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On Bases in $\lambda(P)$-Nuclear Spaces

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A. S. Dynin and B. S. Mitiagin [1], [2] proved that in a nuclear space each equicontinuous basis is absolute. Combining this with the fact that in a Fréchet space, each basis is equicontinuous, we see that in a nuclear Fréchet space each basis is absolute.

Conversely, in his paper [6], W. Wojtyński has shown by the observation concerning bases in non-nuclear Köthe spaces, that a Fréchet space with a basis, where any basis is absolute, is nuclear. Therefore we have the following characterization.

Theorem 1. *Let E be a Fréchet space with a basis. Then E is nuclear if and only if each basis in E is absolute.*

In this note, we shall give some generalizations of Theorem 1 and of the theorem of Dynin and Mitiagin to $\lambda(P)$-nuclear spaces with the concept of $\lambda(P)$-absolute bases, which is a natural extension of that of absolute bases.

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1. Bases in Fréchet spaces

Throughout this note, we denote by E a locally convex space with a basis $\{e_n\}_{n\in N}$, and by $\lambda(P)$ a nuclear smooth sequence space (see [3], [5] for definition) and the notations that will be used without any comments are the same as those in [3].

A sequence $\{e_n\}_{n\in N}$ in E is called a *basis* in E if for each $x \in E$ there exists a unique sequence of scalars $\{f_n(x)\}_{n\in N}$ such that

$$x = \sum_{n=1}^{\infty} f_n(x)e_n .$$

Then $\{f_n\}$ is the well-defined sequence of linear forms on E, and is called the *biorthogonal* sequence to $\{e_n\}$.

For each $n \in N$ we further introduce the linear finite-dimensional mapping

$$S_n : E \longrightarrow E : x \mapsto \sum_{i=1}^{n} f_i(x)e_i .$$

For convenience we put $S_0 := 0$. Then a basis $\{e_n\}$ in E is said to be *equicontinuous* if the corresponding sequence $\{S_n\}$ of projections belongs to $\mathcal{L}(E, E)$, and is equicontinuous. All bases are not equicontinuous in general. We know, however, that each basis in a Fréchet space is equicontinuous.

Now we consider a 0-basis (i.e. a basis of the neighborhood filter of zero) $\mathcal{V}$ in E. A basis $\{e_n\}$ in E is said to be *absolute* if for all $U \in \mathcal{V}$ there exists a $V \in \mathcal{V}$ with $V \subset U$ such that

$$\sum_{n} |f_n(x)|\ p_U(e_n) \leq p_V(x), \qquad \text{for all } x \in E.$$

Certainly all absolute bases are equicontinuous.

Moreover a basis $\{e_n\}$ is said to be $\lambda(P)$-*absolute* if for all $U \in \mathcal{V}$ there exists a $V \in \mathcal{V}$ with $V \subset U$ and an injection $\sigma : N \to N$ with the image $\sigma(N) = N_U := \{n \in N \mid p_U(e_n) \neq 0\}$, such that

$$f_{\sigma(n)} \in E'_{V^o} = (\widetilde{E}_{(V)})', \qquad \text{for all } n \in N$$

and

$$(\| f_{\sigma(n)} \|_V \, p_U(e_{\sigma(n)}))_n \in \lambda(P),$$

where $\| \ \|_V$ is the ordinary norm in E'_{V^o}, and V^o is the polar of V in E'.

The nuclearity of $\lambda(P)$ yields $\lambda(P) \subset (s) \subset \ell_1$ (see [5]). Hence it follows that every $\lambda(P)$-absolute basis is absolute, so equicontinuous. We shall prove later that in a $\lambda(P)$-nuclear space, each equicontinuous basis is $\lambda(P)$-absolute. Here we mention the following good characterization of the sequentially complete locally convex spaces with an absolute basis.

A topological space E is said to be *sequentially complete* if all Cauchy sequences converge in E. On the other hand, we recall that a locally convex space E is called $\lambda(P)$-*nuclear* if it admits a 0-basis $\mathcal{V}$ such that each $U \in \mathcal{V}$ contains a $V \in \mathcal{V}$ and Φ_{UV} is $\lambda(P)$-nuclear, where Φ_{UV} is the extension of the canonical mapping of $E_{(V)}$ on $E_{(U)}$ to their completions. Then we have the following.

