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PDF issue: 2026-11-02

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(Citation)
Mathematics Seminar Notes,5(1):1-21

(Issue Date)
1977

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/0100492004

(URL)
https://hdl.handle.net/20.500.14094/0100492004

KOBE UNIVERSITY

MATHEMATICS SEMINAR NOTES
Vol. 5 (1977) January 1977

Autometrized Griss Algebras

By N. Prabhakara RAO

Introduction. K. Iséki and Y. Imai [4] gave an algebraic formulation for Griss' *negationless intuitionistic mathematics* and introduced the notion of a Griss algebra and observed that any negationless intuitionistic logic is a Griss algebra. But in general every Griss algebra need not be a negationless intuitionistic logic. We slightly change the definition of a Griss algebra so that our Griss algebras and negationless intuitionistic logics are same (for our definition of Griss algebra, see Definition 1 below). Incidentally we observe that the class of Griss algebras subsumes the class of semi-Brouwerian algebras [8] which are duals of implicative semilattices introduced and studied by Nemitz [5]. Since any semi-Brouwerian algebra as studied by Ramana Murthy [8] is an autometrized space generalizing the Boolean geometry of Bluementhal [2] and Brouwerian geometry of Nordhaus and Lapidus [6], the study of Griss algebras as autometrized spaces presents a generalization of Boolean geometry, Brouwerian geometry and semi-Brouwerian geometry. However all the above extensions (i.e. Brouwerian algebras and semi-Brouwerian algebras) of Boolean algebras happen to be semilattices autometrized with a regular metric. So a natural question is to enquire whether there are autometrized semilattices which are not necessarily regular and which contain the Boolean geometry as a special case. Autometrized Griss algebras which we present is a satisfactory an-

swer to this question. In this paper we see that any Griss algebra can be autometrized with symmetric difference as distance and obtain several characterizations of Boolean algebras, semi-Brouwerian algebras in terms of the geometry of Griss algebra. For instance we show that an autometrized Griss algebra turns out to be a Boolean geometry if and only if the symmetric difference is associative (see Theorem 5). Also we show that an autometrized Griss algebra is a semi-Brouwerian algebra if and only if the symmetric difference is regular (see Theorem 2). As another characterization of Boolean geometry we show that an autometrized Griss algebra is a Boolean geometry if and only if semilattice betweenness coincides with metric betweenness (see Theorem 10). Y. Imai and K. Iséki [4] introduced the notion of an ideal (the usual semilattice ideal) in a Griss algebra which we call in this paper, an algebraic ideal. Since we elsewhere [11] introduced the notion of an ideal in certain classes of autometrized algebras we can talk of such ideals (called in this paper metric ideals) in this paper also and we study the equivalence and otherwise of these two notions in this paper. We also show that the set of metric ideals is a complete algebraic lattice. Further results relating to the geometry of Griss algebras, ideals and congruence relations will be communicated soon.

§ 1. We begin with the following

Definition 1. A system $G = (G, \cup, -, 0)$ with two binary operations $\cup$, and a nullary operation 0 is called a *Griss algebra* if and only if

(1.1) $(G, \cup)$ is a join semilattice with least element 0,

(1.2) $(x \cup y) - z = (x - z) \cup (y - z)$ for any x, y, z in G,

(1.3) $x - z \leq (x - y) \cup (y - z)$ for any x, y, z in G,

(1.4) $x - y = 0$ implies $x \leq y$,

(1.5) $x - x = 0$ for any x in G,

where the inclusion $\leq$ is the relation induced by the semilattice operation, i.e. $x \leq y$ if and only if $x \cup y = y$.

Examples 1-4 below show that on any semilattice the axioms (1.2)-(1.5) are independent.

Example 1. Let S be the set consisting of elements $0, a, b, 1$ where $\cup$ and $-$ are defined by the following tables:

$\cup$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$-$	0	a	b	1
0	0	0	0	0
a	b	0	b	0
b	a	a	0	0
1	1	1	1	0

In this, $(a \cup b) - b = 1$ and $(a - b) \cup (b - b) = b$ so that (1.2) fails. It is routine to verify that all the remaining postulates hold.

Example 2. Let S be the set as in Example 1 and let the join operation be defined by the inclusion relation $0 < a < b < 1$. Define $-$ by the following table:

$-$	0	a	b	1
0	0	0	0	0
a	b	0	0	0
b	b	a	0	0
1	1	b	a	0

Here $1 - 0 = 1$ and $(1 - a) \cup (a - 0) = b$ so that (1.3) fails. All the other axioms hold in S.

