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Some Fixed Point Theorems in Banach Spaces

By B. E. RHOADES

In a recent paper [1] Ćirić established the following results:

Theorem C2. *Let K be a nonempty, bounded, closed, convex subset of a uniformly convex Banach space B, and $T:K \to K$ satisfying*

$$\|Tx - Ty\| \le \max\{(\|x - Tx\| + \|y - Ty\|)/2,\ (\|x - Ty\| + \|y - Tx\|)/3,\ (\|x - y\| + \|x - Tx\| + \|y - Ty\|)/3\} \tag{1}$$

for each $x, y \in K$, and

$$\sup_{z \in D} \|z - Tz\| \le \delta(D)/2, \tag{2}$$

where D is any convex subset of K such that $T(D) \subset D$, and where $\delta(D)$ denotes the diameter of D. Then the sequence $\{x_n\}$ defined by $x_0 \in K$,

$$x_{n+1} = (x_n + Tx_n)/2, \quad n \ge 0 \tag{3}$$

converges to the fixed point of T in K.

Theorem C3. *Let B be a uniformly convex Banach space, $T:B \to B$ satisfying (1) on B and (2) for any nonempty convex subset D of B such that $T(D) \subset D$. Then T has a fixed point in B if and only if the sequence $\{x_n\}$ defined by (3) converges in B.*

In this paper we shall extend Theorems C2 and C3 to a wider class of transformations as defined in [3]. Let X denote any convex set, $T:X \to X$. The Ishikawa iterative scheme is defined as follows:

(4) $$x_0 \in X$$

(5) $$y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \quad n \geq 0$$

(6) $$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \quad n \geq 0$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy: $0 \leq \alpha_n \leq \beta_n \leq 1$ for all n, $\lim_n \beta_n = 0$, and $\sum_n \alpha_n\beta_n = \infty$. In this paper we shall make the assumptions that: (i) $0 \leq \alpha_n \leq \beta_n \leq 1$ for all n, (ii) $\sum_n \alpha_n(1 - \alpha_n) = \infty$ and (iii) $\overline{\lim}_n \beta_n = \beta < 1$. If $\beta_n = 0$, the Ishikawa iteration process reduces to the Mann iterative process (For a discussion of and some applications of these two iteration schemes the reader may consult [4] and [5]).

Theorem 1. *Let K be any nonempty bounded, closed convex subset of a uniformly convex Banach space B and $T:K \to K$ satisfying* (1) *and* (2). *Then the sequence $\{x_n\}$ defined by* (4)-(6), *with $\{\alpha_n\}$, $\{\beta_n\}$ satisfying* (i)-(iii), *converges to the fixed point of T in K.*

Proof. From Theorem 1 of [1], T has a unique fixed point u in K. From (1), $\|Tx_n - u\| \leq \|x_n - u\|$. From (6) and (5),

$$\|x_{n+1} - u\| \leq (1 - \alpha_n)\|x_n - u\| + \alpha_n\|Ty_n - u\|$$

and

$$\|Ty_n - u\| \leq \|y_n - u\| \leq (1 - \beta_n)\|x_n - u\| + \beta_n\|Tx_n - u\| \leq \|x_n - u\|,$$

so that $\|x_{n+1} - u\| \leq \|x_n - u\|$. Since $\{\|x_n - u\|\}$ is monotone decreasing, it converges to a number a. If $a = 0$ we are finished. Assume $a > 0$, and suppose there exists a natural number M and an $\varepsilon > 0$ such that $\|x_n - Ty_n\| \geq \varepsilon$ for all $n \geq M$. From Lemma of [2] with

$$x = (x_n - u)/\|x_n - u\|, \quad y = (Tx_n - u)/\|x_n - u\|, \quad b = 2\delta(\varepsilon/\|x_M - u\|),$$

we have, for $n \geq M$,

$$\begin{aligned}\|x_{n+1} - u\| &\leq (1 - \alpha_n(1 - \alpha_n)b)\|x_n - u\| \\ &\leq \|x_{n-1} - u\| - \alpha_{n-1}(1 - \alpha_{n-1})b\|x_{n-1} - u\| - \alpha_n(1 - \alpha_n)b\|x_n - u\|\end{aligned}$$

and, by induction, $a \le \|x_{n+1} - u\| \le \|x_M - u\| - b\|x_n - u\| \sum_{k=M}^{n} \alpha_k(1 - \alpha_k)$. Therefore $a(1 + b\sum_{k=M}^{n} \alpha_k(1 - \alpha_k)) \le \|x_M - u\|$, a contradiction, since the series on the left diverges by (ii). Thus there exists a subsequence of (5) and (6) such that

