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On Ribbon 2-Knots of 1-Fusion

By Yoshihiko MARUMOTO

Concerning an n-sphere S^n embedded in S^{n+2}, it is conjectured that, if $S^{n+2} - S^n$ has the homotopy type of S^1, S^n is unknotted in S^{n+2}. This conjecture is affirmatively answered for $n = 1$ and $n \geq 3$ ([3], [6], [7]). For $n = 2$, it is unsolved. The following is called 'Unknotting Conjecture' : For a 2-knot K^2, if $\pi_1(R^4 - K^2) \cong Z$, then K^2 is unknotted [10]. In this paper, we have a partial answer to the Unknotting Conjecture for ribbon 2-knots of 1-fusion.

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1. Preliminaries

Throughout this paper, we work in the combinatorial category.

By D^n, S^n and R^n, we denote respectively an n-disk, an n-sphere and an n-space, and by I the unit interval $[0, 1]$. We denote the set of integers by Z.

By R^3_t we mean the hyperplane $\{(x_1,x_2,x_3,x_4) \in R^4 \mid x_4 = t\}$ and R^4_+ is the subspace $\{(x_1,x_2,x_3,x_4) \in R^4 \mid x_4 \geq 0\}$ of R^4.

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An *n-knot* is a locally flat n-sphere embedded in R^{n+2}. Two n-knots K_1^n and K_2^n are *equivalent* if there exists an orientation preserving homeomorphism h of R^{n+2} onto itself such that $h(K_1^n) = K_2^n$. An n-knot is *unknotted,* or *trivial* , if it bounds an $(n+1)$-disk in R^{n+2}. Other terminology in knot theory is referred to, for example, S.Suzuki [10].

Definition 1. We say K^n a *ribbon n-knot of m-fusions* if the following condition C_n is satisfied :

C_n : (1) there exists a trivial link $S_0^n \cup S_1^n \cup \cdots \cup S_m^n$ in R^{n+2} ,

(2) there exists an embedding $f_i : D^n \times I \to R^{n+2}$, for each i, such that

(a)
$$f_i(D^n \times I) \cap S_j^n = \begin{cases} f_i(D^n,\ 0) & \text{if } j = 0, \\ f_i(D^n,\ 1) & \text{if } j = i, \\ \emptyset & \text{otherwise,} \end{cases}$$

(b) $f_i(D^n \times I) \cap f_j(D^n \times I) = \emptyset$ for $i \neq j$, and

(3) $K^n = S_0^n \cup \dots \cup S_m^n \cup \bigcup_{i=1}^m f_i(\partial D^n \times I) - int(\bigcup_{j=1}^m f_j(D^n \times \partial I))$.

We will call $f_i(D^n \times I)$ a *band* of K^n.

The following is easily seen [10, 13],

Lemma 1. *Let* K^1 *be a ribbon* 1*-knot of m-fusions with the condition* C_1. *Then there exists a ribbon* 2*-knot* K^2 *of m-fusions with the condition* C_2 *satisfying :* $S_i^1 = S_i^2 \cap R_0^3$, $f_j(D^1 \times I) = g_j(D^2 \times I) \cap R_0^3$ $(i = 0, \dots, m;$ $j = 1, \dots, m)$ *regarding* R^3 *as* R_0^3, *where* $f_j(D^1 \times I)$ *is a band of* K^1 *and* $g_j(D^2 \times I)$ *a band of* K^2. *Moreover this* K^2 *is unique up to ambiant isotopy.*

Definition 2. In Lemma 1, we call K^2 the ribbon 2-knot *associated with* a ribbon 1-knot K^1.

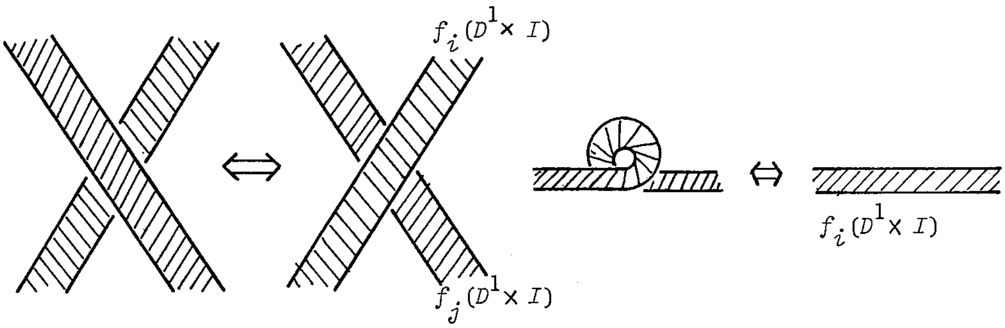
2. Ribbon 2-knots of 1-fusion

Definition 3. Let K^1 be a ribbon 1-knot of m-fusions with the condition C_1 in Definition 1. We define the following :

Operation (A) : To exchange the over and under crossings of bands $f_i(D^1\times I)$ and $f_j(D^1\times I)$. (It may $i = j$.) (Fig.1.A)

Operation (B) : To add the full twist to a band $f_i(D^1\times I)$. (Fig.1.B)

Let $K^{1\prime}$ be a ribbon 1-knot obtained from K^1 by a finite sequence of the operation (A) and (B). Then we say $K^{1\prime}$ is *b-equivalent* to K^1.



