R^{n+2})$, respectively.

An n -knot $(K^n \subset R^{n+2})$ is called *invertible* iff $(K^n \subset R^{n+2})$ is equivalent to $(-K^n \subset R^{n+2})$, and *+amphicheiral* iff $(K^n \subset R^{n+2})$ is equivalent to $-(-K^n \subset R^{n+2})$, and *-amphicheiral* iff $(K^n \subset R^{n+2})$ is equivalent to $-(K^n \subset R^{n+2})$.

In the case $n = 1$, given a regular diagram of a 1-knot $(K^1 \subset R^3)$ we have regular diagrams of $(-K^1 \subset R^3)$, $-(-K^1 \subset R^3)$ and $-(K^1 \subset R^3)$ formally as the following (cf. Fox [3, §7]) :

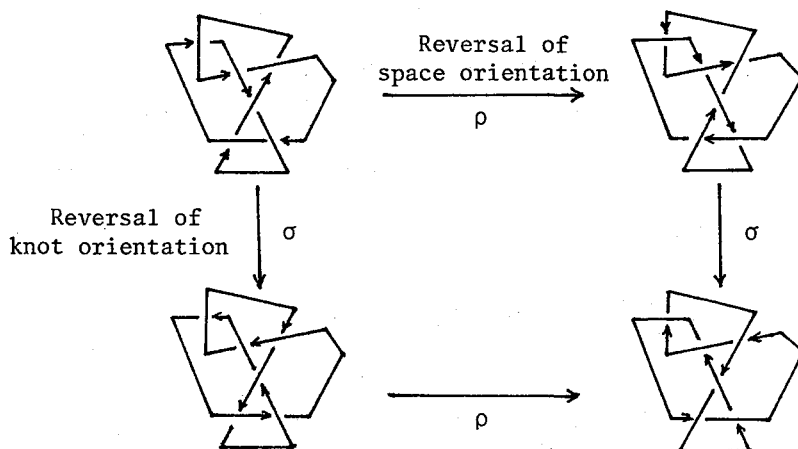


Figure 2.8

Here, $(-K^1 \subset R^3)$ is obtained by reversing the direction of the arrow on K^1 , and $-(-K^1 \subset R^3)$ is obtained by changing the crossing at each double point of K^1 ; where $\rho : R^3 \rightarrow R^3$ is defined by $\rho(x_1, x_2, x_3) = (x_1, x_2, -x_3)$.

Given a standard representative $(K^2 \subset R^4)$ of a 2-knot type $[K^2 \subset R^4]$, we record a formal method to have standard representatives of $[-K^2 \subset R^4]$, $-[-K^2 \subset R^4]$ and $-[K^2 \subset R^4]$ in the similar way as that of 1-knots.

From our promise of the orientation on K^2 (recall Theorem 2.4(4)), a standard representative of $[-K^2 \subset R^4]$ is obtained from $(K^2 \subset R^4)$ by reversing the direction of the arrow on $K^2 \cap R_u^3$ for $-3 \leq u \leq 3$. As an orientation reversing homeomorphism of R^4 onto itself, we take $\rho : R^4 \rightarrow R^4$ defined by $\rho(x_1, x_2, x_3, x_4) = (x_1, x_2, -x_3, x_4)$. We have a standard representative of $-[-K^2 \subset R^4]$ by changing the crossing at each double point of $K^2 \cap R_u^3$ (with respect to a regular projection $q_u : R_u^3 \rightarrow R_u^3$ defined by $q_u(x_1, x_2, x_3, u) = (x_1, x_2, 0, u)$) for $-3 \leq u \leq 3$.

2.18. Theorem. (Y.Marumoto) *Every ribbon 2-knot is -amphicheiral.*

Proof. Let $(K^1 \subset R_0^3)$ be a ribbon 1-knot obtained by a fusion of a trivial 1-link $O_1^1 \cup \dots \cup O_\mu^1$ along embeddings $B_1^+, \dots, B_r^+ : I \times D^1 \rightarrow R_0^3$, $r = \mu - 1$, and let $(K^2(K^1) \subset R^4)$ be an associated ribbon 2-knot. In the same way as that of the proof of Lemma 2.5 (see Figure 2.4), we slide the bands $B_1^+, \dots, B_r^+$, if necessary, so that each B_i spans O_i^1 and O_{i+1}^1 .

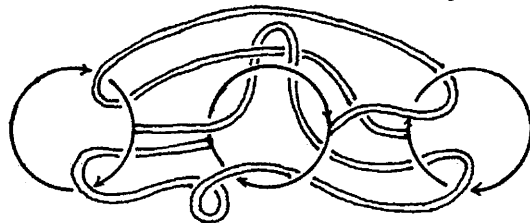


Figure 2.9 : $(K^1 \subset R_0^3) = ((K^2(K^1) \cap R_0^3) \subset R_0^3)$

By the observation in 2.17, we have the following ribbon 1-knot $(K_1^1 \subset R_0^3)$ as shown in Figure 2.10, such that the associated ribbon 2-knot $(K^2(K_1^1) \subset R^4)$ is a standard representative of $-[K^2(K^1) \subset R^4]$.

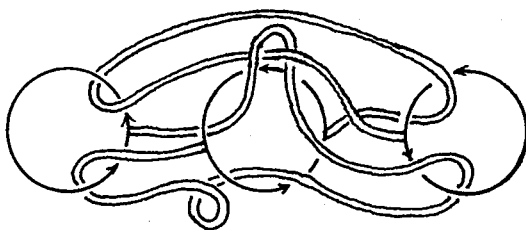


Figure 2.10 : $(K_1^1 \subset R_0^3) = ((K^2(K_1^1) \cap R_0^3 \subset R_0^3)$

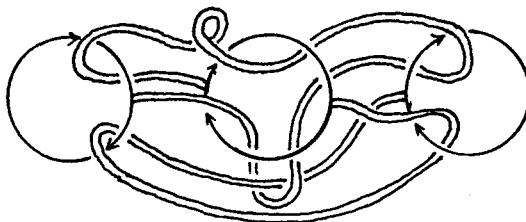


Figure 2.11 : $(K_2^1 \subset R_0^3) = ((K^2(K_2^1) \cap R_0^3 \subset R_0^3)$

In each level R_u^3 , $-3 \leq u \leq 3$, we rotate $K^2(K_1^1) \cap R_u^3$, and have another representative $(K_2^1 \subset R_0^3)$ of $[K_1^1 \subset R_0^3]$ and another standard representative $(K^2(K_2^1) \subset R^4)$ of $-[K^2(K^1) \subset R^4]$, as shown in Figure 2.11. By virtue of Proposition 2.15, we can conclude that $[K^2(K^1) \subset R^4] = [K^2(K_2^1) \subset R^4] = -[K^2(K_1^1) \subset R^4]$. $\square$

We refer the reader to 3.20 and 3.21 in the next Section.

3. Presentations of 2-Knot Groups and Elementary Ideals

The most important topological invariant of an n -link $(L_\mu^n \subset S^{n+2}) = (K_1^n \cup \dots \cup K_\mu^n \subset S^{n+2})$ resp. $(L_\mu^n \subset R^{n+2}) = (K_1^n \cup \dots \cup K_\mu^n \subset R^{n+2})$ is the comple-

mentary domain $S^{n+2} - L_\mu^n$ resp. $R^{n+2} - L_\mu^n$, say simply the *complement* of $(L_\mu^n \subset S^{n+2})$ resp. $(L_\mu^n \subset R^{n+2})$.

The fundamental group $\pi_1(S^{n+2} - L_\mu^n)$ resp. $\pi_1(R^{n+2} - L_\mu^n)$ is called the *group* of $(L_\mu^n \subset S^{n+2})$ resp. $(L_\mu^n \subset R^{n+2})$, and any such group is called an *n-link group with μ components*. In particular, an *n-link group* with 1 component will be called an *n-knot group*. For an *n-link* $(L_\mu^n \subset R^{n+2})$, let $(L_\mu^n \subset S^{n+2})$ be the *n-link* obtained by one-point compactification of R^{n+2} . Then, $\pi_1(R^{n+2} - L_\mu^n) \cong \pi_1(S^{n+2} - L_\mu^n)$, provided that $n \geq 1$.

In this section, we will give a method for presenting the group of a 2-knot $(K^2 \subset R^4)$ (and so $(K^2 \subset S^4)$), using the family of cross sections $K^2 \cap R_\mu^3$, and introduce the so-called elementary ideals.

It is known that :

3.1. Proposition. (Noguchi [1], Zeeman [6], etc.) Let $(K^n \subset S^{n+2})$ and $(K^n \subset R^{n+2})$ be *n-knots*, and let $N(K^n; S^{n+2})$ and $N(K^n; R^{n+2})$ be regular neighborhoods of K^n in S^{n+2} and R^{n+2} , respectively. Then,

$$N(K^n; S^{n+2}) \cong S^n \times D^2 \cong N(K^n; R^{n+2}).$$

We will call $S^{n+2} - {}^oN(L_\mu^n; S^{n+2})$ resp. $R^{n+2} - {}^oN(L_\mu^n; R^{n+2})$ the *exterior* of $(L_\mu^n \subset S^{n+2})$ resp. $(L_\mu^n \subset R^{n+2})$. It is easy to check that the complement of an *n-link* deformation retracts to its exterior.

By Alexander duality and Poincaré duality, we have :

3.2. Proposition. For *n-links* $(L_\mu^n \subset S^{n+2})$ and $(L_\mu^n \subset R^{n+2})$ with μ components

$$H_q(S^{n+2} - L_\mu^n; \mathbb{Z}) \cong H_q(R^{n+2} - L_\mu^n; \mathbb{Z}) = \begin{cases} \mathbb{J} & \text{if } q = 0, \\ \mathbb{J} \times \cdots \times \mathbb{J} \equiv \mathbb{J}^\mu & \text{if } q = 1, \\ 0 & \text{if } q \neq 0, 1, n+1. \end{cases}$$

$$\begin{aligned} H_{n+1}(S^{n+2} - L_\mu^n; \mathbb{Z}) &\cong \mathbb{J} \times \cdots \times \mathbb{J} \cong \mathbb{J}^{\mu-1}, \\ H_{n+1}(R^{n+2} - L_\mu^n; \mathbb{Z}) &\cong \mathbb{J} \times \cdots \times \mathbb{J} \times \mathbb{J} \cong \mathbb{J}^\mu. \end{aligned}$$

3.3. We will discuss a presentation of a 2-knot group $\pi_1(R^4 - K^2)$. The reference for the following is Fox [3, §6] and Kinoshita [1, §4].

Suppose that $x_4 = 0$ is a critical value and a single critical point $v = (0, 0, 0, 0)$ is lying in $K^2 \cap R_0^3$. We also suppose that v is a minimum or saddle-point and that in $H_{(-1,1)}$ there are no critical points. We take the following two open sets :

$$\begin{aligned} U_1 &= \{(x_1, x_2, x_3, x_4) \in H_{(-1,1)} \mid x_4 < \min\{1, (x_1^2 + x_2^2 + x_3^2)^{1/2}\}\}, \\ U_2 &= \{(x_1, x_2, x_3, x_4) \in H_{(-1,1)} \mid (x_1^2 + x_2^2 + x_3^2)^{1/2} < 1\}. \end{aligned}$$

Since U_2 is a neighborhood of v in R^4 , $\pi_1(U_2 - K^2) \cong \mathbb{J}$ because K^2 is locally flat and the 2-disk $K^2 \cap \text{Cl}(U_2)$ is considered to be the join $v * \partial(K^2 \cap \text{Cl}(U_2))$. We apply the well-known van Kampen's theorem to the open sets $U_1 - K^2$ and $U_2 - K^2$.

If v is a minimum, we get no new relation. That is, $\pi_1(H_{(-1,1)} - K^2) \cong \pi_1(R_1^3 - K^2)$.

If v is a saddle-point, it is easy to see that $H_{(-1,1)} - (K^2 \cup U_2)$ deformation retracts to $R_{1/2}^3 - (K^2 \cup U_2)$. Using the Wirtinger presentation in the classical knot theory (cf. Crowell-Fox [C-F]), we have a presentation of the group $G(1/2) \cong \pi_1(R_{1/2}^3 - K^2)$ of the 1-link $(K^2 \cap R_{1/2}^3 \subset R_{1/2}^3)$. Suppose that

$$\begin{aligned} G(1/2) &= \langle x_a, x_b, x_c, x_d, y_1, \dots, y_\alpha \mid x_a = x_b, x_c = x_d, r_1, \dots, r_\beta \rangle \\ &= \langle x_a, x_c, y_1, \dots, y_\alpha \mid r'_1, \dots, r'_\beta \rangle \end{aligned}$$

where x_a, x_b, x_c and x_d are shown in Figure 3.1, and r'_i is obtained from r_i by replacing x_b and x_d with x_a and x_c , respectively.

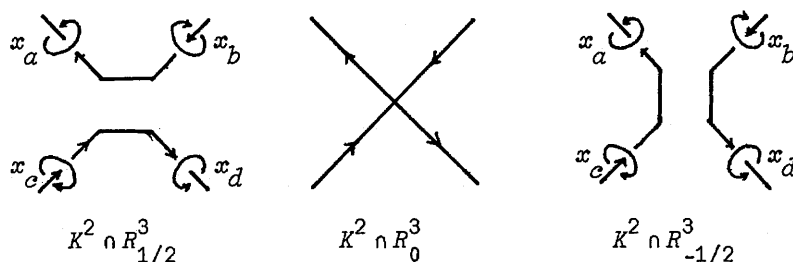


Figure 3.1

Since $\pi_1(U_2 - K^2) \cong J$ has a presentation $\langle x_a, x_b, x_c, x_d \mid x_a = x_b = x_c = x_d \rangle$, we have a presentation of $\pi_1(H_{(-1,1)} - K^2)$ as follows :

$$\begin{aligned} & \langle x_a, x_b, x_c, x_d, y_1, \dots, y_\alpha \mid x_a = x_b = x_c = x_d, r_1, \dots, r_\beta \rangle \\ &= \langle x_a, x_c, y_1, \dots, y_\alpha \mid x_a = x_c, r'_1, \dots, r'_\beta \rangle. \end{aligned}$$

By successive applications of the above, we have :

3.4. Theorem. Let $(K^2 \subset R^4)$ be a standard representative, and let $(K^1 \subset R^3)$ be the equatorial cross section $(K^2 \cap R_0^3 \subset R_0^3)$. Then, from a presentation of $\pi_1(R^3 - K^1)$, we have a presentation of $\pi_1(R^4 - K^2)$ by adding a finite number of relations corresponding to saddle-bands of K^2 .

In particular, there exists a presentation

$$\langle x_1, x_2, \dots, x_\alpha \mid r_1, r_2, \dots, r_{\alpha-1} \rangle$$

of $\pi_1(R^3 - K^1)$ such that the presentation of $\pi_1(R^4 - K^2)$ forms

$$\langle x_1, x_2, \dots, x_\alpha \mid r_1, r_2, \dots, r_{\alpha-1}, s_1^+, \dots, s_{\nu-1}^+, s_1^-, \dots, s_{\lambda-1}^- \rangle,$$

where ν and λ are the numbers of maximum-disks and minimum-disks of K^2 , respectively, and

$$s_i^+ = x_{i_1} x_{i_2}^{-1} \quad \text{for } i = 1, \dots, \nu-1,$$

$$s_j^- = x_{j_1} x_{j_2}^{-1} \quad \text{for } j = 1, \dots, \lambda-1.$$

This remains valid if a standard representative $(K_1^2 \cup \dots \cup K_\mu^2 \subset R^4)$ of a 2-link type and its equatorial cross section $((K_1^2 \cup \dots \cup K_\mu^2) \cap R_0^3 \subset R_0^3)$ are substituted for $(K^2 \subset R^4)$ and $(K^2 \cap R_0^3 \subset R_0^3)$.

3.5. Corollary. Suppose that in Theorem 3.4, $(K^2 \subset R^4)$ is simple symmetric. Then, with $\langle x_1, x_2, \dots, x_\alpha \mid r_1, r_2, \dots, r_{\alpha-1} \rangle$ as in 3.4, we have a presentation

$$\pi_1(R^4 - K^2) = \langle x_1, x_2, \dots, x_\alpha \mid r_1, r_2, \dots, r_{\alpha-1}, s_1^+, \dots, s_{\nu-1}^+ \rangle.$$

3.6. Corollary. Let $(K^2 \subset R^4)$ be a standard representative. If the equatorial cross section $K^2 \cap R_0^3$ is unknotted, then $\pi_1(R^4 - K^2) \cong J$.

Proof. By Theorem 3.4, $\pi_1(R^4 - K^2)$ has a presentation with a single generator. Since the Hurewicz homomorphism $\pi_1(R^4 - K^2) \rightarrow H_1(R^4 - K^2; \mathbb{Z}) \cong J$ is surjective, $\pi_1(R^4 - K^2)$ is not a finite group. $\square$

In Section 10, we shall discuss the 2-knot with unknotted equatorial cross section. See also Example 3.14 below.

3.7. In the classical knot theory, a useful set of invariants of knot types is the sequence of elementary ideals in the integral group ring Λ of J , see Crowell-Fox [C-F]. Using the free differential calculus of Fox [1; 2], Kinoshita [1] defined the Alexander matrix A and a sequence of elementary ideals $E_d(A)$ of A , and gave a characterization of the first elementary ideals. We describe here the sequence of elementary ideals from this point of view. We do not wish to develop the free calculus, so that we may refer the reader to Crowell-Fox [C-F] or Fox [1; 2] for full account of the construction of the Alexander matrix.

We shall give in Section 6 another description of the Alexander matrix and the elementary ideals.

3.8. Definitions. By 3.1, we can take a simple loop $t = \{*\} \times S^1$ on $\partial N(K^2; R^4) \cong S^2 \times S^1$, which defines a conjugacy class in $G \equiv \pi_1(R^4 - K^2)$, any of which elements is called a *meridian* of K^2 . We give an orientation to a meridian t so that t links with K^2 with linking number one; t is a base of the infinite cyclic group $H_1(R^4 - K^2; \mathbb{Z}) \cong G/[G, G] \equiv J$.

Let $\psi: G \rightarrow J$ be the Hurewicz homomorphism, and let $\langle x_1, \dots, x_\alpha \mid r_1, \dots, r_\beta \rangle$ be a finite presentation of G , and let F be a free group on α generators $x_1, \dots, x_\alpha$ and $\phi: F \rightarrow G$ be the natural epimorphism.

Let $\mathbb{Z}F$, $\mathbb{Z}G$ and $\mathbb{Z}J$ be the group-rings of F , G and J with integral coefficients, respectively. Then the homomorphisms $\phi: F \rightarrow G$ and $\psi: G \rightarrow J$ can be linearly extended to ring homomorphisms $\phi: \mathbb{Z}F \rightarrow \mathbb{Z}G$ and $\psi: \mathbb{Z}G \rightarrow \mathbb{Z}J$. Sometimes, we denote by Λ the integral group-ring $\mathbb{Z}J$; we can think of $\mathbb{Z}J = \Lambda$ as the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$. Λ is a Noetherian unique factorization domain.

We define the *Alexander matrix* A of K^2 by

$$A = (r_{ij}) = \left(\psi \phi \left(\frac{\partial r_i}{\partial x_j} \right) \right), \quad i = 1, \dots, \beta; \quad j = 1, \dots, \alpha.$$

For any integer d , the d -th *elementary ideal* $E_d(A)$ of A is the Λ -ideal generated by the determinants of $(\alpha-d) \times (\alpha-d)$ submatrices of A , with the conventions

- (i) $\alpha - d < 1$ then $E_d(A) = \Lambda$,
- (ii) $\alpha - d > \beta$ then $E_d(A) = 0$.

The d -th *Alexander polynomial* $\Delta_d(t) \in \Lambda$ is the generator of the

smallest principal Λ -ideal containing $E_d(A)$.

3.9. Definitions. In the same way as that of 2-knots, we have similar matrices and elementary ideals for 2-link with μ components, $\mu \geq 2$. We refer the reader to Kinoshita [2; 3], Shinohara-Summers [1] for more general arguments. Let $(L_\mu^2 \subset R^4) = (K_1^2 \cup \dots \cup K_\mu^2 \subset R^4)$ be a 2-link with μ components, $\mu \geq 1$. Let t_i be an oriented meridian of K_i^2 , $i=1, \dots, \mu$, such that t_i links with K_i^2 with linking number 1; $\{t_1, \dots, t_\mu\}$ forms a free abelian base of the free abelian group $H_1(R^4 - L_\mu^2; \mathbb{Z}) \cong \mathbb{J}^\mu$.

Let $\langle x_1, \dots, x_\alpha \mid r_1, \dots, r_\beta \rangle$ be a finite presentation of $\pi_1(R^4 - L_\mu^2) = G$ and F be a free group on α generators $x_1, \dots, x_\alpha$. Let $\psi: G \rightarrow \mathbb{J}^\mu$ be the Hurewicz homomorphism and $\phi: F \rightarrow G$ the natural epimorphism. These homomorphism ψ and ϕ are extended to ring homomorphisms of $\mathbb{Z}G$ onto $\mathbb{Z}\mathbb{J}^\mu$ and of $\mathbb{Z}F$ onto $\mathbb{Z}G$, respectively. In particular, $\mathbb{Z}\mathbb{J}^\mu$ is the Laurent polynomial ring $\mathbb{Z}[t_1, t_1^{-1}, \dots, t_\mu, t_\mu^{-1}]$ of μ variables, which is also Noetherian. Now we have the *Alexander matrix* A of L_μ^2 by

$$A = (r_{ij}) = \left[\psi \phi \left(\frac{\partial r_i}{\partial x_j} \right) \right], \quad i = 1, \dots, \beta; j = 1, \dots, \alpha.$$

For any integer d , the d -th *elementary ideal* $E_d(A)$ of A is the $\mathbb{Z}\mathbb{J}^\mu$ -ideal generated by the determinants of $(\alpha-d) \times (\alpha-d)$ submatrices of A with the conventions

$$(i) \quad \alpha - d < 1 \quad \text{then} \quad E_d(A) = \mathbb{Z}\mathbb{J}^\mu,$$

$$(ii) \quad \alpha - d > \beta \quad \text{then} \quad E_d(A) = 0,$$

and the d -th *Alexander polynomial* $\Delta_d(t_1, \dots, t_\mu) \in \mathbb{Z}\mathbb{J}^\mu$ is the generator of the smallest principal $\mathbb{Z}\mathbb{J}^\mu$ -ideal containing $E_d(A)$.

Let $\sigma: \mathbb{J}^\mu \rightarrow \mathbb{J}$ be an epimorphism defined by $\sigma(t_i) = t$, $i=1, \dots, \mu$.

Then σ is naturally extended to a ring homomorphism $\sigma : \mathbb{Z}J^u \rightarrow \mathbb{Z}J$, and let $\tilde{\psi} = \sigma\psi : \mathbb{Z}G \rightarrow \mathbb{Z}J$. Now we have another $\beta \times \alpha$ matrix $\tilde{A}$ defined by

$$\tilde{A} = (\tilde{r}_{ij}) = \left(\tilde{\psi} \phi \left(\frac{\partial r_i}{\partial x_j} \right) \right), \quad i = 1, \dots, \beta; j = 1, \dots, \alpha,$$

which will be called the *reduced Alexander matrix* of L_μ^2 .

In the same way as that of 3.8, we define the *reduced d -th elementary ideal* $E_d(\tilde{A})$ of $\tilde{A}$, and the *reduced d -th Alexander polynomial* $\tilde{\Delta}_d(t)$ for any integer d . If $\mu = 1$, of course, it holds that $\tilde{A} = A$ and so $E_d(\tilde{A}) = E_d(A)$ and $\tilde{\Delta}_d(t) = \Delta_d(t)$.

3.10. Proposition. *The elementary ideals $E_d(A)$'s and the reduced elementary ideals $E_d(\tilde{A})$'s are not affected by choice of presentations of $G = \pi_1(R^4 - L_\mu^2)$ under the above fixed ψ and $\tilde{\psi}$. That is, $E_d(A)$'s and $E_d(\tilde{A})$'s are invariants of 2-link types. The Alexander polynomials $\Delta_d(t_1, \dots, t_\mu)$'s and the reduced Alexander polynomials $\tilde{\Delta}_d(t)$'s are determined up to units in $\mathbb{Z}J^u$ and Λ , respectively.*

Notation. We denote $E_d(A)$ by $E_d(K^2)$, sometimes for $(K^2 \subset R^4)$. For two elements $f(t)$ and $g(t)$ in Λ , " $f(t) \doteq g(t)$ " means that $f(t)$ is equal to $g(t)$ up to units of Λ , see 6.4 below.

3.11. Proposition. *Let $\Delta_1(t)$ be the first Alexander polynomial of a 2-knot. Then, $\Delta_1(1) = \pm 1$.*

This follows from the fact that $A_{t=1}$ is the matrix which determines the first homology group $H_1(R^4; \mathbb{Z}) = 0$.

3.12. Example. Let $(K^2 \subset R^4)$ be the 2-knot given in Example 2.8. As a set of generators of the knot group of the equatorial cross section

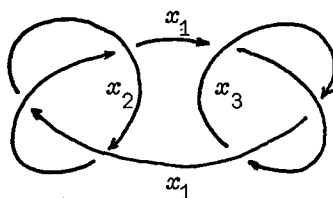


Figure 3.1'

$(K^2 \cap R_0^3 \subset R_0^3) \equiv (K^1 \subset R_0^3)$ (which is the square knot), we take x_1, x_2 and x_3 as shown in Figure 3.1'. $\pi_1(R_0^3 - K^1)$ has a presentation

$$\langle x_1, x_2, x_3 \mid x_1 x_2 x_1 = x_2 x_1 x_2, x_1 x_3 x_1 = x_3 x_1 x_3 \rangle,$$

and so the Alexander matrix is

$$\begin{pmatrix} t^2 - t + 1 & -t^2 + t - 1 & 0 \\ t^2 - t + 1 & 0 & -t^2 + t - 1 \end{pmatrix}.$$

The first elementary ideal is principal and $\Delta_1(t) \doteq (t^2 - t + 1)^2$.

By Corollary 3.5, $\pi_1(R^4 - K^2)$ has a presentation

$$\langle x_1, x_2 \mid x_1 x_2 x_1 = x_2 x_1 x_2 \rangle,$$

and so the Alexander matrix is

$$A = \begin{pmatrix} t^2 - t + 1 & -t^2 + t - 1 \end{pmatrix}.$$

The first elementary ideal $E_1(A)$ is principal and $\Delta_1(t) \doteq t^2 - t + 1$.

This polynomial $\Delta_1(t)$ is the first Alexander polynomial of the trefoil, recall Example 1.13.

We record that :

3.13. Proposition. *The first elementary ideal of any spun 2-knot is principal, and the first Alexander polynomial is reciprocal.*

3.14. Example. (Hosokawa [1]) The 2-knot $(K^2 \subset R^4)$ shown in Figure 3.2 also has the square knot as its equatorial cross section $(K^2 \cap R_0^3 \subset R_0^3)$.

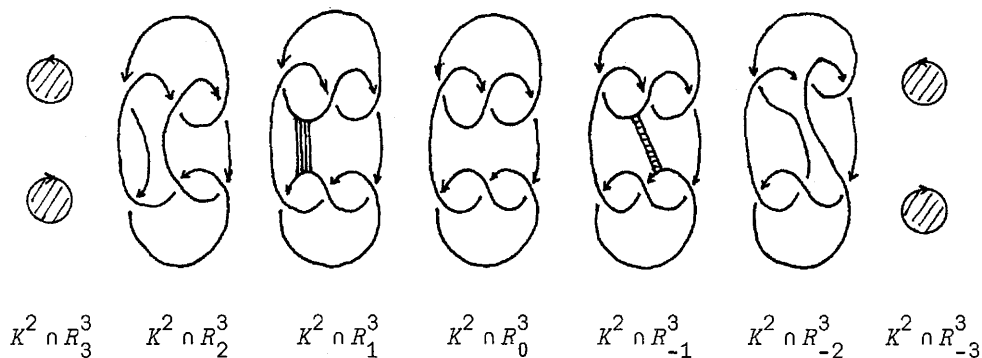


Figure 3.2

Using Lemma 2.5, we change the levels of the saddle-bands B and B' , and we have an equivalent 2-knot $(\tilde{K}^2 \subset R^4)$ given in Figure 3.3. $(\tilde{K}^2 \subset R^4)$ is also a standard representative and the equatorial cross section $\tilde{K}^2 \cap R_0^3$ is a trivial 1-knot. By Corollary 3.6, $\pi_1(R^4 - \tilde{K}^2) \cong \pi_1(R^4 - K^2) \cong J$.

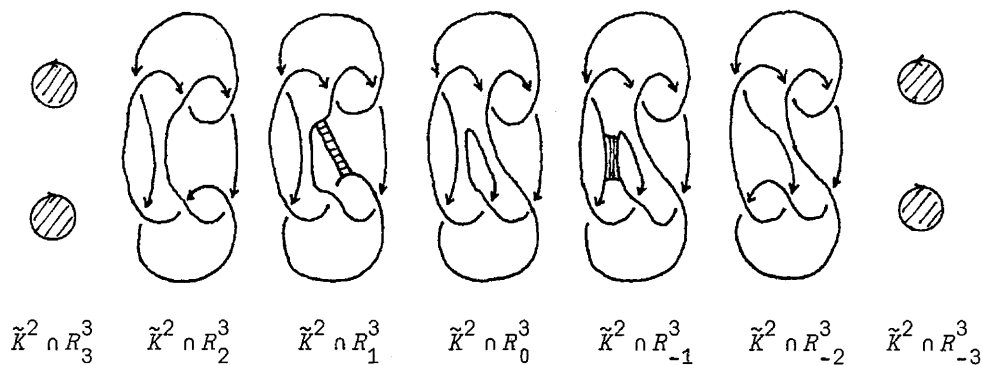


Figure 3.3

In fact, it will be shown in Section 10 that $(\tilde{K}^2 \subset R^4) \cong (K^2 \subset R^4)$ are of trivial 2-knot type, and so the square knot is an equatorial cross-section of a trivial 2-knot; refer to Problem 2.10(a) and Problem 2.11(a).