Example 3. Let S be the same set as in Example 1 with $\cup$ also defined as in Example 1, Define - by $x - y = 0$ for any x, y in S. In this obviously (1.4) fails and all other postulates hold.

Example 4. Let S be the same set as in Example 1 with $\cup$ also defined as in Example 1. Define - by $x - y = 1$ for any x, y in S. Then (1.5) fails and all other postulates hold.

The following system $S = (S, \cup, -, 0, <)$ shows that the truth of (1.2) through (1.5) need not imply $(S, \cup)$ is a join semilattice.

Example 5. Let $S = \{0, a\}$. Let $0 < a$. Define $\cup$ and - by the following tables:

$\cup$	0	a
0	0	a
a	a	0

-	0	a
0	0	0
a	a	0

According to [8] a semi-Brouwerian algebra is by definition an algebra $L = (L, +, -, 0)$ with two binary operations +, - and a nullary operation 0 satisfying the following conditions:

(1) $a + a = a$

(2) $a + b = b + a$

(3) $a - a = 0$

(4) $(a - b) + b = a + b$

(5) $(a - b) - c = a - (b + c)$

for all a, b, c in L.

In any semi-Brouwerian algebra if one define $a \leq b$ to mean $a + b = b$ then by Theorem 1 of [8], any semi-Brouwerian algebra is a Griss algebra.

Since Boolean algebras and Brouwerian algebras are semi-Brouwerian algebras it follows that they are also Griss algebras.

The following is an example of a Griss algebra $G = (G, \cup, -, 0)$ which is not a semi-Brouwerian algebra.

Example 6. Let S be the four element set as in Example 1. Let $\cup$ be defined on S as in Example 1. Define $-$ in the following way.

$-$	0	a	b	1
0	0	0	0	0
a	b	0	b	0
b	a	a	0	0
1	1	a	b	0

Since $(a - b) - a = b - a = a$ and $a - (b \cup a) = a - 1 = 0$, it follows that $(S, \cup, -, 0)$ is not a semi-Brouwerian algebra.

Throughout this paper G stands for a Griss algebra $G = (G, \cup, -, 0)$ and $x, y, z, a, b, c, \cdots$ denote the elements of G.

As in the case of Universal algebra, defining $\cup$ and $-$ componentwisely one can easily observe that the direct product of any family of Griss algebras is a Griss algebra. Moreover under a natural definition, each subalgebra of a Griss algebra is a Griss algebra.

We conclude this section with the following lemmas.

Lemma 1. *$x \leq y$ implies $x - z \leq y - z$.*

Proof. $x \leq y$ implies $x \cup y = y$. Hence by (1.2) $y - z = (x \cup y) - z = (x - z) \cup (y - z)$. Therefore $x - z \leq y - z$.

Lemma 2. $x \leq y$ *implies* $x - y = 0$.

Proof. If $x \leq y$ then by Lemma 1, $x - y \leq y - y = 0$. So $x - y = 0$.

Lemma 3. $x \leq y$ *implies* $z - y \leq z - x$.

Proof. By (1.3) $z - y \leq (z - x) \cup (x - y) = (z - x) \cup 0 = z - x$.

Lemma 4. $(x \cup y) - (x \cup z) \leq y - z$.

Proof.
$$\begin{aligned}(x \cup y) - (x \cup z) &= [x - (x \cup z)] \cup [y - (x \cup z)] \\ &= y - (x \cup z) \qquad \text{(by Lemma 2)} \\ &\leq y - z \qquad \text{(by Lemma 3).}\end{aligned}$$

§ 2. For any a, b in G we shall write $a * b = (a - b) \cup (b - a)$ and we call $a * b$ the symmetric difference of a and b.

The following theorem shows that the symmetric difference satisfies the formal properties of a metric which makes any Griss algebra G into an autometrized algebra (see Definition 1 of [10]).

Theorem 1. $(G, \cup, *, \leq)$ *is an autometrized algebra.*

Proof. First we show that $*$ is a metric operation on G. Since 0 is the least element we have $0 \leq x * y$. Let $x * y = 0$. Then $x - y = 0$ and $y - x = 0$. So by (1.4) it follows that $x \leq y$ and $y \leq x$. Hence $x = y$. Obviously $x * x = 0$. $x * y = y * x$ is clear. By (1.3) we have $x - z \leq (x - y) \cup (y - z)$ and $(z - x) \leq (z - y) \cup (y - x)$. Hence $x * z = (x - z) \cup (z - x) \leq (x - y) \cup (y - z) \cup (z - y) \cup (y - z) = (x * y) \cup (y * z)$. Hence $*$ is a metric operation on G. Since $(G, \cup)$ is join semilattice with least element 0, it follows that $(G, \cup, *, \leq)$ is an autometrized algebra.