Theorem 2. *The sequentially complete locally convex spaces with an absolute basis* (*resp. a* $\lambda(P)$-*absolute basis*) *are precisely identified with the Köthe spaces* (*resp. the* $\lambda(P)$-*nuclear Köthe spaces*) $\lambda(Q)$ *for some*

Köthe set Q.

Proof. Let Q be a Köthe set. Then the sequences $e_n = (\delta_{ni})_i$ constitute obviously a basis in $\lambda(Q)$. For $\lambda = (\lambda_n) = \sum f_n(\lambda)e_n \in \lambda(Q)$, we have

$$\sum_n |f_n(\lambda)|p_a(e_n) = \sum_n |f_n(\lambda)|a_n = \sum |\lambda_n|a_n = p_a(\lambda)$$

for all $a \in Q$, where p_a is a semi-norm in $\lambda(Q)$ defined by $p_a(\xi) = \sum |\xi_n|a_n$, $\xi = (\xi_n) \in \lambda(Q)$.

Further if $\lambda(Q)$ is $\lambda(P)$-nuclear, we can say by Theorem 2.2 of [5] that there exists $b \in Q$ with $a_n \leq b_n$, for all $n \in N$ and an injection $\sigma : N \longrightarrow N$ with its range $\sigma(N) = \{n \in N \mid a_n \neq 0\}$ such that $(a_{\sigma(n)} / b_{\sigma(n)}) \in \lambda(P)$. Thus we have

$$(\| f_{\sigma(n)} \|_b \, p_a(e_{\sigma(n)}))_n \in \lambda(P),$$

where $\| f \|_b = \sup\limits_{p_b(\lambda)\leq 1} |f(\lambda)|$, $f \in \lambda(Q)'$. Consequently $\{e_n\}$ is $\lambda(P)$-absolute.

Now assume that E is a sequentially complete space with an absolute basis $\{e_n\}$, and $\mathfrak{V}$ is a 0-basis in E. Then the set

$$Q := \{(p_V(e_n))_n \mid V \in \mathfrak{V}\}$$

is trivially a Köthe set, and the topology on $\lambda(Q)$ is given by the semi-norms

$$q_V(\lambda) = \sum_n |\lambda_n| p_V(e_n), \qquad V \in \mathfrak{V}, \quad \lambda = (\lambda_n) \in \lambda(Q).$$

Since $\{e_n\}$ is an absolute basis, $(f_n(x))_n \in \lambda(Q)$ for all $x \in E$. Hence we get a well-defined linear mapping

$$T : E \longrightarrow \lambda(Q) : x \longmapsto (f_n(x))_n.$$

Then T is injective, because of the definition of basis. We shall see that T is also surjective.

Let $\lambda \in \lambda(Q)$ be given. Then $q_V(\lambda) < \infty$ for all $V \in \mathfrak{V}$ implies that $(\sum_{n=1}^{k} \lambda_n e_n)_k$ is a Cauchy sequence in E. As E is sequentially complete $x = \sum_n \lambda_n e_n$ exists in E, and $Tx = \lambda$.

For each $U \in \mathfrak{V}$ there exists $V \in \mathfrak{V}$ such that

$$q_U(Tx) = \sum_n |f_n(x)| p_U(e_n) \leq p_V(x), \quad \text{for all } x \in E.$$

Therefore T is continuous. But T is also open, because of

$$p_U(T^{-1}\lambda) = p_U(\sum \lambda_n e_n) \leq \sum |\lambda_n| p_U(e_n) = q_U(\lambda),$$

for all $\lambda \in \lambda(Q)$.