3.15. Example. (Fox [3, Example 10]) The 2-knot $(K_1^2 \subset R^4)$ described in Figure 3.4 is simple symmetric and has the stevedore's knot $(K^1 \subset R_0^3)$ as the equatorial cross section $(K_1^2 \cap R_0^3 \subset R_0^3)$. For the stevedore's knot,

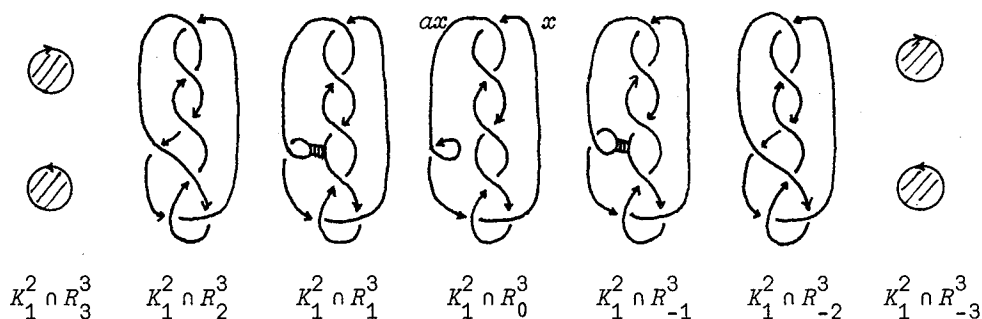


Figure 3.4

$$\pi_1(R_0^3 - K^1) = \langle x, a \mid xa^3xa^{-2}x^{-1} = a^2xa^{-2} \rangle, \text{ and}$$

$$\Delta_1(t) \doteq 2t - 5t + 2 = (1 - 2t)(2 - t).$$

By Corollary 3.5, we have :

$$\begin{aligned} \pi_1(R^4 - K_1^2) &= \langle x, a \mid xa^3xa^{-2}x^{-1} = a^2xa^{-2}, ax = a^2xa^{-2} \rangle \\ &= \langle x, a \mid xa^2 = ax \rangle. \end{aligned}$$

So, the first elementary ideal $E_1(K_1^2)$ of $(K_1^2 \subset R^4)$ is principal, and the first Alexander polynomial is

$$\Delta_1(t) \doteq 2t - 1.$$

Since the polynomial $2t - 1$ is not reciprocal, Example 3.15 shows :

3.16. Proposition. *The first Alexander polynomial of a 2-knot is not always reciprocal, even if the first elementary ideal is principal.*

Combining this with Proposition 3.13, we conclude that the ribbon 2-knot $(K_1^2 \subset R^4)$ is not a spun 2-knot ; this proves Theorem 2.16(2).

For the first Alexander polynomials, we have the following:

3.17. Theorem. (Kinoshita [1]) *Let $f(t)$ be an integral polynomial with $|f(1)| = 1$. Then, there exists a 2-knot whose first elementary ideal E_1 is principal and $E_1 = (\Delta_1(t)) = (f(t))$.*

The proof will be given in Theorem 6.21, see also Theorem 6.17 and Exercise 6.22. By Kinoshita-Terasaka [1] and Whitehead [1], we have infinitely many distinct non-trivial 1-knots with $\Delta_1(t) \neq 1$. By spinning of such the 1-knots, we have :

3.18. Proposition. *There exist infinitely many distinct non-trivial 2-knots with $\Delta_1(t) \neq 1$.*

3.19. Example. (Fox [3, Example 11]) Using the stevedore's knot, we can give another ribbon 2-knot $(K_2^2 \subset R^4)$ shown in Figure 3.5 :

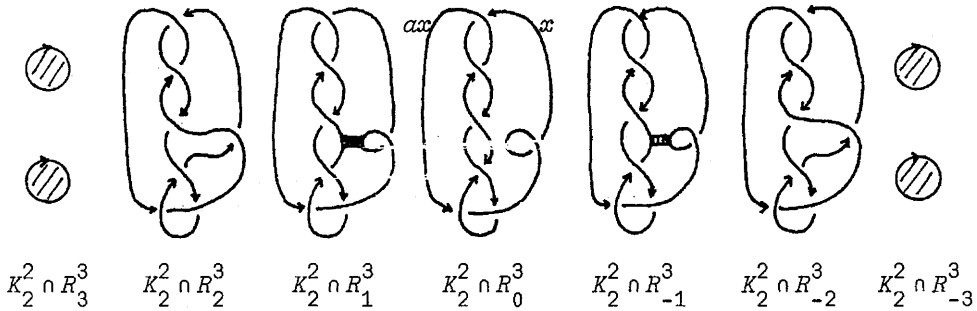


Figure 3.5

$$\begin{aligned} \pi_1(R^4 - K_2^2) &= \langle x, a \mid xa^3xa^{-2}x^{-1} = a^2xa^{-2}, a^2xa^{-1} = x \rangle \\ &= \langle x, a \mid ax^2 = xa \rangle. \end{aligned}$$

The first elementary ideal $E_1(K_2^2)$ is also principal, and the first Alexander polynomial is $\Delta_1(t) \doteq 2 - t$.

Since the stevedore's knot is invertible, we can easily check that $(K_1^2 \subset R^4) \cong (-K_2^2 \subset R^4)$. Since $2t - 1 \neq 2 - t$, $(K_1^2 \subset R^4)$ and so $(K_2^2 \subset R^4)$

are not invertible; recall Definition 2.17. In general, we have :

3.20. Proposition. (Kinoshita) *Suppose that the first elementary ideal E_1 of a 2-knot $(K^2 \subset R^4)$ is principal and the first Alexander polynomial $\Delta_1(t)$ is not reciprocal. Then, $(K^2 \subset R^4)$ is not invertible and not +amphicheiral.*

Proof. Let $G = \pi_1(R^4 - K^2)$. From the definition of invertible and +amphicheiral in 2.17, if $(K^2 \subset R^4)$ is invertible or +amphicheiral, there exists an automorphism of G that maps the generators x_j into their inverses x_j^{-1} , provided that $\psi(x_j) = t$. So, it should hold that $\Delta_1(t) \doteq \Delta_1(t^{-1})$. $\square$

3.21. Problem. (Fox [4, Problem 35]) *Does there exist a non -amphicheiral 2-knot ?*

3.22. Example. (Fox [3, Example 12]) Combining Examples 3.15 and 3.19, we have the following 2-knot $(K_3^2 \subset R^4)$:

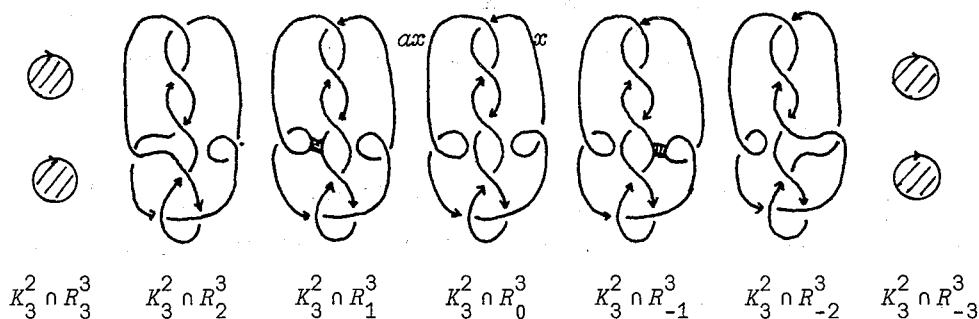


Figure 3.6

$$\begin{aligned}
 G &= \pi_1(R^4 - K_3^2) \\
 &= \langle x, a \mid xa^3xa^{-2}x^{-1} = a^2xa^{-2}, ax = a^2xa^{-2}, a^2xa^{-1} = x \rangle \\
 &= \langle x, a \mid xa^2 = ax, ax^2 = xa \rangle
 \end{aligned}$$

$$= \langle x, a \mid a^3 = 1, xax^{-1} = a^{-1} \rangle.$$

$$[G, G] = \langle a \mid a^3 = 1 \rangle.$$

The first elementary ideal of $(K_3^2 \subset R^4)$ is generated by $2t - 1$ and $2 - t$; $E_1(K_3^2) = (2t - 1, 2 - t)$, which is not principal.

3.23. Proposition. *The first elementary ideal of a 2-knot need not be principal.*

Fox-Milnor [2] has shown that a slice 1-knot (and so a ribbon 1-knot) has the first Alexander polynomial $\Delta_1(t)$ of the form $f(t)f(t^{-1})$, refer also to Terasaka [1]. We have the following :

3.24. Theorem. (Kinoshita [1], Yonebayashi [1]) *Every ribbon 2-knot has the principal first elementary ideal. In particular, let $(K^2 \subset R^4)$ be a simple symmetric representative, and $(K^1 \subset R^3)$ be the equatorial cross section. Then, it holds that*

$$\Delta_{K^1}(t) \doteq \Delta_{K^2}(t)\Delta_{K^2}(t^{-1}),$$

where $\Delta_{K^1}(t)$ and $\Delta_{K^2}(t)$ are the first Alexander polynomials of $(K^1 \subset R^3)$ and $(K^2 \subset R^4)$, respectively.

The proof of Yonebayashi [1] was similar to that of Terasaka [1]. Another proof will be given by the quite similar way to that of the proof of Theorem 2 in Fox-Milnor [2].

Combining Theorem 3.24 with Example 3.22, the 2-knot type of $(K_2^2 \subset R^4)$ in 3.22 is not ribbon type; this proves Theorem 2.16(3).

For the Alexander polynomial of a 2-link with $\mu > 1$ components, we have hardly any information. For the reduced polynomial, we record :

3.25. Theorem. (Kinoshita [3]) Let $f(t)$ be an integral polynomial with $|f(1)| = 1$. Then, there exists a 2-link $(L_2^2 \subset R^4) = (K_1^2 \cup K_2^2 \subset R^4)$ with 2 components such that

- (1) $(K_1^2 \subset R^4)$ and $(K_2^2 \subset R^4)$ are trivial 2-knots,
- (2) the reduced d -th elementary ideal $\tilde{E}_d$ of L_2^2 is as follows:

$$\begin{cases} \tilde{E}_0 = \tilde{E}_1 = 0, \\ \tilde{E}_2 = (\tilde{\Delta}_2(t)) = (f(t)), \text{ and} \\ \tilde{E}_d = \Lambda, \text{ if } d > 2. \end{cases}$$

We shall illustrate other properties of the Alexander polynomial in other Sections.

4. Characterization Problem of 2-Knot Groups

Concerning the 2-knot groups, there are several unsolved problems. In order to motivate the later sections, it will be helpful to state the "characterization problem" at this point.

Let KG_n be a set of all n -knot groups, $n > 0$.

4.1. Theorem. (Gutierrez [5]) Let $G \in KG_n$ and $G = G_1 * G_2$, a free product, then either $G_1 = 1$ or $G_2 = 1$.

Artin's spinning construction (recall Example 1.13) shows inclusions $KG_n \subset KG_{n+1}$ for any $n > 0$. The following is known :

4.2. Theorem. $KG_1 \subsetneq KG_2 \subsetneq KG_3 = KG_4 = \dots$.

By Propositions 3.16 and 3.23, we have $KG_1 \neq KG_2$, because the first elementary ideal of a 1-knot is principal and the first Alexander poly-

nomial is reciprocal. Gutierrez [6] has shown that $KG_2 \neq KG_3$, that will be discussed in a later Section. The fact $KG_n = KG_{n+1}$, $n \geq 3$, is a consequence of the following algebraic characterization of the higher dimensional knot groups due to M.A.Kervaire.

4.3. Theorem. (Kervaire [1, Theorem 1]) *Let G be a finitely presented group. $G \in KG_n$ ($n \geq 3$) if and only if G satisfies the following :*

- (1) $G/[G, G] \cong J$,
- (2) the weight $w(G)$ of G is one,
- (3) $H_2(G; \mathbb{Z}) = 0$.

The weight $w(G)$ of G is defined as follows : Let B be a finite set of G . If the normal closure of B in G equals G , we will call B a *nucleus* of G , and the number of elements of B the *order* of B . We call the minimal order of nuclei of G the *weight* of G , and denote it by $w(G)$. $H_2(G; \mathbb{Z})$ denotes the second homology group of G with integral coefficients and trivial action of G on $\mathbb{Z}$, see Hopf [1].

The "only if" part of 4.3 holds for any $n > 0$.

4.4. Proposition. (Kervaire [1, Lemma 2]) *For any $n > 0$, if $G \in KG_n$, then (1) $G/[G, G] \cong J$, (2) $w(G) = 1$ and (3) $H_2(G; \mathbb{Z}) = 0$.*

4.5. Proposition. (Kervaire [1, pp.116-118]) *The three conditions (1), (2) and (3) in Theorem 4.3 are mutually independent.*

For $n = 1$, further necessary conditions have been given. Refer to Neuwirth [N_1 , Chapter IX]. However, the problem of characterizing 1-knot groups by algebraic conditions remains unsolved.

Kervaire discussed also on 2-knot groups, and proved the following :

4.6. Theorem. (Kervaire [1, Theorem 2]) Let G be a finitely presented group satisfying the following :

- (1) $G/[G, G] \cong J$,
- (2) $w(G) = 1$,
- (4) there exists a presentation of G with deficiency 1.

Then, there exist a homotopy 4-sphere Σ^4 and a locally flat embedding $k : S^2 \rightarrow \Sigma^4$ such that $G \cong \pi_1(\Sigma^4 - k(S^2))$.

If $p = \langle x_1, \dots, x_\alpha \mid r_1, \dots, r_\beta \rangle$ is a finite presentation of a group G , then the deficiency of $p = \text{def} p = \alpha - \beta$. The deficiency of G , $\text{def} G$, is defined as $\sup \{ \text{def} p \}$ over all finite presentations p of G .

Rapaport [1] proved that the condition (4) implies (3) $H_2(G; \mathbb{Z}) = 0$.

Now, the characterization problem of 2-knot groups will be stated in the following way :

4.7. Problem. Characterize, among the finitely presented groups G with (1) $G/[G, G] \cong J$, (2) $w(G) = 1$ and (3) $H_2(G; \mathbb{Z}) = 0$, those which are 2-knot groups.

Using the ribbon 2-knots, Yajima [3] has shown the following :

4.8. Theorem. (Yajima [3]) A finitely presented group G is isomorphic to $\pi_1(R^4 - K^2)$ of some ribbon 2-knot $(K^2 \subset R^4)$ if and only if

- (1) $G/[G, G] \cong J$, and
- (5) G has a Wirtinger presentation of deficiency 1.

4.9. Definition. A finite presentation $\langle x_1, \dots, x_\alpha \mid r_1, \dots, r_\beta \rangle$ of a group is called a Wirtinger presentation iff the relators r_i form

$$\begin{aligned}
r_1 : x_{i_1} &= x_{j_1}^{\varepsilon(1)} x_{k_1} x_{j_1}^{-\varepsilon(1)}, \\
r_2 : x_{i_2} &= x_{j_2}^{\varepsilon(2)} x_{k_2} x_{j_2}^{-\varepsilon(2)}, \\
&\vdots \\
&\vdots \\
r_\beta : x_{i_\beta} &= x_{j_\beta}^{\varepsilon(\beta)} x_{k_\beta} x_{j_\beta}^{-\varepsilon(\beta)}.
\end{aligned}
\quad (\varepsilon(i) = \pm 1, i=1, \dots, \beta),$$

Since any 1-knot group (or 1-link group) has a Wirtinger presentation, as a Corollary to Theorem 3.4, we have :

4.10. Proposition. Any 2-knot group (or 2-link group) G has a Wirtinger presentation.

From Examples given in Section 3, we have :

4.11. Problem. Does there exist a 2-knot $(K^2 \subset R^4)$ such that $\text{def}(\pi_1(R^4 - K^2))$ is neither 0 nor 1?

4.12. Proposition. (Yajima [4]) Let G be a finitely presented group. Then, two conditions (1) $G/[G, G] \cong J$ and

(5') G has a Wirtinger presentation,

imply that (2) $w(G) = 1$. The three conditions (1') $G/[G, G] \cong J^\mu$ with $1 \leq \mu < \infty$, (2') $w(G) = \mu$ and (3) $H_2(G; \mathbb{Z}) = 0$ imply (5').

4.13. Conjecture. (Kawauchi-Maeda) Let G be a finitely presented group with (1) $G/[G, G] \cong J$ and (2) $w(G) = 1$. Then, the followings are equivalent : (3) $H_2(G; \mathbb{Z}) = 0$,

(5') G has a Wirtinger presentation,

(6) There exist a closed, connected, orientable 2-manifold F and a locally flat embedding $h : F \rightarrow R^4$ such that $G \cong \pi_1(R^4 - h(F))$.

By Proposition 4.12, we know that $(3) \Rightarrow (5')$. Using Yajima's argument in Yajima [3], we can easily conclude that $(5') \Rightarrow (6)$. Combining Exercise 2.7 with 3.3, we can deduce that $(6) \Rightarrow (5')$. So, it suffices to check that $(6) \Rightarrow (3)$, that is, whether $H_2(\pi_1(R^4 - h(F)); \mathbb{Z}) = 0$.

From the presentation of a 2-knot group in Theorem 3.4, the following question comes to mind :

4.14. Question. (Fox [4 ; Problem 28]) *If a finitely presented group G is a homomorph of a 1-knot group and $G/[G, G] \cong \mathbb{Z}$, is it a 2-knot group ?*

The answer is negative. Murasugi [2] shows that the group

$$G = \langle a, b \mid a^2b^{-3} = 1, [a, b]^2 = 1 \rangle$$

cannot be an n -knot group for $n \geq 2$, where $[a, b] = aba^{-1}b^{-1}$. G is a homomorph of the group $\langle a, b \mid a^2b^{-3} = 1 \rangle$ of the trefoil, and it holds that (1) $G/[G, G] \cong \mathbb{Z}$ and (2) $w(G) = 1$.

Generally, we have infinitely many such the examples. Let $(K^1 \subset S^3)$ be a non-trivial 1-knot, and $\langle x_1, \dots, x_\alpha \mid r_1 = 1, \dots, r_\beta = 1 \rangle$ a presentation of $\pi_1(S^3 - K^1)$. Let $\ell \subset \partial N(K^1; S^3)$ be a longitude of $(K^1 \subset S^3)$, that is, ℓ is a simple loop on the boundary of a regular neighborhood $N(K^1; S^3) \cong D^2 \times S^1$, such that ℓ is null-homologous in $S^3 - K^1$ and $\ell \neq 1$ on $\partial N(K^1; S^3)$. Then, the factor group

$$G = \langle x_1, \dots, x_\alpha \mid r_1 = 1, \dots, r_\beta = 1, \ell = 1 \rangle$$

of $\pi_1(S^3 - K^1)$ cannot be an n -knot group for $n \geq 1$. In fact, let

$$f : \partial(D^2 \times S^1) \rightarrow \partial(S^3 - \circ N(K^1; S^3)) = \partial N(K^1; S^3)$$

be a homeomorphism such that $f(\partial D^2 \times \{*\}) = \ell$. We have a closed orientable 3-manifold $M = (S^3 - \circ N(K^1; S^3)) \cup_f (D^2 \times S^1)$ with $\pi_1(M) \cong G$ and $H_2(G; \mathbb{Z}) \cong H_2(M; \mathbb{Z}) \neq 0$. $\square$

We consider Example 3.22. Let $\gamma : \pi(R^4 - K_3^2) \rightarrow S_3$ be a map defined by $x \rightarrow (0,1)$, $a \rightarrow (0,1,2)$, where S_3 is the symmetric group of degree 3. It is easy to check that γ is an epimorphism. We conclude that $a \neq 1$ in $\pi_1(R^4 - K_3^2)$, so $\pi_1(R^4 - K_3^2)$ has an element of finite order. (Refer to Papakyriakopoulos [1] for the classical knots.) We record another example due to R.H.Fox.

4.15. Example. (Fox [3, Example 15]) This example $(K_4^2 \subset R^4)$ given in Figure 4.1 is a generalization of Example 3.22. We show only its equatorial cross section $K_4^2 \cap R_0^3$.

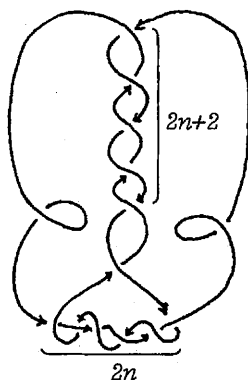


Figure 4.1

The group $\pi_1(R^4 - K_4^2)$ of $(K_4^2 \subset R^4)$ is the metacyclic group

$$\begin{aligned} & \langle x, a \mid xa^{n+1} = a^n x, a^{n+1}x = xa^n \rangle \\ &= \langle x, a, b \mid xa^{n+1} = a^n x, a^{n+1}x = xa^n, b = a^n \rangle \\ &= \langle x, b \mid b^{2n+1} = 1, xbx^{-1} = b^{-1} \rangle. \end{aligned}$$

Thus we have :

4.16. Proposition. (Fox [3]) For every odd integer $2n + 1$, there is a 2-knot whose group has an element of order $2n + 1$.

4.17. Example (*Twisting Spun*) : (Zeeman [6]) E.C. Zeeman introduced a new class of n -knots for every $n \geq 2$, which is called a *twisting spun*. We describe here only twisting spun 2-knots. We use the notation in 1.13. Let B^3 be a 3-disk in the interior of R_+^3 , and let $k^1 : S^1 \rightarrow R^3$ be an embedding such that $k^1(S^1) \cap R^2$ consists of a single arc α , and let $\beta = \text{Cl}(k^1(S^1) - \alpha)$. We now assume that $\beta \cap B^3$ consists of a proper arc γ and that the 1-knots $(k^1(S^1) \subset R^3)$ and $(\gamma \cup c \subset R)$ are equivalent, where c is a simple arc on ∂B^3 which spans the two points $\partial \gamma$. The spinning process of β induces the spun 2-knot $(K^2 \subset R^4)$ of k^1 . The additional refinement that we now add is rotate the knotted arc γ about the polar axis relative B^3 during the spinning. We call this process "twisting" and we twist $|v|$ times during the spinning so that we obtain a 2-knot $(\tilde{K}^2 \subset R^4)$ the v -twist-spun 2-knot of k^1 , see Figure 4.2.

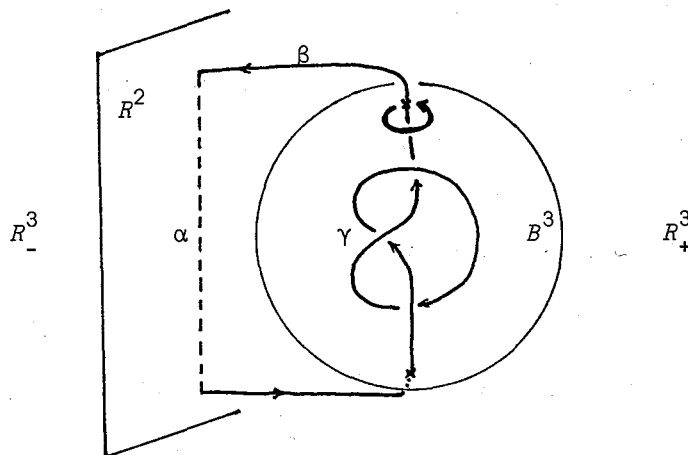


Figure 4.2

Let $(\tilde{K}^2 \subset R^4)$ be the 5-twist-spun 2-knot of the trefoil. We have :

$$\pi_1(R^4 - \tilde{K}^2) = \langle x_1, x_2 \mid x_1 x_2 x_1 = x_2 x_1 x_2, x_2 = x_1^{-5} x_2 x_1^5 \rangle.$$

The relation $x_2 = x_1^{-5} x_2 x_1^5$ comes from the 5-twisting, see Figure 4.3.

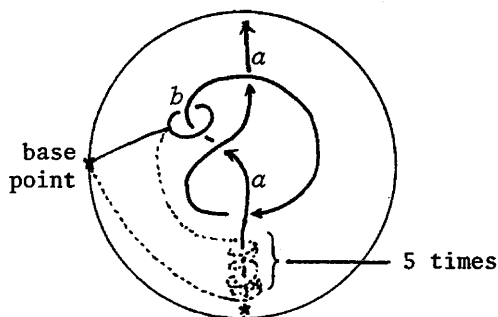


Figure 4.3

As Zeeman indicated in [6, §4], $\pi_1(R^4 - \tilde{K}^2)$ is isomorphic to

$$\mathcal{D} \times \mathbb{Z} = \langle a, b, c \mid a^5 = (ab)^3 = (aba)^2, c^{-1}ac = b, c^{-1}bc = ba^{-1} \rangle$$

by making the substitution $x_1 \rightarrow c$, $x_2 \rightarrow ac$, where $\mathcal{D}$ is the binary dodecahedral group $\langle a, b \mid a^5 = (ab)^3 = (aba)^2 \rangle$. Since $\mathcal{D}$ has order 120, we have a 2-knot whose group has an element of even order.

4.18. Problem. For every even integer $2n$, does there exist a 2-knot whose group has an element of order $2n$?

5. Seifert Manifolds for 2-Knots

5.1. Theorem. For any n -link $K_1^n \cup \dots \cup K_\mu^n$, there exists a compact oriented (connected) $(n+1)$ -manifold V^{n+1} in S^{n+2} (or R^{n+2}) with $\partial V^{n+1} = K_1^n \cup \dots \cup K_\mu^n$.

We call such a manifold V^{n+1} a *Seifert manifold* for $K_1^n \cup \dots \cup K_\mu^n$.

The case $n = 1$ was proved by Flankl-Pontrjagin in 1930 (see Seifert [1]), and the case $n = 2$ by Gluck [2; 3]. In general case, Kervaire [2, Lemma II.10] and Levine [3, Lemma 2] proved by using the Thom-Pontrjagin

construction.

Seifert manifolds carry out an important role in the following, so we give the proof of 5.1 here for 2-knots in the same way as that of Gluck [3] and Yanagawa [2] except for obvious modifications.

5.2. Let V^3 be a Seifert manifold for a 2-knot $(K^2 \subset R^4)$. We assume that K^2 is a standard representative and interior vertices of V^3 are in general position with respect to (x_1, x_2, x_3) and x_4 as in Section 2. Then, an *ordinary* cross section $V^3 \cap R_u^3$ is empty or consists of one or more oriented 2-manifolds (perhaps some of them are with boundary and some of them are without boundary). Since V^3 is compact, the number of *exceptional* cross sections is finite. Let $x_4 = u_0$ be a critical value, and let $v \in V^3 \cap R_{u_0}^3$ be a critical point. Then, a 4-dimensional neighborhood of v is one of the following as shown in Figure 5.1 and Figure 5.2.

The four types in Figure 5.1 correspond to the original critical points of $K^2 = \partial V^3$, and the four types in Figure 5.2 are new critical points contained in $\circ V^3$.