Now, we introduce the following

Definition 2. Any Griss algebra G autometrized with the symmetric difference as distance function is called an *autometrized Griss algebra.*

Any semi-Brouwerian algebra with the symmetric difference $(a - b) + (b - a)$ as metric is a regular autometrized algebra in the sense that for any a, $a * 0 = a$.

Semi-Brouwerian algebras among the class of Griss algebras can be characterized as follows:

Theorem 2. G *is a semi-Brouwerian algebra if and only if it is regular.*

Proof. Suppose G is regular. Hence $a * 0 = a$ for any a in G. By the theorem in [9] it is sufficient to show that G admits a regular metric operation θ satisfying $a \theta (a \cup b) \leq b$ for any a, b in G. From Theorem 1 and the assumed hypothesis it is clear that $*$ is a regular metric operation in G. Now we have

$$\begin{aligned} a * (a \cup b) &= [a - (a \cup b)] \cup [(a \cup b) - a] = (a \cup b) - a \\ &= (a - a) \cup (b - a) = b - a \\ &\leq b - 0 \quad \text{(since } 0 \leq a\text{)} \\ &= b * 0 = b, \end{aligned}$$

proving the theorem.

Lemma 5. $(a \cup b) * (c \cup d) \leq (a * c) \cup (b * d)$ *for any* $a, b, c, d \in G$.

Proof. $$\begin{aligned} (a \cup b) * (c \cup d) &= [(a \cup b) - (c \cup d)] \cup [(c \cup d) - (a \cup b)] \\ &= [a - (c \cup d)] \cup [b - (c \cup d)] \cup [c - (a \cup b)] \cup [d - (a \cup b)] \\ &\leq (a - c) \cup (b - d) \cup (c - a) \cup (d - b) = (a * c) \cup (b * d). \end{aligned}$$

Theorem 3. *For any fixed element* a *of* G *the mapping* $x \mapsto a \cup x$ *is a contraction on* G.

Proof. By Lemma 5, $(a \cup x) * (a \cup y) \leq (a * a) \cup (x * y) = (x * y) \cup 0 = x * y$. Hence $x \mapsto a \cup x$ is a contraction.

The following two theorems give sufficient conditions for an autometrized Griss algebra G to degenerate into a Boolean ring.

Theorem 4. *If* $a - (b - a) = a$ *for all* a, b *in* G *then a multiplication* $\cdot$ *can be introduced in* G *such that* $(G, *, \cdot)$ *is a Boolean ring.*

Proof. Suppose $a - (b - a) = a$ for all a, b in G. Taking $b = 0$ we have $a - (0 - a) = a$ and hence $a * 0 = a - 0 = a$ for every a in G. Hence by Theorem 2, G is a semi-Brouwerian algebra. So the theorem follows from Theorem 4 of [8].

In a Griss algebra G, we write $a \cdot b = (a \cup b) - (a * b)$ for any a, b in G. We usually write ab instead of $a \cdot b$.

Now we have the following

Theorem 5. *In a Griss algebra* G *the following are equivalent*

(1) $a - (b - a) = a$ *for all* a, b *in* G.

(2) $(G, *, \cdot)$ *is a Boolean ring.*

(3) ab *is the greatest lower bound of* a *and* b.

(4) $a \cup (bc) = (a \cup b)(a \cup c)$.

(5) $(G, *)$ *is a loop.*

(6) $*$ *is a group operation.*

(7) $*$ *is associative.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Theorem 4.

Assume (2). Then $aa = a$ for every a in G. Hence $(a \cup a) - (a * a) = a$ so that $a - 0 = a$ i.e., $a * 0 = a$ for every a in G. Hence G is a semi-Brouwerian algebra by Theorem 2. Now by Theorem 5 of [8] ab is the greatest lower bound of a and b. Hence (2) implies (3).

Assume (3). Since a is the greatest lower bound of a and a it follows that $aa = a$. Hence $a * 0 = a$ so that G is a semi-Brouwerian algebra. Hence $a \cup (bc) = (a \cup b)(a \cup c)$ by Theorem 5 of [8]. Thus (3) implies (4).