Moreover if $\{e_n\}$ is $\lambda(P)$-absolute, then for all $(p_U(e_n))_n \in Q$, $U \in \mathfrak{V}$ we can find a $V \in \mathfrak{V}$ and an injection $\sigma : N \longrightarrow N$ with $\sigma(N) = N_U$, such that

$$(\| f_{\sigma(n)} \|_V \, p_U(e_{\sigma(n)}))_n \in \lambda(P).$$

The inequality $\| f_{\sigma(n)} \|_V \geq p_V(e_{\sigma(n)})^{-1}$; for all $n \in N$ yields the $\lambda(P)$-nuclearity of the Köthe space $\lambda(Q)$ (see Theorem 2.2 [5]). This completes the proof.

In the proof of Theorem 2, we even have $T(e_n) = (\delta_{nk})_k$, $n \in N$, which means that our isomorphism preserves bases. Therefore Theorem 2 and the Dynin-Mitiagin theorem give a following generalization of Theorem 1 to $\lambda(P)$-nuclear spaces, because the family of sequentially complete $\lambda(P)$-nuclear spaces with a equicontinuous basis coincides with the family of $\lambda(P)$-nuclear sequence spaces.

Theorem 3. *Let* E *be a Fréchet space with a basis. Then* E *is* $\lambda(P)$-*nuclear if and only if each basis in* E *is* $\lambda(P)$-*absolute.*

2. Bases in $\lambda(P)$-nuclear spaces

Now we shall verify the following main result.

Theorem 4. *All equicontinuous bases* $\{e_n\}$ *in a nuclear space (resp. a* $\lambda(P)$-*nuclear space)* E *are* ℓ_1-*absolute (resp.* $\lambda(P)$-*absolute).*

First of all, we note that in the case when E is nuclear, this theorem follows immediately from the proof of the Dynin-Mitiagin theorem (see [1] [2] and [4]). On the other hand, if E is sequentially complete, Theorem 2 and the Dynin-Mitiagin theorem imply also this theorem.

To prove Theorem 4, we need the following two lemmas. We denote by ℓ_∞ the set of all real-valued bounded sequences, and by c_0 the subset of ℓ_∞ of all sequences (ξ_n) such that ξ_n tends to zero as $n \to \infty$.

Lemma 1. *Let* $D = D[\delta_n] : \ell_\infty \longrightarrow \ell_\infty : (\xi_n) \mapsto (\delta_n \xi_n)$, $(\delta_n) \in c_0$ *be a diagonal operator on* ℓ_∞. *Then there exists an injection* $\sigma : N \longrightarrow N$ *with* $\sigma(N) = \{n \in N \mid \delta_n \neq 0\}$ *such that*

$$\alpha_n(D) = |\delta_{\sigma(n)}|.$$

Where for a $T \in \mathcal{L}(E, F)$ *with Banach spaces* E *and* F, *we define* $\alpha_n(T) = \inf\{\|T - A\| \mid A \in \mathcal{L}(E, F)$ *and* $\dim A(E) \leq n\}$.

The proof of the above lemma will be found in Proposition 8.1.5 of [4]; as it is not difficult, so we omit it.

Lemma 2. *Let E and F be Banach spaces. If $T \in \mathcal{L}(E, F)$ is $\lambda(P)$-nuclear, then*

$$(\alpha_n(T))_n \in \lambda(P).$$

Proof. Since T is $\lambda(P)$-nuclear, it is represented in the form

$$Tx = \sum_n \lambda_n \langle a_n, x\rangle y_n\,, \qquad \text{for all } x \in E,$$

where $(\lambda_n) \in \lambda(P)$, and $\{a_n\}, \{y_n\}$ are bounded sequences in E', F, respectively.

Define $A_i \in \mathcal{L}(E, F)$ by $A_i x := \sum_{n=1}^{i} \lambda_n \langle a_n, x\rangle y_n$, $x \in E$. Then we have clearly $\dim A_i(E) \leq i$ and

$$\| T - A_i \| \leq \sum_{n=i+1}^{\infty} |\lambda_n| \,, \quad \text{therefore} \quad \alpha_i(T) \leq \sum_{n=i+1}^{\infty} |\lambda_n|.$$

For each $b \in P$, we have

$$\sum_i b_i |\alpha_i(T)| \leq \sum_i b_i \Big(\sum_{n=i+1}^{\infty} |\lambda_n| \Big) \leq \sum_n |\lambda_n| \sum_{i=1}^{n} b_i \leq \sum_n |\lambda_n| n b_n.$$