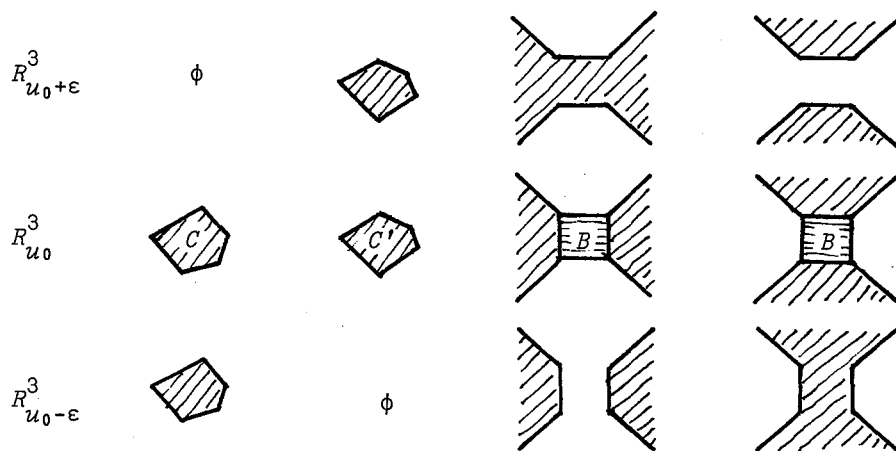


Figure 5.1

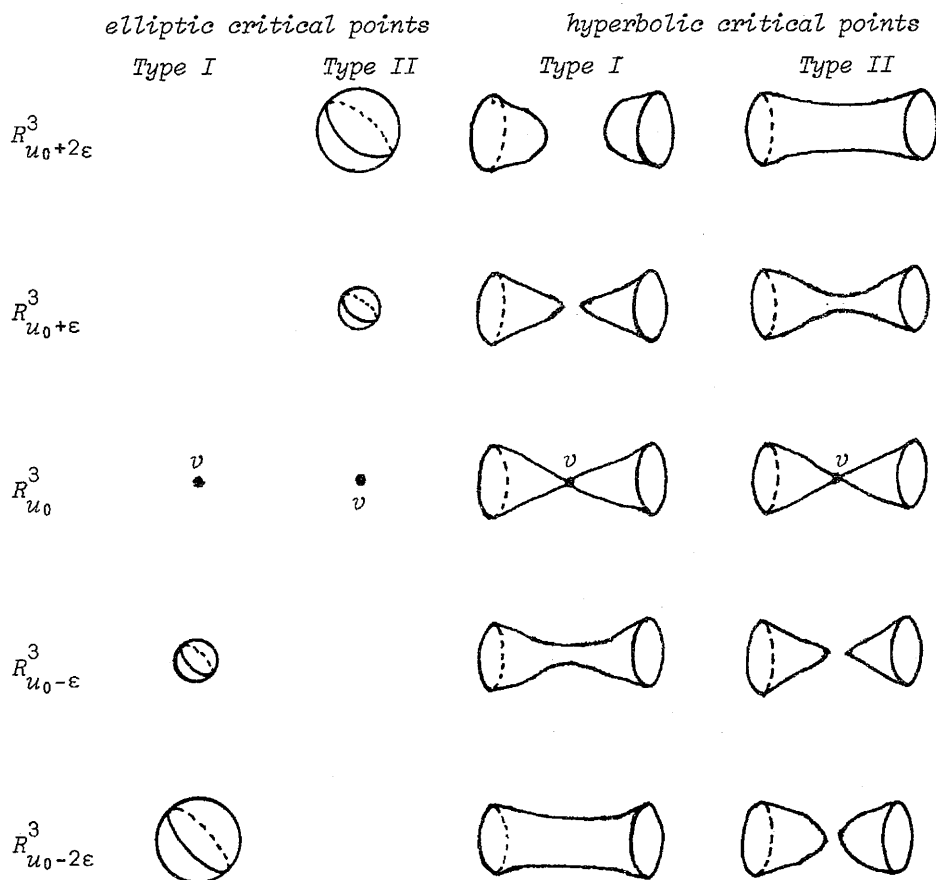


Figure 5.2

We will call the new critical point v in Figure 5.2, respectively, an *elliptic* critical point of type I, an *elliptic* critical point of type II, a *hyperbolic* critical point of type I and a *hyperbolic* critical point of type II, as illustrated in Figure 5.2. A transformation

$$(V^3 \cap R_{u_0+\varepsilon}^3 \subset R_{u_0+\varepsilon}^3) \implies (V^3 \cap R_{u_0}^3 \subset R_{u_0}^3) \implies (V^3 \cap R_{u_0-\varepsilon}^3 \subset R_{u_0-\varepsilon}^3)$$

will be called an *elliptic* transformation of type I, an *elliptic* transformation of type II, a *hyperbolic* transformation of type I or a *hyperbolic* transformation of type II according as v is an elliptic critical point

of type I , an elliptic critical point of type II , a hyperbolic critical point of type I or a hyperbolic critical point of type II.

These critical points, of course, correspond to a 0-handle, a 3-handle, a 1-handle and a 2-handle, respectively, of a handle decomposition of V^3 . In fact, we can make the neighborhood of a critical point v look like Figure 5.3 by an isotopic deformation of R^4 that is the identity outside the neighborhood. We shall discuss frequently critical points of

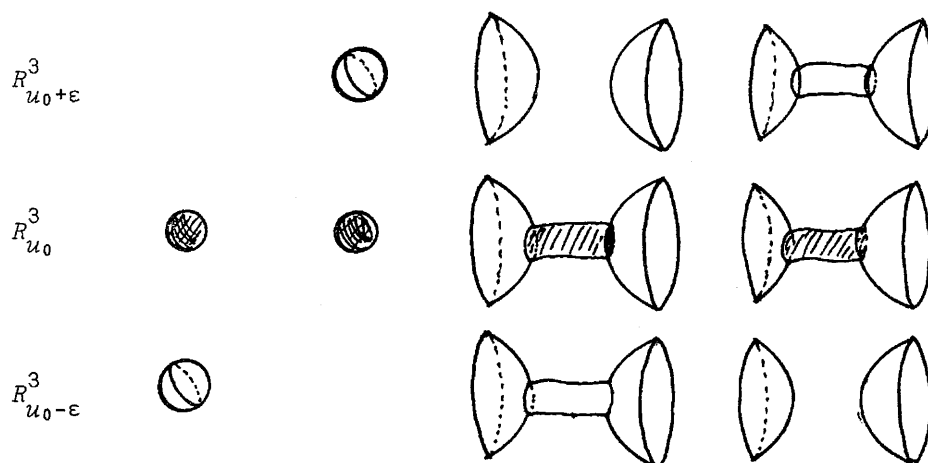


Figure 5.3

3-manifolds in R^4 . For brevity, we shall use the following terminology.

5.3. For an oriented 2-manifold M in a manifold W , let $h^1: D^1 \times D^2 \rightarrow W$ and $h^2: D^2 \times D^1 \rightarrow W$ be embeddings of 3-disks such that

- $$(1) \begin{cases} h^1(D^1 \times D^2) \cap M = h^1(\partial D^1 \times D^2) \subset \circ M, \\ M(h^1) \equiv M - h^1(\partial D^1 \times D^2) \cup h^1(D^1 \times \partial D^2) \text{ is a 2-manifold that} \\ \text{is oriented coherently with that of } M, \end{cases}$$
- $$(2) \begin{cases} h^2(D^2 \times D^1) \cap M = h^2(\partial D^2 \times D^1) \subset \circ M, \\ M(h^2) \equiv M - h^2(\partial D^2 \times D^1) \cup h^2(D^2 \times \partial D^1) \text{ is a 2-manifold that} \\ \text{is oriented coherently with that of } M. \end{cases}$$

We say that $M(h^i)$ is obtained from M by adding an i -handle h^i , $i=1,2$.

Then, for a hyperbolic transformation

$$(V^3 \cap R_{u_0+\varepsilon}^3 \subset R_{u_0+\varepsilon}^3) \implies (V^3 \cap R_{u_0-\varepsilon}^3 \subset R_{u_0-\varepsilon}^3),$$

$V^3 \cap R_{u_0-\varepsilon}^3$ is obtained from $V^3 \cap R_{u_0+\varepsilon}^3$ by adding an appropriate 1-handle or 2-handle according as it is of type I or type II.

5.4. Lemma. Let $(O_1^1 \cup \dots \cup O_\mu^1 \subset R^3)$ be a trivial 1-link with μ components, and let $C = C_1 \cup \dots \cup C_\mu$ and $C' = C'_1 \cup \dots \cup C'_\mu$ be unions of mutually disjoint oriented 1-disks in R^3 with $\partial C_i = O_i^1 = \partial C'_i$, $i=1, \dots, \mu$. Let $A_i = N(O_i^1; C_i)$ and $A'_i = N(O_i^1; C'_i)$ be small regular neighborhoods of $O_i^1 = \partial C_i = \partial C'_i$ in C_i and C'_i , respectively, $i=1, \dots, \mu$. Then, there exists an isotopic deformation $\{\eta_t\}$ of R^3 such that

$$(1) \quad \eta_t | O_1^1 \cup \dots \cup O_\mu^1 = \text{identity for } 0 \leq t \leq 1,$$

$$(2) \quad \eta_1(A_1 \cup \dots \cup A_\mu) = A'_1 \cup \dots \cup A'_\mu,$$

(3) $\eta_1((C_1 - A_1) \cup \dots \cup (C_\mu - A_\mu)) \cap ((C'_1 - A'_1) \cup \dots \cup (C'_\mu - A'_\mu))$ consists of mutually disjoint simple loops.

Proof. Let $\alpha_i = \partial A_i - O_i^1$ and $\alpha'_i = \partial A'_i - O_i^1$, $i=1, \dots, \mu$. Since C_i and C'_i are 2-disks, $\text{Lk}(\alpha_i, O_i^1) = 0 = \text{Lk}(\alpha'_i, O_i^1)$, where Lk is the linking pairing of 1-cycles in R^3 . Now, we have an isotopic deformation $\{\eta_t\}$ of R^3 which satisfies the conditions (1) and (2). The condition (3) follows from the general position argument, and completing the proof. $\square$

The following is a special case of Theorem 5.1.

5.5. Lemma. Let $(K_1^2 \cup \dots \cup K_\mu^2 \subset R^4)$ be a standard representative of a 2-link type with μ components. If it has no saddle-bands, then it is trivial. In particular, there exist mutually disjoint 3-disks $V_1^3, \dots, V_\mu^3$

in R^4 such that $\partial V_i^3 = K_i^2$ and $V_i^3 \cap H_{[3, \infty)} = C_i$, $i = 1, \dots, \mu$. Here, C_i is the maximum-disk of K_i^2 , $i = 1, \dots, \mu$.

5.6. Notation. In this Section, we use frequently the following map. For u with $-\infty < u < \infty$, let $p_u : R^4 \rightarrow R_u^3$ denote a projection defined by $p_u(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, u)$.

Proof of Lemma 5.5. Since $K_i^2 \cap R_0^3$ is connected, each K_i^2 has one maximum-disk C_i and one minimum-disk C'_i ; let denote $C = C_1 \cup \dots \cup C_\mu$, $C' = C'_1 \cup \dots \cup C'_\mu$. Since there are no saddle-bands, $(K_1^2 \cup \dots \cup K_\mu^2) \cap H_{(-3, 3)}$ may be considered as $((K_1^2 \cup \dots \cup K_\mu^2) \cap R_0^3) \times (-3, 3)$ in $H_{(-3, 3)}$. Using the isotopic deformation $\{\eta_t\}$ in Lemma 5.4, we pull down the C continuously with respect to x_4 -axis as far as R_0^3 , so that $C \cap p_0(C')$ satisfies the conditions of Lemma 5.4. That is, we denote by C_u the trace of 2-disks in R_u^3 with $\partial C_u = (K_1^2 \cup \dots \cup K_\mu^2) \cap R_u^3$. Then, $C_u = p_u \eta_t(C)$, where $t = (3-u)/3$ for $3 \leq u \leq 0$. Now, $C_0 \cap p_0(C')$ consists of mutually disjoint simple loops and μ anuli $A_1, \dots, A_\mu$ with

$$(\partial A_1 \cup \dots \cup \partial A_\mu) \supset \partial C_0 = p_0(\partial C') = (K_1^2 \cup \dots \cup K_\mu^2) \cap R_0^3.$$

We choose a loop $e \subset {}^\circ C_0 \cap p_0({}^\circ C')$ which is innermost on $p_0(C')$, and let $E \subset p_0(C')$ be the 2-disk cut off by e with ${}^\circ E \cap C_0 = \emptyset$. With decreasing x_4 , we perform a hyperbolic transformation of type II on C_0 along E . We repeat such the hyperbolic transformations with decreasing x_4 , $0 > x_4 > -1$, and we have a union of μ mutually disjoint 2-disks C_{-1} and some oriented 2-spheres, say $\Sigma_{-1} = \sigma_1 \cup \dots \cup \sigma_\nu$, in R_{-1}^3 with $(C_{-1} \cup \Sigma_{-1}) \cap p_{-1}(C') = p_{-1}(A_1 \cup \dots \cup A_\mu)$ and $p_{-1}(\partial A_i) \supset K_i^2 \cap R_{-1}^3$, $i = 1, \dots, \mu$. Let α_i be the boundary loop of $p_{-1}(A_i)$ with $\alpha_i \cap K_i^2 = \emptyset$, and let $V_i \subset C_{-1}$ be the 2-disk cut off by α_i , $i = 1, \dots, \mu$. With decreasing x_4 , $-1 > x_4 > -2$, we

also perform hyperbolic transformations of type II along $\nabla_1, \dots, \nabla_\mu$; and we have in R_{-2}^3 a union of mutually disjoint μ 2-disks C_{-2} and some oriented 2-spheres, say $\Sigma_{-2} = \sigma_1 \cup \dots \cup \sigma_{\nu+\mu}$ with $C_{-2} = p_{-2}(C')$. Then, we pull down $C_{-2} \cup \Sigma_{-2}$ continuously as far as R_{-3}^3 , so that $C_{-3} = C'$.

When x_u has decreased below -3 , the cross-sectional surface consists of $\nu + \mu$ 2-spheres Σ_u , $u = -3 - \epsilon$, which may be shrink to points as $x_u = u$ decrease further. It is easily seen that the union of all cross-sectional oriented 2-disks C_u and oriented 2-spheres Σ_u forms mutually disjoint 3-disks $V_1, \dots, V_\mu$ with $\partial V_i = K_i^2$ and $V_i \cap H_{[3, \infty)} = C_i$ for $i = 1, \dots, \mu$, and completing the proof of Lemma 5.5. $\square$

We will need the following technical lemmas.

5.7. Lemma. *Let $C = C_1 \cup \dots \cup C_\nu$ be a union of ν mutually disjoint oriented 2-disks in R^3 and let $\partial C_i = O_i$, $i = 1, \dots, \nu$. Let $(K^1 \subset R^3)$ be a ribbon 1-knot obtained by a fusion of $O_1 \cup \dots \cup O_\nu$ along $B_1^+ \cup \dots \cup B_r^+$, $r = \nu - 1$. Then, there exist embeddings $\tilde{B}_1^+, \dots, \tilde{B}_r^+ : I \times D^1 \rightarrow R^3$ and Seifert manifolds V_0 for $O_1 \cup \dots \cup O_\nu$ and V for K^1 such that*

- (0) $\tilde{B}_j^+ | \partial I \times D^1 = B_j^+ | \partial I \times D^1$, $j = 1, \dots, r$,
- (i) $\tilde{B}_1^+ \cup \dots \cup \tilde{B}_r^+$ and $B_1^+ \cup \dots \cup B_r^+$ are ambient isotopic in R^3 keeping $O_1 \cup \dots \cup O_\nu$ fixed,
- (ii) V_0 is obtained from C by adding (step by step) some 1-handles,
- (iii) $V = V_0 \cup \tilde{B}_1^+(I \times D^1) \cup \dots \cup \tilde{B}_r^+(I \times D^1)$.

Proof. We divide the proof into three steps. For brevity, we denote the image $B_j^+(I \times D^1)$ by B_j , $j = 1, \dots, r$.

Step I : We may assume that $\partial C_i \cap {}^\circ B_j = \emptyset$ and ${}^\circ C_i \cap B_j$ consists of

mutually disjoint simple arcs which cross B_j , $i = 1, \dots, v$; $j = 1, \dots, r$.

Now we suppose that B_j spans O_{j_1} and O_{j_2} , and we will give a direction on B_j from O_{j_1} to O_{j_2} .

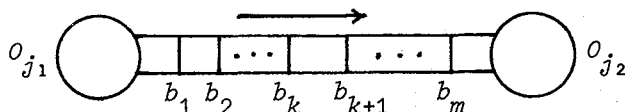


Figure 5.4

Let $b_1 \cup b_2 \cup \dots \cup b_m$ be the segments $B_j \cap C$ ordered according to the direction. We also give a sign $+1$ or -1 to each segment b_k according as B_j goes through C at b_k from its positive side or from its negative side; see Figure 5.5.

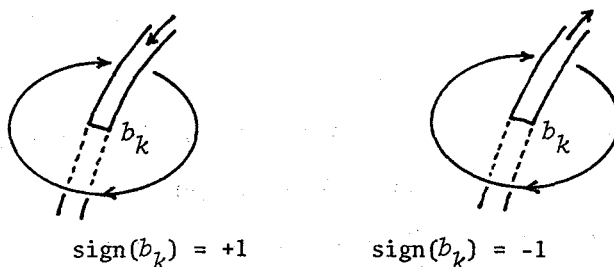


Figure 5.5

Step II : If there exist two segments b_k and b_{k+1} in $B_j \cap C$ with $\text{sign}(b_k) \times \text{sign}(b_{k+1}) = -1$, we add a 1-handle $h^1 : D^1 \times D^2 \rightarrow R^3$ to C along B_j . More precisely, let $E_1 = N(b_k; C)$ and $E_2 = N(b_{k+1}; C)$ be regular neighborhoods, and let $\tilde{b}_k$ be the part of B_j cut off by b_k and b_{k+1} . Then, h^1 is an embedding in R^3 with $h^1(\partial D^1 \times D^2) = E_1 \cup E_2$ and $h^1(D^1 \times D^2) \cap (B_1 \cup \dots \cup B_r) = h^1(D^1 \times D^2) \cap B_j = \tilde{b}_k$. Now we have an oriented 2-manifold $C_1 = C(h^1)$ with $\partial C_1 = O_1 \cup \dots \cup O_v$ and

$$C_1 \cap (B_1 \cup \dots \cup B_r) = C \cap (B_1 \cup \dots \cup B_r) - (b_k \cup b_{k+1}).$$

Step III : We introduce here the following two deformations of bands,

say $(+)$ -twist and $(-)$ -twist ; as illustrated in Figure 5.6. It should be noted that these deformations can be realized by an isotopic deformation of R^3 .

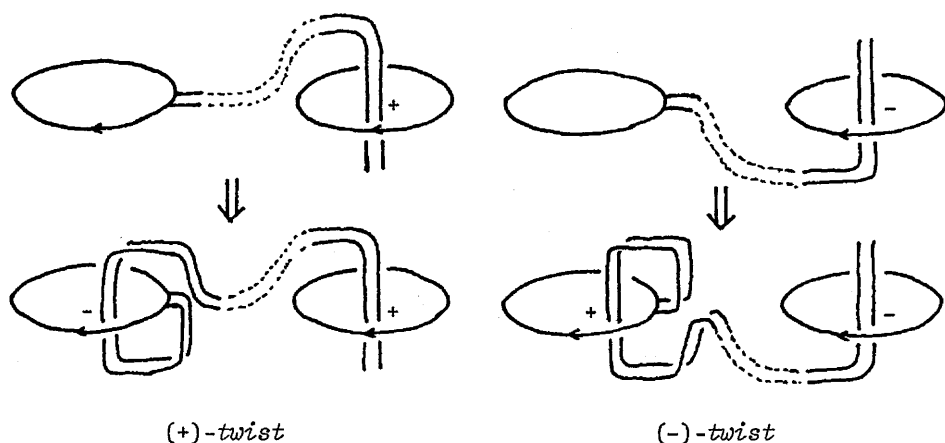


Figure 5.6

Carrying out one of the twists successively, if necessary, we assume that there exist adjacent segments like as Step II. So, we can repeat successively the process in Step II, and we have required Seifert manifolds V_0 for $O_1 \cup \dots \cup O_v$ and $V = V_0 \cup B_1 \cup \dots \cup B_r$ for K^1 . $\square$

By the same way, we have in general the following :

5.8. Lemma. Let $(K_1^1 \cup \dots \cup K_\mu^1 \subset R^3)$ be a 1-link obtained by a fusion (or fission) of a 1-link $k_1 \cup \dots \cup k_v$ along $B_1^+ \cup \dots \cup B_r^+$ (or $B_1^- \cup \dots \cup B_r^-$). Let C be an oriented 2-manifold in R^3 with $\partial C = k_1 \cup \dots \cup k_v$. Then, there exist embeddings $\tilde{B}_1^+, \dots, \tilde{B}_r^+ : I \times D^1 \rightarrow R^3$ (or $\tilde{B}_1^-, \dots, \tilde{B}_r^- : D^1 \times I \rightarrow R^3$) and an oriented 2-manifold V_0 (which may not be connected) and a Seifert manifold V for $K_1^1 \cup \dots \cup K_\mu^1$ such that

$$(0) \quad \tilde{B}_j^+ | \partial I \times D^1 = B_j^+ | \partial I \times D^1 \quad (\text{or} \quad \tilde{B}_j^- | \partial D^1 \times I = B_j^- | \partial D^1 \times I),$$

(i) $\tilde{B}_1^+ \cup \dots \cup \tilde{B}_r^+$ and $B_1^+ \cup \dots \cup B_r^+$ (or $\tilde{B}_1^- \cup \dots \cup \tilde{B}_r^-$ and $B_1^- \cup \dots \cup B_r^-$) are ambient isotopic in R^3 keeping $k_1 \cup \dots \cup k_v$ fixed,

(ii) V_0 is obtained from C by adding (step by step) some 1-handles, and so $\partial V_0 = k_1 \cup \dots \cup k_v$,

(iii) $V = V_0 \cup \tilde{B}_1^+(I \times D^1) \cup \dots \cup \tilde{B}_r^+(I \times D^1)$
(or $V = V_0 \cup \tilde{B}_1^-(D^1 \times I) \cup \dots \cup \tilde{B}_r^-(D^1 \times I)$).

5.9. *Proof of Theorem 5.1 for 2-knots* : The proof of 5.1 is similar to that of Lemma 5.5. Let $(K^2 \subset R^4)$ be a standard representative, and let $C_3 = C_1 \cup \dots \cup C_v$ be the maximum-disks $K^2 \cap R_3^3$. We pull down C_3 continuously with respect to x_4 -axis as far as R_2^3 , and we have a union of oriented 2-disks C_2 in R_2^3 with $\partial C_2 = K^2 \cap R_2^3$.

Let $B_1 \cup \dots \cup B_r$, $r = v - 1$, be the set of saddle-bands of K^2 in R_1^3 and $B'_1 \cup \dots \cup B'_s$ the set of saddle-bands of K^2 in R_{-1}^3 . Let B be $p_2(B_1 \cup \dots \cup B_r)$. (Here, we consider $K^2 \cap H_{(1,3)}$, $K^2 \cap H_{(-1,1)}$ and $K^2 \cap H_{(-3,-1)}$ as $(K^2 \cap R_2^3) \times (1,3)$, $(K^2 \cap R_0^3) \times (-1,1)$ and $(K^2 \cap R_{-2}^3) \times (-3,-1)$, respectively.) By Lemma 5.7, we can construct an oriented 2-manifold F with $\partial F = K^2 \cap R_2^3$ by adding, step by step, some 1-handles on C_2 , so that $F \cup B$ is a Seifert manifold for a 1-knot $(K^2 \cap R_2^3) + \partial B \cong K^2 \cap R_{1-\epsilon}^3$.

With decreasing x_4 , $2 > x_4 > 1$, we perform, step by step, hyperbolic transformations of type I on C_2 corresponding to the 1-handles on F , so that in $R_{1+\epsilon}^3$ we have an oriented 2-manifold $C_{1+\epsilon}$ with $\partial C_{1+\epsilon} = K^2 \cap R_{1+\epsilon}^3$ and $C_{1+\epsilon} = p_{1+\epsilon}(F)$. So we have a Seifert manifold $C_{1-\epsilon}$ for the 1-knot $K^2 \cap R_{1-\epsilon}^3$ with $C_{1-\epsilon} = p_{1-\epsilon}(C_{1+\epsilon}) \cup p_{1-\epsilon}(B_1 \cup \dots \cup B_r)$.

Let $B' = p_{1-\epsilon}(B'_1 \cup \dots \cup B'_s)$. By Lemma 5.8, we can construct a Seifert manifold F' for the 1-knot $K^2 \cap R_{1-\epsilon}^3$ by adding, step by step, some 1-

handles on $C_{1-\varepsilon}$, so that $F' \cup B'$ is a Seifert manifold for the 1-link $(K^2 \cap R_{1-\varepsilon}^3) + \partial B' \cong K^2 \cap R_{-1-\varepsilon}^3$. With decreasing x_4 , $-1-\varepsilon > x_4 > -1+\varepsilon$, we perform, step by step, hyperbolic transformations of type I on $C_{1-\varepsilon}$ corresponding to the 1-handles on F' , so that in $R_{-1+\varepsilon}^3$ we have an oriented 2-manifold $C_{-1+\varepsilon}$ with $C_{-1+\varepsilon} = p_{-1+\varepsilon}(F')$ and $\partial C_{-1+\varepsilon} = K^2 \cap R_{-1+\varepsilon}^3$. Now we have a Seifert manifold $C_{-1-\varepsilon}$ in $R_{-1-\varepsilon}^3$ for the 1-link $K^2 \cap R_{-1-\varepsilon}^3$ with $C_{-1-\varepsilon} = p_{-1-\varepsilon}(C_{-1+\varepsilon}) \cup p_{-1-\varepsilon}(B'_1 \cup \dots \cup B'_g)$.

Let $C' = C'_1 \cup \dots \cup C'_\lambda$, $\lambda = s+1$, be the union of the minimum-disks $K^2 \cap R_{-3}^3$, and let $C'' = p_{-1-\varepsilon}(C')$. Since the 1-link $K^2 \cap R_{-1-\varepsilon}^3$ is trivial and $C_{-1-\varepsilon}$ is oriented, there exists an isotopic deformation $\{\eta_t\}$ of $R_{-1-\varepsilon}^3$ such that $\eta_t|_{\partial C_{-1-\varepsilon}} = \text{identity}$ and $\eta_1(^{\circ}C_{-1-\varepsilon}) \cap ^{\circ}C''$ consists of simple loops. Using loops in $\eta_1(^{\circ}C_{-1-\varepsilon}) \cap ^{\circ}C''$ which are innermost on C'' , we perform hyperbolic transformations of type II on $C_{-1-\varepsilon}$ with decreasing x_4 , $-1-\varepsilon \geq x_4 \geq -2$, so that we have in R_{-2}^3 an oriented 2-manifold C_{-2} with $\partial C_{-2} = K^2 \cap R_{-2}^3$ and $C_{-2} \cap p_{-2}(C') = \partial C_{-2} = p_{-2}(\partial C')$. It will be noticed that in general C_{-2} consists of some closed oriented 2-manifolds and some oriented 2-manifolds with boundary.

Since $C_{-2} \cup p_{-2}(C')$ becomes a closed 2-manifold and C' consists of 2-disks, by Fox-Homma's theorem (Fox [7], Homma [1]) we have a 2-disk e in R_{-2}^3 such that $e \cap (C_{-2} \cup p_{-2}(C')) = \partial e \cap C_{-2} = \partial e$ and $\partial e \neq 1$ on C_{-2} , provided that the total genus of C_{-2} is greater than zero. With decreasing x_4 , $-2 > x_4 \geq -5/2$, we perform, step by step, hyperbolic transformations of type II on C_{-2} along such the 2-disks, so that in $R_{-5/2}^3$ we have a union $C_{-5/2}$ of λ 2-disks and some 2-spheres with $\partial C_{-5/2} = K^2 \cap R_{-5/2}^3$ and $C_{-5/2} \cap p_{-5/2}(C') = \partial C_{-5/2} = p_{-5/2}(\partial C')$. It should be noted that $(K^2 \cap H_{(-\infty, -5/2]}) \cup C_{-5/2}$ is a 2-link which has no saddle-bands.

By Lemma 5.5 and the same argument as that used in the proof of Lemma 5.5, we complete the construction of a required Seifert manifold V^3 for K^2 . $\square$

5.10. Exercise. (Gluck [2; 3]) Prove that : for any locally flat closed oriented 2-manifold M^2 in R^4 , there exists an oriented 3-manifold V^3 in R^4 with $\partial V^3 = M^2$.

5.11. Remark. In Theorem 5.1 and Exercise 5.10, the locally flatness is essential. Refer, for example, to Noguchi [No].

5.12. Remark. In the above proof of Theorem 5.1 for $n = 2$, it is easy to see that $U_1 \equiv V^3 \cap H_{[-1+\epsilon, \infty)}$ and $U_2 \equiv V^3 \cap H_{(-\infty, -1+\epsilon]}$ are 3-dimensional handlebodies of genus g , where g is the number of the hyperbolic transformations of type I in $2 > x_4 > -1 + \epsilon$. That is, (U_1, U_2) is a Heegaard splitting (in certain sense) of V^3 with $U_1 \cap U_2 = V^3 \cap R_{-1+\epsilon}^3$.

5.13. Theorem. (Yanagawa [2, §2]) Let $(K^2 \subset R^4)$ be a ribbon 2-knot. Then there exists a Seifert manifold W^3 for $(K^2 \subset R^4)$ such that W^3 is homeomorphic to D^3 or a connected-sum of some copies of $S^2 \times S^1$ with an open 3-disk missing ; $(S^2 \times S^1) \# \dots \# (S^2 \times S^1) - \circ \Delta^3$. Moreover, if $W^3 \neq D^3$, then W^3 has a trivial system of 2-spheres which will be defined as below. (Recall Definition 2.12 for ribbon 2-knots.)

5.14. Definition. A collection $\{S_1^2, S_2^2, \dots, S_{2r-1}^2, S_{2r}^2\}$ of mutually disjoint 2-spheres in a 3-manifold $W^3 \cong (S^2 \times S^1) \# \dots \# (S^2 \times S^1) - \circ \Delta^3$ in R^4 is called a trivial system of 2-spheres in W^3 iff it satisfies the following :

- (1) $S_1^2 \cup S_2^2 \cup \dots \cup S_{2r-1}^2 \cup S_{2r}^2$ is a trivial 2-link in R^4 ,
- (2) $S_i^2 \cup S_{r+i}^2$ bounds a spherical-shell $N_i \cong S^2 \times I$ in W^3 with

$N_i \cap N_j = \emptyset$ for $i \neq j$, $1 \leq i, j \leq r$,

(3) $Cl(W^3 - N_1 - \dots - N_r)$ is homeomorphic to the 3-sphere S^3 with $2r + 1$ open 3-disks missing.