Assume (4). Taking $b = c = 0$ we have $a \cup (00) = (a \cup 0)(a \cup 0)$. Hence $a \cup 0 = aa$, i.e. $aa = a$. Hence $a * 0 = a$ for every a in G so that G is a semi-Brouwerian algebra. Hence (G, $*$) is a loop by Theorem 5 of [8]. This establishes the implication (4) = (5).

Assume (5). Suppose $\alpha \in G$ be the two sided identity in G, i.e. $a * \alpha$ $\alpha * a = a$ for every a in G. Then in particular for $a = \alpha$ we have $\alpha * \alpha = \alpha$ which implies $\alpha = 0$. Hence $a * 0 = a$ for every a in G so that G is a semi-Brouwerian algebra by Theorem 2. Hence $*$ is a group operation on G by Theorem 5 of [8]. Thus (5) implies (6).

It suffices to prove that (7) implies (1). Assume (7). Then for every a in G, we have $(a * 0) * a = (a * a) * 0 = 0 * 0 = 0$. Hence $a * 0 = a$ for every a in G so that G is a semi-Brouwerian algebra. Therefore $a - (b - a) = a$ for every a, b in G, by Theorem 5 of [8].

It is shown by Ramana Murthy [8] that a semi-Brouwerian algebra is a Boolean ring if and only if it admits a metric group operation. Also the metric group operation must be the symmetric difference by the well known result (Theorem 6.1 of [2]) of Blumenthal. Eventhough Theorem 4 says that

any Griss algebra is a Boolean ring if the metric operation 'symmetric difference' is a group operation, however there exists a Griss algebra $(G, \cup, -, 0)$ on which there can be defined a metric group operation θ which does not coincide with the symmetric difference in the Griss algebra as the following example shows:

Example 7. Let G be the Griss algebra of Example 6, and let (G, θ) be the Klein's Four group. It is routine to verify that θ is a metric group operation on G. But $a \theta 0 = a$ whereas $a * 0 = b$.

Following the usual terminology, a distance preserving surjection from G onto G is called an *isometry* of G. The following theorem is another characterization of a Boolean ring.

Theorem 6. *An autometrized Griss algebra* G *is a Boolean geometry if and only if for each* a *in* G *the mapping* $x \mapsto a * x$ *is an isometry.*

Proof. Suppose $x \mapsto a * x$ is an isometry for every a in G. Since $0 \in G$, the mapping $g: x \mapsto 0 * x$ is an isometry. Hence $x * 0 = g(x) * g(0) = (x * 0) * (0 * 0) = (x * 0) * 0$. Hence $g(x) = g(x * 0)$. Since every isometry is an injection, we have $x = x * 0$. This being true for every x in G, we see that G is a semi-Brouwerian algebra. Now we show that $*$ is cancellative. Let $a * b = a * c$. Since $g': x \mapsto a * x$ is an isometry, it follows that $g'(b) = g'(c)$. Hence $b = c$. Thus $*$ is cancellative. Now from Theorem 5 of [8], it follows that G is a Boolean geometry. The other part is obvious.

We shall treat any three distinct elements a, b, c as a *triangle* denoted by $\Delta(a, b, c)$ with vertices a, b, c and sides $a*b, b*c, c*a$.

Now we have

Theorem 7. *If* G *is a chain, all triangles are isosceles.*

Proof. Let $\Delta(a, b, c)$ be any triangle in G. Since G is a chain, either $a * b \leq b * c$ or $b * c \leq a * b$. Suppose $a * b \leq b * c$.

Case 1. Suppose $b * c \leq c * a$. If $a * b \leq c * a$, then since $a * b \leq c * a \leq (c * b) \cup (b * a) = c * b \leq c * a$, we have $c * a = c * b$, which shows that $\Delta(a, b, c)$ is an isosceles triangle. If $c * a \leq a * b$, then since $a * b \leq b * c \leq c * a \leq a * b$, we have $a * b = b * c$, and therefore $\Delta(a, b, c)$ is an isosceles triangle.

Case 2. Suppose $c * a \leq b * c$. If $a * b \leq c * a$, then since $a * b \leq c * a \leq b * c \leq (b * a) \cup (a * c) = a * c$, we have $a * c = b * c$, and so $\Delta(a, b, c)$ is an isosceles triangle. If $c * a \leq a * b$, then since $c * a \leq a * b \leq b * c \leq (b * a) \cup (c * a) = b * a$, we have $a * b = b * c$, which shows that $\Delta(a, b, c)$ is an isosceles triangle.