(A) (B)

Fig.1

The following is easily proved [10, 13],

Theorem 1. *Let K_i^1 be a ribbon 1-knot of m-fusions (i = 1, 2) and K_i^2 be the ribbon 2-knot associated with K_i^1. If K_1^1 is b-equivalent to K_2^1, then K_1^2 and K_2^2 are equivalent.*

Question 1. Is the inverse of Theorem 1 is true ?

Definition 4. Let K^1 be a ribbon 1-knot. We say K^1 *doubly-null-cobordant in the strong sense* if the ribbon 2-knot, associated with K^1, is unknotted (c.f. [7, 11]).

By Theorem 1, we have the following :

Corollary 1.1. *The Kinoshita-Terasaka knot* [2] *is doubly-null-cobordant in the strong sense.* (Fig.3)

Proof. The Kinoshita-Terasaka knot is

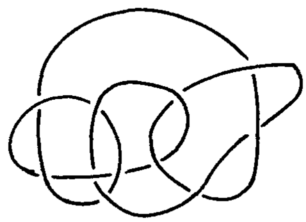
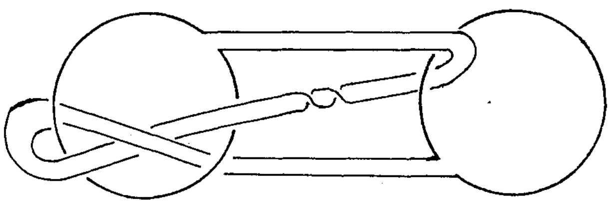


Fig.3

This is, as a 1-knot, equivalent to the following knot :



Clearly this is b-equivalent to a trivial knot, hence doubly-null-cobordant in the strong sense.

Especially, in this section, we consider a ribbon knot of 1-fusion. Let K^1 be a ribbon 1-knot of 1-fusion with the condition C_1 in Definition 1. Then we can find a ribbon 1-knot $K^{1\prime}$, b-equivalent to K^1, in

Fig.4 :