Proof of Theorem 5.13. By Theorem 2.13, we may assume that $(K^2 \subset R^4)$ is a simple symmetric representative. In the proof of Theorem 5.1 (5.9), we need not add 1-handles which depend on $B'_1, \dots, B'_s$. That is, we pull down the cross-sectional 2-manifold $C_{1-\epsilon}$ continuously as far as $R^3_{-1+\epsilon}$. By the symmetricity, we can take $B'_1 \cup \dots \cup B'_s$ on C_{-1} in R^3_{-1} , and we perform the original hyperbolic transformations on K^2 along $B'_1 \cup \dots \cup B'_s$ and in $R^3_{-1-\epsilon}$ we have an oriented 2-manifold $C_{-1-\epsilon}$ such that $\partial C_{-1-\epsilon} = K^2 \cap R^3_{-1-\epsilon}$ and $C_{-1-\epsilon} = p_{-1-\epsilon}(C_{1+\epsilon})$. Now we can perform hyperbolic transformations of type II on $C_{-1-\epsilon}$ which cut the 1-handles added in $H_{(1,2)}$. That is, we can construct a Seifert manifold W^3 for $(K^2 \subset R^4)$ such that $W^3 \cap R_u = p_u(W^3 \cap R_{-u})$.

From the traces of the hyperbolic transformation in $R^3_{u_0}$, $2 > u_0 > 1$, and the corresponding hyperbolic transformation in $R^3_{-u_0}$, we obtain a 2-sphere S^2 in W^3 . Let m be the number of the hyperbolic transformations in $H_{(1,2)}$, and let $\{S^2_1, S^2_2, \dots, S^2_m\}$ be a collection of such the 2-spheres in W^3 . It is easily checked that $W^3 - (S^2_1 \cup S^2_2 \cup \dots \cup S^2_m)$ is homeomorphic to a 3-disk with $2m$ holes. This shows that W^3 is homeomorphic to a connected-sum of m copies of $S^2 \times S^1$ with an open 2-disk missing.

By Lemma 5.5, the 2-link $(S^2_1 \cup S^2_2 \cup \dots \cup S^2_m \subset R^4)$ is trivial. Let N_i be a regular neighborhood $N(S^2_i; W^3)$ of S^2_i in W^3 ; $N_i \cong S^2 \times I$; and let $S^2_{i1} \cup S^2_{i2} = \partial N_i$, $i = 1, \dots, m$. Then, by Lemma 5.5, the 2-link $(S^2_{11} \cup S^2_{12} \cup \dots \cup S^2_{m1} \cup S^2_{m2} \subset R^4)$ is also trivial, completing the proof. $\square$

5.15. Theorem. (Yanagawa [2]) Suppose that a 2-knot $(K^2 \subset R^4)$ has a Seifert manifold W^3 homeomorphic to a connected-sum of r copies of $S^2 \times S^1$ with an open 3-disk missing, and that W^3 has a trivial system $\{S_1^2, \dots, S_{2r}^2\}$ of 2-spheres. Then, $(K^2 \subset R^4)$ is a ribbon 2-knot.

Proof. Let $N_1, \dots, N_r$ be spherical-shells in Definition 5.14-(2), and let $Q = \text{Cl}(W^3 - N_1 - \dots - N_r)$. We take a point x in $^\circ Q$. Then, by 5.14-(3), we can choose simple arcs $\alpha_1, \dots, \alpha_{2r}$ in Q which satisfy the following :

- (a) $\alpha_j \cap \alpha_k = \partial \alpha_j \cap \partial \alpha_k = x, \quad j \neq k,$
- (b) $\alpha_j \cap (S_1^2 \cup \dots \cup S_{2r}^2) = \partial \alpha_j \cap S_j^2 \neq \emptyset, \text{ and } \alpha_j \cap K^2 = \emptyset,$
- (c) $\bar{Q} \equiv \text{Cl}(Q - N(\alpha_1 \cup \dots \cup \alpha_{2r}; Q)) = S^2 \times I$, where $N(\alpha_1 \cup \dots \cup \alpha_{2r}; Q)$

is a regular neighborhood of $\alpha_1 \cup \dots \cup \alpha_{2r}$ in Q .

Then, by (b), one of the boundary components of $\bar{Q}$ is K^2 , let $\partial \bar{Q} = K^2 \cup K'$. By (c), K' is equivalent to K^2 , and it is clear that K' is obtained by a fusion of the trivial 2-link $S_1^2 \cup \dots \cup S_{2r}^2$. $\square$

5.16. Exercise. Let $(K^1 \subset R^3)$ be a 1-knot with a Seifert manifold V^2 of genus m , and let $(K^2 \subset R^4)$ be the spun of $(K^1 \subset R^3)$. Prove that : there is a Seifert manifold V^3 for K^2 such that V^3 is a connected sum of m lens spaces (including $S^2 \times S^1$) with an open 3-disk missing.

5.17. Definition. Throughout the note, we use the following notation.

Let V be a Seifert manifold for an n -knot $(K^n \subset S^{n+2})$. Let Y be the $(n+2)$ -manifold obtained from S^{n+2} by cutting along V ; ∂Y consists of two copies of V , say V_+ and V_- , with $V_+ \cap V_- = \partial V_+ \cap \partial V_- = K^n$. Let $\nu_\pm : V \rightarrow V_\pm \subset Y$ be the compositions. We also denote by $\nu_\pm$ the induced homomorphisms $\nu_{\pm\#} : \pi_q(V) \rightarrow \pi_q(Y)$ and $\nu_{\pm*} : H_q(V; G) \rightarrow H_q(Y; G)$.

If both maps $v_{\pm} : \pi_1(V) \rightarrow \pi_1(Y)$ are monomorphisms, we say that V is a *minimal* Seifert manifold.

5.18. Proposition. (Gutierrez [2], Neuwirth [N_1 , Chapter IV]) *For an n -knot $(K^n \subset S^{n+2})$ with $n \neq 2$, there exists a minimal Seifert manifold.*

5.19. Problem. *Does there exist a minimal Seifert manifold V^3 for any 2-knot $(K^2 \subset S^4)$?*

This is very difficult and important ; it implies, among other things, the unknotting theorem (see 10.2 and 10.3 in Section 10) modulo Poincaré conjecture.

5.20. Theorem. (Zeeman [6]) *For every v -twist-spun (including 0-twist-spun = spun) 2-knot, there exists a minimal Seifert manifold.*

For spun 2-knots, this is a good exercise. For v -twist-spun 2-knots, a more sharp statement will be given later.

Gutierrez [4] settled for a somewhat weaker property :

5.21. Definition. Let G be a group and $B \subset G$ a subset. We write N_G^B for the normal closure of B in G . Let $f : H \rightarrow G$ be a group homomorphism, and let x and x' be any finite subsets of H . We say that f has no elbow iff $f(x) \subset N_G(f(x'))$ implies $x \subset N_H(x') \cdot \text{Ker}(f)$.

Let V be a Seifert manifold for a 2-knot $(K^2 \subset S^4)$. We say that V is a *minimal* Seifert manifold in the weak sense iff at least one of $v_{\pm} : \pi_1(V) \rightarrow \pi_1(Y)$ is a monomorphism without elbow.

5.22. Theorem. (Gutierrez [4]) *Every 2-knot has a minimal Seifert manifold in the weak sense.*

6. Alexander Invariants

In this section, we will give another description of elementary ideals expressed as invariant of homology of the universal abelian covering of the complement (or exterior) of a knot. The basic reference for the following discussion is Kervaire [2, Chapter II], Levine [5]. For the classical knots, we also refer to Seifert [1] and Neuwirth [N_1 , Chapter IV].

6.1. Let $(K^n \subset S^{n+2})$ be an n -knot and $X = S^{n+2} - K^n$. The *universal abelian covering space* $\tilde{X}$ of X is the covering space associated with the commutator subgroup of $\pi_1(X)$. By Alexander duality (see 3.2), the group of covering translations of $\tilde{X}$ is an infinite cyclic group, say J , generated by t which is an oriented meridian of K^n . Thus, $H_q(\tilde{X}; \mathbb{Z})$ has a well-defined action by J .

Let Λ be the integral group ring of J (viewed as $\mathbb{Z}[t, t^{-1}]$), and $\Gamma = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ the rational group ring of J . Note that Λ is a Noetherian unique factorization domain. Since the rational polynomial ring $\mathbb{Q}[t]$ is a principal ideal domain, Γ is a principal ideal domain (PID) by a similar argument to that of Crowell-Fox [C-F].

For all q , the integral chain group $C_q(\tilde{X}; \mathbb{Z})$ are finitely generated Λ -modules because X is homotopy equivalent to the exterior $E = S^{n+2} - \circ N(K^n; S^{n+2})$. Since Λ is Noetherian, $H_q(\tilde{X}; \mathbb{Z})$ is a finitely generated Λ -module, and also $H_q(\tilde{X}; \mathbb{Q}) \cong H_q(\tilde{X}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a finitely generated torsion Γ -module.

We use here the notions of presentation of a module and elementary ideals of a finitely generated module. We refer the reader, for example, to Zassenhaus [Za, p.117] for details.

6.2. Let R be a ring and A an abelian group which is finitely generated as a R -module. Then there exist free R -modules F_1 and F_2 on the entries $\{x_1, \dots, x_\alpha\}$ and $\{r_1, \dots, r_\beta\}$, respectively, and an exact sequence of R -modules

$$F_2 \xrightarrow{\tau} F_1 \longrightarrow A \longrightarrow 0$$

such that $\tau(r_i) = \sum_{j=1}^{\alpha} m_{ij} x_j$, $m_{ij} \in R$.

We refer to the $\beta \times \alpha$ matrix $M = (m_{ij})$ with entries R as a *presentation matrix* of A .

Let M be an $\beta \times \alpha$ matrix with entries in R , and d an integer. The d -th *elementary ideal* E_d of M is the ideal in R generated by the determinants of $(\alpha-d+1) \times (\alpha-d+1)$ minors of M , with the conventions

$$(i) \quad \alpha - d + 1 < 1 \quad \text{then} \quad E_d = R,$$

$$(ii) \quad \alpha - d + 1 > \beta \quad \text{then} \quad E_d = 0.$$

If M is a presentation matrix of A , it is known that the elementary ideals of M are invariant of R -isomorphism class of A .

6.3. From 6.1, we have a presentation matrix $M_q = (m_{ij}(t))$, $m_{ij}(t) \in \Lambda$, of $H_q(\tilde{X}; Z)$. Since $-\otimes_Z Q$ is a right exact functor, we can take $M'_q = (m_{ij}(t) \otimes 1)$ to be a presentation matrix of $H_q(\tilde{X}; Q)$ as a Γ -module.

The d -th *Alexander polynomial* $\Delta_d^q(t) \in \Lambda$ of $\tilde{X}$ (or of $(K^n \subset S^{n+2})$) in dimension q is the generator of the smallest principal ideal containing the d -th elementary ideal E_d^q of M_q . Of course, $\Delta_d^q(t)$ is determined up to unit in Λ .

Similarly, the d -th *rational Alexander polynomial* $\Delta_d'^q(t) \in \Gamma$ of $\tilde{X}$ (or of $(K^n \subset S^{n+2})$) in dimension q is the generator of the d -th elementary ideal $E_d'^q$ of M'_q . Since Γ is PID, $E_d'^q$ is a principal ideal.

6.4. Two elements λ, μ of a ring R are *associate* in R if there exists a unit $\alpha \in R$ such that $\lambda = \alpha \cdot \mu$, and we write $\lambda \doteq \mu$ (in R). If λ is a multiple of μ by an element of R , we write $\mu \mid \lambda$ (in R).

An element $\lambda(t) \in \Lambda$ is *primitive* if all its coefficients are relatively prime.

6.5. Proposition. (Levine [5, (1.4) & (1.5)]) (i) Let $\lambda, \mu \in \Lambda$ and λ primitive. Then, $\lambda \mid \mu$ in Λ if and only if $\lambda \mid \mu$ in Γ .

(ii) Given $\mu(t) \in \Lambda$, suppose $\mu(t) \doteq \mu(t^{-1})$ in Λ and $\mu(1)$ is odd. Then, the associate class in Λ containing $\mu(t)$ contains a unique element $\lambda(t)$ satisfying $\lambda(t) = \lambda(t^{-1})$ and $\lambda(1) > 0$. (We call such the element $\lambda(t)$ *normal*.)

Hence, every associate class in Γ contains a primitive element of Λ , unique up to associate class in Λ . Since the generator of E_d^q in Γ is defined up to associate class in Γ , we can take $\Delta_d'^q(t)$ to be primitive element of Λ . There exists a simple relation between Δ_d^q and $\Delta_d'^q$.

6.6. Proposition. (Shinohara-Summers [1, Lemma 2.2]) $\Delta_d'^q(t) = r \cdot \Delta_d^q(t)$, where $0 \neq r \in \mathbb{Q}$ is the unique rational number such that $r \cdot \Delta_d^q(t)$ is primitive. (If $\Delta_d^q(t) = 0$, we take $r = 1$.) So, if $\Delta_d^q(t)$ is primitive (as in the case when $\Delta_d^q(1) = \pm 1$), $\Delta_d'^q(t) = \Delta_d^q(t)$.

6.7. Now we will show that for a 2-knot the d -th elementary ideal E_d and so the d -th Alexander polynomial $\Delta_d(t)$ given in Section 3 coincide with the above E_d^1 and $\Delta_d^1(t)$ in dimension 1.

By virtue of Proposition 3.10, we choose a pre-abelian presentation (cf. Magnus-Karrass-Solitor [M-K-S, p.140]) :

$$\langle x_1, x_2, \dots, x_\alpha \mid r_1, r_2, \dots, r_\beta \rangle^{\psi\phi}$$

of $\pi_1(X)$ so that $\psi\phi(x_1) = t$ and $\psi\phi(x_j) = 1, j = 2, \dots, \alpha$. Let $F_1 = F[x_1^*, x_2^*, \dots, x_\alpha^*]$ and $F_2 = F[r_1^*, r_2^*, \dots, r_\beta^*]$ be free Λ -modules on the entries $\{x_1^*, x_2^*, \dots, x_\alpha^*\}$ and $\{r_1^*, r_2^*, \dots, r_\beta^*\}$, respectively. Define

$$F_2 \xrightarrow{\tau_2} F_1 \xrightarrow{\tau_1} \Lambda$$

as follows :

$$\tau_2(r_i^*) = \sum_{j=1}^{\alpha} \psi\phi\left(\frac{\partial r_i}{\partial x_j}\right) x_j^*, \quad \tau_1(x_j^*) = \psi\phi(x_j) - 1.$$

It is easy to see that $\tau_1\tau_2 : F_2 \rightarrow \Lambda$ is a zero map.

Since $\tau_1(x_1^*) = \psi\phi(x_1) - 1 = t - 1$ and

$$\tau_1(x_j^*) = \psi\phi(x_j) - 1 = 1 - 1 = 0 \quad \text{for } j = 2, \dots, \alpha,$$

it holds that

$$\psi\phi\left(\frac{\partial r_i}{\partial x_1}\right) = 0 \quad \text{for } i = 1, \dots, \beta, \text{ and}$$

$\text{Ker } \tau_1 = F[x_2^*, \dots, x_\alpha^*]$; free Λ -module on the entries $\{x_2^*, \dots, x_\alpha^*\}$.

Let $\tau_2' : F_2 \rightarrow \text{Ker } \tau_1$ be the map defined by τ_2 . By Crowell [1, p.39],

$$H_1(\tilde{X}; Z) \cong_{\Lambda} \text{Ker } \tau_1 / \text{Image } \tau_2'.$$

So the sequence

$$F_2 \xrightarrow{\tau_2'} F[x_2^*, \dots, x_\alpha^*] \xrightarrow{\tau_1'} H_1(\tilde{X}; Z) \longrightarrow 0$$

is exact as Λ -modules, where τ_1' is the restriction of τ_1 . Hence

$$M = \left[\psi\phi\left(\frac{\partial r_i}{\partial x_j}\right) \right], \quad 1 \leq i \leq \beta, \quad 2 \leq j \leq \alpha,$$

is a presentation matrix of $H_1(\tilde{X}; Z)$. Since $\psi\phi\left(\frac{\partial r_i}{\partial x_j}\right) = 0, 1 \leq i \leq \beta,$
 $E_d(A)$ coincides with E_d^1 , where $A = \left[\psi\phi\left(\frac{\partial r_i}{\partial x_j}\right) \right], 1 \leq i \leq \beta, 1 \leq j \leq \alpha$.

6.8. There exists a better way of thinking of the rational Alexander

polynomials of $(K^n \subset S^{n+2})$. Since $H_q(\tilde{X}; Q)$ is a finitely generated torsion Γ -module and Γ is PID, by the structure theorem $H_q(\tilde{X}; Q)$ is decomposed into cyclic Γ -modules :

$$H_q(\tilde{X}; Q) = \Gamma/(\lambda_1^q(t)) \oplus \cdots \oplus \Gamma/(\lambda_m^q(t)),$$

$$\lambda_i^q(t) \in \Lambda, \quad \lambda_{i+1}^q(t) \mid \lambda_i^q(t) \text{ in } \Lambda \text{ for all } i.$$

That is, $H_q(\tilde{X}; Q)$ has a diagonal presentation matrix

$$M'_q(t) = \begin{pmatrix} \lambda_1^q(t) & & & & \\ & \lambda_2^q(t) & & & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & \lambda_m^q(t) \end{pmatrix}$$

over Γ . Levine [5] called $\{\lambda_i^q(t)\}$ the *Alexander invariants* of an n -knot. The d -th rational Alexander polynomials are given by the formula :

$$\Delta_d^q(t) = \begin{cases} \prod_{i=d}^m \lambda_i^q(t), & 1 \leq d \leq m, \quad 1 \leq q \leq n+1, \\ 1, & m+1 < d, \quad 1 \leq q \leq n+1. \end{cases}$$

6.9. Theorem. (Levine [5, Theorem 1]) Let $\{\lambda_i^q(t)\}$, $0 < q < n+1$, $0 < i \leq m$, be the family of Alexander invariants of an n -knot, $n \geq 1$.

Then, the following properties must be satisfied :

- (a) $\lambda_{i+1}^q(t) \mid \lambda_i^q(t)$ in Λ , (a') $\Delta_{d+1}^q(t) \mid \Delta_d^q(t)$ in Λ ,
 (b) $\lambda_i^q(1) = \pm 1$, (b') $\Delta_d^q(1) = \pm 1$,
 (c) $\lambda_i^q(t) \doteq \lambda_i^{n+1-q}(t^{-1})$ in Λ , (c') $\Delta_d^q(t) \doteq \Delta_d^{n+1-q}(t^{-1})$ in Λ .

Sketch Proof. By 6.8, (a) is obvious. Following Levine [5], we will give the outline of the proof of 6.9-(b) and (c). We refer the reader to Shinohara-Summers [1].

(6.10) *Step I* : Let V be a Seifert manifold for an n -knot $(K^n \subset S^{n+2})$.

In the notation of 5.17, the universal abelian covering space $\tilde{X}$ of $X = S^{n+2} - K^n$ may be constructed from Y in the following way :

Let (Y_i, V_-^i, V_+^i) , $-\infty < i < \infty$, be an infinite number of copies of $(Y-K^n, V_-K^n, V_+K^n)$. Then, $\tilde{X}$ is obtained as a quotient space of the disjoint union of the Y_i by identifying V_+^i and V_-^{i+1} for every i . The generator t of the covering translation group maps Y_i onto Y_{i+1} in the obvious manner.

Let $Y^0 = \bigcup_i Y_{2i}$, $Y^1 = \bigcup_i Y_{2i+1}$; then $(\tilde{X}, Y^0, Y^1)$ is a proper triad, and $Y^0 \cap Y^1 = \bigcup_i V_+^i$. By the Mayer-Vietoris theorem, we have :

$$\rightarrow H_q(\bigcup_i V_+^i; Q) \rightarrow H_q(Y^0; Q) \oplus H_q(Y^1; Q) \rightarrow H_q(\tilde{X}; Q) \rightarrow H_{q-1}(\bigcup_i V_+^i; Q) \rightarrow .$$

Since $\bigcup_i V_+^i$ is invariant under the covering translation of $\tilde{X}$,

$$\begin{aligned} H_q(\bigcup_i V_+^i; Q) &\cong \sum_i H_q(V_+^i; Q) \cong_{\Lambda} H_q(V; Q) \otimes_Q \Gamma, \\ H_q(V_+^i; Q) &\mapsto H_q(V; Q) \otimes_Q t^i. \end{aligned}$$

Also we have :

$$\begin{aligned} H_q(Y^0; Q) \oplus H_q(Y^1; Q) &\cong \sum_i H_q(Y_i; Q) \cong_{\Lambda} H_q(Y; Q) \otimes_Q \Gamma, \\ H_q(Y_i; Q) &\mapsto H_q(Y; Q) \otimes_Q t^i. \end{aligned}$$

So, we obtain an exact sequence of Γ -modules :

$$\longrightarrow H_q(V; Q) \otimes_Q \Gamma \xrightarrow{\tau} H_q(Y; Q) \otimes_Q \Gamma \xrightarrow{e} H_q(\tilde{X}; Q) \longrightarrow \quad (1)$$

Here τ can be described as follows. Let t be the element of Γ corresponding to the covering translation $Y_i \rightarrow Y_{i+1}$. Then, for $\alpha \in H_q(V; Q)$,

$$\begin{aligned} \tau(\alpha \otimes 1) &= v_-(\alpha) \otimes t - v_+(\alpha) \otimes 1 \\ &= t(v_-(\alpha) \otimes 1) - v_+(\alpha) \otimes 1. \end{aligned}$$

We will show below that τ is a monomorphism for all q . By exactness, this implies e is onto for every q . Therefore, we have a presentation matrix of $H_q(\tilde{X}; Q)$ from (1) as in 6.2.

(6.11) *Step II* : For any space T , let $B_q(T)$ denote the image of the natural inclusion $H_q(T; Z) \rightarrow H_q(T; Q)$. $B_q(T)$ is free abelian, and $H_q(T; Q) = B_q(T) \otimes_Z Q$. For brevity, let $p = n + 1 - q$. Then, there is a homomorphism $\mathcal{L}k : B_q(V) \otimes_Z B_p(S^{n+2} - V) \rightarrow Z$, which is defined by linking numbers (cf. Seifert-Threlfall [S-T, p.277]). By Alexander duality, given basis $a_1, \dots, a_\alpha$ of $B_q(V)$, there exists a dual basis $\hat{a}_1, \dots, \hat{a}_\alpha$ of $B_p(S^{n+2} - V)$ such that $\mathcal{L}k(a_i \otimes \hat{a}_j) = \delta_{ij}$. By $\mathcal{L}k$, we also denote the natural extension

$$\mathcal{L}k : H_q(V; Q) \otimes_Q H_p(S^{n+2} - V; Q) \rightarrow Q.$$

Given $a^q \in H_q(V; Q)$ and $a^p \in H_p(V; Q)$, we have :

$$\mathcal{L}k(a^q \otimes v_-(a^p)) = (-1)^{pq+1} \mathcal{L}k(a^p \otimes v_+(a^q)) \quad (2)$$

using the isomorphism $H_q(S^{n+2} - V; Q) \cong H_q(Y; Q)$ induced by inclusion.

Also we have :

$$\mathcal{L}k(a^q \otimes v_+(a^p)) - \mathcal{L}k(a^q \otimes v_-(a^p)) = a^q \cdot a^p \quad (3)$$

where $a^q \cdot a^p$ is the homology intersection number in V .

(6.12) *Step III* : For every q , choose a basis $\{a_j^q\}$ of $B_q(V)$ and a dual basis $\{\hat{a}_j^p\}$ of $B_p(S^{n+2} - V) \cong B_p(Y)$. Then $\{a_j^q \otimes 1\}$ and $\{\hat{a}_j^p \otimes 1\}$ form bases of the free Γ -modules $H_q(V; Q) \otimes_Q \Gamma$ and $H_p(Y; Q) \otimes_Q \Gamma$. Set

$$v_-(a_j^q) = \sum_i \lambda_{ji}^q \hat{a}_i^q, \quad v_+(a_j^q) = \sum_i \mu_{ji}^q \hat{a}_i^q.$$

Note that λ_{ji}^q and μ_{ji}^q are integers. Thus,

$$\tau(a_j^q \otimes 1) = \sum_i (t\lambda_{ji}^q - \mu_{ji}^q)(\hat{a}_i^q \otimes 1),$$

and we can define a presentation matrix $P_q(t)$ of $H_q(\tilde{X}; Q)$ with entries in Λ by

$$P_q(t) = (t\lambda_{ji}^q - \mu_{ji}^q).$$

It will be noticed that $P_q(t)$ is a square matrix since

$$H_q(S^{n+2} - V; Q) \cong H_p(V; Q) \quad \text{by Alexander duality, and}$$

$$H_p(V; Q) \cong H_q(V; Q) \quad \text{by Poincaré duality}$$

for $0 < q < n + 1$.

(6.13) *Step IV* : We translate (2) and (3) in *Step II* into information about $P_q(t)$. We have :

$$\mathcal{L}k(a_j^q \otimes v_-(a_k^p)) = \mathcal{L}k(a_j^q \otimes (\sum_i \lambda_{ki}^p \hat{a}_i^p)) = \sum_i \lambda_{ki}^p \cdot \mathcal{L}k(a_j^q \otimes \hat{a}_i^p) = \lambda_{kj}^p \quad (4)$$

$$\mathcal{L}k(a_k^p \otimes v_+(a_j^q)) = \mathcal{L}k(a_k^p \otimes (\sum_i \mu_{ji}^q \hat{a}_i^q)) = \sum_i \mu_{ji}^q \cdot \mathcal{L}k(a_k^p \otimes \hat{a}_i^q) = \mu_{jk}^q \quad (5)$$

By (2), we have $\lambda_{kj}^p = (-1)^{pq+1} \mu_{jk}^q$. This implies

$$(-1)^{pq} P_q(t) = t [P_p(t^{-1})]', \quad (6)$$

where ' indicates transpose.

As a consequence of (3), (4) and (5), we have :

$$\mu_{ji}^q - \lambda_{ji}^q = \hat{a}_i^p \cdot a_j^q \quad (7)$$

Thus $P_q(1)$ is, up to sign, the matrix of intersection pairing

$$B_q(V) \otimes_{\mathbb{Z}} B_p(V) \rightarrow \mathbb{Z}.$$

Since $\partial V = K^n$ is an n -sphere, $P_q(1)$ is unimodular, i.e.

$$\det P_q(1) = \pm 1 \quad (8)$$

As a consequence of (8), $P_q(t)$ is non-singular, and so the homomor-

phism τ in Step I-(1) is injective. This implies e in (1) is surjective, and so $P_q(t)$ is, indeed, a presentation matrix of $H_q(\tilde{X}; \mathbb{Q})$.

(6.14) Step V : $\Delta_d'^q$ can be chosen as a greatest common divisor of the determinants of $(\alpha-d+1) \times (\alpha-d+1)$ minors of $P_q(t)$; so $\Delta_d'^q$ is an element of Λ . By (8), $\Delta_1'^q(1) = \det P_q(1) = \pm 1$, and therefore

$$\Delta_d'^q(1) = \pm 1$$

from 6.8. Furthermore, by Step IV-(6), we conclude

$$\Delta_d'^q(t) \doteq \Delta_d'^p(t^{-1}) = \Delta_d'^{n+1-q}(t^{-1}) \quad \text{in } \Lambda.$$

From 6.8, we complete the proof of (b) and (c). $\square$

6.15. We have discussed only the Alexander polynomials of an n -knot. We will consider the *reduced* Alexander polynomials of an n -link $(K_1^n \cup \dots \cup K_\mu^n \subset S^{n+2}) = (L_\mu^n \subset S^{n+2})$ with $\mu \geq 2$ components. Let $X = S^{n+2} - L_\mu^n$, and let $\bar{X}$ be a *infinite cyclic covering space* of X associated with the epimorphism

$$\tilde{\psi} = \sigma\psi : \pi_1(X) \xrightarrow{\psi} H_1(X; \mathbb{Z}) \cong \mathbb{J}^\mu \xrightarrow{\sigma} \mathbb{J}$$

defined in Definition 3.9. Then, 6.1 remains valid if L_μ^n and $\bar{X}$ are substituted for K^n and $\tilde{X}$. From a presentation matrix of $H_q(\bar{X}; \mathbb{Q})$, we have the *reduced d -th Alexander polynomial* $\tilde{\Delta}_d^q(t) \in \Lambda$ and the *reduced rational d -th Alexander polynomial* $\tilde{\Delta}_d'^q(t) \in \Gamma$ of $\bar{X}$ (or $(L_\mu^n \subset S^{n+2})$) in dimension q , by 6.3. Proposition 6.6, 6.7 and 6.8 remain valid if $\tilde{\Delta}_d^q$, $\tilde{\Delta}_d'^q$ and $\tilde{\psi}$ etc. are substituted for Δ_d^q , $\Delta_d'^q$ and ψ etc..