Hence in any case if $a * b \leq b * c$ then $\Delta(a, b, c)$ is an isosceles triangle. Similarly we can show $\Delta(a, b, c)$ is an isosceles triangle if $b * c \leq a * b$.

The following example shows that the converse of the above theorem is not true in Griss algebras eventhough it is true in semi-Brouwerian algebras.

Example 8. Let S be the set as in Example 6 and let $\cup$ be defined as in example 6. Define $-$ in S by the table at right.

$-$	0	a	b	1
0	0	0	0	0
a	1	0	1	0
b	1	1	0	0
1	1	1	1	0

In fact, in this example all the triangles

are equilateral.

As another characterization of Boolean geometry we have the following.

Theorem 8. *Suppose* G *is an autometrized Griss algebra in which* $a * b = (a * b) * 0$ *for any* a, b *in* G. *Then* G *is a Boolean geometry if and only if it is free of isosceles triangles.*

Proof. The necessity is obvious. Suppose G is free of isosceles triangles. By Theorem 7 of [8] it is enough to show that G is a semi-Brouwerian algebra. Suppose $a * 0 \neq a$ for some a in G. Consider the triangle $\Delta(0, a, a * 0)$. Since $a * 0 = (a * 0) * 0$ it follows that it is an isosceles triangle, which contradicts the assumption. Hence $a * 0 = a$ for every a in G.

By Theorem 7 of [8], a semi-Brouwerian algebra which is free of isosceles triangles is a Boolean geometry. However, Example 6 is an instance of an autometrized Griss algebra which is free of isosceles triangles but not a Boolean geometry. In this example the condition $a * b = (a * b) * 0$ fails.

Following [6] a triangle is said to have *fixity* if each vertex is equal to the opposite side. We have now the following.

Theorem 9. *If* G *is a Griss algebra which is a chain, then* G *is free of triangles with fixity.*

Proof. Let $\Delta(a, b, c)$ be a triangle with fixity in G which is a chain. We can assume without loss of generality that $a < b < c$. Since $\Delta(a, b, c)$ has fixity $a * b = c$, $b * c = a$, $c * a = b$. Hence $b - a = c$, $c - b = a$ and $c - a = b$. Now $c = b \cup c = (b - a) \cup (c - a) = (b \cup c) - a$

$= c - a = b$. Hence $b = c$, i.e. the triangle $\Delta(a, b, c)$ degenerates. So G is free of triangle with fixity.

The converse of the above theorem is not true since example 6 is a Griss algebra which is free of triangles with fixity but noy a chain.

In a semi-Brouwerian algebra, Ramana Murthy [8] introduced the notion of semilattice betweenness (which is a generalization of betweenness introduced by Blumenthal) and has proved that a semi-Brouwerian algebra is a Boolean geometry if and only if semilattice betweenness coincides with metric betweenness. Following Ramana Murthy in any Griss algebra G, for any a, b, c in G we say b *lies between* a *and* c *in terms of semilattice betweenness* (in symbols (a, b, c)L) if and only if $(ab) \cup (bc) = b = (a \cup b)(b \cup c)$. We say b *lies metrically between* a *and* c (in symbols (a, b, c)M) if and only if $(a * b) \cup (b * c) = a * c$.

Now we have the following theorem.

Theorem 10. *An autometrized Griss algebra* G *is a Boolean geometry if and only if semilattice betweenness coincides with metric betweenness.*

Proof. If G is a Boolean geometry then G is a semi-Brouwerian algebra. Hence semilattice betweenness coincides with metric betweenness. Conversely suppose in G semilattice betweemmess coincides with metric betweenness. For any a in G we have $(a, a, 0)$M. Hence $(a, a, 0)$L. Therefore $(a \cup a)(a \cup 0) = a$ which implies $aa = a$. Hence $a * 0 = a$. So G is a semi-Brouwerian algebra. Hence by Theorem 20 of [8], G is a Boolean geometry.

We also have the following

Theorem 11. *Suppose* G *is an autometrized Griss algebra in which* $a * 0 \leq a$ *for every* a *in* G. *Then* G *is a Boolean geometry if and only if* $ac \leq b \leq a \cup c$ *implies* (a, b, c)L.