$$\lim_i \|x_{n_i} - Ty_{n_i}\| = 0. \tag{7}$$

We wish to show that

$$\lim_i \|x_{n_i} - Tx_{n_i}\| = 0. \tag{8}$$

It will be sufficient, with (7), to show that $\lim_i \|Tx_{n_i} - Ty_{n_i}\| = 0$. From (1) we have, for any integer n,

$$\|Tx_n - Ty_n\| \le \max\{ \|x_n - Tx_n\| + \|y_n - Ty_n\|)/2,\ (\|x_n - Ty_n\| + \|y_n - Tx_n\|)/3, \tag{9}$$
$$(\|x_n - y_n\| + \|x_n - Tx_n\| + \|y_n - Ty_n\|)/3 \}.$$

$\|x_n - Tx_n\| \le \|x_n - Ty_n\| + \|Ty_n - Tx_n\|$ and, from (5),

$\|y_n - Ty_n\| \le (1 - \beta_n)\|x_n - Ty_n\| + \beta_n\|Tx_n - Ty_n\|$. Hence, if $\|Tx_n - Ty_n\| \le (\|x_n - Tx_n\| + \|y_n - Ty_n\|)/2$, then

$$\|Tx_n - Ty_n\| \le (2 - \beta_n)\|x_n - Ty_n\|/(1 - \beta_n). \tag{10}$$

Since $\|y_n - Tx_n\| = (1 - \beta_n)\|x_n - Tx_n\| \le (1 - \beta_n)(\|x_n - Ty_n\| + \|Ty_n - Tx_n\|)$, it follows that $\|Tx_n - Ty_n\| \le (\|x_n - Ty_n\| + \|y_n - Tx_n\|)/3$ implies

$$\|Tx_n - Ty_n\| \le (2 - \beta_n)\|x_n - Ty_n\|/(2 + \beta_n). \tag{11}$$

Since $\|x_n - y_n\| = \beta_n\|x_n - Tx_n\|$, it follows therefore that $\|Tx_n - Ty_n\| \le (\|x_n - y_n\| + \|x_n - Tx_n\| + \|y_n - Ty_n\|)/3$ implies

$$\|Tx_n - Ty_n\| \le \|x_n - Ty_n\|/(1 - \beta_n). \tag{12}$$

Applying (10)-(12) to (9) we obtain $\|Tx_n - Ty_n\| \le 2\|x_n - Ty_n\|/(1 - \beta_n)$.

Therefore $\|Tx_{n_i} - Ty_{n_i}\| \le 2\|x_{n_i} - Ty_{n_i}\|/(1 - \beta_{n_i})$ and (7) implies $\lim_i \|Tx_{n_i} - Ty_{n_i}\| = 0$, and hence (8). From [1, p. 47], $\{Tx_{n_i}\}_{i=1}^{\infty}$ is Cauchy. Call the limit v. Then, as in [1, p.48], $v = Tv$. Since u is unique fixed point of T in K, $v = u$.

Theorem 2. *Let K be a closed convex subset of a Banach space B. Let $T:K \to K$ satisfying*

(13) $\|Tx - Ty\| \leq \max\{\|x - y\|, (\|x - Tx\| + \|y - Ty\|)/2, (\|x - Ty\| + \|y - Tx\|)/2\}$

for all $x, y \in K$. If the sequence $\{x_n\}$ defined by (4)-(6), with $\{\alpha_n\}$, $\{\beta_n\}$ satisfying (i)-(iii) and (iv): $\lim_n \alpha_n = \alpha > 0$, converges to a point v in K, then v is a fixed point of T in K.

Proof. The condition $\lim_n x_n = v$ implies $\|x_{n+1} - x_n\| \to 0$. From (6), $\|x_{n+1} - x_n\| = \alpha_n\|x_n - Ty_n\|$, so that, from (iv), $\|x_n - Ty_n\| \to 0$. From (13),

(14) $\|Tx_n - Ty_n\| \leq \max\{\|x_n - y_n\|, (\|x_n - Tx_n\| + \|y_n - Ty_n\|)/2, (\|x_n - Ty_n\| + \|y_n - Tx_n\|)/2\}$.