We have the following :

6.16. Theorem. (Shinohara-Summers [1]) Let $(L_\mu^n \subset S^{n+2})$ be an n -link with μ components, and $\mu \geq 2$, $n \geq 2$. Then,

$$(a'') \quad \tilde{\Delta}_{d+1}^q(t) \mid \tilde{\Delta}_d^q(t) \text{ in } \Lambda,$$

$$(b'') \quad \tilde{\Delta}_d^q(1) = \begin{cases} 0 & (q=0; d=1), (q=1, n+1; d \leq \mu-1), \\ \pm 1 & \text{otherwise,} \end{cases}$$

$$(c'') \quad \tilde{\Delta}_d^1(t) = \begin{cases} 0 & d < \mu, \\ \tilde{\Delta}_{d-\mu+1}^n(t^{-1}) & d \geq \mu, \end{cases}$$

$$\tilde{\Delta}_d^q(t) = \tilde{\Delta}_d^{n+1-q}(t^{-1}), \quad 2 \leq q \leq n-1.$$

6.17. Theorem. (Levine [5, Theorem 2]) Let $\{\lambda_i^q(t)\}$, $0 < q < n+1$, $0 < i \leq m$, be a set of primitive elements of Λ satisfying (a), (b) and (c) of Theorem 6.9. Then, if n is even, there exists an n -knot with $\{\lambda_i^q(t)\}$ as its Alexander invariants.

For odd dimensional cases, we need some additional conditions (refer to Levine [5]). For $n=1$, 6.17 is proved in Seifert [1], Levine [4]. The case $n=2$ is Theorem 3.17 (Kinoshita [1]).

The proof of Theorem 6.9 leads to a method for computing the Alexander matrix and the Alexander polynomials from given a Seifert manifold. In particular, this method is useful for ribbon 2-knots, because a ribbon 2-knot bounds a special Seifert manifold W given in Theorem 5.13. Both $H_1(W; \mathbb{Z})$ and $H_2(W; \mathbb{Z})$ are torsion free, and we can choose free bases of $H_1(W; \mathbb{Z})$ and $H_2(W; \mathbb{Z})$, respectively, by mutually disjoint simple loops and mutually disjoint 2-spheres in W in the obvious manner. Here we will give another description of a ribbon 2-knot due to Rolfsen [Ro, Chapter 7], Hudson-Summers [1], Summers [1; 2] and Yanagawa [2].

6.18. Definition. Here we will only define a *plumbing* of a 2-sphere and a 1-sphere. For a more general definition of a plumbing of a p -sphere and a q -sphere, see Rolfsen [Ro, Chapter 7].

Let $S_+^2 \subset S^2$ and $S_+^1 \subset S^1$ be upper hemispheres, and let $h_1: S_+^2 \rightarrow D^2$ and $h_2: S_+^1 \rightarrow D^1$ be homeomorphisms. A *plumbing* of S^2 and S^1 is an orientable 3-manifold P obtained from disjoint union of $S^2 \times D^1$ and $D^2 \times S^1$ with $S_+^2 \times D^1$ and $D^2 \times S_+^1$ identified via the homeomorphism

$$h_1 \times h_2^{-1} : S_+^2 \times D^1 \rightarrow D^2 \times S_+^1.$$

Then, P is homeomorphic to $S^2 \times S^1$ with an open 3-disk missing, and so $\partial P \cong S^2$. It should be noted that there exists a natural inclusion of the wedge $S^2 \vee S^1$ in P . We will consider embeddings of the plumbing P in S^4 (or R^4), so that P is a Seifert manifold for ∂P .

6.19. Example. (Rolfsen [Ro, 7F4-7F6]) Let $(K_0^2 \subset S^4)$ be a trivial 2-knot, and $V \subset S^4$ be a 3-disk with $\partial V = K_0^2$; V has a bicollar neighborhood $B^4 \cong V \times [-1, 1] = V \times D^1$. Restricting B^4 to the boundary $\partial V = K_0^2$, we have $N_2 \cong K_0^2 \times D^1$. We take $c+1$ points $x_0 \in K_0^2 = \partial V$, $x_1, \dots, x_c \in {}^\circ V$. Let K^1 be a simple oriented loop in S^4 such that $K^1 \cap V = x_0 \cup x_1 \cup \dots \cup x_c$ and K^1 runs through B^4 along $x_0 \times D^1, x_1 \times D^1, \dots, x_c \times D^1$ in the direction $-1 \rightarrow 1$. We extend this to $N_1 \cong D^2 \times K^1$ in S^4 in a natural way, so that $\{0\} \times K^1 = K^1$ and $N_2 \cup N_1$ is the plumbing of K_0^2 and K^1 . Let $W = N_2 \cup N_1$ and $K^2 = \partial W$, see Figure 6.1.

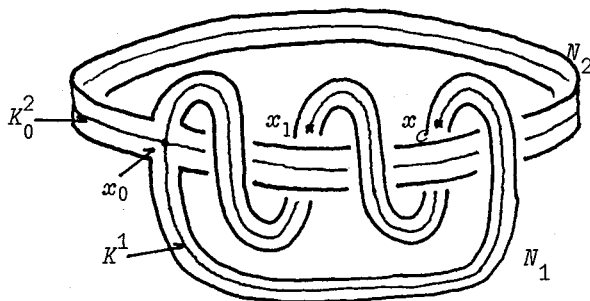


Figure 6.1

Let $N \cong W \times [-1, 1]$ be a bicollar neighborhood of W in S^4 . By N^- and N^+ we denote $W \times [-1, 0]$ and $W \times [0, 1]$, respectively, via the homeomorphism. We specify the positive side N^+ of N to be one which intersects B^4 near x_0 . Let a^2 be the 2-cycle carried by K_0^2 , and b^1 the 1-cycle carried by K^1 . It will be noticed that a^2 is a free (abelian) basis of $H_2(W; \mathbb{Z}) \cong J$, and b^1 is a free (abelian) basis of $H_1(W; \mathbb{Z}) \cong J$. Pushing a^2 and b^1 off W and into N^+ or N^- , we denote them by a_+^2 , b_+^1 or a_-^2 , b_-^1 , respectively. Then,

$$\begin{aligned}\mathcal{Lk}(a_+^2, b_+^1) &= c + 1 = \mathcal{Lk}(a_-^2, b_-^1), \\ \mathcal{Lk}(a_+^2, b_-^1) &= c = \mathcal{Lk}(a_-^2, b_+^1),\end{aligned}$$

where $\mathcal{Lk}$ is the linking pairing of 2-cycles and 1-cycles in S^4 .

Using W we construct a universal abelian covering space $\tilde{X}$ of $X = S^4 - K^2$. By Alexander duality, we have :

$$\begin{aligned}H_1(S^4 - W) &\cong \mathbb{Z} \text{ generated by } \hat{a}^1 \text{ (dual to } a^2), \\ H_2(S^4 - W) &\cong \mathbb{Z} \text{ generated by } \hat{b}^2 \text{ (dual to } b^1).\end{aligned}$$

Then, we have :

$$\begin{aligned}v_-(b^1) &= c \hat{a}^1, & v_+(b^1) &= (c+1) \hat{a}^1, \\ v_-(a^2) &= (c+1) \hat{b}^2, & v_+(a^2) &= c \hat{b}^2,\end{aligned}$$

and so presentation matrices of $H_1(\tilde{X}; \mathbb{Q})$ and $H_2(\tilde{X}; \mathbb{Q})$ are

$$P_1(t) = \{ct - (c+1)\}, \quad P_2(t) = \{(c+1)t - c\}.$$

Thus,

$$\begin{aligned}\Delta_1^1(t) &= \lambda_1^1(t) = ct - (c+1), \\ \Delta_1^2(t) &= \lambda_1^2(t) = (c+1)t - c.\end{aligned}$$

Since K^2 is a trivial 2-knot, it is easily checked that W has a trivial system $\{S_1^2, S_2^2\}$ of 2-spheres, so $(K^2 \subset S^4)$ is a ribbon 2-knot by Theorem 5.13.

6.20. Exercise. Describe a standard (simple symmetric) representative of the 2-knot $(K^2 \subset R^4)$ given in 6.19.

By the same type of argument as that used in Example 6.19, we give the proof of Theorem 3.17. This proof follows from Rolfsen [Ro, 7H1].

6.21. Theorem (= Theorem 3.17). Let $\Delta(t)$ be an integral polynomial with $|\Delta(1)| = 1$. Then, there exists a 2-knot $(K^2 \subset S^4)$ whose Alexander polynomials are

$$\begin{aligned}\Delta_1^1(t) &= \Delta(t) = \Delta_1(t), \\ \Delta_1^2(t) &= \Delta(t^{-1}).\end{aligned}$$

Proof. We may suppose that $\Delta(1) = -1$. Then, we may set

$$\Delta(t) = -1 + c_1(t-1) + c_2(t-1)^2 + \dots + c_r(t-1)^r.$$

Let $(K_1^2 \cup \dots \cup K_r^2 \subset S^4)$ be a *trivial* 2-link with r components. We take r simple oriented loops $K_1^1, \dots, K_r^1$ in S^4 so that $K_1^1 \cup \dots \cup K_r^1$ and $K_1^2 \cup \dots \cup K_r^2$ link according to the schematic :

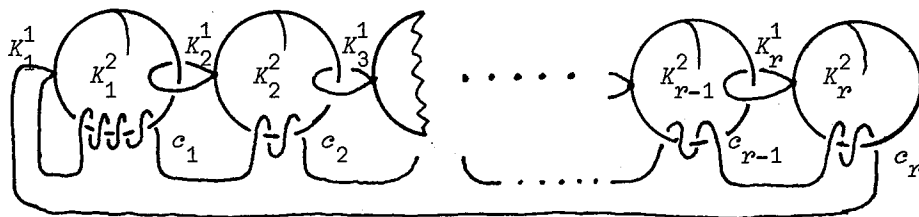


Figure 6.2

Then we have r mutually disjoint plumbings $W_1, \dots, W_r$ of S^2 and S^1 in S^4 by the same way as that of W in Example 6.19. Let a_i^2 and b_i^1 be 2- and 1-cycles in W_i carried by K_i^2 and K_i^1 , respectively, $i = 1, \dots, r$. Then, we require the following relations :

$$(*) \left\{ \begin{array}{l} lk(a_{1+}^2, b_{1+}^1) = c_1 + 1 = lk(a_{1-}^2, b_{1-}^1), \\ lk(a_j^2, b_{j+}^1) = 1 = lk(a_{j-}^2, b_{j-}^1) \quad \text{for } j=2, \dots, r, \\ lk(a_{1+}^2, b_1^1) = c_1 = lk(a_1^2, b_{1-}^1), \\ lk(a_{j+}^2, b_j^1) = 0 = lk(a_j^2, b_{j-}^1) \quad \text{for } j=2, \dots, r, \\ lk(a_j^2, b_1^1) = c_j \quad \text{for } j=2, \dots, r, \\ lk(a_{j-1}^2, b_j^1) = 1 \quad \text{for } j=2, \dots, r, \\ lk(a_\ell^2, b_j^1) = 0 \quad \text{for } j=2, \dots, r; \ell \neq j-1; \ell=1, \dots, r. \end{array} \right.$$

Let $\partial W_i = \tilde{K}_i^2$, $i=1, \dots, r$. We consider a fusion of $\tilde{K}_1^2 \cup \dots \cup \tilde{K}_r^2$. Let

$$B_j^+ : I \times D^2 \rightarrow S^4, \quad j=2, \dots, r,$$

be embeddings of a 3-disks $I \times D^2$ into S^4 such that

$$\begin{aligned} B_j^+(I \times D^2) \cap (W_1 \cup \dots \cup W_r) &= B_j^+(\partial I \times D^2) \cap (\tilde{K}_{j-1}^2 \cup \tilde{K}_j^2), \\ B_j^+({0} \times D^2) &\subset \tilde{K}_{j-1}^2, \quad B_j^+({1} \times D^2) \subset \tilde{K}_j^2. \end{aligned}$$

Let $W = W_1 \cup B_2^+(I \times D^2) \cup W_2 \cup B_3^+(I \times D^2) \cup \dots \cup B_r^+(I \times D^2) \cup W_r$; W is homeomorphic to a connected sum of r copies of $S^2 \times S^1$ with an open 3-disk missing and ∂W is a fusion of $\tilde{K}_1^2 \cup \dots \cup \tilde{K}_r^2$ along $B_2^+ \cup \dots \cup B_r^+$. Let $K^2 = \partial W$; since $K_1^2 \cup \dots \cup K_r^2$ is a trivial 2-link, W has a trivial system of 2-spheres, and so K^2 is a ribbon 2-knot by Theorem 5.13.

It is easy to see that $\{a_1^2, \dots, a_r^2\}$ and $\{b_1^1, \dots, b_r^1\}$ form free abelian bases for $H_2(W; \mathbb{Z})$ and $H_1(W; \mathbb{Z})$, respectively, and the equations $(*)$ hold for W .

Using W we construct a universal abelian covering space $\tilde{X}$ of $X = S^4 - K^2$. By Alexander duality, we have :

$$\begin{aligned} H_1(S^4 - W; \mathbb{Z}) &\cong r\mathbb{Z} \text{ generated by } \hat{a}_1^1, \dots, \hat{a}_r^1 \text{ (dual to } a_1^2, \dots, a_r^2), \\ H_2(S^4 - W; \mathbb{Z}) &\cong r\mathbb{Z} \text{ generated by } \hat{b}_1^2, \dots, \hat{b}_r^2 \text{ (dual to } b_1^1, \dots, b_r^1), \end{aligned}$$

where $r\mathbb{Z}$ denotes a free abelian group of rank r .

Now, from the equations (*), we have :

$$\begin{aligned}
 v_-(b_1^1) &= c_1 \hat{a}_1^1 + c_2 \hat{a}_2^1 + \dots + c_r \hat{a}_r^1, \\
 v_+(b_1^1) &= (c_1 + 1) \hat{a}_1^1 + c_2 \hat{a}_2^1 + \dots + c_r \hat{a}_r^1, \\
 v_-(b_2^1) &= \hat{a}_1^1, & v_+(b_2^1) &= \hat{a}_1^1 + \hat{a}_2^1, \\
 v_-(b_3^1) &= \hat{a}_2^1, & v_+(b_3^1) &= \hat{a}_2^1 + \hat{a}_3^1, \\
 &\vdots & &\vdots \\
 v_-(b_r^1) &= \hat{a}_{r-1}^1, & v_+(b_r^1) &= \hat{a}_{r-1}^1 + \hat{a}_r^1.
 \end{aligned}$$

So we have a presentation matrix $P_1(t)$ of $H_1(\tilde{X}; \mathbb{Q})$:

$$\begin{pmatrix}
 c_1 t - (c_1 + 1) & c_1 t - c_1 & c_2 t - c_2 & \dots & c_{r-1} t - c_{r-1} & c_r t - c_r \\
 t - 1 & -1 & 0 & \dots & 0 & 0 \\
 0 & t - 1 & -1 & \dots & 0 & 0 \\
 \vdots & \vdots & \vdots & & \vdots & \vdots \\
 0 & 0 & 0 & \dots & -1 & 0 \\
 0 & 0 & 0 & \dots & t - 1 & -1
 \end{pmatrix}$$

Thus, the first Alexander polynomial $\Delta_1^1(t)$ in dimension 1 is

$$\Delta_1^1(t) = -1 + c_1(t - 1) + c_2(t - 1)^2 + \dots + c_r(t - 1)^r = \Delta(t).$$

The calculation of $H_2(\tilde{X}; \mathbb{Q})$ is similar, so we leave it to the reader to verify the formula : $\Delta_1^2(t) \doteq \Delta(t^{-1})$. $\square$

6.22. Exercise. Prove Theorem 6.17 for 2-knots. That is, let $\{\lambda_1(t), \dots, \lambda_m(t)\}$ be a family of integral polynomials satisfying

$$\text{(a) } \lambda_{i+1}(t) \mid \lambda_i(t) \text{ in } \Lambda, \quad \text{(b) } \lambda_i(1) = \pm 1.$$

Then, there exists a 2-knot $(K^2 \subset S^4)$ with

$$H_1(\tilde{X}; \mathbb{Q}) = \Gamma/(\lambda_1(t)) \oplus \cdots \oplus \Gamma/(\lambda_m(t)),$$

$$H_2(\tilde{X}; \mathbb{Q}) = \Gamma/(\lambda_1(t^{-1})) \oplus \cdots \oplus \Gamma/(\lambda_m(t^{-1})).$$

6.23. Exercise. Let $(K_0^2 \subset S^4)$ be any 2-knot, and let V be a Seifert manifold for K_0^2 . We can construct in S^4 a plumbing W of K_0^2 and a simple oriented loop K^1 by the same way as that used in Example 6.19. Show that : if K_0^2 is a ribbon 2-knot, then ∂W is also a ribbon 2-knot. More generally, let $(K_1^2 \cup \cdots \cup K_r^2 \subset S^4)$ be a 2-link. Then, we can construct r mutually disjoint plumbings W_i 's of K_i^2 's and 1-spheres K_i^1 's and a disk-sum W of the plumbings W_i by the same way as that used in the proof of 6.21. Show that : if each component K_i^2 is a ribbon 2-knot, $i = 1, \dots, r$, and if there exist r mutually disjoint 4-disks $B_1^4, \dots, B_r^4$ in S^4 with $B_i^4 \cap (K_1^2 \cup \cdots \cup K_r^2) = K_i^2$, $i = 1, \dots, r$, then ∂W is a ribbon 2-knot.

6.24. Problem. Does there exist a non-ribbon 2-knot whose Seifert manifold V is homeomorphic to connected-sum of r copies of $S^2 \times S^1$ with an open 3-disk missing ?

7. Local Alexander Invariants

For an n -knot $(K^n \subset S^{n+2})$, let $\tilde{X}$ be the universal abelian covering space of the complement $X = S^{n+2} - K^n$. In the preceding Section, we have studied the Γ -modules $\{H_q(\tilde{X}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}\}$ and actually characterized their structures. In this section, we will be concerned with the integral homology groups $\{H_q(\tilde{X}; \mathbb{Z})\}$, which are finitely presented Λ -modules. In particular, we will introduce another polynomial invariants of $H_q(\tilde{X}; \mathbb{Z})$,

called *local Alexander polynomials*. For 2-knots, by duality $H_1(\tilde{X}; Z)$ is essential as in Section 6. We refer the reader to Kervaire [2], Summers [6], Gutierrez [3] and Levine [10].

7.1. As in (6.10), we have the following exact sequence of Λ -modules:

$$\cdots \rightarrow H_q(V; Z) \otimes_Z \Lambda \xrightarrow{\tau} H_q(Y; Z) \otimes_Z \Lambda \xrightarrow{e} H_q(\tilde{X}; Z) \otimes_Z \Lambda \rightarrow H_{q-1}(V; Z) \otimes_Z \Lambda \rightarrow \cdots$$

where τ is defined by

$$\tau(a \otimes 1) = v_-(a) \otimes t - v_+(a) \otimes 1 = t(v_-(a) \otimes 1) - v_+(a) \otimes 1$$

for $a \in H_q(V; Z)$.

Let $V^* = S^{n+2} - Y$; then V^* is a deformation retraction of V and $V^* \cap Y = V_- \cup V_+$. From the Mayer-Vietoris sequence for the proper triad (S^{n+2}, Y, V^*) , we obtain

$$0 \rightarrow H_q(V; Z) \oplus H_q(V; Z) \xrightarrow{e'} H_q(Y; Z) \oplus H_q(V; Z) \rightarrow 0,$$

where $e'(a, b) = (v_-(a) - v_+(b), a - b)$. Let

$$\Delta : H_q(V; Z) \rightarrow H_q(V; Z) \oplus H_q(V; Z)$$

be the diagonal map. Then we can define a monomorphism

$$\bar{\tau} = e' \Delta : H_q(V; Z) \rightarrow H_q(Y; Z)$$

by $a \mapsto v_-(a) - v_+(a)$. Since Λ is Noetherian, τ is an isomorphism. So we have a short exact sequence

$$0 \rightarrow H_q(V; Z) \otimes_Z \Lambda \rightarrow H_q(Y; Z) \otimes_Z \Lambda \xrightarrow{e} H_q(\tilde{X}; Z) \rightarrow 0 \quad (1')$$

which corresponds to (1) in (6.10).

Let θ be the map induced by multiplication by $(t-1)$. Then we have :

$$\begin{array}{ccccccc} 0 & \rightarrow & H_q(V; Z) \otimes_Z \Lambda & \xrightarrow{\tau} & H_q(Y; Z) \otimes_Z \Lambda & \xrightarrow{e} & H_q(\tilde{X}; Z) \rightarrow 0 \\ & & \downarrow \theta_1 & & \downarrow \theta_2 & & \downarrow \theta_3 \\ 0 & \rightarrow & H_q(V; Z) \otimes_Z \Lambda & \xrightarrow{\tau} & H_q(Y; Z) \otimes_Z \Lambda & \xrightarrow{e} & H_q(\tilde{X}; Z) \rightarrow 0. \end{array}$$

Then the kernel-cokernel sequence is the follows :

$$0 \longrightarrow \text{Ker}(\theta_3) \longrightarrow H_q(V; \mathbb{Z}) \xrightarrow[e' \Delta]{\begin{smallmatrix} \text{Coker}(\theta_1) \\ \parallel \end{smallmatrix}} H_q(Y; \mathbb{Z}) \xrightarrow[\begin{smallmatrix} \text{Coker}(\theta_2) \\ \parallel \end{smallmatrix}]{\quad} \text{Coker}(\theta_3) \longrightarrow 0.$$

Since $\bar{\tau} = e' \Delta$ is isomorphism, $\text{Coker}(\theta_3) = 0$.

Let T_q be the $\mathbb{Z}$ -torsion Λ -submodule of $H_q(\tilde{X}; \mathbb{Z})$ and F_q the torsion free part of $H_q(\tilde{X}; \mathbb{Z})$. Then, it follows from Kervaire [2, Lemma II.8] that T_q is a finite group since $\text{Coker}(\theta_3) = 0$. Thus, $H_q(\tilde{X}; \mathbb{Z})$ splits as a group into $F_q \oplus T_q$ by Fuchs [Fu, Theorem 24.5].

7.2. As in (6.11), we define a homomorphism

$$\mathcal{L}k : H_q(V; \mathbb{Z}) \otimes_{\mathbb{Z}} H_p(S^{n+2} - V; \mathbb{Z}) \rightarrow \mathbb{Z} \quad (\text{or } \mathbb{Q}/\mathbb{Z})$$

which is defined by linking numbers, where $p = n + 1 - q$.

Let $H_q(V; \mathbb{Z}) = F \oplus T$ where F is the free part and T the torsion part. Let $a_1^q, \dots, a_r^q$ be a basis for F , and suppose that T has generators $b_1^q, \dots, b_s^q$ with elementary divisors $d_1 | d_2 | \dots | d_s$. As in (6.11) (cf. Seifert-Threlfall [S-T, §77 & p.432]), we can find a basis $\{\hat{a}_j^p\}$ for the free part of $H_p(Y; \mathbb{Z})$, and generators $\{\hat{b}_\ell^{p-1}\}$ for the torsion part of $H_{p-1}(Y; \mathbb{Z})$ and chains $B_\ell^{q+1} \in C_{q+1}(V; \mathbb{Z})$ with $\partial B_\ell^{q+1} = d_\ell b_\ell^q$ ($j=1, \dots, r; \ell=1, \dots, s$) such that

$$\begin{aligned} \mathcal{L}k(a_j^q \otimes \hat{a}_k^p) &= \delta_{jk} \in \mathbb{Z}, \\ \mathcal{L}k(B_\ell^{q+1} \otimes \hat{b}_k^{p-1}) &= \delta_{\ell k} / d_k \in \mathbb{Q}/\mathbb{Z}. \end{aligned}$$

7.3. As in (6.12), we set :

$$\begin{aligned} v_-(a_j^q) &= \sum_i \lambda_{ji}^q \hat{a}_i^q + \sum_k \psi_{jk}^q \hat{b}_k^q, & v_+(a_j^q) &= \sum_i \mu_{ji}^q \hat{a}_i^q + \sum_k \omega_{jk}^q \hat{b}_k^q, \\ v_-(b_\ell^q) &= \sum_k \sigma_{\ell k}^q \hat{b}_k^q, & v_+(b_\ell^q) &= \sum_k \tau_{\ell k}^q \hat{b}_k^q. \end{aligned}$$

where all the coefficients are integers. Then, the presentation matrix

of $H_q(\tilde{X}; Z)$ corresponding to (1') in 7.1 will have the form

$$\begin{pmatrix} P_q & N_q \\ 0 & Q_q \end{pmatrix},$$

where $P_q = \begin{pmatrix} t\lambda_{ji}^q & -\mu_{ji}^q \end{pmatrix}$, $Q_q = \begin{pmatrix} t\sigma_{\ell k}^q & -\tau_{\ell k}^q \end{pmatrix}$ and $N_q = \begin{pmatrix} t\psi_{jk}^q & -\omega_{jk}^q \end{pmatrix}$,

$1 \leq i, j \leq r$ and $1 \leq k, \ell \leq s$, and 0 is a $s \times r$ zero matrix.

The following sequences of Λ -modules are exact :

$$0 \longrightarrow F \otimes_Z \Lambda \xrightarrow{P_q} F \otimes_Z \Lambda \longrightarrow H_q(\tilde{X}; Z)/T_q \longrightarrow 0 \quad (1^*)$$

$$0 \longrightarrow T \otimes_Z \Lambda \xrightarrow{Q_q} T \otimes_Z \Lambda \longrightarrow T_q \longrightarrow 0 \quad (1^{**})$$

Since P_q is a square matrix, $\text{Coker}(P_q)$ is a Z -torsion free module, so (1*) is clear, and (1**) follows as in Kervaire [2, Lemma III14]. In fact, P_q coincides with the $P_q(t)$ in (6.12).

From (1*), $H_q(\tilde{X}; Z)/T_q$ is a Z -torsion free Λ -module, and (1**) presents the Z -torsion Λ -module T_q .

7.4. As in (6.13), we have the following :

$$lk(a_j^q \otimes v_-(a_k^p)) = lk(a_j^q \otimes (\sum_i \lambda_{ki}^p \hat{a}_i^p)) = \sum_i \lambda_{ki}^p \cdot lk(a_j^q \otimes \hat{a}_i^p) = \lambda_{kj}^p \quad (4')$$

$$lk(a_j^q \otimes v_+(a_k^p)) = lk(a_j^q \otimes (\sum_i \mu_{ki}^p \hat{a}_i^p)) = \sum_i \mu_{ki}^p \cdot lk(a_j^q \otimes \hat{a}_i^p) = \mu_{kj}^p \quad (5')$$

$$lk(B_\ell^{q+1} \otimes v_-(a_j^{p-1})) = lk(B_\ell^{q+1} \otimes (\sum_k \psi_{jk}^{p-1} \hat{a}_k^{p-1})) = \psi_{j\ell}^{p-1} \quad (4^*)$$

$$lk(B_\ell^{q+1} \otimes v_+(a_j^{p-1})) = lk(B_\ell^{q+1} \otimes (\sum_k \omega_{jk}^{p-1} \hat{a}_k^{p-1})) = \omega_{j\ell}^{p-1} \quad (5^*)$$

$$lk(B_\ell^{q+1} \otimes v_-(b_k^{p-1})) = lk(B_\ell^{q+1} \otimes (\sum_i \sigma_{ki}^{p-1} \hat{b}_i^{p-1})) = \sigma_{k\ell}^{p-1}/d_\ell \quad (4^{**})$$

$$lk(B_\ell^{q+1} \otimes v_+(b_k^{p-1})) = lk(B_\ell^{q+1} \otimes (\sum_i \tau_{ki}^{p-1} \hat{b}_i^{p-1})) = \tau_{k\ell}^{p-1}/d_\ell \quad (5^{**})$$

From the commutativity of linking numbers, we have :

$$\lambda_{kj}^p = (-1)^{pq+1} \mu_{jk}^q \quad (2')$$

$$\left. \begin{aligned} \psi_{j\ell}^{p-1} - \omega_{j\ell}^{p-1} &= B_{\ell}^{q+1} \cdot a_j^{p-1} = d_{\ell} \cdot \ell k (b_{\ell}^q \otimes a_j^{p-1}) \equiv 0 \pmod{d_{\ell}}, \\ \text{where } B_{\ell}^{q+1} \cdot a_j^{p-1} &\text{ is the homology intersection number.} \\ d_{\ell} \sigma_{\ell k}^{p-1} &= (-1)^{(q+1)p+1} d_k \tau_{k\ell}^q \end{aligned} \right\} \quad (2^*)$$

As a corollary, we have :

$$\left. \begin{aligned} P_q(1) &\text{ is unimodular,} \\ N_q(t) &\text{ is the form } \{(t-1)\psi_{j\ell}^q\}, \\ Q_q(1) &\text{ induces an automorphism of } T. \end{aligned} \right\} \quad (7^*)-(8^*)$$

Now, we have a presentation of $H_q(\tilde{X}; \mathbb{Z})$:

$$\langle x_i^q, y_j^q \mid \sum_k (t\lambda_{ik}^q - \mu_{ik}^q) x_k^q = (t-1) \sum_j \psi_{ij}^q y_j^q, \sum_j (t\sigma_{\ell j}^q - \tau_{\ell j}^q) y_j^q = 0, d_j y_j^q = 0 \rangle.$$

As a corollary to (2*), we have :

7.5. Proposition. $H_q(\tilde{X}; \mathbb{Z})/T_q$ and $H_p(\tilde{X}; \mathbb{Z})/T_p$ have dual Λ -presentation matrices ; similarly T_q and T_{p-1} have dual matrices. Moreover, these dual matrices establish isomorphisms of $F \otimes_{\mathbb{Z}} \Lambda$ and $T \otimes_{\mathbb{Z}} \Lambda$, respectively. In particular, $T_n = T_{n+1} = 0$.