Proof. The necessity follows from the fact that in a distributive lattice $(L, \cup, \cdot)$, $ac \leq b \leq a \cup c$ if and only if (a, b, c)L. Conversely, suppose $ac \leq b \leq a \cup c$ implies (a, b, c)L. Since $aa = (a \cup a) - (a * a) = a - 0 = a * 0 \leq a$, we have $aa \leq a \leq a \cup a$ for every a in G. Hence (a, a, a)L, and so $(a \cup a)(a \cup a) = a$. Hence $a = aa = a * 0$ for every a in G so that G is a semi-Brouwerian algebra by Theorem 2. It follows from Theorem 18 of [8] that G is a Boolean geometry.

According to Pitcher and Smiley [7] we say that any betweenness relation (a, b, c)R has *transitivity* t_1 if and only if (a, b, c)R and (a, x, b)R imply (x, b, c)R; and (a, b, c)R has *transitivity* t_2 if and only if (a, b, c)R and (a, x, b)R imply (a, x, c)R. Now we have

Theorem 12. *In an autometrized Griss algebra* G, *metric betweenness has transitivity* t_2.

Proof. Suppose (a, b, c)M and (a, x, b)M. Then $(a * b) \cup (b * c) = a * c$ and $(a * x) \cup (x * b) = a * b$. Now $(a * x) \cup (b * x) \cup (b * c) = (a * b) \cup (b * c) = a * c$. Since $c * x \leq (c * b) \cup (b * x)$ it follows that $(c * x) \cup (a * x) \leq (c * b) \cup (b * x) \cup (a * x) = a * c$. Hence $a * c = (a * x) \cup (c * x)$, and thus (a, x, c)M.

Also we have the following

Theorem 13. *If in an autometrized Griss algebra* G *metric betweenness has transitivity* t_1, *then* G *is free of isosceles triangles.*

Proof. Suppose $\Delta(a, b, c)$ is an isosceles triangle in which $a * b = a * c$. Since $b * c \leq (b * a) \cup (a * c) = a * b$, we have $(a * b) \cup (b * c) = a * b = a * c$. Hence (a, b, c)M. Since $b * c \leq a * b = a * c$, we also have $(a * c) \cup (b * c) = a * b$. Hence (a, c, b)M. Since metric betweenness has transitivity t_1, (a, b, c)M and (a, c, b)M imply (c, b, c)M, i.e., $c * b = 0$. Hence $b = c$ i.e., the triangle $\Delta(a, b, c)$ degenerates. So G is free of isosceles triangles.

Theorem 14. *Let* G *be an autometrized Griss algebra in which* $a * b = (a * b) * 0$ *for any* a, b *in* G. *Then* G *is a Boolean geometry if and only if metric betweenness has transitifity* t_1.

Proof. The necessity is obvious. Suppose metric betweenness has transitivity t_1. Then by Theorem 13, G is free of isosceles triangles. Now from Theorem 8 it follows that G is a Boolean geometry.

Since we have introduced a multiplication $\cdot$ in a Griss algebra G, following [6] we say that G is *symmetric* if and only if $(a \cup b) * (ab) = a * b$ for all a, b in G. We immmediately have the following.

Theorem 15. *A Griss algebra* G *is symmetric if and only if it is a semi-Brouwerian algebra.*

Proof. Suppose G is symmetric. Then $(a \cup b) * (ab) = a * b$ for all a, b in G. Taking $b = a$ we have $(a \cup a) * (aa) = a * a$. Hence $a * (aa) = 0$, and so $a * 0 = aa = a$. Therefore G is a semi-Brouwerian algebra by Theorem 2. Now suppose G is a semi-Brouwerian algebra. First observe that

$$(a \cup b) * (ab) = (a \cup b) * [(a \cup b) - (a * b)]$$

$$= \{(a \cup b) - [(a \cup b) - (a * b)]\} \cup \{[(a \cup b) - (a * b)] - (a \cup b)\}$$

$$= (a \cup b) - [(a \cup b) - (a * b)]$$

since $(a \cup b) - (a * b) \leq a \cup b$. Hence

$$[(a \cup b) * (ab)] - (a * b) = \{(a \cup b) - [(a \cup b) - (a * b)]\} - (a * b)$$

$$= (a \cup b) - \{[(a \cup b) - (a * b)] \cup (a * b)\}$$

$$= (a \cup b) - [(a \cup b) \cup (a * b)]$$

$$= 0.$$

Hence $(a \cup b) * (ab) \leq a * b$. Since $(a \cup b) - (a * b) = ab = a - a * b$ (by Lemma 5 of [8]) $\leq a$ we have $b - a \leq b - [(a \cup b) - (a * b)]$. By symmetry $a - b \leq a - [(a \cup b) - (a * b)]$. Hence

$$a * b = (a - b) \cup (b - a)$$

$$\leq \{a - [(a \cup b) - (a * b)]\} \cup \{b - [(a \cup b) - (a * b)]\}$$

$$= (a \cup b) - [(a \cup b) - (a * b)]$$

$$= (a \cup b) * (ab).$$

Therefore $(a \cup b) * (ab) = a * b$. Thus G is symmetric.