On the other hand, it follows from Proposition 2.1 of [5] and the condition (A2)' in the definition of smooth sequence spaces [5], that there exists a $c \in P$ and a constant $\rho > 0$ such that

$$n\, b_n \leq \rho\, c_n \qquad \text{for all } n \in N.$$

Thus we have

$$\sum_i b_i |\alpha_i(T)| \leq \rho \sum_n |\lambda_n| c_n < \infty, \quad \text{for all } b \in P.$$

Hence $(\alpha_n(T))_n \in \lambda(P)$, and this proves the lemma.

Now we are ready to show our theorem.

Proof of Theorem. From the above observation, it suffices to consider only the case when E is $\lambda(P)$-nuclear and a basis $\{e_n\}$ is ℓ_1-absolute. Let $\mathfrak{V}$ be a 0-basis in E, and let $\{f_n\}$ and $\{S_n\}$ be the biorthogonal sequence and the sequence of projections associated with $\{e_n\}$, respectively.

Assume that $U \in \mathfrak{V}$ is given. Then since $\{S_n\}$ is equicontinuous in $\mathcal{L}(E, E)$, there exists a $U_0 \in \mathfrak{V}$ with $U_0 \subset U$ such that

$$p_U(S_n x) \leq \frac{1}{2} p_{U_0}(x), \quad \text{for all} \quad x \in E, \ n \in N_0,$$

where $N_0 = N \cup \{0\}$. Thus we have

$$|\langle f_n, x\rangle| p_U(e_n) = p_U(S_n x - S_{n-1} x) \leq p_{U_0}(x), \quad \text{for all} \quad x \in E, \ n \in N.$$

Next define a linear mapping

$$A : E \longrightarrow \ell_\infty : x \mapsto (\langle f_n, x\rangle p_U(e_n))_n.$$

Then it satisfies $\| Ax \| \leq p_{U_0}(x)$ for all $x \in E$, and so it factors through Φ_{U_0} into a continuous linear mapping

$$\tilde{A} : \tilde{E}_{(U_0)} \longrightarrow \ell_\infty,$$

with $\| \tilde{A} \| \leq 1$.

Since E is $\lambda(P)$-nuclear, we can find a $V_0 \in \mathfrak{V}$ with $V_0 \subset U_0$ such that $\Phi_{U_0 V_0}$ is $\lambda(P)$-nuclear.

On the other hand, as $\{e_n\}$ is ℓ_1-absolute, there exists a $V \in \mathfrak{V}$ such that

$$\sum_{n \in N_{V_0}} \| f_n \|_V \, p_{V_0}(e_n) \leq 1, \quad \text{and so} \quad \sum_{n \in N_U} \| f_n \|_V \, p_{V_0}(e_n) \leq 1,$$

where we define $N_V = \{n \in N \mid p_V(e_n) \neq 0\}$ for each $V \in \mathfrak{V}$. Moreover we define a continuous mapping R by

$$R : \ell_\infty \longrightarrow E_{V_0} : (\xi_n) \mapsto \sum_{n \in N_U} \| f_n \|_V \, \xi_n e_n$$

with $\| R \| \leq 1$.

If we consider the product $T := \tilde{A} \cdot \Phi_{U_0 V_0} \cdot R : \ell_\infty \longrightarrow \ell_\infty$, then it precisely coincides with a $\lambda(P)$-nuclear diagonal mapping $D[\delta_n]$ with

$$\delta_n = \begin{cases} \| f_n \|_V \, p_U(e_n), & n \in N_U \\ 0 & \text{otherwise.} \end{cases}$$

Thus Lemma 1 and 2 imply that there exists an injection $\sigma : N \longrightarrow N$, $\sigma(N) = N_U$ such that $(\delta_{\sigma(n)})_n \in \lambda(P)$ i.e.

$$(\| f_{\sigma(n)} \|_V \, p_U(e_{\sigma(n)}))_n \in \lambda(P).$$

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