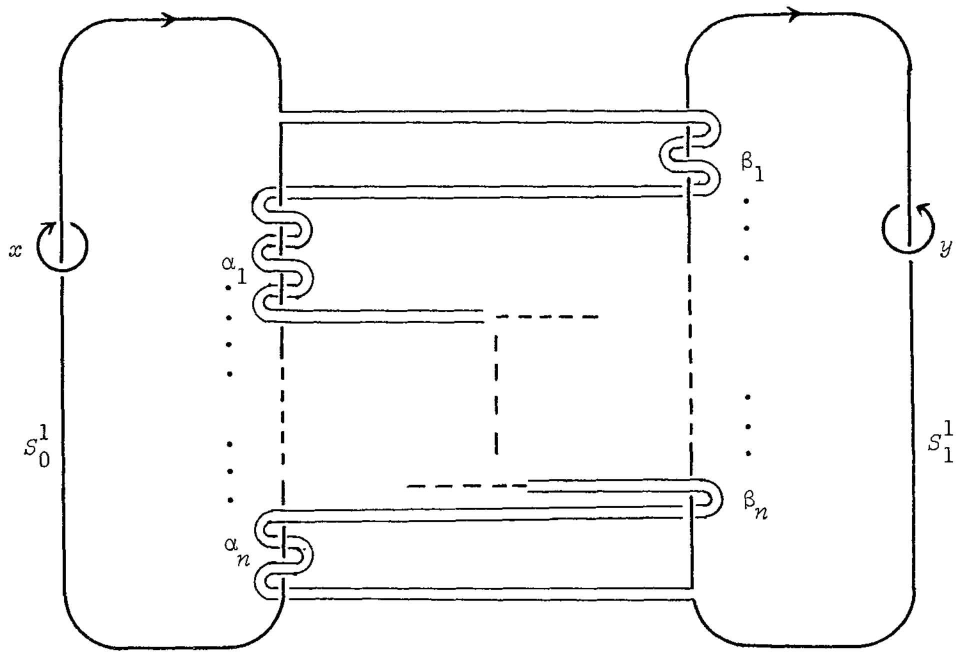


Fig.4

The integers $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n$ are decided as follows : Let D^2_i be a 2-disk bounded by S^1_i $(i = 0,1)$ with $D^2_0 \cap D^2_1 = \emptyset$ and the band of $K^{1\prime}$ is $f_1(D^1 \times I)$. We give an orientation to S^1_i in Fig.4. Let $\ell = f_1(p \times I)$ be a center-line of the band where $p \in \mathit{int}\, D^1$, and we give an orientation to ℓ by passing from $f_1(p, 0)$ to $f_1(p, 1)$. We put $f(t) = f_1(p, t)$ for $0 \leq t \leq 1$. Let $r_0, r_1, \cdots, r_n, s_1, \cdots, s_n$ be real numbers such that

(1) $0 < r_0 < s_1 < r_1 < s_2 < \cdots < s_n < r_n < 1$,

(2) $$f([r_\lambda, s_{\lambda+1}]) \cap D^2_i = \begin{cases} \emptyset & \text{if } i = 0, \\ \text{non-empty} & \text{if } i = 1, \end{cases}$$

$$f([s_\mu,\ r_\mu]) \cap D_i^2 = \begin{cases} \emptyset & \text{if } i = 1, \\ \text{non-empty} & \text{if } i = 0, \end{cases}$$

for $\lambda = 0, 1, \cdots, n-1$ and $\mu = 1, 2, \cdots, n,$ and

(3) $f((0,\ r_0]) \cap D_i^2 = f([r_n,\ 1)) \cap D_i^2 = \emptyset$ for $i = 1, 2.$ (Fig.5)

Then we define α_i to be the intersection number of $f([s_i,\ r_i])$ and D_0^2, β_i of $f([r_{i-1},\ s_i])$ and D_1^2 $(i = 1, \cdots, n)$. (Fig.6)

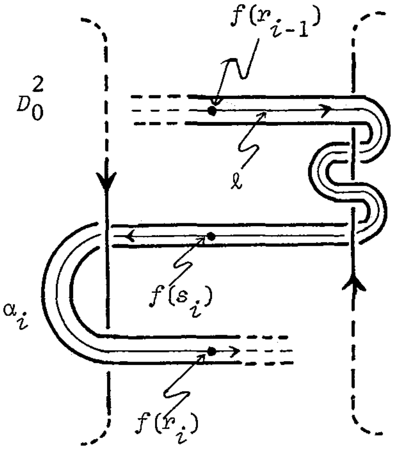


Fig.5

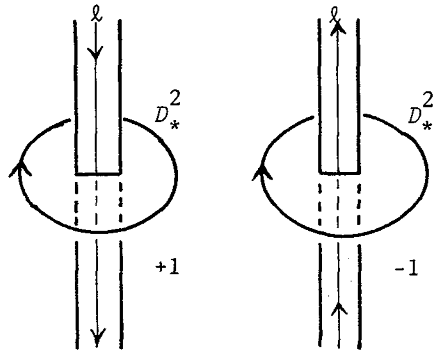


Fig.6

And let K^2 be the ribbon 2-knot associated with K^1, then we have ([12]):

Lemma 3. *The fundamental group of* $R^4 - K^2$ *has the presentation*

$$\langle x,\ y\ ;\ y = W\,x\,W^{-1} \rangle$$

where $W = x^{\alpha_n} y^{\beta_n} x^{\alpha_{n-1}} \cdots x^{\alpha_1} y^{\beta_1}$. *So the 1-st Alexander polynomial of* K^2 *is equal to* $t^{\phi(1)} - t^{\psi(1)} + t^{\phi(2)} - t^{\psi(2)} + \cdots + t^{\phi(n)} - t^{\psi(n)} + 1$

where $\phi(m) = \alpha_n + \beta_n + \alpha_{n-1} + \beta_{n-1} + \cdots + \alpha_m + \beta_m$

and $\psi(m) = \phi(m) - \beta_m$ *for* $1 \leq m \leq n$.

Proof. By Theorem 1, we calculate the fundamental group of $R^4 - K^2$ by using $K^{1}{}'$ in Fig.4. Let x and y be, respectively, meridional loops of S_0^1 and S_1^1 in Fig.4. In Fig.7, a_i, b_i and c_i (i=1,2) are generators of $\pi_1(R_0^3 - K^{1}{}')$. In Fig.7, we add a relation $a_1 = b_1$, then we get relations $a_2 = b_2$, $c_1 = c_2$ and $b_2 = c_1^{\lambda}\cdot b_1\cdot c_1^{-\lambda}$. By easy computing, we have the desired presentation of $\pi_1(R^4 - K^2)$. From the presentation of $\pi_1(R^4 - K^2)$, the 1-st Alexander polynomial is easily calculated.