§ 3. Y. Imai and K. Iséki [4] introduced the notion of an ideal in a Griss algebra G in the following way:

A nonempty subset I of G is called an *ideal* if and only if

(3.1) $x \in I$, $y \in I$ imply $x \cup y \in I$,

(3.2) $x \in I$, $y \leq x$ imply $y \in I$.

We call these ideals *algebraic ideals*. Now since G is an autometrized algebra and since there is the notion of an ideal (see Definitions 1 and 2 of [11]) in any normal autometrized algebra (which we refer in this paper as metric ideals) and eventhough Griss algebras are not in general normal autometrized algebras (see Example 6) and since there are several Griss algebras which are normal autometrized algebras (for instance semi-

Brouwerian algebras) we introduce the notion of metric ideal in a Griss algebra as follows.

Definition 3. A nonempty subset I of a Griss algebra is called a *metric ideal* if and only if

(3.1) $x \in I$ and $y \in I$ imply $x \cup y \in I$,

(3.3) $x \in I$ and $y * 0 \leq x * 0$ imply $y \in I$.

In general the notions of metric ideal and algebraic ideal are different. Obviously every metric ideal in a Griss algebra is an algebraic ideal. However there are Griss algebras in which every algebraic ideal is not a metric ideal (for instance in Example 8, $\{0, a\}$ is an algebraic ideal but not a metric ideal). We have the following characterization.

Theorem 16. *In a Griss algebra* G *every algebraic ideal is a metric ideal if and only if* $x - 0 \leq y - 0$ *imply* $x \leq y$ *for all* x, y *in* G.

We need the following lemma.

Lemma 6. *For any nonempty subsets* S *of* G, $\{x \in G \mid x - 0 \leq (a_1 \cup \cdots \cup a_n) - 0$ *for some* $a_1, \cdots, a_n$ *in* $S\}$ *is the smallest metric ideal containing* S.

Proof. Let $J = \{x \in G \mid x - 0 \leq (a_1 \cup \cdots \cup a_n) - 0$ for some $a_1, \cdots, a_n$ in $S\}$. Since S is nonempty, J is nonempty. Let $x, y \in J$. Then $x - 0 \leq (a_1 \cup \cdots \cup a_n) - 0$ for some $a_1, \cdots, a_n$ in S and $y - 0 \leq (b_1 \cup \cdots \cup b_m) - 0$ for some $b_1, \cdots, b_m$ in S. Hence $(x \cup y) - 0 = (x - 0) \cup (y - 0) \leq [(a_1 \cup \cdots \cup a_n) - 0] \cup [(b_1 \cup \cdots \cup b_m) - 0] = (a_1 \cup \cdots \cup a_n \cup b_1 \cup \cdots \cup b_m) - 0$ where $a_1, \cdots, a_n, b_1, \cdots, b_m \in S$. Hence $x \cup y \in J$. Let $x \in J$ and $y * 0 \leq x * 0$. Since $x \in J$, $x - 0 \leq (a_1 \cup \cdots \cup a_n) - 0$ for

some $a_1, \cdots, a_n$ in S. Now $y - 0 = y * 0 \leq x * 0 = x - 0 \leq (a_1 \cup \cdots \cup a_n) - 0$. Hence $y \in J$. Thus J is a metric ideal. Clearly $S \subset J$. Now let J' be any metric ideal containing S. Let $x \in J$. Then $x - 0 \leq (a_1 \cup \cdots \cup a_n) - 0$ for some $a_1, \cdots, a_n$ in S. Hence $x * 0 = x - 0 \leq (a_1 \cup \cdots \cup a_n) - 0 = (a_1 \cup \cdots \cup a_n) * 0$ for $a_1, \cdots, a_n$ in J'. Since J' is a metric ideal it follows that $x \in J'$. Hence we have $J \subset J'$, proving Lemma.