7.6. Let $\pi \in \mathbb{Z}$ be a prime. The π -primary component $T_{q,\pi}$ of the $\mathbb{Z}$ -torsion part T_q of $H_q(\tilde{X}; \mathbb{Z})$ is a Λ -submodule as in Kervaire [2, Lemma III15]. By virtue of (2*), we know that $T_{q,\pi}$ admits a presentation

$$T_{q,\pi} = \langle y_1, \dots, y_s \mid \sum_j (ta_{\ell j}^q - b_{\ell j}^q) y_j = 0, (\pi^{\varepsilon_j}) y_j = 0 \rangle,$$

$$\text{where } (\pi^{\varepsilon_{\ell}}) a_{\ell j}^q + (-1)^{(q+1)p} (\pi^{\varepsilon_j}) b_{j\ell}^{p-1} = 0 \quad (9)$$

Let $D_q(t) = (ta_{\ell j}^q - b_{\ell j}^q)$, where the j -th column is determined modulo π^{ε_j} . If $\ell < j$, the (ℓ, j) -entry of $D_q(t)$ is $ta_{\ell j}^q - b_{\ell j}^q$, whereas the (j, ℓ) -entry will be $(-1)^{(q+1)p+1} (\pi^{\varepsilon_j - \varepsilon_{\ell}}) (tb_{j\ell}^{p-1} - a_{j\ell}^{p-1})$. So, it will be

remarked that $D_q(t) \neq (-1)^{(q+1)p+1} t [D_{p-1}(t^{-1})]'$, where ' indicates transpose. Let

$$\nabla = \begin{pmatrix} \pi^{\varepsilon_1} & & & & \\ & \pi^{\varepsilon_2} & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \pi^{\varepsilon_s} \end{pmatrix}.$$

Then, we have :

$$D_q(t) \cdot \nabla = \nabla \cdot (-1)^{(q+1)p+1} t [D_{p-1}(t^{-1})]'$$

We consider a map $\gamma : \Lambda \rightarrow \Lambda_\pi = \mathbb{Z}_\pi \otimes \Lambda$; it should be noticed that Λ_π is PID. We can now consider the image of $D_q(t)$ under γ , and can consider the elementary ideals of $\gamma D_q(t)$. That is ;

7.7. Definition. Let $T_{q,\pi}$ be the π -primary component of the $\mathbb{Z}$ -torsion part T_q of $H_q(\tilde{X}; \mathbb{Z})$. The polynomials $\lambda_{\pi i}^q \in \Lambda_\pi$ extracted from $T_{q,\pi}$ (i.e. $\gamma D_q(t)$) are called the π -local Alexander polynomials of $(K^n \subset S^{n+2})$ in dimension q .

Corresponding to Theorem 6.9 and Theorem 6.17, we have the following :

7.8. Theorem. (Gutierrez [3, Theorem 1]) Associated to every n -knot $(K^n \subset S^{n+2})$, any integer $1 \leq q \leq n-1$ and every prime π , there exists a family $\{\lambda_{\pi i}^q\}$ of polynomials in Λ_π , determined up to units in Λ_π and satisfying :

(a*) $\lambda_{\pi, i+1}^q \mid \lambda_{\pi i}^q$ in Λ_π and $\lambda_{\pi i}^q = 1$ for large i and for almost all primes π 's,

(b*) $\lambda_{\pi i}^q(1) = 1$ in $\mathbb{Z}_\pi$,

(c*) $\lambda_{\pi i}^q(t) \doteq \lambda_{\pi i}^{n-q}(t^{-1})$ in Λ_π ,

(d*) If $n = 2q$, we can find $\Delta_{\pi 1}^q \in \Lambda_{\pi}$ with $\Delta_{\pi 1}^q(t) = \Delta_{\pi 1}^q(t^{-1})$ and $\Delta_{\pi 1}^q(1) = 1$. If q is odd, $\Delta_{\pi 1}^q(-1)$ is a square in $\mathbb{Z}_{\pi}$.

7.9. Proposition. (Gutierrez [3, Theorem 2]) Let $\{\lambda_{\pi i}^q\}$ be a family of polynomials in Λ_{π} satisfying (a*) ~ (d*) of Theorem 7.8. Then, there exists an n -knot $(K^n \subset S^{n+2})$, $n \geq 3$, with $\{\lambda_{\pi i}^q\}$ as its π -local Alexander polynomials in dimension q .

7.10. Example. Theorem 7.8 asserts that any π -local Alexander polynomial of a 2-knot is reciprocal in Λ_{π} . Using this property, we can prove that $\text{KG}_2 \neq \text{KG}_3$ in Theorem 4.2. In fact, by Proposition 7.9 we can construct a group $G \in \text{KG}_3$ with non-reciprocal polynomials. There are some other methods for proving that $\text{KG}_2 \neq \text{KG}_3$, refer to Gutierrez [6], Farber [1], Kawauchi [3]. Now we consider a group

$$G = \langle a, b \mid ab = ba^2, ba^5 = a^5b \rangle,$$

given by Hosokawa-Kawauchi [1], Farber [1]. Since $a = (ba)^{-1}b^{-1}(ba)b$, it is not difficult to verify that $H_1(G; \mathbb{Z}) \cong \mathbb{Z}$ and $w(G) = 1$. By G' and G'' we denote the commutator subgroups of G and G' . Since

$$G' = \langle a \mid a^5 = 1 \rangle \cong \mathbb{Z}_5,$$

it follows that $G' = G'/G'' \cong_{\Lambda} \Lambda/(2t-1, 5-5t)$

$$\cong_{\Lambda} \Lambda_5/(2t-1),$$

because $5 = 5(2t-1) + 2(5-5t)$. If G is the knot group of a 2-knot $(K^2 \subset S^4)$, then it has the 5-local Alexander polynomial $\lambda_{5,1}^1 = 2t-1$ for $H_1(\tilde{X}; \mathbb{Z}) \cong G'/G''$. Since $2t-1$ is not reciprocal in Λ_5 , this contradicts to Theorem 7.8.

To show that $G \in \text{KG}_3$, the only thing remaining is to prove $H_2(G; \mathbb{Z})$

$= 0$, by Theorem 4.3. We use the so-called Wang exact sequence (cf. Cartan-Eilenberg [C-Ei]) of homology.

7.11. Proposition. Let E be a finite complex and let $p : \tilde{E} \rightarrow E$ be an infinite cyclic covering space. Then we have the following long exact sequence of homology with an arbitrary coefficient :

$$\rightarrow H_q(\tilde{E}) \xrightarrow{t-1} H_q(\tilde{E}) \xrightarrow{p_*} H_q(E) \xrightarrow{\partial} H_{q-1}(\tilde{E}) \xrightarrow{t-1} H_{q-1}(\tilde{E}) \rightarrow ,$$

where t denotes the isomorphism induced from a generator t of the covering translation group of $\tilde{E}$.

Proof. Consider the short exact sequence of chain complexes :

$$0 \longrightarrow C_*(\tilde{E}) \xrightarrow{t-1} C_*(\tilde{E}) \xrightarrow{p_*} C_*(E) \longrightarrow 0.$$

By taking homology, we obtain the desired sequence. $\square$

7.12. (Continued from 7.10) As a corollary to 7.11, we have the Wang exact sequence of homology :

$$\rightarrow H_q(G') \xrightarrow{t-1} H_q(G') \xrightarrow{i_*} H_q(G) \xrightarrow{\partial} H_{q-1}(G') \xrightarrow{t-1} H_{q-1}(G') \rightarrow$$

where i is the inclusion homomorphism. From this sequence, we have the exact sequence

$$0 \longrightarrow H_2(G; \mathbb{Z}) \xrightarrow{\partial} H_1(G'; \mathbb{Z}) \xrightarrow{t-1} H_1(G'; \mathbb{Z}) \longrightarrow 0 ,$$

because $H_2(G'; \mathbb{Z}) = 0$ and $H_1(G; \mathbb{Z}) \cong \mathbb{Z}$. Since $(t-1) : \Lambda_5/(2t-1) \rightarrow \Lambda_5/(2t-1)$ is an isomorphism between abelian groups and $H_1(G'; \mathbb{Z})$ is identical with $G'/G'' = \Lambda_5/(2t-1)$, we conclude that $(t-1) : H_1(G'; \mathbb{Z}) \rightarrow H_1(G'; \mathbb{Z})$ is an isomorphism of abelian group. Thus, $H_2(G; \mathbb{Z}) = 0$. $\square$

In Milnor [3] a duality for the rational homology of X is found; this is based on the work of Blanchfield [1]. These dualities are reflected in

the polynomial invariants in Theorems 6.9 and 7.8. In the remainder of this section, we will introduce a generalization of the Milnor's duality theorem for infinite cyclic covering spaces due to Kawauchi [1; 2]. Here we will formulate the result only for the universal abelian coverings for exteriors of n -knots.

Let E be an exterior $S^{n+2} - {}^{\circ}N(K^n; S^{n+2})$ of an n -knot $(K^n \subset S^{n+2})$ and $\tilde{E}$ be the universal abelian covering space of E associated with the commutator subgroup of $\pi_1(E) \cong \pi_1(S^{n+2} - K^n)$. $\tilde{E}$ is constructed by the same way as that of the universal abelian covering space $\tilde{X}$ of the complement $X = S^{n+2} - K^n$ in (6.10). Let t be the generator of the infinite cyclic group of covering translations. Let

$$Y_0 = E - {}^{\circ}N(V; S^{n+2}) = S^{n+2} - {}^{\circ}N(V; S^{n+2}),$$

where V is a Seifert manifold for K^n . We set :

$$N_p = t^p Y_0 \cup t^{p+1} Y_0 \cup t^{p+2} Y_0 \cup \dots$$

and
$$\bar{N}_q = t^{-q} Y_0 \cup t^{-q-1} Y_0 \cup t^{-q-2} Y_0 \cup \dots.$$

From the construction of E , clearly N_p and $\bar{N}_q$ are connected. Note that the families

$$\{H^*(\tilde{E}, N_p; G)\}_{p=0,1,2,3,\dots},$$

$$\{H^*(\tilde{E}, \bar{N}_q; G)\}_{q=0,1,2,3,\dots}, \text{ and}$$

$$\{H^*(\tilde{E}, N_p \cup \bar{N}_q; G)\}_{p,q=1,2,3,\dots}$$

are directed systems directed by the inclusion homomorphisms. Since the family $\{N_p \cup \bar{N}_q\}$ is cofinal in the family of cobounded systems of $\tilde{E}$, we have :

7.13. Lemma. $\varinjlim \{H^*(\tilde{E}, N_p \cup \bar{N}_q; G)\} = H^*_c(\tilde{E}; G)$, where H^*_c denotes the cohomology with compact support.

7.14. Proposition. (Milnor [3]) Let $\tilde{K}$ be an infinite cyclic covering space of a finite complex K . For some field F and for fixed i , if $H_i(K; F) \cong_F H_i(S^1; F)$, then $H_i(\tilde{K}; F)$ is finitely generated over F .

Proof. For $i \geq 2$, since $H_i(K; F) = 0$, we have the exact sequence

$$H_i(\tilde{K}; F) \xrightarrow{t-1} H_i(\tilde{K}; F) \longrightarrow 0$$

from the exact sequence of Proposition 7.11. So, every element of $H_i(\tilde{K}; F)$ is divisible by $t-1$. Since $H_i(\tilde{K}; F)$ is finitely generated over FJ and FJ is PID, it follows that $H_i(\tilde{K}; F)$ is a torsion module over FJ . By the structure theorem, $H_i(\tilde{K}; F)$ is a finite dimensional vector space over F . For $i = 1$, we only remark that $H_0(K; F) \cong H_1(K; F) \cong H_0(\tilde{K}; F) \cong F$. $\square$

By Propositions 3.2 and 7.14, we have :

7.15. Proposition. $H_*(\tilde{E}; Q)$ is finitely generated over Q .

7.16. Lemma. If $H_{i-1}(\tilde{E}; Z)$ is finitely generated over Z , then for any torsion free abelian group A ,

$$\varinjlim \{H^i(\tilde{E}, N_p; A)\} = \varinjlim \{H^i(\tilde{E}, \bar{N}_q; A)\} = 0.$$

Proof. We shall only show that $\varinjlim \{H^i(\tilde{E}, N_p; A)\} = 0$.

In the universal coefficient theorem for cohomology

$$0 \rightarrow \text{Ext}[H_{i-1}(\tilde{E}, N_p; Z), A] \rightarrow H^i(\tilde{E}, N_p; A) \rightarrow \text{Hom}[H_i(\tilde{E}, N_p; Z), A] \rightarrow 0,$$

the direct limit commutes the exact sequence. So it suffices to show that

$$\varinjlim \{\text{Ext}[H_{i-1}(\tilde{E}, N_p; Z), A]\} = \varinjlim \{\text{Hom}[H_{i-1}(\tilde{E}, N_p; Z), A]\} = 0.$$

We divide the rest of the proof into two steps.

Step I : We show that $H_{i-1}(\tilde{E}, N_0; Z)$ is finitely generated over Z ,

and that $H_i(\tilde{E}, N_0; Z) \otimes Q$ is finitely generated over Q .

Consider the Mayer-Vietoris sequence

$$\longrightarrow H_j(N_0 \cap \bar{N}_{-1}) \longrightarrow H_j(N_0) \oplus H_j(\bar{N}_{-1}) \longrightarrow H_j(\tilde{E}) \longrightarrow.$$

Since $N_0 \cap \bar{N}_{-1} = V_-^0$ is a finite complex and $H_{i-1}(\tilde{E}; Z)$ is finitely generated over Z , and $H_i(\tilde{E}; Q)$ is finitely generated over Q by 7.15, it follows that $H_{i-1}(\bar{N}_{-1}; Z)$ and $H_i(\bar{N}_{-1}; Q)$ are finitely generated, for Q is Noetherian. By the exact sequence of the pair $(\bar{N}_{-1}, \bar{N}_{-1} \cap N_0) = (\bar{N}_{-1}, V_-^0)$, $H_{i-1}(\bar{N}_{-1}, V_-^0; Z)$ and $H_i(\bar{N}_{-1}, V_-^0; Q)$ are finitely generated. Using the excision isomorphism $H_*(\tilde{E}, N_0) = H_*(N_{-1}, V_-^0)$, we obtain that $H_{i-1}(\tilde{E}, N_0; Z)$ and $H_i(\tilde{E}, N_0; Q)$ are finitely generated. By the universal coefficient theorem for homology, $H_i(\tilde{E}, N_0; Z) \otimes Q$ is finitely generated over Q .

Step II : Since $H_i(\tilde{E}, N_0; Z) \otimes Q$ is finitely generated and A is torsion free, there exists a finite collection $\{z_0^1, \dots, z_0^m\}$ of elements of $H_i(\tilde{E}, N_0; Z)$ such that $\text{Hom}[H_i(\tilde{E}, N_p; Z)/(z_0^1, \dots, z_0^m), A] = 0$, where $(z_0^1, \dots, z_0^m)$ is the subgroup of $H_i(\tilde{E}, N_0; Z)$ generated by $z_0^1, \dots, z_0^m$. Since $H_{i-1}(\tilde{E}, N_0; Z)$ is finitely generated, there is a positive integer s so that all homology classes of $H_{i-1}(\tilde{E}, N_0; Z)$ and the elements $z_0^1, \dots, z_0^m$ are representable by cycles in N_{-s} . So, the inclusion homomorphism $H_{i-1}(N_{-s}, N_0; Z) \rightarrow H_{i-1}(\tilde{E}, N_0; Z)$ is onto and $z_0^1, \dots, z_0^m$ are contained in $\text{Image}[H_i(N_{-s}, N_0; Z) \rightarrow H_i(\tilde{E}, N_0; Z)]$. From the exact sequence of the triple $\tilde{E} \supset N_{-s} \supset N_0$, we know that $H_{i-1}(\tilde{E}, N_0; Z) \rightarrow H_{i-1}(\tilde{E}, N_{-s}; Z)$ is trivial and $z_0^1, \dots, z_0^m$ are contained in $\text{Ker}[H_i(\tilde{E}, N_0; Z) \rightarrow H_i(\tilde{E}, N_{-s}; Z)]$. Applying t^{p+s} for any p , the inclusion homomorphism $H_{i-1}(\tilde{E}, N_{p+s}; Z) \rightarrow H_{i-1}(\tilde{E}, N_p; Z)$ is trivial and $z_{p+s}^1, \dots, z_{p+s}^m$ are contained in $\text{Ker}[H_i(\tilde{E}, N_{p+s}; Z) \rightarrow H_i(\tilde{E}, N_p; Z)]$, where $z_{p+s}^k = t^{p+s}(z_0^k)$,

$k = 1, \dots, m$. Now the induced homomorphism

$$\text{Ext}[H_{i-1}(\tilde{E}, N_p; Z), A] \longrightarrow \text{Ext}[H_{i-1}(\tilde{E}, N_{p+s}; Z), A]$$

is trivial, and so we conclude that $\varinjlim \{\text{Ext}[H_{i-1}(\tilde{E}, N_p; Z), A]\} = 0$.

Moreover, we know that $\text{Hom}[H_i(\tilde{E}, N_{p+s}; Z)/(z_{p+s}^1, \dots, z_{p+s}^m), A] = 0$. Because $z_{p+s}^1, \dots, z_{p+s}^m$ belong to $\text{Ker}[H_i(\tilde{E}, N_{p+s}; Z) \rightarrow H_i(\tilde{E}, N_p; Z)]$, this implies that the induced homomorphism

$$\text{Hom}[H_i(\tilde{E}, N_p; Z), A] \longrightarrow \text{Hom}[H_i(\tilde{E}, N_{p+s}; Z), A]$$

is also trivial, and so $\varinjlim \{\text{Hom}[H_i(\tilde{E}, N_p; Z), A]\} = 0$. $\square$

7.17. Theorem. (Kawauchi [2, Theorem 2.3]) *For fixed i , if $H_{i-1}(\tilde{E}; Z)$ and $H_i(\tilde{E}; Z)$ are finitely generated, then for any torsion free abelian group A , there exists a duality*

$$H^i(\tilde{E}; A) \cong H_{n-i+1}(\tilde{E}, \partial\tilde{E}; A).$$

(Recall that $\tilde{E}$ is the universal abelian covering of an exterior of an n -knot, so $\dim(\tilde{E}) = n + 2$.)

Proof. Consider the Mayer-Vietoris sequence

$$\begin{aligned} \longrightarrow H^i(\tilde{E}, N_p; A) \oplus H^i(\tilde{E}, \bar{N}_q; A) &\longrightarrow H^i(\tilde{E}, N_p \cap \bar{N}_q; A) \xrightarrow{\delta} \\ H^{i+1}(\tilde{E}, N_p \cup \bar{N}_q; A) &\longrightarrow H^{i+1}(\tilde{E}, N_p; A) \oplus H^{i+1}(\tilde{E}, \bar{N}_q; A) \longrightarrow. \end{aligned}$$

By Lemmas 7.13 and 7.16, we obtain the exact sequence

$$\longrightarrow 0 \oplus 0 \longrightarrow H^i(\tilde{E}; A) \xrightarrow{\cong} H_c^{i+1}(\tilde{E}; A) \longrightarrow 0 \oplus 0 \longrightarrow.$$

This implies that the composition

$$H^i(\tilde{E}; A) \xrightarrow{\cong} H_c^{i+1}(\tilde{E}; A) \xrightarrow{\cong} H_{n-i+1}(\tilde{E}, \partial\tilde{E}; A)$$

is an isomorphism. $\square$

7.18. Corollary. (cf. Kawauchi [1]) *Let E be the exterior of a 2-knot $(K^2 \subset S^4)$. If $H_1(\tilde{E}; Z)$ is finitely generated over Z , then $H_*(\tilde{E}; Z)$ is finitely generated over Z .*

Proof. Since $H^0(\tilde{E}; Z)$ and $H_1(\tilde{E}; Z)$ are finitely generated, it follows from the universal coefficient theorem for cohomology that $H^0(\tilde{E}; Z)$ and $H^1(\tilde{E}; Z)$ are finitely generated. Also by the duality theorem 7.17,

$$H^i(\tilde{E}; Z) \cong H_{2-i+1}(\tilde{E}, \partial\tilde{E}; Z) = H_{3-i}(\tilde{E}, \partial\tilde{E}; Z), \quad i = 1, 2.$$

So, $H_2(\tilde{E}, \partial\tilde{E}; Z)$ and $H_3(\tilde{E}, \partial\tilde{E}; Z)$ are finitely generated.

On the other hand, $H_i(\partial\tilde{E}; Z) \cong H_i(S^2 \times R^1; Z)$, $i \geq 2$, are also finitely generated. By the exact sequence of the pair $(\tilde{E}, \partial\tilde{E})$, we know $H_i(\tilde{E}; Z)$ is finitely generated for $i \geq 2$, and completing the proof. $\square$

8. The Second Homotopy Groups

For a classical knot $(K^1 \subset S^3)$, the complement $X = S^3 - K^1$ is aspherical, that is $\pi_i(X) = 0$ for all $i \geq 2$, Papakyriakopoulos [1]. On the contrary, Andrews-Curtis [1] have shown that :

8.1. Theorem. (Andrews-Curtis [1]) *For a 2-knot $(K^2 \subset S^4)$, $\pi_2(S^4 - K^2)$ may not be trivial.*

This was shown to be the case if K^2 is the spun of the trefoil. In general, Epstein proved the following :

8.2. Theorem. (Epstein [1]) *If $(K^2 \subset S^4)$ is a non-trivial spun 2-knot, then $\pi_2(S^4 - K^2)$ is a free abelian group of infinite rank.*

8.3. Theorem. (Epstein [1]) *The following conditions are equivalent for a spun 2-knot $(K^2 \subset S^4)$:*

- (i) $X = S^4 - K^2$ is a $K(\pi_1, 1)$,
- (ii) $(K^2 \subset S^4)$ is unknotted,
- (iii) $X = S^4 - K^2$ is a $K(\mathbb{Z}, 1)$.

This Theorem 8.3 means that a spun 2-knot $(K^2 \subset S^4)$ is aspherical if and only if $(K^2 \subset S^4)$ is unknotted. More generally, Dyer-Vasquez have shown that :

8.4. Theorem. (Dyer-Vasquez [1]) For an n -knot $(K^n \subset S^{n+2})$, $n \geq 2$, if $S^{n+2} - K^n$ is aspherical, then $\pi_1(S^{n+2} - K^n) \cong J$. (Moreover, if $n \geq 3$, then $(K^n \subset S^{n+2})$ is unknotted. Refer to 10.2 below.)

This is an answer (modulo unknotting conjecture) to Problem 37 of Fox [4] : *Do there exist aspherical 2-knots ?*

Theorem 8.4 has been generalized in several ways. We refer the reader to Eckmann [1], Kawauchi [3], Swarup [1].

8.5. Theorem. (Eckmann [1], cf. Kawauchi [3]) Let E be the exterior of an n -knot $(K^n \subset S^{n+2})$. If the homomorphism $\pi_n(\partial E) \rightarrow \pi_n(E)$ induced by inclusion is a zero map and if $\pi_{n+1}(E) = 0$, then $S^{n+2} - K^n \simeq E \simeq S^1$ (homotopy equivalence).

The converse of 8.4 has also proved by Kawauchi.

8.6. Theorem. (Kawauchi [1]) For a 2-knot $(K^2 \subset S^4)$, if $\pi_1(S^4 - K^2) \cong J$, then $S^4 - K^2 \simeq S^1$ (homotopy equivalence). (Refer to Giffen [1, Conjecture B'].)

Proof. Let E be an exterior of $(K^2 \subset S^4)$. It suffices to show that $E \simeq S^1$ because $E \simeq S^4 - K^2$. It should be noticed that the universal

abelian covering space $\tilde{E}$ of E is simply connected. Since $H_1(\tilde{E}; Z) = H_2(\tilde{E}; Z) = 0$, it follows from Corollary 7.18 that $H_*(\tilde{E}; Z)$ is finitely generated. So, by Theorem 7.17, for every i we have :

$$H^i(\tilde{E}; Z) \cong H_{3-i}(\tilde{E}, \partial\tilde{E}; Z).$$

Now we have :

$$H_3(\tilde{E}, \partial\tilde{E}; Z) \cong H^0(\tilde{E}; Z) \cong J \cong H_2(S^2 \times R^1; Z) \cong H_2(\partial\tilde{E}; Z),$$

$$\text{and } H_2(\tilde{E}, \partial\tilde{E}; Z) \cong H^1(\tilde{E}; Z) = 0 = H_1(\tilde{E}; Z).$$

In particular, we can check that

$$H_3(\tilde{E}, \partial\tilde{E}; Z) \xrightarrow{\partial_*} H_2(\partial\tilde{E}; Z)$$

is onto. Using the Mayer-Vietoris sequence of the pair $(\tilde{E}, \partial\tilde{E})$, we conclude that $H_3(\tilde{E}; Z) = 0 = H_2(\tilde{E}; Z)$, since $H_3(\partial\tilde{E}; Z) = 0$. Since $H_q(\tilde{E}; Z) = 0$ for $q \geq 4$, $H_*(\tilde{E}; Z) = 0$. Because $\tilde{E}$ is simply connected, $\pi_q(\tilde{E}) = 0$ for all $q \geq 2$. That is, $\tilde{E}$ is a $K(Z, 1)$. $\square$

As R.H.Fox suggested in [4, Problem 36], $\pi_2(S^4 - K^2) = \pi_2(X)$ should be described as a $Z\pi_1(X)$ -module under the action, for example, described in Hu [Hu, p.125~]. For the case of spun 2-knots, an answer has been found in Andrews-Lomonaco [1; 2] and Lomonaco [1].

8.7. Theorem. (Andrews-Lomonaco [2]) If $(K^2 \subset S^4)$ is a spun 2-knot, then $\pi_2(S^4 - K^2) \cong K/[K, K]$, where K is the kernel of the homomorphism

$$i_* : \pi_1(S_0^3 - K^2) \rightarrow \pi_1(S^4 - K^2)$$

induced by inclusion.

There is the same result for a class of ribbon 2-knots in Yanagawa [3, Theorem 4.3], but in the proof of Lemma (3.2) there is a slight gap.

8.8. Theorem. (Andrews-Lomonaco [2], Lomonaco [1]) Let $(K^2 \subset S^4)$ be a

spun 2-knot obtained from a 1-knot $(k^1(S^1) \subset S^3)$. In the notation of Example 1.13, let

$$\langle x_0, x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$$

be the presentation of $\pi_1(S^3 - k^1(S^1))$ with x_0 the image of the generator of $\pi_1(S_0^2 - \beta)$ under the inclusion map. Then,

$$\langle x_1, \dots, x_n \mid \sum_{i=1}^n \left(\frac{\partial r_j}{\partial x_i} \right) x_i = 0 \quad (j=1, \dots, m) \rangle$$

is a presentation of $\pi_2(S^4 - K^2)$ as a $\mathbb{Z}\pi_1(X)$ -module.

8.9. Definition. Let $(K^n \subset S^{n+2})$ be an n -knot. Proposition 3.1 provides us with a map $p: X = S^{n+2} - K^n \rightarrow S^1$, which in some case is a fiber map. We call $(K^n \subset S^{n+2})$ a *fibred n -knot* iff there exists a fiber map $p: X \rightarrow S^1$. A fibred 1-knot will be called, sometimes, a *Neuwirth knot*.

The higher dimensional fibred knots were first studied by Zeeman [6]. For general theorems on fibred knots, we refer to Andrews-Summers [1] and Summers [6]. For a fibred n -knot $(K^n \subset S^{n+2})$, from the homotopy exact sequence of the fibration $F^{n+1} \rightarrow X \xrightarrow{p} S^1$, we know that:

$$\begin{aligned} \pi_1(F^{n+1}) &\cong [\pi_1(X), \pi_1(X)], & \pi_i(F^{n+1}) &\cong \pi_i(X), \quad i \geq 2, \\ \pi_1(X) &\cong \pi_1(F^{n+1}) \times J, \text{ the semidirect product.} \end{aligned}$$

8.10. Theorem. (Andrews-Summers [1, Lemma 4]) Let $(K^2 \subset S^4)$ be a spun 2-knot obtained from a 1-knot $(K^1 \subset S^3)$. Then, $(K^2 \subset S^4)$ is a fibred 2-knot if and only if $(K^1 \subset S^3)$ is a *Neuwirth knot*.

8.11. Theorem. (Andrews-Summers [1, Theorem 5]) Let $(K^2 \subset S^4)$ be a spun 2-knot obtained from a *Neuwirth knot* $(K^1 \subset S^3)$ of genus $g \geq 1$. Then, the universal covering space U of $X = S^4 - K^2$ is homotopy equivalent to $V^+ \vee V^+ \vee \dots \vee V^+$ ($2g$ -fold wedge product), where $V^+ = \bigvee_{\alpha \in \pi_1(X)} S_{\alpha}^2$.

So, $U \simeq \bigvee_{i=1}^{\infty} S_i^2$, the infinite wedge product of 2-spheres.

8.12. Corollary. (Andrews-Summers [1]) Let $(K^2 \subset S^4)$ be a spun 2-knot obtained from a Neuwirth knot of genus g . Then, $\pi_2(S^4 - K^2)$ is a free $\mathbb{Z}\pi_1(X)$ -module on $2g$ generators.

8.13. Corollary. For a 2-knot $(K^2 \subset S^4)$, $\pi_3(S^4 - K^2)$ may not be trivial. (Refer to Andrews-Curtis [1, p.570, Question].)