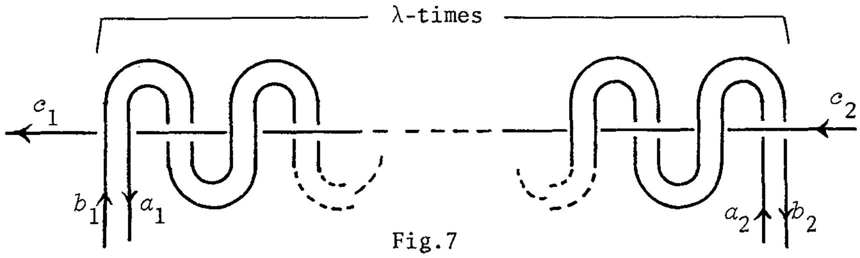


Fig.7

3. Main results

In group theory, we refer to Magnus, Karrass and Solitar [4].

For a word W in $x_1, \cdots, x_n$, $\sigma(W; x_i)$ is the sum of the exponent of the letter x_i in W. Other terminology is referred to [4].

Theorem 2. *Let* $G = \langle a_1, a_2 ; R\rangle$ *such that* $R = a_1^m\cdot W$, $\sigma(W; a_1) = 1$, $\sigma(W; a_2) = 0$ *and* R *is cyclically reduced. Then* G *is isomorphic to* Z *if and only if* $m = \pm 1$ *and* $W = 1$.

Proof. Sufficiency is clear. For necessity, suppose G is isomorphic to Z. Then $G \cong G/G'$ where G' is the commutator subgroup of G, and

$$\begin{aligned} G &= \langle a_1, a_2 ; R, a_1a_2a_1^{-1}a_2^{-1} \rangle \\ &= \langle a_1, a_2 ; a_1 \rangle \end{aligned} .$$

By Conjugacy theorem [4, p.261], R and a_1^{ε} are conjugate in the free group on a_1, a_2 for $\varepsilon = \pm 1$. Therefore we get $m = \pm 1$ and $W = 1$.

Then we have the following corollary :

Corollary 2.1. *Let* $G = \langle x, y\ ;\ R \rangle$ *such that* $R(x,y) = y \cdot W \cdot x^{-1} \cdot W^{-1}$ *is cyclically reduced. Then* G *is isomorphic to* Z *iff* $W = 1$.

Proof. We put $x = a_2$, $y = a_1 a_2$, then by a Tietze transformation, $G = \langle a_1, a_2\ ;\ R' \rangle$ where $R'(a_1, a_2) = R(a_1, a_1 a_2)$. Now we may assume $R' = a_1^m \cdot V$ and is cyclically reduced satisfying the assumption of Theorem 2. Clearly $W = 1$ if and only if $V = 1$. This completes the proof.

In other words, by Lemma 3,

Corollary 2.2. *For a ribbon* 1*-knot* K^2 *of* 1*-fusion,* K^2 *is unknotted if and only if* $\pi_1(R^4 - K^2) \cong Z$.

D.W.Sumners constructed some class of 2-knots [9], and $\pi_1(R^4 - K^2)$,for Sumners' 2-knot K^2, has a presentation satisfying the hypothsis of Theorem 2. Hence we get the same result as in Corollary 2.2 for Sumners' 2-knots. But A.Omae showed that every Sumners' 2-knot is a ribbon 2-knot[5] and we can easily check that it is of 1-fusion.

By Theorem 1, Lemma 3 and Corollary 2.1, we aave :

Theorem 3. *Let* K^1 *be a ribbon* 1*-knot of* 1*-fusion. Then* K^1 *is doubly-null-cobordant in the strong sense if and only if* K^1 *is b-equivalent to a trivial knot.*

This is a partial answer to Question 1.

Remark 2. There exists a knotted 2-sphere in R^4 with trivial 1-st Alexander polynomial. For example, let K^1 be a ribbon 1-knot obtained from Fig.4 with $\alpha_1 = \alpha_2 = \beta_1 = 1$ and $\beta_2 = -2$, and K^2 the ribbon 2-knot associated with K^1. Then, by Lemma 3 and Corollary 2,1, K^2 is unknotted and has the trivial 1-st Alexander polynomial ([10, p.278]).

Remark 3. If a ribbon 1-knot is doubly-null-cobordant in the strong sense, its 1-st Alexander polynomial is trivial. So, 9_{46}-knot is doubly-null-cobordant [8, 11] but not in the strong sense.

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