Proof of Theorem 16. Suppose $x - 0 \leq y - 0$ implies $x \leq y$ for any x, y in G. Let I be any algebraic ideal. To show I is a metric ideal we have to show that (3.3) is satisfied. Suppose $y \in I$ and $x * 0 \leq y * 0$. Then $x - 0 = x * 0 \leq y * 0 = y - 0$. Hence by the assumption it follows that $x \leq y$. Since $y \in I$ we have $x \in I$. Hence I is a metric ideal. Now suppose every algebraic ideal is a metric ideal. Let $x - 0 \leq y - 0$. Clearly $I = \{z \in G \mid z \leq y\}$ is an algebraic ideal. Hence from Lemma 6 it follows that $J = \{t \in G \mid t - 0 \leq z - 0$ for some $z \in I\}$ is the smallest metric ideal containing I, Since every algebraic ideal is a metric ideal, I is a metric ideal containing I. Hence $J \subset I$. Since $x - 0 \leq y - 0$ and $y \in I$ it follows that $x \in J$. Hence $x \in I$, and so $x \leq y$. This completes the proof.

However in semi-Brouwerian algebras the notions of metric ideal and algebraic ideal are same. But Example 6 shows that Griss algebra need not be a semi-Brouwerian algebra even if these two notions coincide.

Now we prove the following theorem which generalize the corresponding theorem for Boolean, Brouwerian and semi-Brouwerian algebras.

Theorem 17. *The set of metric ideals of a Griss algebra* G *is a complete algebraic lattice ordered by set inclusion.*

This theorem follows at once from the following lemmas.

Lemma 7. *The intersection of any nonempty collection of metric ideals is a metric ideal.*

Proof. Obvious.

Lemma 8. *If* I *and* J *are metric ideals of* G, $\{a \in G \mid a - 0 \leq (x \cup y) - 0$ *for some* $x \in I$ *and* $y \in J\}$ *is the join of* I *and* J.

Proof. From Lemma 6, this lemma follows.

From Lemma 6, it is clear that the principal ideal generated by a (denoted by (a)), i.e. the smallest metric ideal containing a, is $\{x \in G \mid x - 0 \leq a - 0\}$.

Lemma 9. *Th join of any two principal metric ideals is a principal metric ideal.*

Proof. We will show that $(a) \vee (b) = (a \cup b)$ where (a) and (b) are principal metric ideals. Let $x \in (a)$. Then $x - 0 \leq a - 0$. Since $a \leq a \cup b$, we have $x - 0 \leq a - 0 \leq (a \cup b) - 0$. Hence $x \in (a \cup b)$. Hence $(a) \subset (a \cup b)$ and on similar lines we can show $(b) \subset (a \cup b)$. Let I be any metric ideal containing (a) and (b). Let $z \in (a \cup b)$. Then $z - 0 \leq (a \cup b) - 0$. $a \in (a) \subset I$ and $b \in (b) \subset I$ show that $a \in I$ and $b \in I$. Hence $a \cup b \in I$. Since $z * 0 = z - 0 \leq (a \cup b) - 0 = (a \cup b) * 0$, it follows that $z \in I$. Hence $(a \cup b) \subset I$. Thus $(a) \vee (b) = (a \cup b)$.

Lemma 10. *In a Griss algebra, compact metric ideals are just the principal metric ideals.*

Proof. Let (a) be a principal metric ideal. Let $(a) \subset \bigvee\{I_\alpha \mid \alpha \in \Delta\}$. Then $a \in \bigvee\{I_\alpha \mid \alpha \in \Delta\}$ which implies $a - 0 \leq (a_1 \cup \cdots \cup a_n) - 0$ where $a_1 \in I_{\alpha_1}, \cdots, a_n \in I_{\alpha_n}$ for some $\alpha_1, \cdots, \alpha_n$. Hence $a \in I_{\alpha_1} \vee \cdots \vee I_{\alpha_n}$. So $(a) \subset I_{\alpha_1} \vee \cdots \vee I_{\alpha_n}$. Hence (a) is a compact metric ideal. Now let I be any compact metric ideal. Since $I \subset \bigvee\{(a) \mid a \in I\}$, it follows that $I \subset (a_1) \vee \cdots \vee (a_n)$ for some $a_1, \cdots, a_n \in I$. Hence $I = (a_1) \vee \cdots \vee (a_n) = (a_1 \cup \cdots \cup a_n)$ (by Lemma 7). Hence I is a principal metric ideal.

In conclusion I thank Dr. K. L. N. Swamy for his valuable guidance and encouragement throughout the preparation of this paper.

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