For a spun 2-knot $(K^2 \subset S^4)$, a finite presentation of $\pi_3(S^4 - K^2)$, as a $\mathbb{Z}\pi_1(X)$ -module, has been given by Lomonaco [2].

Another well-known fibered 2-knots are twist-spun 2-knots given in Example 4.17. We record here the main theorem of Zeeman [6] intuitively.

8.14. Definition. In the notation of 5.17, we will define the v -fold cyclic branched covering space Σ_v of an n -knot $(K^n \subset S^{n+2})$ by the same way as that used in (6.10). For the rigorous definition, refer to Zeeman [6, §5]. For an integer $v \geq 1$, let $(\bar{Y}_i, \bar{V}_-^i, \bar{V}_+^i)$, $i = 1, \dots, v$, be v copies of (Y, V_-, V_+) . Then, Σ_v is obtained as a quotient space of the disjoint union of the Y_i identifying $\bar{V}_+^i$ and $\bar{V}_-^{i+1}$ for every $i \pmod{v}$. So, the 1-fold branched covering space Σ_1 is S^{n+2} .

In the notation of Example 4.17, let $(K^1 \subset S^3)$ and $(\tilde{K}^2 \subset S^4)$ be 1- and 2-knots obtained from $(k^1(S^1) \subset R^3)$ and $(\tilde{K}^2 \subset R^4)$, respectively, by one-point compactifications of R^3 and R^4 . We still call $(\tilde{K}^2 \subset S^4)$ the v -twist-spun 2-knot of $(K^1 \subset S^3)$.

8.15. Theorem. (Zeeman [6]) Given $(K^1 \subset S^3)$, let $(\tilde{K}^2 \subset S^4)$ be the v -twist-spun 2-knot. Let Σ_v be the v -fold cyclic branched covering space

of $(K^1 \subset S^3)$, and Σ_v° the Σ_v with a (closed) 3-disk missing. Then :

(1) If $v \neq 0$, there exists a bundle

$$\Sigma_v^\circ \longrightarrow S^4 - \tilde{K}^2 \longrightarrow S^1$$

with group Z_v .

(2) The closure of each fiber is the Σ_v with an open 3-disk missing, and it is a minimal Seifert manifold for $\tilde{K}^2$.

8.16. Corollary. If $v = \pm 1$, then $(\tilde{K}^2 \subset S^4)$ is unknotted.

8.17. Corollary. The Σ_v° can be embedded in S^4 and Σ_v in S^5 . In particular, if p odd, then the lens spaces $L(p, q)$ with an open 3-disk missing can be embedded in S^4 . (Refer to Schubert [2, Satz 6].)

On the other hand, Sumners [2] has constructed a 2-knot $(K^2 \subset S^4)$ for which $\pi_2(X)$ has non-trivial $Z\pi_1(X)$ -torsion.

8.18. Theorem. (Sumners [2, Theorem 1]) Given any polynomial $\Delta(t) \in \Lambda = ZJ$ with $\Delta(1) = \pm 1$, there exists a 2-knot $(K^2 \subset S^4)$ such that

$$(i) \quad H_1(\tilde{X}; Z) \cong_{\Lambda} \Lambda/(\Delta(t)), \quad H_2(\tilde{X}; Z) \cong_{\Lambda} \Lambda/(\Delta(t^{-1})),$$

(ii) $\pi_2(X)$ has $Z\pi_1(X)$ -torsion $w \cdot c = 0$ where $0 \neq c \in \pi_2(X)$ and $0 \neq w \in Z\pi_1(X)$,

(iii) $\bar{h}_1 : Z\pi_1(X) \rightarrow ZH_1(X; Z) \cong ZJ$ the unique extension of the Hurewicz homomorphism $h_1 : \pi_1(X) \rightarrow H_1(X; Z)$, then $\bar{h}_1(w) = \Delta(t^{-1})$,

(iv) $h_2 : \pi_2(X) \rightarrow H_2(X; Z)$ the Hurewicz homomorphism, then $h_2(c)$ is the Λ -generator of $H_2(X; Z)$.

Refer to Giffen [1]. It is known by Ōmae[1] that the 2-knot given by Sumners is a ribbon 2-knot.

Let U be the universal covering space of $X = S^4 - K^2$. Then, the description of $\pi_2(X)$ in Theorem 8.8 (or 8.7) is obtained from $\pi_2(X) \cong \pi_2(U) \cong H_2(U; \mathbb{Z})$. Using the universal abelian covering space $\tilde{X}$ of X , Gutierrez followed a different approach. That is, he obtained that $\pi_2(X) \cong \pi_2(\tilde{X})$ is the group which $\mathbb{Z}$ acts on (as covering transformations) by the following natural exact sequence.

For a space S , by $\pi_2(S) // \pi_1(S)$ we denote the quotient of $\pi_2(S)$ by the action of $\pi_1(S)$.

8.19. Theorem. (Gutierrez[2, Theorem 0]) *Let $(K^n \subset S^{n+2})$ be an n -knot with a minimal Seifert manifold V . Then, the following is an exact sequence of Λ -modules :*

$$(\pi_2(V) // \pi_1(V)) \otimes_{\mathbb{Z}} \Lambda \xrightarrow{\tau} (\pi_2(Y) // \pi_1(Y)) \otimes_{\mathbb{Z}} \Lambda \longrightarrow \pi_2(X) // \pi_1(X) \longrightarrow 0,$$

where τ is defined by $\tau(\alpha \otimes 1) = v_-(\alpha) \otimes t - v_+(\alpha) \otimes 1$ with $v_{\pm}$ being the homomorphisms of second homotopy induced by the map $v_{\pm}$ given in 5.17.

Recall Proposition 5.18 and Theorem 5.20. Recently, Gutierrez[6] also announced that Theorem 8.19 holds for a minimal Seifert manifold in the weak sense.

9. The Reducibility of 2-Knots

If $(K_1^2 \subset S^4) \cong (K_2^2 \subset S^4)$, then, of course, $X_1 = S^4 - K_1^2 \cong X_2 = S^4 - K_2^2$, and $E_1 = S^4 - {}^{\circ}N(K_1^2; S^4) \cong E_2 = S^4 - {}^{\circ}N(K_2^2; S^4)$. But the converse is an open problem ; that is,

9.1. Problem. *For two 2-knots $(K_1^2 \subset S^4)$ and $(K_2^2 \subset S^4)$, if $X_1 \cong X_2$ (or $E_1 \cong E_2$), then are $(K_1^2 \subset S^4)$ and $(K_2^2 \subset S^4)$ equivalent ?*

Concerning the problem, there are some results due to Gluck [1; 3; 4]. The following is a main result postponing to 9.1.

9.2. Theorem. (Gluck [3, Theorem(15.2)]) *There exist at most two inequivalent 2-knots with a given exterior.*

This holds for every $n \geq 3$, Browder[1], Kato[1], Lashof-Shaneson[1].

To verify this result, we need the following fundamental proposition. Let $H(S^2 \times S^1)$ be the group of homeomorphisms of $S^2 \times S^1$ onto itself, and $\mathcal{D}(S^2 \times S^1)$ the normal subgroup of $H(S^2 \times S^1)$ consisting of those homeomorphisms which are isotopic to the identity. The quotient group $M(S^2 \times S^1) = H(S^2 \times S^1)/\mathcal{D}(S^2 \times S^1)$ will be called the *homeotopy group* of $S^2 \times S^1$. For a map $f \in H(S^2 \times S^1)$, we denote by $[f]$ the isotopy class of f in $M(S^2 \times S^1)$. Since $[f] \in M(S^2 \times S^1)$ induces an automorphism of $H_1(S^2 \times S^1; \mathbb{Z}) \cong \mathbb{Z}$ and an automorphism of $H_2(S^2 \times S^1; \mathbb{Z}) \cong \mathbb{Z}$, we have a homomorphism $\phi : M(S^2 \times S^1) \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Let $r : S^2 \rightarrow S^2$ be the antipodal map and $s : S^1 \rightarrow S^1$ the map defined by $s(x_1, x_2) = (x_1, -x_2)$. Let $M' \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ be the subgroup of $M(S^2 \times S^1)$ consisting of the isotopy classes of the maps $(1, 1), (1, s), (r, 1)$ and (r, s) .

9.3. Proposition. (Gluck [3]) $M(S^2 \times S^2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

To prove this proposition, it suffices to show that $\text{Ker } \phi \cong \mathbb{Z}_2$, since a normal subgroup of order two is central and ϕ splits.

Let Φ_α denote a rotation of S^2 about the x_3 -axis (of R^3) through an angle $2\pi\alpha$ in some fixed direction. Define the homeomorphism

$$T : S^2 \times S^1 \rightarrow S^2 \times S^1 \quad \text{by } T(x, t) = (\Phi_t(x), t),$$

where S^1 will be considered as the space of real numbers modulo 1. The

type of $f \in H(S^2 \times S^1)$ is defined by the image of $[f]$ under ϕ . Thus the kernel of ϕ consists of the isotopy classes of the homeomorphisms of type $(1,1)$. It will be remarked that the identity map 1 on $S^2 \times S^1$ and T are of type $(1,1)$. Gluck proved that any element of $H(S^2 \times S^1)$ of type $(1,1)$ is isotopic either to 1 or T , and also T is not homotopic to 1 , and a fortiori, not isotopic to 1 .

Now, Theorem 9.2 is a corollary to the following lemma.

9.4. Lemma. (Gluck[3, (15.1)]) Let E be a given exterior of a 2-knot. For $f, g \in H(S^2 \times S^1)$, let $(K_1^2 \subset M^4) = (S^2 \times \{0\} \subset E \cup_f (S^2 \times D^2))$ and $(K_2^2 \subset N^4) = (S^2 \times \{0\} \subset E \cup_g (S^2 \times D^2))$, where 0 is the origin of D^2 . If $[g^{-1}][f] = [g^{-1}f]$ in M' , then $(K_1^2 \subset M^4) \cong (K_2^2 \subset N^4)$.

Proof. It should be noted that the homeomorphisms $(1,1)$, $(1,s)$, $(r,1)$ and (r,s) of $S^2 \times S^1$ extend to homeomorphisms of the pair $(S^2 \times \{0\} \subset S^2 \times D^2)$ onto itself. By the isotopy extension theorem, if $[h] \in M'$, the $h : S^2 \times S^1 \rightarrow S^2 \times S^1$ also extends to a homeomorphism of the pair $(S^2 \times \{0\} \subset S^2 \times D^2)$. We construct a homeomorphism $\psi : M^4 \rightarrow N^4$ with $\psi(K_1^2) = K_2^2$ as follows: On E , let ψ be the identity. On $S^2 \times S^1$ viewed as the boundary of E , ψ is still identity, but on $S^2 \times S^1$ viewed as the boundary of $S^2 \times D^2$, $\psi = g^{-1}f$. By the above remarks, $g^{-1}f$ extends to a homeomorphism ψ of $(S^2 \times \{0\} \subset S^2 \times D^2)$, which we take as the definition of ψ on $S^2 \times D^2$. Thus ψ is a required homeomorphism. $\square$

9.5. Corollary. Let $(K^2 \subset S^4)$ be a 2-knot with an exterior E . For $f \in H(S^2 \times S^1)$, let $(K_0^2 \subset M^4) = (S^2 \times \{0\} \subset E \cup_f (S^2 \times D^2))$. If $[f] \in M'$, then $(K_0^2 \subset M^4) \cong (K^2 \subset S^4)$.

Let $(K^2 \subset S^4)$ be a 2-knot with an exterior E , and let $(K_0^2 \subset M^4) = (S^2 \times \{0\} \subset E \cup_T (S^2 \times D^2))$. Proposition 9.3 and Lemma 9.4 imply that the original 2-knot $(K^2 \subset S^4)$ and the new pair $(K_0^2 \subset M^4)$ give the only possible locally flat embeddings of a 2-sphere in a 4-manifold with an exterior homeomorphic to E . The followings have been proved :

9.6. Proposition. (Gluck [3, §16]) *The above $M^4 = E \cup_T (S^2 \times D^2)$ is a homotopy 4-sphere.*

9.7. Proposition. (Gluck [3, (15.3)]) *The homeomorphism $T : S^2 \times S^1 \rightarrow S^2 \times S^1$ does not extend to a continuous map of $S^2 \times D^2$ onto itself.*

Now we consider the problem of extending T to a homeomorphism of an exterior E .

The family $\{\Phi_t\}$, $t \in I$, is a one-parameter family of homeomorphisms of S^2 beginning and ending at the identity. Let M^3 be a compact orientable 3-manifold with $\partial M^3 \cong S^2$. If $\{\Phi_t\}$ can be extended to a one-parameter family of homeomorphisms $\{\tilde{\Phi}_t\}$ of M^3 , we say that M^3 is *spinable*.

9.8. Theorem. (Gluck [3, (17.1)]) *Let $(K^2 \subset S^4)$ be a 2-knot with an exterior E , and let V be a Seifert manifold for it. If V is spinable, then the homeomorphism T extends to a homeomorphism of E .*

Proof. Since T is defined by $T(x, t) = (\Phi_t(x), t)$, T is clearly isotopic to the homeomorphism T^* defined by

$$\begin{aligned} T^*(x, t) &= (\Phi_{2t+1/2}(x), t) & \text{for } 0 \leq t \leq 1/4 \text{ and } 3/4 \leq t \leq 1; \\ T^*(x, t) &= (x, t) & \text{for } 1/4 \leq t \leq 3/4. \end{aligned}$$

So, it suffices to extend T^* to a homeomorphism of E . Now we identify

$\partial N(K^2; S^4)$ with $S^2 \times S^1$ and K^2 with $S^2 \times \{0\}$. Let $\bar{V} = V \cap E$ and let $N(\bar{V}; E)$ be a regular neighborhood of $\bar{V}$ in E . Since $\bar{V}$ is locally flat, $N(\bar{V}; E)$ is homeomorphic to $\bar{V} \times [-1/4, 1/4]$, and we may assume that $N(\bar{V}; E) \cap (S^2 \times S^1) \cong S^2 \times [-1/4, 1/4]$. Since V is spinable and $\bar{V} \cong V$, the one-parameter family of homeomorphisms $\{\Phi_t\}$ of $S^2 = S^2 \times \{0\}$ extend to a one-parameter family of homeomorphisms $\{\tilde{\Phi}_t\}$ of $\bar{V}$ beginning and ending at the identity. Now we can extend T^* on $S^2 \times S^1$ to T^* on $(S^2 \times S^1) \cup \{\bar{V} \times [-1/4, 1/4]\}$ by defining

$$T^*(x, t) = (\tilde{\Phi}_{2t+1/2}(x), t) \text{ for } x \in \bar{V} \text{ and } -1/4 \leq t \leq 1/4.$$

Let $W = S^4 - (\bar{V} \cup N(K^2; S^4))$; which is open in E . Since on the boundary of W , T^* has been defined as the identity, we can extend T^* over W via the identity. $\square$

9.9. Corollary. (Gluck [3, p.362]) *Under the same condition in 9.8, let $(K_0^2 \subset M^4) = (S^2 \times \{0\} \subset E \cup_T (S^2 \times D^2))$. Then, $(K_0^2 \subset M^4) \cong (K^2 \subset S^4)$.*

9.10. Proposition. (i) *Lens spaces (including $S^2 \times S^1$) with an open 3-disk missing are spinable.*

(ii) *Let M_i be a spinable 3-manifold, and $\hat{M}_i$ a closed 3-manifold obtained from M_i by pasting a 3-disk D_i^3 on ∂M_i , $i = 1, 2$. Then, the connected-sum $\hat{M}_1 \# \hat{M}_2$ with an open 3-disk missing is spinable.*

(i) is obvious, and (ii) is obtained by the same argument of Gluck[5, (3.2)]. Combining this with Theorem 5.13 (cf. Exercise 5.16), we have :

9.11. Theorem. (cf. Gluck [3, (22.1)]) *Every spun 2-knot is uniquely determined by its exterior. Moreover, every ribbon 2-knot is uniquely determined by its exterior.*

For higher dimensional knots, refer to Cappell [1] and Levine [8].

9.12. Problem. Does it hold that *the complements of two 2-knots are homeomorphic if and only if their exteriors are homeomorphic*?

This problem is affirmative for the case $n = 1$ by Edwards [1] and the case $n \geq 3$ by González-Acuña [1, §10].

Added in Proof. Recently, Cappell-Shaneson have shown that :

9.13. Theorem. (Cappell-Shaneson[2]) *For $n = 3, 4$, there are infinitely many pairs of smooth n -knots such that each pair consists of two inequivalent knots with diffeomorphic exteriors. The pairs are distinguished by their first Alexander polynomials.*

They also announced that " C.McA. Gordon has given examples of 2-knots that are not determined by their complements ".

10. Unknotting Conjecture and Prime Decompositions

From Definition 1.3, the following is obvious.

10.1. Proposition. *If $(K^n \subset S^{n+2})$ is unknotted, then $S^{n+2} - K^n \approx S^1$.*

The converse is affirmative except for $n = 2$; that is,

10.2. Proposition. (Papakyriakopoulos [1], Stallings [2], Levine [3], Shaneson [1]) *Given an n -knot $(K^n \subset S^{n+2})$ with $n \neq 2$, if $S^{n+2} - K^n \approx S^1$, then $(K^n \subset S^{n+2})$ is unknotted.*

More strictly, if $\pi_i(S^{n+2} - K^n) \cong \pi_i(S^1)$, $i \leq \left[\frac{n+1}{2} \right]$, then $(K^n \subset S^{n+2})$ is unknotted.

10.3. Unknotting Conjecture. For a 2-knot $(K^2 \subset S^4)$, if $S^4 - K^2 \simeq S^1$, then $(K^2 \subset S^4)$ is unknotted.

By Theorem 8.6, this is equivalent to :

10.3'. Unknotting Conjecture. For a 2-knot $(K^2 \subset S^4)$, if $\pi_1(S^4 - K^2) \cong J$, then $(K^2 \subset S^4)$ is unknotted.

By Corollary 3.6, the following question comes to mind :

10.4. Conjecture. Let $(K^2 \subset R^4)$ be a standard representative. If the equatorial cross section $(K^2 \cap R_0^3 \subset R_0^3)$ is unknotted, then $(K^2 \subset R^4)$ is unknotted.

There exist two papers relevant to this conjecture; one is Hosokawa[1] and the other is Yanagawa [4]. Yanagawa [4, Theorem (2.2)] announced an affirmative answer for ribbon 2-knots, but there is a gap in the proof of Lemma (2.4). We record the theorem as a conjecture :

10.5. Conjecture. (Yanagawa [4]) Let $(K^2 \subset R^4)$ be a simple symmetric representative. If the equatorial cross section $(K^2 \cap R_0^3 \subset R_0^3)$ is unknotted, then $(K^2 \subset R^4)$ is unknotted.

Before discussing the unknotting conjecture, we introduce here a fundamental concept of "composition" (and so "decomposition") of n -knots due to Schubert [1], Mazur [2], (see also Noguchi [3], Kervaire [2], etc.), and give some elementary properties.

10.6. Definition. Let $(K_1^n \subset S^{n+2})$ and $(K_2^n \subset S^{n+2})$ be n -knots, and let B_1^{n+2} and B_2^{n+2} be $(n+2)$ -disks in S^{n+2} such that $(K_1^n \cap B_1^{n+2} \subset B_1^{n+2})$ and $(K_2^n \cap B_2^{n+2} \subset B_2^{n+2})$ are equivalent to the standard disk-pair $(D^n \subset D^{n+2})$.

(Such the $(n+2)$ -disk B_i^{n+2} is easily obtained by taking a regular neighborhood of a point on K_i^n .) Then, the *composition* $(K_1^n \subset S^{n+2}) \# (K_2^n \subset S^{n+2})$ of two n -knots $(K_1^n \subset S^{n+2})$ and $(K_2^n \subset S^{n+2})$ is a new n -knot $(K^n \subset S^{n+2})$ obtained by matching the boundaries $\partial(S^{n+2} - \circ B_1^{n+2})$ and $\partial(S^{n+2} - \circ B_2^{n+2})$ using an orientation reversing homeomorphism ζ such that $\zeta \partial(K_1^n - \circ B_1^{n+2}) = \partial(K_2^n - \circ B_2^{n+2})$ and $\zeta|_{(K_1^n - \circ B_1^{n+2})}$ is also orientation reversing. (Recall that S^{n+2} and K_i^n are oriented.)

Conversely, we will say that $(K_1^n \subset S^{n+2}) \# (K_2^n \subset S^{n+2})$ is a *decomposition* for $(K^n \subset S^{n+2})$, and we denote

$$(K^n \subset S^{n+2}) \cong (K_1^n \subset S^{n+2}) \# (K_2^n \subset S^{n+2}).$$

By the homogeneity theorem of Newman-Gugenheim (cf. Gugenheim [1]), the operation $\#$ of composition is well-defined up to equivalence, and also it is associative and commutative. So the *composition* (and *decomposition*)

$$[K^n \subset S^{n+2}] = [K_1^n \subset S^{n+2}] \# [K_2^n \subset S^{n+2}] = [(K_1^n \subset S^{n+2}) \# (K_2^n \subset S^{n+2})]$$

of two n -knot types is well-defined.

For n -knots $(K^n \subset R^{n+2})$'s in R^{n+2} , we can also define, in a natural way, a *composition* (and *decomposition*).

10.7. Theorem. (Mazur [2], etc.) Given $n \geq 1$. The set of all n -knot types $K_n = \{[K^n \subset S^{n+2}]\}$ is a commutative semi-group under the composition $\#$. In particular, the trivial n -knot type $[S^n \subset S^{n+2}]$ is a unit.

To give a basic theorem due to B. Mazur, we need a definition.

10.8. Definition. A $*$ -homeomorphism $\psi^*: S^{n+2} \rightarrow S^{n+2}$ will be an orientation preserving homeomorphism which is piecewise linear except at a finite number of points $v_1, \dots, v_m \in S^{n+2}$. It is a homeomorphism such

that $\psi^*|_{S^{n+2}} - \cup v_i$ is a piecewise linear map — simplicial with respect to a possibly infinite subdivision of the open complexes involved.

Two n -knots $(K_1^n \subset S^{n+2})$ and $(K_2^n \subset S^{n+2})$ will be called **-equivalent*, denoted by $(K_1^n \subset S^{n+2}) \stackrel{*}{\cong} (K_2^n \subset S^{n+2})$, iff there exists a **-homeomorphism* $\psi^* : S^{n+2} \rightarrow S^{n+2}$ with $\psi^*(K_1^n) = K_2^n$.

An element $[K^n \subset S^{n+2}] \in K_n$ is *invertible* iff there exists an element $[\tilde{K}^n \subset S^{n+2}] \in K_n$ such that $[K^n \subset S^{n+2}] \# [\tilde{K}^n \subset S^{n+2}] = [S^n \subset S^{n+2}]$.

10.9. Inverse Theorem. (Mazur [2]) $[K^n \subset S^{n+2}] \in K_n$ is invertible if and only if $(K^n \subset S^{n+2}) \stackrel{*}{\cong} (S^n \subset S^{n+2})$.

(10.10) Sketch Proof of "only if" part of 10.9. Let $[\tilde{K}^n \subset S^{n+2}]$ be the knot type with $[K^n \subset S^{n+2}] \# [\tilde{K}^n \subset S^{n+2}] = [S^n \subset S^{n+2}]$.

Let $\{r_i\}_{i=1}^\infty$ be a monotonic convergent sequence of positive numbers with $\lim_{i \rightarrow \infty} r_i = 0$. We compactify R^{n+2} with a point ∞ to give S^{n+2} , and we take an infinite number of concentric $(n+1)$ -spheres $\Sigma(r_i)$ in S^{n+2} with common center ∞ such that the diameter of $\Sigma(r_i) = 2r_i$. We consider two infinite compositions of $(K^n \subset S^{n+2})$'s and $(\tilde{K}^n \subset S^{n+2})$'s as illustrated in Figure 10.1.

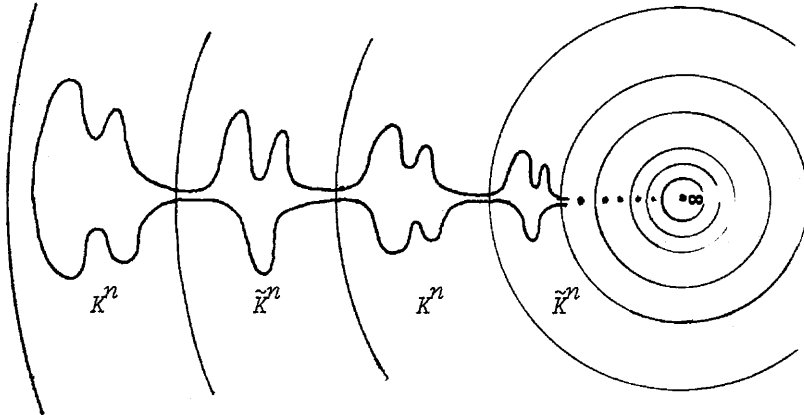


Figure 10.1

$$(K^* \subset S^{n+2}) \cong (K^n \subset S^{n+2}) \# (\tilde{K}^n \subset S^{n+2}) \# (K^n \subset S^{n+2}) \# \dots \cup \{\infty\},$$

$$(K^{**} \subset S^{n+2}) \cong (\tilde{K}^n \subset S^{n+2}) \# (K^n \subset S^{n+2}) \# (\tilde{K}^n \subset S^{n+2}) \# \dots \cup \{\infty\}.$$

$$\text{Since } (K^n \subset S^{n+2}) \# (\tilde{K}^n \subset S^{n+2}) \cong (S^n \subset S^{n+2}) \cong (\tilde{K}^n \subset S^{n+2}) \# (K^n \subset S^{n+2})$$

from our assumption,

$$(K^* \subset S^{n+2}) \stackrel{*}{\cong} (S^n \subset S^{n+2}) \stackrel{*}{\cong} (K^{**} \subset S^{n+2}).$$

From the constructions, we know that :

$$(K^* \subset S^{n+2}) \cong (K^n \subset S^{n+2}) \# (K^{**} \subset S^{n+2}) \stackrel{*}{\cong} (K^n \subset S^{n+2}),$$

$$\text{and } (K^{**} \subset S^{n+2}) \cong (\tilde{K}^n \subset S^{n+2}) \# (K^* \subset S^{n+2}) \stackrel{*}{\cong} (\tilde{K}^n \subset S^{n+2}),$$

so we conclude that :

$$(K^n \subset S^{n+2}) \stackrel{*}{\cong} (S^n \subset S^{n+2}) \stackrel{*}{\cong} (\tilde{K}^n \subset S^{n+2}). \quad \square$$

To prove the "if" part of 10.9, we need a lemma :

10.11. Lemma. Suppose that $(K^n \subset S^{n+2}) \stackrel{*}{\cong} (S^n \subset S^{n+2})$ and that there exists a $*$ -homeomorphism $\psi^* : S^{n+2} \rightarrow S^{n+2}$ such that $\psi^*(K^n) = S^n$ and ψ^* is piecewise linear except at a point $v \in S^{n+2}$ with $v \notin K^n$. Then, $(K^n \subset S^{n+2}) \cong (S^n \subset S^{n+2})$.

Proof. We subdivide S^{n+2} so that v must be a vertex. Let $St(v)$ be a star neighborhood of v , and let B^{n+2} be $S^{n+2} - \circ St(v)$, which is an $(n+2)$ -disk. From our assumption, we may assume that $K^n \subset \circ B^{n+2}$ and $\psi^*|_{B^{n+2}}$ is piecewise linear. By Gugenheim [1], $\psi^*|_{B^{n+2}}$ can be extended to an orientation preserving piecewise linear homeomorphism $\tilde{\psi}^*$ of S^{n+2} onto itself. This $\tilde{\psi}^*$ yields the equivalence $(K^n \subset S^{n+2}) \cong (S^n \subset S^{n+2})$. $\square$

(10.12) *Sketch Proof of "if" part of 10.9.* By Lemma 10.11, we assume

that there exists a $*$ -homeomorphism $\psi^*: S^{n+2} \rightarrow S^{n+2}$ such that $\psi^*(K^n) = S^n$ and ψ^* is piecewise linear except at a point $v \in K^n$. We subdivide S^{n+2} so that v is a vertex. Let $St(v)$ be a star neighborhood of v . Since K^n is locally flat, $C = K^n \cap St(v)$ is unknotted in $St(v)$, that is, $(C \subset St(v))$ is equivalent to the standard disk-pair $(D^n \subset D^{n+2})$, and $(\partial C \subset \partial St(v)) \cong (S^{n-1} \subset S^{n+1})$.

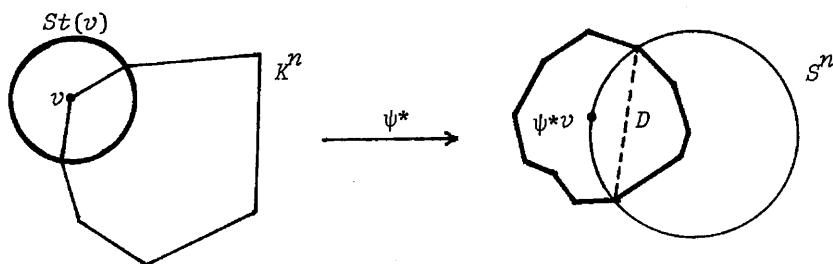


Figure 10.2

Let D be a proper n -disk in $\psi^*(St(v))$ with the properties :

- (i) $\partial D = \psi^*(\partial C)$,
- (ii) $\psi^*(v) \notin D$,
- (iii) the n -knot $K = D \cup (S^n - \psi^*(C))$ is still trivial.