If $\|Tx_n - Ty_n\| \leq \|x_n - y_n\|$ then, from (5), $\|x_n - y_n\| = (1 - \beta_n)\|x_n - Tx_n\|$. Therefore

(15) $\|Tx_n - Ty_n\| \leq \beta_n\|x_n - Ty_n\|/(1 - \beta_n)$.

If $\|Tx_n - Ty_n\| \leq (\|x_n - Tx_n\| + \|y_n - Ty_n\|)/2$, then, from (5), $\|y_n - Ty_n\| \leq (1 - \beta_n)\|x_n - Ty_n\| + \beta_n\|Tx_n - Ty_n\|$, so that

(16) $\|Tx_n - Ty_n\| \leq (2 - \beta_n)\|x_n - Ty_n\|/(1 - \beta_n)$.

If $\|Tx_n - Ty_n\| \leq (\|x_n - Ty_n\| + \|y_n - Tx_n\|)/2$, then, from (5), $\|y_n - Tx_n\| = (1 - \beta_n)\|x_n - Tx_n\| \leq (1 - \beta_n)(\|x_n - Ty_n\| + \|Ty_n - Tx_n\|)$, so that

(17) $\|Tx_n - Ty_n\| \leq (2 - \beta_n)\|x_n - Ty_n\|/(1 + \beta_n)$.

Applying (15)-(17) to (14) we obtain

$$\|Tx_n - Ty_n\| \leq (2 - \beta_n)\|x_n - Ty_n\|/(1 - \beta_n)$$

which, with (iii) and the condition $\|x_n - Ty_n\| \to 0$ yields $\|Tx_n - Ty_n\| \to 0$ and thus $\|x_n - Tx_n\| \to 0$. Thus $\|y_n - x_n\| = \alpha_n\|Tx_n - x_n\| \to 0$,

$$\|y_n - v\| \leq \|y_n - x_n\| + \|x_n - v\| \to 0, \text{ and}$$

$$\|y_n - Ty_n\| \leq (1 - \beta_n)\|x_n - Ty_n\| + \beta_n\|Tx_n - Ty_n\| \to 0.$$

From (6)

$$\|x_{n+1} - Tv\| \leq (1 - \alpha_n)\|x_n - v\| + (1 - \alpha_n)\|v - Tv\| + \alpha_n\|Ty - Tv\|.$$

Using (13), either

$$\|x_{n+1} - Tv\| \leq (1 - \alpha_n)\|x_n - v\| + (1 - \alpha_n)\|v - Tv\| + \alpha_n\|x_n - y_n\|$$

or

$$\|x_{n+1} - Tv\| \leq (1 - \alpha_n)\|x_n - v\| + (1 - \alpha_n/2)\|v - Tv\| + (\alpha_n/2)\|x_n - Tx_n\|$$

or

$$\|x_{n+1} - Tv\| \leq (1 - \alpha_n)\|x_n - v\| + (1 - \alpha_n/2)\|v - Tv\| + (\alpha_n/2)\,(2\|x_n - v\| + \|x_n - Tx_n\|).$$

In either case,

$$\|x_{n+1} - Tv\| \leq \|x_n - v\| + \alpha_n\|x_n - y_n\| + (\alpha_n/2)\|x_n - Tx_n\| + (1 - \alpha_n/2)\|v - Tv\|.$$

Taking limits and using (iv) we obtain $\|v - Tv\| \leq (1 - \alpha/2)\|v - Tv\|$, which implies $v = Tv$.

Theorem 3. *Let B be a uniformly convex Banach space, $T:B \to B$ satisfying* (1) *and* (2) *on B. Then T has a fixed point in B if and only if the sequence $\{x_n\}$ of* (4)-(6), *with $\{\alpha_n\}$, $\{\beta_n\}$ satisfying* (i)-(iv) *converges in B.*

Proof. If T has a fixed point u in B, then $x_n \to u$ by Theorem 1. Note that

$$(\|x - y\| + \|x - Tx\| + \|y - Ty\|)/3 \leq \max\{\|x - y\|,\ (\|x - Tx\| + \|y - Ty\|)/2\},$$

so that, if T satisfies (1) over B it automatically satisfies (13). The converse then follows from Theorem 2.

Example 5.1.5 of [6] shows that a mapping T satisfying (13) need not possess a fixed point.

If T satisfies the hypothesis of Theorem 2, with (iv) omitted, and if T has a fixed point in K, then, using the techniques of Theorem 1 it can

be shown that $\lim_{n} \|x_n - Tx_n\| = 0$.

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