Then, we have an n -knot $(\psi^{*-1}(K) \subset S^{n+2}) \cong (K' \subset S^{n+2})$. $\partial St(v)$ gives a decomposition $(K' \subset S^{n+2}) \cong (K^n \subset S^{n+2}) \# (\tilde{K}^n \subset S^{n+2})$, where K^n is represented by $K^n \cap (S^{n+2} - \circ St(v))$ and $\tilde{K}^n$ by $\psi^{*-1}(D) \subset St(v)$. It will be noticed that ψ^* gives a $*$ -equivalence $(K' \subset S^{n+2}) \stackrel{*}{\cong} (K \subset S^{n+2})$ such that ψ^* is piecewise linear except at the point $v \notin K'$. Because $(K \subset S^{n+2}) \cong (S^n \subset S^{n+2})$, by Lemma 10.11 we conclude that

$$(K^n \subset S^{n+2}) \# (\tilde{K}^n \subset S^{n+2}) \cong (K' \subset S^{n+2}) \cong (K \subset S^{n+2}) \cong (S^n \subset S^{n+2}). \quad \square$$

10.13. Proposition. *Given $n \neq 2$. An n -knot type $[K^n \subset S^{n+2}]$ is invertible if and only if $[K^n \subset S^{n+2}] = [S^n \subset S^{n+2}]$.*

Proof. The "if" part is obvious (even if $n = 2$), and so we observe the "only if" part. From the definition of $*$ -equivalence, in (10.10) we have *topological* homeomorphisms $\psi_1, \psi_2 : S^{n+2} \rightarrow S^{n+2}$ with $\psi_1(S^n) = K^*$ and $\psi_2(S^n) = K^{**}$. So S^{n+2} has piecewise linear structures induced from the standard structure of S^{n+2} via ψ_1 and ψ_2 , so that K^* and K^{**} are finite subcomplexes, respectively. From the uniqueness of the piecewise linear structure of S^{n+2} with $n \neq 2$, we conclude this 10.13.

10.14. Conjecture. A 2-knot type $[K^2 \subset S^4]$ is invertible if and only if $[K^2 \subset S^4] = [S^2 \subset S^4]$.

10.15. Definition. An n -knot type $[K^n \subset S^{n+2}]$ is said to be *prime* if there exist no decompositions $[K^n \subset S^{n+2}] = [K_1^n \subset S^{n+2}] \# [K_2^n \subset S^{n+2}]$ with both $[K_1^n \subset S^{n+2}]$ and $[K_2^n \subset S^{n+2}]$ non-trivial.

10.16. Proposition. (Schubert[1], Sosinskii[3]) Given $n \neq 2$. For any $[K^n \subset S^{n+2}] \in K_n$, there exists a prime decomposition

$$[K^n \subset S^{n+2}] = [K_1^n \subset S^{n+2}] \# [K_2^n \subset S^{n+2}] \# \dots \# [K_u^n \subset S^{n+2}].$$

For the case $n = 1$, the summands $[K_i^n \subset S^{n+2}]$ are uniquely determined.

Now we have the following important problem.

10.17. Problem. (i) Does there exist a prime decomposition for any 2-knot type $[K^2 \subset S^4]$?

(ii) If (i) is affirmative, is the prime decomposition unique?

The Sosinskii's proof of the existence of prime decomposition remains valid for 2-knot types if Unknotting Conjecture 10.3 (or 10.3') is affirmative.

We return to the unknotting conjecture. From the decomposition of a 2-knot, we can restate Conjecture 10.5 as follows :

10.18. Problem. Let $(K^2 \subset R^4)$ be a standard representative. If it has one maximum (or one minimum), then is $(K^2 \subset R^4)$ unknotted ?

Hosokawa [1] discussed about a special case of this problem, and noted a sufficient condition for a 2-knot to be unknotted, but in the proof of Lemma 2, there is a slight gap.

10.19. Conjecture. (Hosokawa [1]) Let $(K^2 \subset R^4)$ be a standard representative. If it has one maximum and two minima, then it is unknotted.

We reformulate Lemma 2 in Hosokawa [1] as follows :

10.20. Problem. Let $(O_1^1 \cup O_2^1 \subset R^3)$ be a trivial 1-link with 2 components, such that $(O_1^1 \cup O_2^1) \cap R_+^3 = O_1^1$ and $(O_1^1 \cup O_2^1) \cap R_-^3 = O_2^1$. Let $(K^1 \subset R^3)$ be a 1-knot obtained from $O_1^1 \cup O_2^1$ by a fusion along $B^+ : I \times D^1 \rightarrow R^3$, and we suppose that $(K^1 \subset R^3)$ is trivial. Does there exist an isotopic deformation $\{\eta_t\}$ of R^3 such that $\eta_t|_{O_1^1 \cup O_2^1} = \text{identity}$ for $0 \leq t \leq 1$ and $\eta_1 B^+(I \times D^1) \cap R^2$ consists of a single arc ?

About most concrete examples, it is easily checked that Problem 10.20 is affirmative. So, the 2-knot $(K^2 \subset R^4)$ given in Example 3.14 is, in fact, unknotted. We record here two examples of unknotted 2-knots.

10.21. Example. (Terasaka-Hosokawa[1], Hosokawa[1]) The 2-knot $(K^2 \subset R^4)$ shown in Figure 10.3 has 9_{46} (see Reidemeister [Re]) as the equatorial cross section $(K^2 \cap R_0^3 \subset R_0^3)$. By Lemma 2.5, we can change the levels of the saddle-bands B_1 and B_2 , and we have an equivalent 2-knot $(K'^2 \subset R^4)$

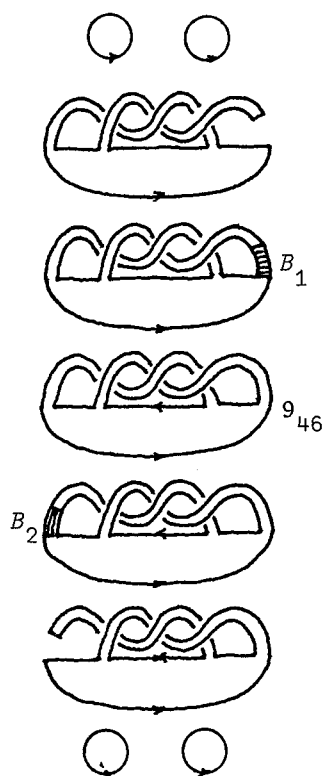


Figure 10.3

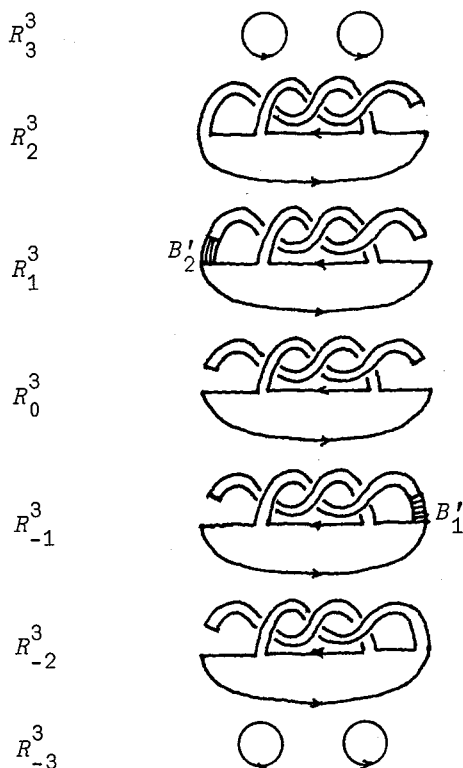


Figure 10.4

shown in Figure 10.4 whose equatorial cross section is unknotted. Since we have isotopic deformations for B'_1 and B'_2 which satisfy the condition of 10.20, we conclude that $(K'^2 \subset R^4) \cong (K^2 \subset R^4) \cong (S^2 \subset R^4)$.

Unknottedness of Example 3.14 and Example 10.21 have been checked by changing the levels of saddle-bands. The following can not be checked to be unknotted by such the method.

10.22. Example. (Yanagawa [4, p.171]) *There exists a simple symmetric representative $(K^2 \subset R^4)$ of the trivial 2-knot type $[S^2 \subset R^4]$ such that the equatorial cross section $(K^2 \cap R_0^3 \subset R_0^3)$ is a non-trivial 1-knot.*

Let $(K^1 \subset R^3)$ be the ribbon 1-knot given in Figure 10.5. The associated ribbon 2-knot $(K^2(K^1) \subset R^4)$ is one of such the example. (Recall Remark 2.14.) $(K^1 \subset R^3)$ is the so-called Kinoshita-Terasaka knot whose first Alexander polynomial $\Delta_1^1(t) = 1$; Kinoshita-Terasaka [1, Fig.(13)].

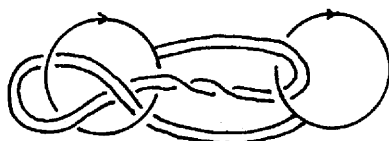


Figure 10.5

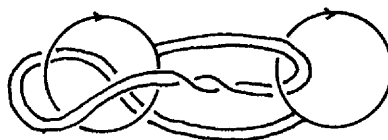


Figure 10.6

To see that $(K^2(K^1) \subset R^4)$ is unknotted, we change the crossing of B as shown in Figure 10.6, and obtain a trivial 1-knot. It follows from Proposition 2.15 that $(K^2(K^1) \subset R^4) \cong (S^2 \subset R^4)$. By the same way, we can construct infinitely many such the representatives.

10.23. Exercise. Construct a Seifert manifold V^3 for $(K^2(K^1) \subset R^4)$ such that $V^3 \cong D^3$ and $V^3 \cap R_0^3$ is a 2-manifold of genus 3.

10.24. Problem. Find a sufficient condition for a 2-knot to be trivial.

Recall the classical Dehn's lemma, Papakyriakopoulos[1]. We will concern a generalization of the lemma.

10.25. Definition. An n -knot $(K^n \subset S^{n+2})$ is called to be an n -knot of Dehn's type iff there exists a map $f: D^{n+1} \rightarrow S^{n+2}$ with $S(f) \subset \partial D^{n+1}$ and $f(\partial D^{n+1}) = K^n$, where $S(f) = \text{Cl}\{x \in D^{n+1} \mid f^{-1}f(x) \neq \{x\}\}$.

10.26. Theorem. (Kawauchi [3]) Let E be the exterior of a 2-knot $(K^2 \subset S^4)$. Then, $(K^2 \subset S^4)$ is of Dehn's type if and only if the homomorphism $\pi_2(\partial E) \rightarrow \pi_2(E)$ induced by inclusion is a zero map.

Recall Theorem 8.5. For higher dimensional case, refer to Kawauchi[3].

10.27. Problem. Does any 2-knot of Dehn's type have the complement X homotopy equivalent to S^1 ?

10.28. Definition. By a 3-dimensional handlebody U_g of genus g , we mean an oriented 3-manifold which is a disk-sum of g copies of $D^2 \times S^1$; and by a surface F_g of genus g we mean the oriented boundary surface ∂U_g .

Let $\mathfrak{f} : F_g \rightarrow R^4$ be a locally flat embedding. The embedded surface $(\mathfrak{f}(F_g) \subset R^4)$ is said to be *unknotted* iff there exists an embedding $\tilde{\mathfrak{f}} : U_g \rightarrow R^4$ with $\tilde{\mathfrak{f}}|_{\partial U_g} = \mathfrak{f}$. This is a natural generalization of Definition 1.3 and Proposition 1.4. In fact, we have the following :

10.29. Theorem. (cf. Hosokawa-Kawauchi [1], Klingmann [1]) Let $\mathfrak{f}_1, \mathfrak{f}_2$ be embeddings of F_g into R^4 . If $(\mathfrak{f}_1(F_g) \subset R^4)$ and $(\mathfrak{f}_2(F_g) \subset R^4)$ are unknotted, then there exists an isotopic deformation $\{\eta_t\}$ of R^4 with $\eta_1 \mathfrak{f}_1(F_g) = \mathfrak{f}_2(F_g)$.

By Theorem 5.1, we have the following, which corresponds, in certain sense, to the unknotting number for the classical knots.

10.30. Theorem. (Hosokawa-Kawauchi[1]) For any 2-knot $(K^2 \subset R^4)$, there exists a finite number of 1-handles $h_1^1, \dots, h_u^1 : D^1 \times D^2 \rightarrow R^4$ such that $(K^2(h_1^1, \dots, h_u^1) \subset R^4)$ is an unknotted surface of genus u .

(Recall the notation 5.3.)

11. Concepts of Cobordism

Fox-Milnor [1; 2] introduced a concept of knot cobordism for classical knots, and higher dimensional cases have studied in Haefliger[1], Kervaire [2], Noguchi [3], Levine [6; 7], etc..

11.1. Definition. Two n -knots $(K_1^n \subset S^{n+2})$ and $(K_2^n \subset S^{n+2})$ (and their knot types $[K_1^n \subset S^{n+2}]$ and $[K_2^n \subset S^{n+2}]$) are said to be *cobordant*, denoted by $(K_1^n \subset S^{n+2}) \sim (K_2^n \subset S^{n+2})$ (and $[K_1^n \subset S^{n+2}] \sim [K_2^n \subset S^{n+2}]$), iff there is a proper locally flat oriented submanifold W of $S^{n+2} \times I$, such that

$$(i) \quad W = S^n \times I, \text{ and}$$

$$(ii) \quad (W \cap S^{n+2} \times \{0\} \subset S^{n+2} \times \{0\}) \cong (K_1^n \subset S^{n+2}),$$

$$(W \cap S^{n+2} \times \{1\} \subset S^{n+2} \times \{1\}) \cong -(K_2^n \subset S^{n+2}).$$

An n -knot $(K^n \subset S^{n+2})$ (and its knot type $[K^n \subset S^{n+2}]$) is said to be *null-cobordant* iff $(K^n \subset S^{n+2}) \sim (S^n \subset S^{n+2})$.

11.2. Theorem. The cobordant relation $\sim$ is an equivalence relation in K_n . By $\langle K^n \subset S^{n+2} \rangle = \langle [K^n \subset S^{n+2}] \rangle$, we denote the equivalence class determined by the type $[K^n \subset S^{n+2}] \in K_n$.

We have the following lemma :

11.3. Lemma. (1) For any $[K^n \subset S^{n+2}]$, $[K^n \subset S^{n+2}] \# (-[K^n \subset S^{n+2}])$ is null-cobordant.

(2) If $[K_1^n \subset S^{n+2}]$ and $[K_2^n \subset S^{n+2}]$ are both null-cobordant, then so is $[K_1^n \subset S^{n+2}] \# [K_2^n \subset S^{n+2}]$.

(3) If $[K_1^n \subset S^{n+2}]$ and $[K_1^n \subset S^{n+2}] \# [K_2^n \subset S^{n+2}]$ are both null-cobordant, then so is $[K_2^n \subset S^{n+2}]$.

11.4. Definition. It follows from Lemma 11.3(2) the sum operation

$$\langle K_1^n \subset S^{n+2} \rangle \# \langle K_2^n \subset S^{n+2} \rangle = \langle [K_1^n \subset S^{n+2}] \# [K_2^n \subset S^{n+2}] \rangle$$

is well-defined. The set $C_n = K_n / \sim$ of the equivalence classes inherits the operation $\#$ from the semi-group K_n , and with this operation forms an abelian group, which will be called the *n-dimensional knot cobordism group*. The identity element of this group is the class $\langle S^n \subset S^{n+2} \rangle$ of null-cobordant knots, and the inverse of a class $\langle K^n \subset S^{n+2} \rangle$ is the class $-\langle K^n \subset S^{n+2} \rangle = \langle -[K^n \subset S^{n+2}] \rangle$.

11.5. Theorem. (Fox-Milnor [2], Kervaire [2, Theorem III6]) :

For any $n \geq 1$, $C_{2n-1} \neq 0$, and $C_{2n} = 0$. In particular, $C_2 = 0$.

More precisely, Levine [6] gave a faithful representation of C_{2n-1} onto a matrix-group.

11.6. Definition. In this section, all matrices will be integral.

A square matrix A is *null-cobordant* iff it is congruent to a matrix of the form $\begin{pmatrix} 0 & N_1 \\ N_2 & N_3 \end{pmatrix}$, where N_i are square matrices of the same size. If A_1 and A_2 are matrices, we define the *block sum* $A_1 \oplus A_2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$.

We will say square matrices A_1 and A_2 are *cobordant* iff $A_1 \oplus -A_2$ is null-cobordant. We say that a square matrix A has property ϵ ($=\pm 1$) iff $A \oplus \epsilon A$ is unimodular.

11.7. Proposition. (Levine [6]) The cobordism classes G_ϵ of matrices with property ϵ form an abelian group under block sum. The zero class in G_ϵ consists of the null-cobordant matrices of type ϵ .

11.8. If a square matrix A has property $+$, then $A + A'$ has signature a multiple of 8 (see Milnor [1, Theorem 1]). Let G_+^0 be the subset of

G_+ , defined by matrices A with signature $(A+A')$ a multiple of 16. It will be noticed that signature $(A+A')$ depends only on the cobordism class of A . G_+^0 is a subgroup of G_+ of index 2.

11.9. Definition. Let $n = 2m - 1$. Let $(K^n \subset S^{n+2})$ be an n -knot and V a Seifert manifold for K^n ; V is of dimension $2m$.

We define a pairing :

$$\Theta : H_m(V; \mathbb{Z}) \otimes H_m(V; \mathbb{Z}) \longrightarrow \mathbb{Z}$$

by $\Theta(a \otimes b) = \text{lk}(a \otimes v_+(b))$. By (6.11)-(3), we have the formula :

$$\Theta(a \otimes b) + (-1)^m \Theta(b \otimes a) = -a \cdot b \quad (3')$$

A basis for the torsion-free part of $H_m(V; \mathbb{Z})$ determines a matrix S representing Θ — such a matrix S will be called a *Seifert matrix* for $(K^n \subset S^{n+2})$ (associated with V). It follows from (3') that S satisfies property $(-1)^m$, since the intersection pairing of $H_m(V; \mathbb{Z})$ is unimodular.

If S_1, S_2 are Seifert matrices for $(K_1^n \subset S^{n+2}), (K_2^n \subset S^{n+2})$ (associated with V_1, V_2), respectively, then $S_1 \oplus S_2$ is a Seifert matrix for $(K_1^n \subset S^{n+2}) \# (K_2^n \subset S^{n+2})$, since the disk-sum of V_1 and V_2 in S^{n+2} is a Seifert manifold for $(K_1^n \subset S^{n+2}) \# (K_2^n \subset S^{n+2})$, and then $-S_1$ is also a Seifert matrix for $-(K_1^n \subset S^{n+2})$.

From the definition, $tS + (-1)^m S'$ is a presentation matrix $P_m(t)$ of $H_m(\tilde{X}; \mathbb{Q})$, $n = 2m - 1$, recall (6.11).

11.10. Theorem. (Levine [6]) If $(K^n \subset S^{n+2})$ is null-cobordant and S is a Seifert matrix for $(K^n \subset S^{n+2})$, $n=2m-1$. Then S is null-cobordant. The implication is reversible for $m \geq 2$.

11.11. Theorem. (Levine [6]) Let $\epsilon_m = (-1)^m$, and let $\phi_m : C_{2m-1} \rightarrow G_{\epsilon_m}$ be a homomorphism defined by assigning to the knot cobordism class, the

cobordism class of any Seifert matrix. Then, ϕ_m is

- (a) an isomorphism onto G_{ε_m} for $m \geq 3$,
- (b) an isomorphism onto G_+^0 for $m = 2$,
- (c) an epimorphism for $m = 1$.

11.12. Exercise. Prove that $C_2 = 0$ (in Theorem 11.5). (Hint: Use the construction of a Seifert manifold V in 5.9. For each hyperbolic critical point v of V , we can take a simple loop on ${}^{\circ}C_u \subset {}^{\circ}V$, which intersects once with the 1-handle corresponding to v . The loop may bound in S_+^5 a locally flat 2-disk.)

We will give here an answer to Problem 2.10(a) due to Summers [7].

11.13. Definition. We identify two cobordisms $(W_1 \subset S^{n+2} \times I)$ and $(W_2 \subset S^{n+2} \times I)$ between $(K_1^n \subset S^{n+2})$ and $(K_2^n \subset S^{n+2})$ if there exists an orientation preserving homeomorphism $\xi : S^{n+2} \times I \rightarrow S^{n+2} \times I$ with $\xi(W_1) = W_2$. If $(W \subset S^{n+2} \times I)$ and $(W' \subset S^{n+2} \times I)$ are cobordisms between $(K_1^n \subset S^{n+2})$ and $(K_2^n \subset S^{n+2})$, and $-(K_2^n \subset S^{n+2})$ and $(K_3^n \subset S^{n+2})$, respectively, we can paste $(W' \cap S^{n+2} \times \{0\} \subset S^{n+2} \times \{0\})$ to $(W \cap S^{n+2} \times \{1\} \subset S^{n+2} \times \{1\})$ along K_2^n to get a cobordism, say $(W \cup W' \subset S^{n+2} \times I)$, between $(K_1^n \subset S^{n+2})$ and $(K_3^n \subset S^{n+2})$.

A null-cobordant n -knot $(K^n \subset S^{n+2})$ is said to be *doubly-null-cobordant* iff there exist two cobordisms $(W \subset S^{n+2} \times I)$ and $(W' \subset S^{n+2} \times I)$ between $(S^n \subset S^{n+2})$ and $(K^n \subset S^{n+2})$, and $-(K^n \subset S^{n+2})$ and $(S^n \subset S^{n+2})$, respectively, such that $(W \cup W' \subset S^{n+2} \times I) = (S^n \times I \subset S^{n+2} \times I)$, the product cobordism. It is obvious that $(K^n \subset S^{n+2})$ is doubly-null-cobordant if and only if it is a cross section of a trivial $(n+1)$ -knot.

A square matrix A (over $\mathbb{Z}$) is said to be *doubly-null-cobordant* iff

it is congruent to $\begin{pmatrix} 0 & N_1 \\ N_2 & 0 \end{pmatrix}$, where N_i are square matrices of the same size.

11.14. Theorem. (Summers [7]) If $(K^n \subset S^{n+2})$ is doubly-null-cobordant and S is a Seifert matrix for $(K^n \subset S^{n+2})$, $n=2m-1$. Then S is doubly-null-cobordant.

Sketch Proof. For brevity, we consider in the euclidean space. From our assumption, there exists a trivial $(n+1)$ -knot $(O^{n+1} \subset R^{n+3})$ such that $(O^{n+1} \cap R_0^{n+2} \subset R_0^{n+2}) \cong (K^n \subset R^{n+2})$. Let B be an $(n+2)$ -disk in R^{n+3} with $\partial B = O^{n+1}$. We may assume that $B \cap R_0^{n+2} = V$ is an $(n+1)$ -manifold with $\partial V = K^n$. Note that V may not be connected and $n+1 = 2m$. Let $U_{\pm} = B \cap R_{\pm}^{n+3}$; $U_+ \cap U_- = V$. Let $\iota_{\pm} : H_m(V; \mathbb{Z}) \rightarrow H_m(U_{\pm}; \mathbb{Z})$ be inclusion homomorphisms.

We will show that the Seifert matrix associated with V is doubly-null-cobordant. It suffices to find a free basis $a_1, \dots, a_r, b_1, \dots, b_r$ for the free part of $H_m(V; \mathbb{Z})$ such that $\Theta(a_i, a_j) = 0 = \Theta(b_i, b_j)$, $1 \leq i \neq j \leq r$.

Consider the Mayer-Vietoris sequence for the triad (B, U_-, U_+) :

$$0 \longrightarrow H_m(V; \mathbb{Z}) \xrightarrow{(-\iota_-) \oplus \iota_+} H_m(U_-; \mathbb{Z}) \oplus H_m(U_+; \mathbb{Z}) \longrightarrow 0.$$

We have $H_m(V; \mathbb{Z}) = \text{Ker}(\iota_-) + \text{Ker}(\iota_+)$. Levine [6, Lemma 2] asserted that $\text{rank}(\text{Ker}(\iota_-)) = \frac{1}{2} \text{rank}(H_m(V; \mathbb{Z})) = \text{rank}(\text{Ker}(\iota_+))$, so let $a_1, \dots, a_r$ freely generate the free part of $\text{Ker}(\iota_-)$, and $b_1, \dots, b_r$ freely generate the free part of $\text{Ker}(\iota_+)$. Since $a_i, a_j \in \text{Ker}(\iota_-)$, a_i and $v_+(a_j)$ bound disjoint chains in B , hence $\Theta(a_i, a_j) = 0$. $\square$

11.15. Corollary. (Summers [7]) If an n -knot $(K^n \subset S^{n+2})$, $n=2m-1$, is doubly-null-cobordant, then $H_m(\tilde{X}; \mathbb{Q})$ is presented as a Γ -module by a

diagonal matrix $P_m(t)$ of the form

$$\text{diag}(\lambda_1^m(t), \dots, \lambda_s^m(t), \lambda_1^m(t^{-1}), \dots, \lambda_s^m(t^{-1})),$$

where $\lambda_i^m \in \Lambda$, $\lambda_{i+1}^m | \lambda_i^m$ in Λ , and $\lambda_i^m(1) = \pm 1$ for all i .

Proof. By Theorem 11.14, we have now a doubly-null-cobordant Seifert matrix $S = \begin{pmatrix} 0 & N_1 \\ N_2 & 0 \end{pmatrix}$. Hence

$$P_m(t) = tS + (-1)^m S' = \begin{pmatrix} 0 & tN_1 + (-1)^m N_2' \\ tN_2 + (-1)^m N_1' & 0 \end{pmatrix}.$$

Let $L(t) = tN_1 + (-1)^m N_2'$. Then $\begin{pmatrix} 0 & L(t) \\ L(t^{-1}) & 0 \end{pmatrix}$ presents a Γ -module isomorphic to $H_m(\tilde{X}; \mathbb{Q})$. Since Γ is PID, there exists a diagonal matrix

$$\text{diag}(\lambda_1^m(t), \dots, \lambda_s^m(t)), \lambda_i^m \in \Lambda, \lambda_{i+1}^m | \lambda_i^m \text{ in } \Lambda,$$

which presents the same Γ -module as $L(t)$, recall 6.8. $\square$

11.16. Corollary. (Summers [7]) If an n -knot $(K^n \subset S^{n+2})$, $n = 2m - 1$, is doubly-null-cobordant, then the torsion numbers of the 2-fold cyclic branched covering space Σ_2 of $(K^n \subset S^{n+2})$ appear in pair.

Proof. Let S be a Seifert matrix of $(K^n \subset S^{n+2})$. Then, it follows from Shinohara-Summers [1, Corollary 5.3] that $H_m(\Sigma_2; \mathbb{Z})$ is presented by $S + (-1)^{m+1} S'$ as an abelian group. From the proof of Corollary 11.15 it

follows that $H_m(\Sigma_2; \mathbb{Z})$ is presented by a matrix of the form $\begin{pmatrix} 0 & L(1) \\ L(-1) & 0 \end{pmatrix}$,

from which the result immediately follows. $\square$

Using these results, Summers [7] concluded that :

11.17. Theorem. The only doubly-null-cobordant 1-knot included in

Reidemeister's table [Re] of prime 1-knots is 9_{46} .

The proof that 9_{46} is doubly-null-cobordant has been given in 10.21.
As a corollary to 8.16, we have :

11.18. Corollary. (Summers [7]) $(K^n \subset S^{n+2}) \# (-(K^n \subset S^{n+2}))$ is doubly-null-cobordant, for every n -knot $(K^n \subset S^{n+2})$.

On the other hand, since $C_2 = 0$ by Theorem 11.5, Yanagawa [2] introduced another concept of cobordism in K_2 .

11.19. Definition. Two cobordant n -knots $(K_1^n \subset S^{n+2})$ and $(K_2^n \subset S^{n+2})$ (and their knot types $[K_1^n \subset S^{n+2}]$ and $[K_2^n \subset S^{n+2}]$) are said to be r -cobordant, denoted by $(K_1^n \subset S^{n+2}) \sim (K_2^n \subset S^{n+2})$, iff there exists a cobordism $(W \subset S^{n+2} \times I)$ between them such that

(iii) $W \cap S^{n+2} \times \{t\}$ is connected for each $t \in I$.

The following is obvious :

11.20. Theorem. The r -cobordant relation $\sim$ is an equivalence relation in K_n .

We have the following two lemmas :

11.21. Lemma. (Yanagawa [2]) If a 2-knot $(K^2 \subset S^2)$ is a ribbon 2-knot, then $(K^2 \subset S^4) \sim (S^2 \subset S^4)$, r -null-cobordant.

11.22. Lemma. (Yanagawa [2]) For any $(K^2 \subset S^4)$, $(K^2 \subset S^4) \# (-(K^2 \subset S^4))$ is r -null-cobordant.

As the consequence of Lemma 11.21, we have :

11.23. Theorem. (Yanagawa [2]) *The set $C_2^x = K_2 / \mathbb{Z}$ of the $\mathbb{Z}$ equivalence classes forms an abelian group, under the operation $\#$ inherited from the semi-group K_2 .*

11.24. Problem. *Determine the group structure of C_2^x . In particular, does there exist a $[K^2 \subset S^4] \in K_2$ with $[K^2 \subset S^4] \not\sim [S^2 \subset S^4]$?*

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BOOKS and MONOGRAPHS

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